Math 8212 **Commutative and Homological Algebra 2** **Spring 2022**

Homework Assignment 4 Due **Saturday 5/7/2022**, uploaded to Gradescope.
Each question part is worth 1 point. There are 8 question parts. You are on target for an A if you make a genuine attempt on at least half of them. This homework has fewer parts than previous homeworks. If you can find a way to do the calculation of your overall score so that it comes to be more than 50% (e.g. by weighting each of the four homeworks so that they count equally, or by something else), I will accept that.

1. Although we did not define explicitly the higher differentials in a spectral sequence, it is possible to deduce what they must be in this example. Consider the double complex

$$\begin{array}{ccccc} \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} & \longleftarrow & 0 \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} & & \big\downarrow \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} \end{array}$$

with terms that are 0 except as shown, the nonzero terms being in bidegrees

$$(0,0),(1,0),(-1,1) \text{ and } (0,1).$$

a. Compute the homology of the total complex of this double complex. [You may want to use standard facts having to do with structure of finitely generated modules over a Euclidean domain, Smith Normal Form etc, including the fact that the order of a quotient of a free abelian group by the subgroup spanned by columns of a square matrix is the determinant of that matrix.]

b. Filtering the double complex by rows (so that the modules in each row describe the quotient of two consecutive terms in the filtration), we get a spectral sequence. Find all the numbers n for which $E^n = E^{n+1}$. Determine whether the naturally given grading on the E^∞ term is the same as the naturally given grading on the homology of the total complex from a.

c. Filtering the double complex by columns (so that the modules in each column describe the quotient of two consecutive terms in the filtration), we get a spectral sequence. Find all the numbers n for which $E^n = E^{n+1}$. Determine whether the naturally given grading on the E^∞ term is the same as the naturally given grading on the homology of the total complex from a.

d. Describe how to construct a spectral sequence for which $E^5 \neq E^6 = E^\infty$.

2. (Similar to Exercise 10.12 of Eisenbud) Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k in r variables, where the indeterminate x_i has degree d_i. Let M be a finitely generated graded S-module and put $H_M(n) = \dim_k M_n$ and $h_M(t) = \sum_{n\geq 0} H_M(n)t^n$.

a. Show that $h_M(t)$ is a rational function of t, and that in fact $h_M(t)$ may be written as a polynomial divided by $\prod_{i=1}(1-t^{d_i})$.

b. Show that there is a number d (which may be taken to be the least common multiple of the degrees d_i) such that for each s, $H_M(dn+s)$ agrees with a polynomial in n for all $n >> 0$; that is, $H_M(n)$ is an 'almost PORC function' of n.

3. Let k be a field. The ring $A = k[x,y]/(x^2-y^3)$ that we studied in class has maximal ideal $\mathfrak{m} = (\bar{x}, \bar{y})$, where bars denote the images of elements in A. We saw in class that the ideal $(\bar{y})$ is $\mathfrak{m}$-primary. For any $\mathfrak{m}$-primary ideal J we may form the (Hilbert-)Samuel function $\chi_J(n) = \dim_k(A/J^n)$. Note that if we were to localize at $\mathfrak{m}$ we would have $A_{\mathfrak{m}}/(J_{\mathfrak{m}})^n \cong A/J^n$, so we don't have to localize before computing the dimension.

a. Compute the values of $\chi_{\mathfrak{m}}(n)$. Find the polynomial $f(t)$ such that $\chi_{\mathfrak{m}}(n) = f(n)$ for large n.

b. Compute the values of $\chi_{(\bar{y})}(n)$. Find the polynomial $f(t)$ such that $\chi_{(\bar{y})}(n) = f(n)$ for large n.