Math 8246 **Homework 2** **PJW**
Date due: Monday February 21, 2011

1. Suppose that we have two commutative diagrams of group homomorphisms

$$\begin{array}{ccccccccc} 1 & \to & L & \xrightarrow{\gamma} & J & \to & G & \to & 1 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\phi_i} & & \downarrow{\scriptstyle 1_G} & & \\ 1 & \to & M & \xrightarrow{\alpha_i} & E_i & \to & G & \to & 1 \end{array}$$

where $i = 1, 2$, the maps labeled without the suffix i are the same in both diagrams, L and M are abelian and the two rows are group extensions (i.e. short exact sequences of groups). Assume that the two module actions of G on M given by conjugation within E_1 and E_2 are the same. Show that the two bottom extensions are equivalent. [Hint: one way to proceed is to show that they are both equivalent to a third extension which you construct.]

2. Show that the two extensions $\mathbb{Z}\xrightarrow{\mu}\mathbb{Z}\xrightarrow{\epsilon}\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}\xrightarrow{\mu'}\mathbb{Z}\xrightarrow{\epsilon'}\mathbb{Z}/3\mathbb{Z}$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod 3$ and $\epsilon'(1) \equiv 2 \pmod 3$.

3. (D&F 17.1, 8) Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}^n_R(N, D) \to \mathrm{Ext}^n_R(M, D) \to \mathrm{Ext}^n_R(L, D) \to 0$ is also short exact and split. [Use a splitting homomorphism and Proposition 5, which says that Ext is functorial in each variable.]

4. (D&F 17.1, 19) Suppose $r \neq 0$ is not a zero divisor in the commutative ring R.
   (a) Prove that multiplication by r gives a free resolution $0 \to R\xrightarrow{r}R \to R/rR \to 0$ of the quotient R/rR.
   (b) DO NOT ATTEMPT THIS PART. WE DID IT IN CLASS MORE-OR-LESS. Prove that $\mathrm{Ext}^0_R(R/rR, B) = {}_rB$ is the set of elements $b \in B$ with $rb = 0$, that $\mathrm{Ext}^1_r(R/rR, B) \cong B/rB$, and that $\mathrm{Ext}^n_r(R/rR, B) = 0$ for $n \geq 2$ for evry R-module b.
   (c) Prove that $\mathrm{Tor}^R_0(A, R/rR) = A/rA$, that $\mathrm{Tor}^R_1(A, R/rR) = {}_rA$ is the set of elements $a \in A$ with $ra = 0$, and that $\mathrm{Tor}^R_n(A, R/rR) = 0$ for $n \geq 2$ for every R-module A.

5. If N is a right $\mathbb{Z}G$-module and M is a left $\mathbb{Z}G$-module we may make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = ng^{-1} \otimes gm$, extended linearly to the whole of $N \otimes_{\mathbb{Z}} M$. Show that $N \otimes_{\mathbb{Z}G} M \cong (N \otimes_{\mathbb{Z}} M)_G$.
   [Not part of the question, just information: if N and M are two left modules we make $N \otimes_{\mathbb{Z}} M$ into a left $\mathbb{Z}G$-module via $g(n \otimes m) = gn \otimes gm$. This is called the *diagonal action* on the tensor product.]

6. (a) Let M and N be $\mathbb{Z}G$-modules and suppose that N has the trivial G-action. Show that $\mathrm{Hom}_{\mathbb{Z}G}(M, N) \cong \mathrm{Hom}_{\mathbb{Z}G}(M/(IG \cdot M), N)$.

(b) Show that for all groups G, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, IG) = 0$; and that if we suppose that G is finite then $\mathrm{Hom}_{\mathbb{Z}G}(IG, \mathbb{Z}) = 0$.
(c) By applying the functor $\mathrm{Hom}_{\mathbb{Z}G}(IG, \quad)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ show that for all finite groups G, if $f : IG \to \mathbb{Z}G$ is any $\mathbb{Z}G$-module homomorphism then $f(IG) \subseteq IG$.
(d) Show that if G is finite and $d : G \to \mathbb{Z}G$ is any derivation then $d(G) \subseteq IG$. Is the same true for arbitrary groups G?

7. Let G be a finite group. Show that the endomorphism ring $\mathrm{Hom}_{\mathbb{Z}G}(IG, IG)$ is isomorphic to $\mathbb{Z}G/(N)$ where $N = \sum_{g\in G} g$ is the norm element which generates $(N) = (\mathbb{Z}G)^G$.
[You may assume that every $\mathbb{Z}G$-module homomorphism $IG \to \mathbb{Z}G$ has image contained in IG. Apply the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, \mathbb{Z}G)$ to the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$. You may assume for a finite group G that $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = 0$.]
8. Show that for every group G:
(a) all derivations $d : G \to M$ satisfy $d(1) = 0$, and
(b) the mapping $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$ is a derivation.

9. (a) Show that the short exact sequence $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$ is split as a sequence of $\mathbb{Z}G$-modules if and only if $G = 1$. Deduce that the identity group is the only group of cohomological dimension 0.
(b) Show that if G is a free group then $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) \neq 0$.

10. Let V and W be vector spaces over $\mathbb{R}$. Given vector space endomorphisms $\alpha : V \to V$ and $\beta : W \to W$ we may make V and W into $\mathbb{R}[X]$-modules by defining $X \cdot v = \alpha(v)$, $X \cdot w = \beta(w)$ for $v \in V$ and $w \in W$. In each of the following cases where we specify α, β by means of matrices with respect to some bases, compute $\dim_{\mathbb{R}} \mathrm{Ext}^1_{\mathbb{R}[X]}(V, W)$.
(i) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \beta = (1)$
(ii) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
(iii) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \beta = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$
(iv) $\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
In case (i) above, exhibit a non-split extension of V by W (i.e. a non-split short exact sequence $0 \to W \to M \to V \to 0$ of $\mathbb{R}[X]$-modules).

11. (i) Suppose that A, B, and C are R-modules and that there are homomorphisms

$$\begin{tikzcd} A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & C \arrow[l, shift left, "\gamma"] \end{tikzcd}$$

such that $\beta\alpha = 0$ and such that the identity map on B can be written $1_B = \alpha\delta + \gamma\beta$. Show that $\beta = \beta\gamma\beta$. Suppose in addition to all this that $\alpha = \alpha\delta\alpha$. Show that $B \cong \alpha\delta(B) \oplus \gamma\beta(B)$.

(ii) A chain complex $\mathcal{C}$ of R-modules is called *contractible* if it is chain homotopy equivalent (by R-module homomorphisms) to the zero chain complex. Prove that $\mathcal{C}$ is contractible if and only if $\mathcal{C}$ can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where α is an isomorphism.

Extra questions: do not hand in!

12. (D&F 17.1, 12) Prove that $\mathrm{Tor}_0^R(D, A) \cong D \otimes_R A$.

Given a homomorphism of chain complexes of R-modules $\phi : \mathcal{C} \to \mathcal{D}$ we may define $E_n = C_{n-1} \oplus D_n$, and a mapping $e_n : E_n \to E_{n-1}$ by $e_n(a, b) = (-\partial a, \phi a + \partial b)$, where we denote the boundary maps on $\mathcal{C}$ and $\mathcal{D}$ by ∂. The specification $\mathcal{E}(\phi) = \{E_n, e_n\}$ is called the *mapping cone* of ϕ.

13. Show that $\mathcal{E} = \{E_n, e_n\}$ is indeed a chain complex.

14. Show that there is a short exact sequence of chain complexes $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{C}[1] \to 0$ where $\mathcal{C}[1]$ denotes the chain complex with the same R-modules and boundary maps as $\mathcal{C}$ but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

$$\cdots \to H_n(\mathcal{C}) \to H_n(\mathcal{D}) \to H_n(\mathcal{E}(\phi)) \to H_{n-1}(\mathcal{C}) \to \cdots$$

Show that $\mathcal{E}(\phi)$ is acyclic if and only if ϕ induces an isomorphism $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ for every n.

15. Show that if $\phi \simeq \psi : \mathcal{C} \to \mathcal{D}$ then $\mathcal{E}(\phi) \cong \mathcal{E}(\psi)$.

16. Let $0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/16\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0$ be a short exact sequence.
   (i) Construct its inverse under the group operation in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$ with sufficient precision that you can determine by examination of the two sequences whether or not they are equivalent.
   (ii) Determine the isomorphism type of middle term of the sum of the sequence with itself. [By 'the sum' is meant the addition in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.]

17. Let $G = \langle g \rangle$ be an infinite cyclic group. Consider an extension of $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_2} \mathbb{Z} \to 0$$

' in which the maps are inclusion into the first summand and projection onto the second summand, and where g acts on $\mathbb{Z}\oplus\mathbb{Z}$ as the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to the basis given by this direct sum decomposition. In the identification $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$, determine the Ext class of this extension, and conclude that the extension is not split. Find a description of an extension represented by $5 \in \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$.

18. Let $0 \to \mathbb{Z} \to E \to \mathbb{Z}/n\mathbb{Z} \to 0$ be an extension of abelian groups represented by $r + n\mathbb{Z} \in \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$ under the identification of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$ with $\mathbb{Z}/n\mathbb{Z}$, where $r \in \mathbb{Z}$. Show that $E \cong \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ where $d = h.c.f.\{r, n\}$ and identify the morphisms $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ in this extension, giving their components with respect to this direct sum decomposition of E.