Math 8300 **Homework 1** **PJW**

Date due: Wednesday September 18, 2019.

1. (a) Describe all the isomorphism classes of representations of $\mathbb{C}[X]$ of dimension 1. How many are there?

   (b) Describe also the isomorphism classes of representations of $\mathbb{C}[X]$ of dimension 2. Can they all be generated by a single element? If not, identify the representations that can be generated by a single element. Are any of these representations of dimension 2 simple?

2. (a) Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial. Show that every finitely generated module for the ring $A = \mathbb{Q}[X]/(f^r)$ is a direct sum of modules isomorphic to $V_s := \mathbb{Q}[X]/(f^s)$, where $1 \le s \le r$. Show that A has only one simple module up to isomorphism. When $r = 5$, calculate $\dim \operatorname{Hom}_A(V_2, V_4)$ and $\dim \operatorname{Hom}(V_4, V_2)$.

   (b) Show that $\mathbb{Q}[X]/((X-1)^5) \cong \mathbb{Q}[X]/((X-2)^5)$ as algebras.

3. Let A be a ring and let V be an A-module.

   (a) Show that V is simple if and only if for all nonzero $x \in V$, x generates V.

   (b) Show that V is simple if and only if V is isomorphic to A/I for some maximal left ideal I.

   (c) Show that if A is a finite dimensional algebra over a field then every simple A-module is a composition factor of the free rank 1 module ${}_AA$, and hence that a finite dimensional algebra only has finitely many isomorphism classes of simple modules.

4. Let K be a field, and let $Q_2 = y\bullet \stackrel{\beta}{\leftarrow} \bullet x$ be the quiver in the notes with representations $S_x = 0 \stackrel{0}{\leftarrow} K$, $S_y = K \stackrel{0}{\leftarrow} 0$ and $V = K \stackrel{1}{\leftarrow} K$.

   (a) Compute $\dim \operatorname{Hom}_{K(F(Q_2))}(S_x, V)$, $\dim \operatorname{Hom}_{K(F(Q_2))}(V, S_x)$ and $\dim \operatorname{Hom}_{K(F(Q_2))}(V, V)$.

   (b) Determine whether or not the path algebra $K(F(Q_2))$ is isomorphic to either $K[X]/(X^2)$ or $K[X]/(X^3)$.

5. Show that the path algebras of the two quivers $\bullet \to \bullet \leftarrow \bullet$ and $\bullet \leftarrow \bullet \to \bullet$ over R are isomorphic to the algebras of 3×3 matrices over R of the form

$$\begin{bmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ 0 & 0 & * \end{bmatrix},$$

   determining which path algebra is isomorphic to which algebra of matrices. Show that these two algebras are the opposite of each other.

6. Let K be a field. Show that the space of column vectors K^n is a simple module for $M_n(K)$. Show that, as a left module, $M_n(K)$ is the direct sum of n modules each isomorphic to K^n. Show that, up to isomorphism, $M_n(K)$ has only one simple module.

7. Let $\mathcal{C}_n$ denote the category with n objects, labeled $a_1, \ldots, a_n$, and where there is a unique homomorphism $a_i \to a_j$ for every ordered pair of numbers (i, j). Note that this defines the composition of morphisms in the category. Show that the category algebra $R\mathcal{C}_n$ is isomorphic to the algebra of $n \times n$-matrices $M_n(R)$.

8. Let x be an object of a finite category $\mathcal{C}$.

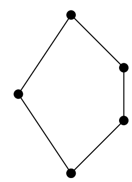
   (a) Show that the subset $R\mathcal{C} \cdot 1_x$ of the category algebra $R\mathcal{C}$ is the span of the morphisms whose domain is x, and that $1_x \cdot R\mathcal{C}$ is the span of the morphisms whose codomain is x.

   (b) Show that $R\mathcal{C} = \bigoplus_{x \in \mathrm{Ob}\mathcal{C}} R\mathcal{C} \cdot 1_x$ as left $R\mathcal{C}$-modules.

   (c) Let $R\,\mathrm{Hom}_{\mathcal{C}}(x, -)$ denote the functor $\mathcal{C} \to R$-mod that sends an object y to the free R-module with the set of homomorphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ as a basis. Under the correspondence between representations of $\mathcal{C}$ over R and $R\mathcal{C}$-modules, show that the functor $R\,\mathrm{Hom}_{\mathcal{C}}(x, -)$ corresponds to the left $R\mathcal{C}$ module $R\mathcal{C} \cdot 1_x$ and that $R\,\mathrm{Hom}(-, x)$ corresponds to the right $R\mathcal{C}$ module $1_x \cdot R\mathcal{C}$.

Extra questions: do NOT hand in

9. What is the dimension of the category algebra $K\mathcal{P}$ when $\mathcal{P}$ is the poset with Hasse diagram



Find the dimensions of the spaces $K\mathcal{P} \cdot 1_x$.

10. We say that a diagram of A-modules $U \xrightarrow{\alpha} V \xrightarrow{\beta} W$ is *exact* at V if $\ker\beta = \operatorname{Im}\alpha$.

   (a) Using the correspondence between representations of a category $\mathcal{C}$ and $R\mathcal{C}$-modules, show that a diagram $L \to M \to N$ of representations of $\mathcal{C}$ is exact at M if and only if for all objects x of $\mathcal{C}$ the sequence of R-modules $L(x) \to M(x) \to N(x)$ is exact at $M(x)$.

   (b) Is it true that a short exact sequence of representations of $\mathcal{C}$ is split if and only if for all objects x of $\mathcal{C}$, the sequence of evaluations at x is split?

11. (a) Show that the simple representations of a quiver Q over a field K are in bijection with the vertices x of Q, and have the form $S_x(x) = K$, $S_x(y) = 0$ if $y \neq x$, and where all arrows in the quiver act as 0. (Pay special attention to the part of the argument that says every simple representation must have this form.)

   (b) For representations of a category $\mathcal{C}$, is it always the case that for each simple representation S of $\mathcal{C}$ there is an object x so that that $S(y) = 0$ for all objects $y \neq x$?

12. Let $U = S_1 \oplus \cdots \oplus S_r$ be an A-module that is the direct sum of finitely many simple modules $S_1, \ldots, S_r$. Show that if T is any simple submodule of U then $T \cong S_i$ for some i.

13. Let V be an A-module for some ring A and suppose that V is a sum $V = V_1 + \cdots + V_n$ of simple submodules. Assume further that the V_i are pairwise non-isomorphic. Show that the V_i are the only simple submodules of V and that $V = V_1 \oplus \cdots \oplus V_n$ is their direct sum.