Math 8300 **Homework 4** **PJW**

Date due: Monday December 2, 2019

1. Let P_S be an indecomposable projective module for a finite dimensional algebra over a field. Show that every non-zero homomorphic image of P_S
   (a) has a unique maximal submodule,
   (b) is indecomposable, and
   (c) has P_S as its projective cover.

2. Let A be a finite dimensional algebra.
   (a) Show that if $f : U \to V$ is a homomorphism of A-modules for which the restriction $f|_{\operatorname{Soc} U} : \operatorname{Soc} U \to V$ is one-to-one then f is one-to-one.
   (b) Show that the injective envelope of $A/\operatorname{Rad} A$ has the same dimension as A. (You may assume question 7 from homework 3.)
   (c) Show that if $A/\operatorname{Rad} A \cong \operatorname{Soc} A$ as left A modules then the left regular representation ${}_AA$ is injective as a left A module; and also that the right regular representation A_A is injective as a right A module. (An algebra satisfying this condition is called *self-injective*)
   (d) Give an example of a self-injective algebra that is not semisimple.

3. Let A be a finite dimensional algebra and let U be an A-module.
   (a) Prove that if U is indecomposable then $\operatorname{Rad}_A(U, U) = \operatorname{Rad}^2_A(U, U)$.
   (b) Find an example of an algebra A and a module U for which $\operatorname{Rad}_A(U, U) \neq \operatorname{Rad}^2_A(U, U)$.

4. Let A be a finite dimensional commutative algebra. Show that A is a finite product of commutative local algebras. (The product of algebras A and B is often written as a direct sum $A \oplus B = \{(a, b) \mid a \in A,\, b \in B\}$.)

5. Let $\alpha : U \to V_1 \oplus V_2$ be a homomorphism of finite dimensional A-modules where A is a finite dimensional algebra over a field, so that we can write $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ where $\alpha_i = p_i \circ \alpha : U \to V_i$ are the component maps of α, the $p_i : V_1 \oplus V_2 \to V_i$ being the projections with respect to the direct sum decomposition. Suppose that U is indecomposable, so that $\operatorname{End}_A(U)$ is a local ring, and that $V_1 \neq 0 \neq V_2$.
   (a) Show that if α_1 is split mono then α is split mono.

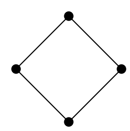
(b) Show that if α is split mono then one of α_1 and α_2 is split mono.

(c) Show that if α is an irreducible morphism then neither of α_1, α_2 is split epi.

(d) Show that if α is an irreducible morphism then each of α_1 and α_2 is an irreducible morphism

6. Let A be a finite dimensional K-algebra such that $\mathrm{Rad}_A^m(-,-) = 0$ for some $m \geq 1$. Prove that any nonzero nonisomorphism between indecomposable modules in A-mod is a sum of compositions of irreducible morphisms.

7. In this question, describe modules by showing their composition factors, in such a way that we can also see the composition factors of their radical and socle series. A diagrammatic notation (as done in class) is sufficient to achieve this. Let P be the poset with four elements $\{1, 2, 3, 4\}$ and partial order $1 < 2 < 4, 1 < 3 < 4$ so that the Hasse diagram is:



We consider representations of this poset over $\mathbb{Q}$, namely, modules for the category algebra $\mathbb{Q}\mathcal{P}$ of P regarded as a category $\mathcal{P}$, over $\mathbb{Q}$, which are the same thing as functors from $\mathcal{P}$ to $\mathbb{Q}$-vector spaces.

(a) Describe (in the sense just explained) the four indecomposable projective representations, and also the four indecomposable injective representations.

(b) For each of the four simple modules S, compute $DTr(S)$.

(c) Write down all almost split sequences that have as a middle term a module that is both projective and injective.

(d) Complete the calculation of the Auslander-Reiten quiver of $\mathbb{Q}P$, giving a justification for each calculation made.

8. Find the error in the following argument (perhaps showing by example what is wrong) and then give an example as requested at the very end:

Theorem 0.1. *Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be an almost split sequence. If $V = V_1 \oplus V_2$ is the direct sum of two non-zero submodules then the restriction of β to each of V_1 and V_2 is one-to-one.*

Proof. Let $V = V_1 \oplus V_2$ and let $p_i : V \to V_i$ be projection and $\iota_i : V_i \to V$ be inclusion with respect to this direct sum decomposition, $i = 1, 2$. Suppose one of the component maps $\beta \circ \iota_i = \beta|_{V_i} : V_i \to W$ is epi. Then it is not split epi, because otherwise β would be split epi. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "\alpha"] & V \arrow[r, "\beta"] & W \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta|_{V_i} \arrow[u, "\iota_i|_{\operatorname{Ker}\beta|_{V_i}}"] \arrow[r] & V_i \arrow[u, "\iota_i"] \arrow[r, "\beta|_{V_i}"] & W \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Because ι_i is split by p_i, the restriction $\iota_i|_{\operatorname{Ker}\beta|_{V_i}}$ is split by $p_i|_U$, so $\iota_i|_{\operatorname{Ker}\beta|_{V_i}}$ is split mono. Now U is indecomposable, so $\operatorname{Ker}\beta|_{V_i}$ is either isomorphic to U or is 0. In the first case ι_i is an isomorphism, so V does not have two summands. In the second case $V_i \cong W$ and β is split epi. Both of these are contradictions, so each restriction $\beta|_{V_i}$ is one-to-one. $\square$

Give an example of an almost split sequence where the middle term has two direct summands and the restriction of β to one of them is epi.