Math 8211 **Commutative and Homological Algebra I** **Fall 2021**

Homework Assignment 4 Due Saturday 12/18/2021, uploaded to Gradescope.

1. Prove that if $0 \to L \to M \to N \to 0$ is a split short exact sequence of R-modules, then for every $n \geq 0$ the sequence $0 \to \mathrm{Ext}^n_R(D, L) \to \mathrm{Ext}^n_R(D, M) \to \mathrm{Ext}^n_R(D, N) \to 0$ is also short exact and split. [Use a splitting homomorphism and the fact that Ext is functorial in each variable.]

Solution. Labelling the arrows $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ we also have splitting morphisms $L \xleftarrow{f} M \xleftarrow{g} N$ so that $f\alpha = 1_L$ and $\beta g = 1_N$. Functoriality gives us morphisms

$$\mathrm{Ext}^n_R(D, L) \xrightarrow{\alpha_*} \mathrm{Ext}^n_R(D, M) \xrightarrow{\beta_*} \mathrm{Ext}^n_R(D, N)$$

and

$$\mathrm{Ext}^n_R(D, L) \xleftarrow{f_*} \mathrm{Ext}^n_R(D, M) \xleftarrow{g_*} \mathrm{Ext}^n_R(D, N)$$

so that $f_*\alpha_* = (f\alpha)_* = 1_* = 1_L$ and $\beta_* g_* = (\beta g)_* = 1_* = 1$. This means that α_* is split mono and β_* is split epi, so the short sequence in question is exact (it is always exact in the middle) and split.

2. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of right R-modules where both A and C are flat. Prove that B is flat.

Solution. For any left R-module N the long exact sequence

$$\cdots \to \mathrm{Tor}^R_1(A, N) \to \mathrm{Tor}^R_1(B, N) \to \mathrm{Tor}^R_1(C, N) \to A \otimes_R N \to \cdots$$

has $\mathrm{Tor}^R_1(A, N) = \mathrm{Tor}^R_1(C, N) = 0$, which forces $\mathrm{Tor}^R_1(B, N) = 0$, and hence B is flat.

3. (a) Suppose that U, V, and W are R-modules and that there are homomorphisms

$$U \underset{\delta}{\overset{\alpha}{\rightleftarrows}} V \underset{\gamma}{\overset{\beta}{\rightleftarrows}} W$$

such that $\beta\alpha = 0$ and such that the identity map on V can be written $1_V = \alpha\delta + \gamma\beta$. Show that $\beta = \beta\gamma\beta$. Suppose in addition to all this that $\alpha = \alpha\delta\alpha$. Show that $V \cong \alpha\delta(V) \oplus \gamma\beta(V)$.

(b) Recall that a chain complex C of R-modules is called *contractible* if it is chain homotopy equivalent to the zero chain complex. Prove that C is contractible if and only if C can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where α is an isomorphism.

Solution. (a) We calculate $\beta = \beta 1_V = \beta\alpha\delta + \beta\gamma\beta = \beta\gamma\beta$ because $\beta\alpha = 0$.
We see that $(\beta\gamma)^2 = (\beta\gamma\beta)\gamma = \beta\gamma$ and similarly $(\alpha\delta)^2 = \alpha\delta$ so that $\alpha\delta$ and $\beta\gamma$ are idempotent. They are orthogonal because $\beta\gamma = 1 - \alpha\delta$ so $\alpha\delta\beta\gamma = \alpha\delta - (\alpha\delta)^2 = 0$

and similarly $\beta\gamma\alpha\delta = 0$. We have seen before in a different exercise that this implies $V \cong \alpha\delta(V) \oplus \gamma\beta(V)$.
(b) Suppose that C is contractible. This means there are maps $T_n : C_n \to C_{n+1}$ so that for all n we have $1_{C_n} = T_{n-1}d_n + d_{n+1}T_n$. By part (a) this implies that $d_nT_{n-1}d_n = d_n$ for all n and $C_n \cong d_{n+1}T_n(C_n) \oplus T_{n-1}d_n(C_n)$. Now each d_n is zero on $d_{n+1}T_n(C_n)$ so $\operatorname{Ker} d_n \supseteq d_{n+1}T_n(C_n)$ and also $d_{n+1}(C_{n+1}) = d_{n+1}T_nd_{n+1}(C_{n+1}) \subseteq d_{n+1}T_n(C_n)$. Because C is contractible it is acyclic, so $\operatorname{Ker} d_n = d_{n+1}T_n(C_n) = d_{n+1}(C_{n+1})$ and d_n sends the summand $T_{n-1}d_n(C_n)$ isomorphically to the summand $d_nT_{n-1}(C_{n-1})$. From this we see that C is the direct sum of complexes

$$\cdots \to 0 \to T_{n-1}d_n(C_n) \to d_nT_{n-1}(C_{n-1}) \to 0 \to \cdots$$

where the middle morphism is an isomorphism.
Conversely, complexes $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$, where α is an isomorphism, are contractible, using the degree $+1$ map which, on B, is the inverse of the isomorphism. A direct sum of contractible complexes is contractible.

4. Let $R = k[X]/(X^3)$ where k is a field. Let C be the complex $R \xrightarrow{X^2} R$.
(a) Find $\dim_k \operatorname{Hom}(C, C)$, the dimension of the space of chain maps from C to C.
(b) Find the dimension of the subspace of chain maps $C \to C$ which are homotopic to zero. Hence find the dimension of the space $\underline{\operatorname{Hom}}(C, C)$ of homotopy classes of chain maps $C \to C$.

Extra question parts for question 4: do **not** hand in parts (c), (d), (e) or (f).
(c) Show that, for this complex C, the set of chain maps $C \to C$ that are non-isomorphisms forms a vector subspace of the space of all endomorphisms of C. Find the dimension of this subspace.
(d) Show that it is possible to find another complex D for which the set of non-isomorphisms $D \to D$ does not form a vector subspace of all endomorphisms.
(e) Show that, for this complex C, all chain maps $C \to C$ which are equivalences are, in fact, automorphisms
(f) Determine, for this complex C, whether or not all invertible chain maps $C \to C$ are homotopic to each other.

Solution (a) The symbols X that follow should mostly be $\bar{X}$ to indicate that we are really working with their images in R. A chain map $C \to C$ is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{X^2} & R \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ R & \xrightarrow{X^2} & R \end{array}$$

where the vertical maps are multiplication by a and b in R, respectively. Thus $X^2a = bX^2$ in R, and because R is commutative, $(a - b)X^2 \in (X^3)$, so $a = b + c$ for some $c \in (X)$.

Thus a is determined once we have determined b (three dimensions) and c (two dimensions). Thus the space of chain maps has dimension 5.
(b) Such a chain map is homotopic to 0 if and only if it has the form $Td+dT$ for some degree 1 map T, whose only non-zero component will be a map $R \to R$ that is multiplication by some $t \in R$. Thus $a = tX^2$ and $b = X^2 t$, so that $a = b$ and this element is divisible by X^2. A basis for such maps is $(a, b) = (X^2, X^2)$, so the maps homotopic to 0 have dimension 1. It follows that $\underline{\mathrm{Hom}}(C, C)$ has dimension $5 - 1 = 4$.

5. Given a homomorphism of chain complexes of R-modules $\phi : \mathcal{C} \to \mathcal{D}$ we may define $E_n = C_{n-1} \oplus D_n$, and a mapping $e_n : E_n \to E_{n-1}$ by $e_n(a, b) = (-\partial a, \phi a + \partial b)$, where we denote the boundary maps on $\mathcal{C}$ and $\mathcal{D}$ by ∂. The specification $\mathcal{E}(\phi) = \{E_n, e_n\}$ is called the *mapping cone* of ϕ.
(a) Show that $\mathcal{E} = \{E_n, e_n\}$ is indeed a chain complex.
(b) Show that there is a short exact sequence of chain complexes $0 \to \mathcal{D} \to \mathcal{E} \to \mathcal{C}[1] \to 0$ where $\mathcal{C}[1]$ denotes the chain complex with the same R-modules and boundary maps as $\mathcal{C}$ but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

$$\cdots \to H_n(\mathcal{C}) \to H_n(\mathcal{D}) \to H_n(\mathcal{E}(\phi)) \to H_{n-1}(\mathcal{C}) \to \cdots$$

(c) Show that $\mathcal{E}(\phi)$ is acyclic if and only if ϕ induces an isomorphism $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ for every n.
Extra question part: do **not** hand in part (d).
(d) Show that if $\phi \simeq \psi : \mathcal{C} \to \mathcal{D}$ then $\mathcal{E}(\phi) \cong \mathcal{E}(\psi)$.

Solution (a) We check that $e_{n-1} e_n = 0$. Thus

$$e_{n-1}(-\partial a, \phi a + \partial b) = (-\partial(-\partial a), \phi(-\partial a) + \partial(\phi a + \partial b)) = (0, 0)$$

because $\partial\partial = 0$ and $\phi\partial = \partial\phi$..
(b) The mapping $\mathcal{D} \to \mathcal{E}$ specified by $b \mapsto (0, b)$ in each degree is a chain map because $e_n(0, b) = (0, \partial b)$, and it is a monomorphism. The surjective map $\mathcal{E} \to \mathcal{C}[1]$ specified by $(a, b) \mapsto (-1)^n a$ if $a \in C_{n-1}$ is also a chain map, with kernel the previous map $\mathcal{D} \to \mathcal{E}$, and so we have a short exact sequence of chain complexes as claimed. The exact sequence follows, noting that $H_n(\mathcal{C}[1]) = H_{n-1}(\mathcal{C})$ because the term in degree n of $\mathcal{C}[1]$ is C_{n-1}.
(c) This is immediate from the long exact sequence: $\mathcal{E}(\phi)$ is acyclic if and only if all the terms $H_n(\mathcal{E}(\phi))$ in the sequence are 0, which happens if and only if all the maps $H_n(\mathcal{C}) \to H_n(\mathcal{D})$ are isomorphisms.

6. (a) Suppose that we have chain maps $C \xrightarrow{f} D \xrightarrow{g} E$ and suppose that D is a contractible complex. Show that the composite gf is homotopic to zero (i.e. null homotopic).

(b) Consider the diagram

$$\begin{array}{ccccccccc}
C: & \cdots & \xrightarrow{d} & C_2 & \xrightarrow{d} & C_1 & \xrightarrow{d} & C_0 & \xrightarrow{d} & \cdots \\
\downarrow i_C & & & \downarrow \binom{d}{1} & & \downarrow \binom{d}{1} & & \downarrow \binom{d}{1} & & \\
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_1 \oplus C_0 & \xrightarrow{\delta} & \cdots
\end{array}$$

where $\delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Show that I_C is contractible and that i_C is a one-to-one chain map.
(c) Show that if $f = Td + eT : C \to D$ is any null-homotopic map of complexes then f defines a chain map $I_C \to D$ as follows:

$$\begin{array}{ccccccccc}
I_C: & \cdots & \xrightarrow{\delta} & C_1 \oplus C_2 & \xrightarrow{\delta} & C_0 \oplus C_1 & \xrightarrow{\delta} & C_{-1} \oplus C_0 & \xrightarrow{\delta} & \cdots \\
\downarrow & & & \downarrow (T,eT) & & \downarrow (T,eT) & & \downarrow (T,eT) & & \\
D: & \cdots & \xrightarrow{e} & D_2 & \xrightarrow{e} & D_1 & \xrightarrow{e} & D_0 & \xrightarrow{e} & \cdots
\end{array}$$

such that the composite of this morphism with i_C is f. Deduce that any null-homotopic map factors through a contractible complex.

Solution. (a) One approach is to quote that if $u_1 \simeq u_2$ and $v_1 \simeq v_2$ then $u_1 v_1 \simeq u_2 v_2$ (where these maps can be suitably composed). If D is contractible then the identity map $1_D \simeq 0$ so $gf = g1_D f \simeq g0f = 0$. Writing this out more fully, let the differentials on C, D and E be denoted c, d and e. The identity on D can be written $1_D = Td + dT$. Now $gf = g1_D f = gTdf + gdTf = (gTf)c + e(gTf)$ which shows that $gf \simeq 0$ using the degree 1 map gTf.
(b) We verify that the squares commute to see that i_C is a chain map, and it is one-to-one because the second component of i_C is the identity. To show that I_C is contractible let $T : I_C \to I_C$ be the degree 1 map that in degree n maps C_n identically to C_n in degree $n+1$, and is zero on C_{n-1}. Then the identity on I_C has the form $\delta T + T\delta$.
(c) We check that the squares in the complex commute. Going round the left side gives $(eT, e^2T) = (eT, 0)$, and round the right side we get $(T, eT)\delta = (eT, 0)$. The composite of the vertical morphism with i_C is $Td + eT$, which is f. Any null-homotopic map f can be written the way f is, and so factors through I_C.

7. Show that the two extensions $0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/3\mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \xrightarrow{\mu'} \mathbb{Z} \xrightarrow{\epsilon'} \mathbb{Z}/3\mathbb{Z} \to 0$ are not equivalent, where $\mu = \mu'$ is multiplication by 3, $\epsilon(1) \equiv 1 \pmod{3}$ and $\epsilon'(1) \equiv 2 \pmod{3}$.

Solution. If the extensions are equivalent there is an automorphism $f : \mathbb{Z} \to \mathbb{Z}$ so that $f|_{3\mathbb{Z}}$ is the identity on $3\mathbb{Z}$, and $\epsilon = \epsilon' f$. Such f must be multiplication by 1 or by -1, and the restriction to $3\mathbb{Z}$ being the identity forces $f = 1$. This f does not satisfy $\epsilon = \epsilon' f$, so no such f exists.