Math 8246 **Homework 4** **PJW**

Date due: April 4, 2011.

Notation: F is a field, M is an FG-module. Generally we follow the notation of James for things like U° (on p.2).

1. Show that M admits a non-singular G-invariant symmetric bilinear form if and only if $M \cong M^*$ as FG-modules.

2. Let U be an FG-submodule of M. Show that U° is an FG-submodule of M^*. Suppose further that M comes supplied with a non-singular G-invariant symmetric bilinear form. Show that $U^\perp \cong U^\circ$ as FG-modules. Deduce that the isomorphism type of $U^\perp$ is independent of the choice of non-singular G-invariant symmetric bilinear form.

3. Let V be the subspace of the 10-dimensional space $\mathbb{F}^{10}$ over the field $\mathbb{F}$ which has as a basis the vectors

$$\begin{matrix} (0, & 1, & -1, & -1, & 1, & 0, & 0, & 0, & 0, & 0) \\ (1, & 0, & -1, & -1, & 0, & 1, & 0, & 0, & 0, & 0) \\ (0, & 1, & -1, & 0, & 0, & 0, & -1, & 1, & 0, & 0) \\ (1, & 0, & -1, & 0, & 0, & 0, & -1, & 0, & 1, & 0) \\ (1, & 0, & 0, & 0, & -1, & 0, & -1, & 0, & 0, & 1). \end{matrix}$$

   With respect to this basis of V, write down the Gram matrix for the bilinear form on V which is the restriction of the standard bilinear form on $\mathbb{F}^{10}$. Supposing further that $\mathbb{F}$ has characteristic 3, determine the dimension of the space $V/(V \cap V^\perp)$.

4. Let H be a subgroup of a group G, and write

$$H\backslash G = \{Hg \mid g \in G\}$$

   for the set of right cosets of H in G. There is a permutation action of G on this set from the right, namely $(Hg_1)g_2 = Hg_1g_2$. Let $\overline{H} = \sum_{h\in H} h \in FG$ denote the sum of the elements of H, as an element of the group ring of G. Show that the permutation module $F[H\backslash G]$ is isomorphic as an FG-module to the submodule $\overline{H} \cdot FG$ of FG.

5. Find all pairs of partitions of 7 which are not comparable in the dominance ordering, i.e. pairs (λ, μ) for which it is neither true that $\lambda \rhd \mu$ nor $\mu \rhd \lambda$.

6. Show that if $\lambda > \mu$ in the dictionary order then the number of λ-tabloids is greater than the number of μ-tabloids. Deduce that if $\lambda \rhd \mu$ then also the number of λ-tabloids is greater than the number of μ-tabloids. Determine all natural numbers n and partitions λ of n for which the number of λ-tabloids is 12 or fewer (and hence gain an impression of the examples that it is feasible to work by hand).

7. a) Show that the generators $(1,2)$ and $(1,2,3,4)$ of S_4 act on the set of $(2,2)$-tabloids as permutations with cycle types $(2^2,1^2)$ and $(4,2)$ respectively. Write down matrices giving the action of these generators on the permutation module $M^{(2,2)}$.
   b) Find the cycle types of the permutations by which $(1,2)$ and $(1,2,3,4)$ act on the set of $(2,1,1)$-tabloids.