Math 8212 **Commutative and Homological Algebra I** **Spring 2022**

Homework Assignment 1 Due Saturday 2/12/2022, uploaded to Gradescope.
Each question part is worth 1 point.

1. Let $R \subseteq S \subseteq T$ be commutive rings and let M be an S-module.
(a) (4.1 of Eisenbud) Show that if S is finite over R and M is finitely generated as an S-module, then M is finitely generated as an R-module.
(b) Suppose that S is integral over R and T is integral over S. Show that T is integral over R.

2. (4.2 of Eisenbud with R and S interchanged.) Let k be a field, $R = k[t]$ and suppose $R \subseteq S$ is a containment of rings, where S is supposed to be a domain.
(a) Show that if S is finitely generated as an R-module, then S is free as an R-module.
(b) Show by giving a basis that if $S = k[x,y]/(x^2 - y^3)$ and $t = x^m y^n$, then the rank of S as an R-module is $3m + 2n$.
(c) Assuming again only that the domain S is finitely generated as an R-module, let $\bar{S}$ be the integral closure of S in its field of fractions. Assume Noether's theorem 4.14 that $\bar{S}$ is again finitely generated (and thus free) as an R-module. Show that it has the same rank as S.
[Feel free to make use of the structure theorem for finitely generated modules over a PID.]

3. (4.7 of Eisenbud) Show that the Jacobson radical of R is

$$J = \{r \in R \mid 1 + rs \text{ is a unit for every } s \in R\}.$$

4. (4.11 of Eisenbud minus the graded bit)
(a) Use Nakayama's lemma to show that if R is a commutative local ring and M is a finitely generated projective module, then M is free.
[Identify the radical, consider factoring out its action, produce a map from a free module that is an isomorphism with M.]
(b) Use Proposition 2.10 to show that a finitely presented module M is projective if and only if M is locally free, in the sense that the localization M_P is free over R_P for every maximal ideal P of R (and then of course M_P is free over R_P for every prime ideal P of R).

5. (4.20 of Eisenbud) For each $n \in \mathbb{Z}$, find the integral closure of $\mathbb{Z}[\sqrt{n}]$ as follows:
(a) Reduce to the case where n is square-free.
(b) $\sqrt{n}$ is integral, so what we want is the integral closure R of $\mathbb{Z}$ in the field $\mathbb{Q}[\sqrt{n}]$. If $\alpha = a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$, then the minimal polynomial of α is $x^2 - \text{Trace}(\alpha)x + \text{Norm}(\alpha)$ where $\text{Trace}(\alpha) = 2a$ and $\text{Norm}(\alpha) = a^2 - b^2 n$. Thus $\alpha \in R$ if and only if $\text{Trace}(\alpha)$ and $\text{Norm}(\alpha)$ are integers.

(c) Show that if $\alpha \in R$ then $a \in \frac{1}{2}\mathbb{Z}$. If $a = 0$, show $\alpha \in R$ iff $b \in \mathbb{Z}$. If $a = \frac{1}{2}$ and $\alpha \in R$, show that $b \in \frac{1}{2}\mathbb{Z}$. Thus, subtracting a multiple of $\sqrt{n}$, we may assume $b = 0$ or $\frac{1}{2}$. Observe $b = 0$ is impossible.
(d) Conclude that the integral closure is $\mathbb{Z}[\sqrt{n}]$ if $n \not\equiv 1 \pmod 4$, and is $\mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{n}]$ if $n \equiv 1 \pmod 4$.

6. (1.3 of Matsumura plus) Let A and B be rings, and $f : A \to B$ a surjective homomorphism.
(a) Prove that $f(\operatorname{Jac} A) \subseteq \operatorname{Jac} B$, and construct an example where the inclusion is strict.
(b) Prove that if A is a semilocal ring (a ring with only finitely many maximal ideals) then $f(\operatorname{Jac} A) = \operatorname{Jac} B$.
(c) Continue to assume that A is a semilocal ring. Show that, as an A-module, $A/\operatorname{Jac}(A)$ is a direct sum of finitely many simple A-modules, and that $\operatorname{Jac}(A)$ is the smallest ideal with this property. (That is, if J is an ideal so that A/J is a direct sum of simple A-modules, then $J \supseteq \operatorname{Jac}(A)$.)

Extra question: do not upload to Gradescope.

7. Show that the Jacobson radical of $k[x_1, \ldots, x_n]$ is 0.