Math 8212 Commutative and Homological Algebra 2 Spring 2022

Homework Assignment 3 - Solutions Due **Sunday** 4/17/2022, uploaded to Gradescope.
Each question part is worth 1 point. There are 17 question parts. You are on target for an A if you make a genuine attempt on at least half of them. We define $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be the category whose objects are functors $\mathcal{C}\to\mathcal{D}$ and whose morphisms are natural transformations.

In these questions p is a prime. We will write an element $a_0+a_1p+a_2p^2+\cdots$ of the p-adic integers $\mathbb{Z}_p^\wedge$, where $0\le a_i\le p-1$, as a string $\cdots a_3a_2a_1a_0.$ with a point to the right of a_0.

1. a. Calculate the 3-adic expansion of $\frac{1}{2}$ in $\mathbb{Z}_3^\wedge$.

b. What fraction does the recurring 3-adic integer $\cdots\overline{0121}01211.$ represent?

c. Show that a p-adic integer is a negative (rational) integer if and only if it is of the form $\overline{(p-1)}a_n\cdots a_3a_2a_1a_0.$

d. Show that the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at (p) is the subset of $\mathbb{Z}_p^\wedge$ consisting of strings

$$\overline{a_m\cdots a_n}\cdots a_3a_2a_1a_0.$$

that eventually recur to the left.

Solution: a. The multiplication sum

$$\begin{array}{rrrrr} \cdots & 1 & 1 & 1 & 2.\\ \times & & & & 2.\\ \hline & {}_10 & {}_10 & {}_10 & {}_11.\end{array}$$

shows that $\cdots\bar{1}2.$ multiplied by 2 equals 1, so $\cdots\bar{1}2.=\frac{1}{2}$.
b. Let $x=\cdots\overline{0121}01211.$ The subtraction $\cdots\overline{0121}10000.-\cdots\overline{0121}01211.=1012.$, which is $27+3+2$ in decimal notation, shows that $3^4x-x=32$. Thus $x=80/32=2/5$.
c. The positive integers are precisely the p-adic integers that are eventually 0 to the left. Any subtraction sum of the form

$$\begin{array}{rrrrrrr} & \cdots & {}^10 & {}^10 & {}^10 & {}^10 & {}^10.\\ - & \cdots & 0 & 0 & a & b & c.\\ \hline & \cdots & p-1 & p-1 & d & e & f.\end{array}$$

finishes with recurring $p-1$ in the answer, because each 0 in the top lline has to borrow 1 from the next place, causing 1 to be added in the column to the left in the second row, producing a sum $10-1=p-1$ in p-adic notation. Conversely any subtraction sum

$$\begin{array}{rrrrrrr} & \cdots & {}^10 & {}^10 & {}^10 & {}^10 & {}^10.\\ - & \cdots & p-1 & p-1 & a & b & c.\\ \hline & \cdots & 0 & 0 & d & e & f.\end{array}$$

finishes with 0 to the left because 1 must always be borrowed, increasing $p-1$ to 10, giving an eventual computation $10-10=0$ in each place.

d. The computation of the p-adic expansion of a/b where $p \not| b$ always gives a recurring string, by the pigeon hole principle, because at each stage in the division the p-adic remainder is one of the digits $\{1, \dots, p-1\}$ and the calculation must repeat after some time. Equally, every p-adic integer x with a recurring expansion of length n is a rational integer because $p^n x - x = a$ is an integer, and now $x = \frac{a}{p^n - 1}$. The map $\mathbb{Z}_{(p)} \to \mathbb{Z}_p^\wedge$ specified by $\frac{a}{b} \mapsto$ (p-adic expansion of $\frac{a}{b}$) is an injective ring homomorphism.

2. In this question consider the 10-adic topology on $\mathbb{Z}$, determined by the powers of the ideal (10), with completion the 10-adic integers $\mathbb{Z}_{(10)}^\wedge$, and also the 2-adic topology on $\mathbb{Z}$ with completion $\mathbb{Z}_{(2)}^\wedge$

a. Show that a sequence of integers that is a Cauchy sequence in the 10-adic topology is also a Cauchy sequence in the 2-adic topology.

b. Show that the identity map $1 : \mathbb{Z} \to \mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}_{(10)}^\wedge \to \mathbb{Z}_{(2)}^\wedge$.

c. Determine whether the identity map $1 : \mathbb{Z} \to \mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}_{(2)}^\wedge \to \mathbb{Z}_{(10)}^\wedge$.

d. Using the fact that $\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a product of rings, show that $\mathbb{Z}_{(10)}^\wedge \cong A \times B$ for certain rings A, B that are also ideals of $\mathbb{Z}_{(10)}^\wedge$, with $A/(A \cap (10)) \cong \mathbb{Z}/2\mathbb{Z}$ and $B/(B \cap (10)) \cong \mathbb{Z}/5\mathbb{Z}$.

e. Show that $\mathbb{Z}_{(10)}^\wedge$ has just two maximal ideals, generated by 2 and 5.

f. Show that the composite morphism specified as the inclusion of the ideal $A \hookrightarrow \mathbb{Z}_{(10)}^\wedge$, followed by the ring homomorphism $\mathbb{Z}_{(10)}^\wedge \to \mathbb{Z}_{(2)}$ of part b, is surjective. (Consider using Nakayama's lemma.)

Solution: a. Taking the distance in the m-adic topology to be $d_m(a, b) = \frac{1}{m^t}$ if m^t is the largest power of m that divides $a - b$, if (a_n) is a Cauchy sequence in the 10-adic topology then, given $\epsilon > 0$, we can find u so that $\frac{1}{2^u} < \epsilon$. Now find N so that $i, j \geq N$ implies $d_{10}(a_i, a_j) < \frac{1}{10^u}$, that is, $10^u | (a_i - a_j)$. Now $2^u | (a_i - a_j)$ so $d_2(a_i, a_j) \leq \frac{1}{2^u} < \epsilon$ for all $i, j \geq N$. This shows that (a_n) is a Cauchy sequence in the 2-adic topology.
b. Regarding the completion as the set of equivalence classes of Cauchy sequences, the identity provides a map of sets

$$\{\text{10-adic Cauchy sequences}\} \to \{\text{2-adic Cauchy sequences}\} \to \mathbb{Z}_{(2)}^\wedge$$

by part a. Equivalent 10-adic Cauchy sequences are also 2-adic equivalent by a similar argument, so we get a map of sets $\mathbb{Z}_{(10)}^\wedge \to \mathbb{Z}_{(2)}^\wedge$, and it is a ring homomorphism because the identity map is.
c. The identity on $\mathbb{Z}$ does not extend to a ring homomorphism $f : \mathbb{Z}_{(2)}^\wedge \to \mathbb{Z}_{(10)}^\wedge$. Consider the composite of such an f with the quotient map $\mathbb{Z}_{(10)}^\wedge \to \mathbb{Z}_{(10)}^\wedge / 10\mathbb{Z}_{(10)}^\wedge \cong \mathbb{Z}/10\mathbb{Z}$ (the last isomorphism was done in class). Under this map 1 is sent to 1, which generates $\mathbb{Z}/10\mathbb{Z}$ as a ring, so the composite is surjective. The kernel contains 10, and $\mathbb{Z}_{(2)}^\wedge / 10\mathbb{Z}_{(2)}^\wedge =$

$\mathbb{Z}^\wedge_{(2)}/2\mathbb{Z}^\wedge_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ because 5 is invertible in $\mathbb{Z}^\wedge_{(2)}$. This ring has size 2, so the composite cannot be surjective. Thus no such f can exist.
d. The decomposition of $\mathbb{Z}/10\mathbb{Z}$ (assumed, but FYI it is a consequence of the Chinese Remainder Theorem) gives an expression $1 = e+(1-e)$ as a sum of two non-zero orthogonal idempotents, where e is the identity in $\mathbb{Z}/2\mathbb{Z}$ and $1-e$ is the identity in $\mathbb{Z}/5\mathbb{Z}$. We did in class that $\mathbb{Z}^\wedge_{(10)}/10\mathbb{Z}^\wedge_{(10)} \cong \mathbb{Z}/10\mathbb{Z}$, and we also did in class using Hensel's lemma that there exists an idempotent $f \in \mathbb{Z}^\wedge_{(10)}$ with $f + 10\mathbb{Z}^\wedge_{(10)} = e$, giving a ring decomposition $\mathbb{Z}^\wedge_{(10)} = A\times B$ where $A = \mathbb{Z}^\wedge_{(10)}f$ and $B = \mathbb{Z}^\wedge_{(10)}(1-f)$. The quotient map $A\times B \to \mathbb{Z}/10\mathbb{Z}$ has kernel $A\cap(10)\times B\cap(10)$ with the summands mapping to $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$, respectively, so $A/(A\cap(10)) \cong \mathbb{Z}/2\mathbb{Z}$ and $B/(B\cap(10)) \cong \mathbb{Z}/5\mathbb{Z}$.
e. Every element of $\mathbb{Z}^\wedge_{(10)}$ not in (2) or (5) is invertible, by the same argument that showed that that the completion at a maximal ideal is a local ring: if x is not in either ideal we can find y so that $xy-1 \in (10)$, so $xy = 1+a$ with $a \in (10)$. Now $(xy)^{-1} = 1-a+a^2-a^3+\cdots$ and $x^{-1} = y(xy)^{-1}$. From this it follows that if I is an ideal then $I \subset (2)\cup(5)$. Now if I contains an element a not in (2) and b not in (5) then it contains $a+b$ which lies in neither (2) nor (5), so is invertible, and I is the whole ring. This means that every ideal is contained in either (2) or (5) so these ideals are maximal and are the only such.
f. The composite $\mathbb{Z}^\wedge_{(10)} \to \mathbb{Z}^\wedge_{(2)} \to \mathbb{Z}/2\mathbb{Z}$ is surjective because 1 is sent to 1, and this generates $\mathbb{Z}/2\mathbb{Z}$. It gives rise to a surjective map of groups $A/(A\cap(10))\times B/(B\cap(10)) \to \mathbb{Z}/2\mathbb{Z}$, and the component $B/(B\cap(10)) \cong \mathbb{Z}/5\mathbb{Z}$ it goes to 0. Thus the map $A/(A\cap(10)) \to \mathbb{Z}/2\mathbb{Z}$ is surjective, as is $A \to \mathbb{Z}/2\mathbb{Z}$. Now $\mathbb{Z}^\wedge_{(2)}$ is a local ring, so that its Jacobson radical is $2\mathbb{Z}^\wedge_{(2)}$. Together with this radical, the image of A generates $\mathbb{Z}^\wedge_{(2)}$. By Nakayama's lemma, the image of A equals $\mathbb{Z}^\wedge_{(2)}$. and the map is surjective.

3. Find how many cube roots each of the following numbers has in $\mathbb{Z}^\wedge_{(7)}$: 1, 9, -4, 4, 12, 6. Also find how many cube roots each of the following numbers has in $\mathbb{Z}^\wedge_{(5)}$: 1, 2, 3, 4, 5.

Solution. We find roots of $f(x) = x^3 - t$ where t is prime to 7. Now $f'(x) = 3x$ so if a in $\mathbb{Z}^\wedge_{(7)}$ has $a^3 \equiv t$ (prime to 7) then a is a unit (mod 7), as is $f'(a) = 3a$. For such a, Hensel's lemma applies and there is a cube root b of t with $b \equiv a$ (mod 7). This means the number cube roots of t in $\mathbb{Z}^\wedge_{(7)}$ equals the number of cube roots of t in $\mathbb{Z}/7\mathbb{Z}$. In $\mathbb{Z}/7\mathbb{Z}$ the cubes of $1,2,3,4,5,6$ are $1,1,6,1,6,6$. This means the numbers $1,6$ both have 3 cube roots in v and the other numbers 9, -4, 4, 12 have no cube root in $\mathbb{Z}^\wedge_{(7)}$.
Doing the same thing module 5, the cubes of $1,2,3,4$ are $1,3,2,4$. This means that each $1,2,3,4$ has a unique cube root in $\mathbb{Z}^\wedge_{(5)}$. The question probably should not have asked about cube roots of 5, but if $x \in \mathbb{Z}^\wedge_{(5)}$ lies in $(5)^d$ then x^3 lies in $(5)^{3d}$. From this we see that $x^3 = 5$ has no solutions, because $5 \notin (5)^{3d}$ with $d \geq 1$.

4. Let I be an ideal of R. Consider the polynomial $f(x) = 3x^4 + x^2 + 5$ as a function $R \to R$. Show that f is continuous in the I-adic topology on R. (The I-adic topology on R is given by the distance function determined by the powers of I.)

Solution. We use the distance function $d(u,v) = \frac{1}{2^n}$ if $u - v \in I^n - I^{n+1}$, and write $|u| = d(u,0)$. We show first that the function x^r is continuous. Given $\epsilon > 0$ take $\delta = \epsilon$. Now if $|u| < \delta$ then $u \in I^N$ where $\frac{1}{2^n} < \delta$, and $d(x^r, (x+u)^r) = |(x+u)^r - x^r| = |uv| < \epsilon$ (for some v) because $uv \in I^N$ also (I^N is an ideal). This shows that x^r is continuous.
We next show that if f and g are continuous functions then $f+g$ is continuous. For each x, given $\epsilon > 0$ we can find δ so that $|u| < \delta$ implies both $|f(x+u) - f(x)| < \epsilon$ and $|g(x+u) - g(x)| < \epsilon$. This means that $f(x+u) - f(x) \in I^N$ and $g(x+u) - g(x) \in I^N$ for some N with $\frac{1}{2^N} < \epsilon$ and now $f(x+u) + g(x+u) - (f(x) + g(x)) = (f(x+u) - f(x)) + (g(x+u) - g(x)) \in I^N$ so $|f(x+u) + g(x+u) - (f(x) + g(x))| < \epsilon$. This shows that $f+g$ is continuous. Scalar multiplication is continuous, similarly. Putting this together we see that polynomials are continuous.

5. For a category $\mathcal{C}$ and commutative ring R we may take the R-linear category $R\mathcal{C}$ to have the same objects as $\mathcal{C}$, and with $\mathrm{Hom}_{R\mathcal{C}}(x,y) = R\,\mathrm{Hom}_{\mathcal{C}}(x,y)$, the set of formal linear combinations of morphism $x \to y$ in $\mathcal{C}$. Composition is R-bilinear. The constant functor $\underline{R} : R\mathcal{C} \to R\text{-mod}$ is the functor that assigns R to each object of $\mathcal{C}$, and the identity map 1_R to each morphism of $\mathcal{C}$.

a. Let $\mathcal{C}$ be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the constant functor on $\mathcal{C}$ is representable as a linear functor $R\mathcal{C} \to R$-mod.

b. Let $\mathcal{D}$ be the category $\bullet \rightarrow \bullet \leftarrow \bullet$ with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show that the constant functor is not representable.

c. Show that the inverse limit functor $\varprojlim : \mathrm{Fun}(\mathcal{D}, R\text{-mod}) \to R\text{-mod}$ is representable, represented by the constant functor.

Solution. a. Label the three objects a, b, c from left to right, and the non-identity morphisms $\alpha : b \to a$ and $\beta : b \to c$. We claim that the constant functor is represented by object b. This is because $\mathrm{Hom}_{R\mathcal{C}}(b,x) \cong R$ for each object x, and each morphism of $\mathcal{C}$ is sent by this functor to an isomorphism. Specifically, $\mathrm{Hom}_{R\mathcal{C}}(b,a) = R\alpha$, $\mathrm{Hom}_{R\mathcal{C}}(b,b) = R1_b$ and $\mathrm{Hom}_{R\mathcal{C}}(b,c) = R\beta$. The functorial effect on α is postcomposition with α, namely $\alpha_* : \mathrm{Hom}_{R\mathcal{C}}(b,b) \to \mathrm{Hom}_{R\mathcal{C}}(b,a)$, so $\alpha_*(1_x) = \alpha$, and it is similar with β. This functor is thus naturally isomorphic to the constant functor, by a natural isomorphism that sends each of $\alpha, 1_b, \beta$ to 1 in R.
b. Label the three objects a, b, c from left to right. If the constant functor were representable, it would be representable by one of a, b, c. The representable functor at a is non-zero only on a and b, the representable functor at b is non-zero only on b, and the representable functor at c is non-zero only on b and c. None of these is the constant functor, so it is not representable.

c. We have seen in class exactly that $\varprojlim F \cong \mathrm{Hom}_{\mathrm{Fun}}(\underline{R}, F)$ where $\underline{R}$ is the constant functor on $\mathcal{D}$, Fun is short for Fun($\mathcal{D}$, R-mod) and the Hom denotes natural transformations. Thus $\varprojlim$ and $\mathrm{Hom}_{\mathrm{Fun}}(\underline{R}, -)$ are naturally isomorphic functors, and are representable.

6. Let $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$ be the category of functors from $\mathcal{C}$ to abelian groups, with natural transformations as morphisms. We may take as a definition that a sequence $F_1 \to F_2 \to F_3$ in $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$ is exact if and only if, for all objects X in $\mathcal{C}$, the sequence of abelian groups $F_1(X) \to F_2(X) \to F_3(X)$ is exact. This is equivalent to other possible definitions of exactness. We may regard the inverse limit construction as a functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{abgps}) \to \mathrm{abgps}$.

a. Let $\mathcal{C}$ be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{abgps}) \to \mathrm{abgps}$ is exact.

b. Let $\mathcal{D}$ be the category $\bullet \rightarrow \bullet \leftarrow \bullet$ with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show (by example, or by giving a reason) that the functor $\varprojlim : \mathrm{Fun}(\mathcal{D}, \mathrm{abgps}) \to \mathrm{abgps}$ is not exact in general.

Solution. a. The constant functor $\underline{R}$ is representable and hence projective in $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$, by something we did in class. This means that $\mathrm{Hom}_{\mathrm{Fun}}(\underline{R}, -) \simeq \varprojlim$ is exact.
b. We have seen that $\underline{R}$ is not representable in this case, and in fact it is not projective. We could see this from our knowledge of representations of the quiver $\bullet \rightarrow \bullet \leftarrow \bullet$. A more rudimentary approach is to product a short exact sequence of functors $0 \to F_1 \to F_2 \to F_3 \to 0$ on which $\varprojlim$ is not exact. Let F_1 be the functor described by $0 \to R \leftarrow 0$, meaning that $F_1(a) = 0$, $F_1(b) = R$ and $F_1(c) = 0$. Similarly, let F_2 be $(R \to R \leftarrow 0) \oplus (0 \to R \leftarrow R)$ and let F_3 be $\underline{R} = R \to R \leftarrow R$. All morphisms in describing these functors and the short exact sequence are either 1_R or 0. Now $\varprojlim F_3 = R$ and $\varprojlim F_2 = 0$, so $\varprojlim F_2 \to \varprojlim F_3$ is not surjective. This shows that $\varprojlim$ is not exact.