Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 2 Due Thursday 10/21/2021, uploaded to Gradescope.

1. (2.10 of Eisenbud) Let R be a commutative ring. Show that every finitely generated module over $R[U^{-1}]$ is the localization of a finitely generated module over R. (Eisenbud also notes that the same implication without the condition finitely generated looks deeper but is a triviality. Do not address this comment.)

Solution: Let M be a module for $R[U^{-1}]$ generated by elements $x_1, \ldots, x_d$ over $R[U^{-1}]$. Thus every element of M can be expressed as a sum $\frac{a_1}{u_1}x_1 + \cdots + \frac{a_d}{u_d}x_d$ for some elements $a_i \in R$ and $u_i \in U$. Let N be the R-submodue of M generated by $x_1, \ldots, x_d$. The inclusion of R-modules $N \to M$ gives an inclusion of $R[U^{-1}]$-modules $N[U^{-1}] \to M[U^{-1}] \cong M$ (by exactness of localization), and it is surjective because the element $\frac{a_1}{u_1}x_1 + \cdots + \frac{a_d}{u_d}x_d$ can be written with a common denominator $\frac{a_1}{u_1}x_1 + \cdots + \frac{a_d}{u_d}x_d = \frac{1}{v}(b_1x_1 + \cdots + b_dx_d)$ where v is the product $u_1 \cdots u_d$ and $b_i \in R$, so it lies in $N[U^{-1}]$.

2. Let $A = M_{m,m}(R)$ and $B = M_{n,n}(R)$ be matrix rings over a commutative ring R. Show that $A \otimes_R B \cong M_{mn,mn}(R)$, where the multiplication giving the ring structure on the tensor product is determined by $(a \otimes b)(c \otimes d) := ac \otimes bd$ as in class and on page 65 of Eisenbud.

Solution. Let A and B have bases denoted $E_{i,j}$, where this is the matrix with 1 in row i and column j, and 0 elsewhere. Then $A \otimes B$ has a basis of tensors $E_{i,j} \otimes E_{k,p}$ and we define a map of vector spaces $A \otimes_R B \to M_{mn,mn}(R)$ by specifying it on the basis elements as $E_{i,j} \otimes E_{k,p} \mapsto E_{i+km,j+pm}$. To check that it preserves multiplication, we only need to check it on the basis elements, and $(E_{i,j} \otimes E_{k,p})(E_{a,b} \otimes E_{c,d}) = 0$ unles $(j,p) = (a,c)$, when it equals $E_{i,b} \otimes E_{k,d}$. This product is mapped to $E_{i+km,j+pm} \cdot E_{a+cm,b+dm}$ which equals 0 unless $j + pm = a + cm$ or, in other words, $j = a$ and $p = c$. Thus multiplication is preserved and we have an isomorphism of algebras.

3. If R is any integral domain with quotient field Q, prove that

$$(Q/R) \otimes_R (Q/R) = 0.$$

Solution. The typical element of Q/R can be written as a coset $\frac{a}{b} + R$ with $b \neq 0$ and

$$\begin{aligned}
(\frac{a}{b} + R) \otimes_R (\frac{c}{d} + R) &= (\frac{a}{bd}d + R) \otimes_R (\frac{c}{d} + R) \\
&= (\frac{a}{bd} + R) \otimes_R (d\frac{c}{d} + R) \\
&= (\frac{a}{bd} + R) \otimes_R (c + R) \\
&= (\frac{a}{bd} + R) \otimes_R 0 \\
&= 0.
\end{aligned}$$

Thus the tensor product is 0.

4. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

Solution. If it could be so written it would be

$$e_1 \otimes e_2 + e_2 \otimes e_1 = (ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + bde_2 \otimes e_2.$$

The four tensors $e_i \otimes e_j$ on the right are a basis for the tensor product, so we deduce $ac = 0 = bd$ and $ad = 1 = bc$. One of a and c must be 0, so the second equations cannot be satisfied if this is so. The tensor cannot be so written.

5. (a) Let $K \supseteq \mathbb{Q}$ be a field containing $\mathbb{Q}$. Show that $K \otimes_{\mathbb{Q}} \mathbb{Q}[x] \cong K[x]$ as rings
(b) Show that, as a ring, $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2})$ is the direct sum of two fields.
[The ring multiplication is $(a \otimes b)(c \otimes d) := ac \otimes bd$ on basic tensors. Use the isomorphism $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2)$.]

Solution. (a) We define a map $K \otimes_{\mathbb{Q}} \mathbb{Q}[x] \to K[x]$ by the specification $a \otimes_{\mathbb{Q}} f \mapsto af$, which is well defined because it is balanced. We define a map $K[x] \to K \otimes_{\mathbb{Q}} \mathbb{Q}[x]$ on monomials bx^n by $bx^n \mapsto b \otimes x^n$ extended by linearity because these monomials span $K[x]$. These two maps are ring homomorphisms and are mutually inverse.
(b) $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}(\sqrt{2})[x]/(x^2 - 2)$ by an extension of part (a). In $\mathbb{Q}(\sqrt{2})[x]$ the polynomial $x^2 - 2$ factors as $(x - \sqrt{2})(x + \sqrt{2})$, so by the Chinese Remainder Theorem $\mathbb{Q}(\sqrt{2})[x]/(x^2 - 2) \cong \mathbb{Q}(\sqrt{2})[x]/(x - \sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})[x]/(x + \sqrt{2}) \cong \mathbb{Q}(\sqrt{2}) \oplus \mathbb{Q}(\sqrt{2})$.

6. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of R-modules, for some ring R. Suppose that A can be generated as an R-module by a subset $X \subseteq A$ and that C can be generated as an R-module by a subset $Y \subseteq C$. For each $y \in Y$, choose $y' \in B$ with $\beta(y') = y$. Prove that B is generated by the set $\alpha(X) \cup \{y' \mid y \in Y\}$.

Solution. Let B_1 be the submodule of B generated by $\alpha(X) \cup \{y' \mid y \in Y\}$. Now B_1 contains $\alpha(A)$ and by the correspondence theorem corresponds to a submodule of $B/\alpha(A)$, which is isomorphic to C via a map induced by β. Because C is generated by Y that submodule is $B/\alpha(A)$. Because B also corresponds to this submodule, $B = B_1$.

7. Let A, B be left R-modules and let $r \in Z(R) = \{s \in R \mid st = ts \text{ for all } t \in R\}$, the center of R. Let $\mu_r : B \to B$ be multiplication by r. Prove that the induced map $(\mu_r)_* : \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, B)$ is also multiplication by r.

Solution. Let $f : A \to B$ be an R-module homomorphism. The effect of $(\mu_r)_* f$ on an element $a \in A$ is $((\mu_r)_* f)(a) = \mu_r(f(a)) = rf(a)$. This equals the effect of rf on a by the definition of the R-module structure on $\mathrm{Hom}_R(A, B)$.

Extra questions: do not upload to Gradescope.

8. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.

9. (Part of 2.4 of Eisenbud) Let k be a field and let m, n be integers. Describe as explicitly as possible the following. (For example, if the object is a finite-dimensional vector space, what is its dimension?)
a. $\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m))$
b. $k[x]/(x^n), \otimes_{k[x]} k[x]/(x^m)$
c. $k[x] \otimes_k k[x]$ (describe this as an algebra).

10. Eisenbud question 2.11 (It is very similar for modules to something we did for rings.

11. Let $U \subset R$ be a multiplicative subset not containing any zero divisors of a commutative ring R. We can regard R as the set of elements $\frac{r}{1}$ of $R[U^{-1}]$ where r ranges through R. If S is a ring with $R \subseteq S \subseteq R[U^{-1}]$, show that $S[U^{-1}] = R[U^{-1}]$.

12. Suppose that U and V are two multiplicative subsets of the commutative ring R with $U \subseteq V$. Writing V' for the image of V in $R[U^{-1}]$, show that $R[U^{-1}][V'^{-1}] = R[V^{-1}]$.