Math 8300 **Homework 4** **due Saturday 12/18/2021**
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Questions with more parts count for more.

1. Let $G = C_2 \times C_2$ be the Klein four group with generators a and b, and $k = \mathbb{F}_2$ the field of two elements. Let V be a 3-dimensional space on which a and b act via the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Determine (with an argument) whether or not this representation is indecomposable.
(b) Draw a diagram to represent this module, where the nodes in the diagram biject with the vectors in a basis for this module, and there are arrows between the nodes corresponding to the action of $a - 1$ and $b - 1$.

Solution. (a) We claim that the radical of V has codimension 1. This is because $\operatorname{Rad} V = IG \cdot V$ and IG is spanned by $a-1, b-1, (a-1)(b-1)$, so $\operatorname{Rad} V$ is generated as a kG-module by the images of $a - 1$ and $b - 1$, so is the span of the standard basis vectors e_2 and e_3. Thus V has a unique simple image. If $V = U \oplus W$ then each of U and W would have a simple image and V would have more than one simple image. Thus V is indecomposable.
(b) The diagram is $\circ \xleftarrow{a-1} \circ \xrightarrow{b-1} \circ$ because $a - 1$ sends basis vector 1 to basis vector 2 and is zero on basis vector 3, while $b - 1$ sends basis vector 1 to basis vector 3 and is zero on basis vector 2.

2. (a) Prove that if N is a normal subgroup of G and k is a field then $\operatorname{Rad}(kN) = kN \cap \operatorname{Rad}(kG)$. (Consider using Clifford's theorem and the various things we know about the radical.)
(b) Show by example that if H is a subgroup of G that is not normal then it need not be true that $\operatorname{Rad} kH \subseteq \operatorname{Rad} kG$ (in which case $\operatorname{Rad}(kH) \neq kH \cap \operatorname{Rad}(kG)$).

Solution. (a) Because $\operatorname{Rad}(kG)$ is a 2-sided ideal of kG, and is nilpotent, so $kN \cap \operatorname{Rad}(kG)$ is a 2-sided ideal of kN, and is nilpotent, so $kN \cap \operatorname{Rad}(kG) \subseteq \operatorname{Rad}(kN)$. On the other hand $kG/\operatorname{Rad}(kG)$ is semisimple, so it is also semisimple as a kN-module by Clifford's theorem. This means $\operatorname{Rad}(kN) \cdot kG \subseteq \operatorname{Rad}(kG)$, so $\operatorname{Rad}(kN) \subseteq \operatorname{Rad}(kG)$, from which we see $\operatorname{Rad}(kN) \subseteq kN \cap \operatorname{Rad}(kG)$, and we have equality.
(b) Let $G = S_3$, $H = S_2$ and $k = \mathbb{F}_2$. Then $\operatorname{Rad}(kH)$ is the span of $() + (1, 2)$, and we know $\operatorname{Rad}(kS_3)$ is the span of the sum of the elements of G. Thus $\operatorname{Rad}(kH) \not\subseteq \operatorname{Rad}(kG)$.

3. Show that the following conditions are equivalent for a module U that has a composition series.
(a) U is uniserial (i.e. U has a unique composition series).
(b) The set of all submodules of U is totally ordered by inclusion.
(c) $\operatorname{Rad}^r U/\operatorname{Rad}^{r+1} U$ is simple for all r.

(d) $\operatorname{Soc}^{r+1} U/\operatorname{Soc}^r U$ is simple for all r.

Solution. (a) implies (b). Suppose U has a unique composition series. Any submodule not in this series is also part of a composition series, which must be a second one, so no such submodule exists. This means all submodules of U appear in the composition series and they are totally ordered.
(b) implies (c). Suppose the set of submodules of U is totally ordered. If $\operatorname{Rad}^r U/\operatorname{Rad}^{r+1} U$ is not simple then it decomposes as $(L/\operatorname{Rad}^{r+1} U) \oplus (M/\operatorname{Rad}^{r+1} U)$ for some submodules L, M of U, which are not comparable. This would mean the submodules are not totally ordered, so cannot happen. Thus $\operatorname{Rad}^r U/\operatorname{Rad}^{r+1} U$ is simple.
(b) implies (d) is similar, and also (d) implies (a) is similar to (c) implies (a).
(c) implies (a). Suppose that $\operatorname{Rad}^r U/\operatorname{Rad}^{r+1} U$ is simple for all r and let $0 = U_n \subset U_{n-1} \subset \cdots \subset U_0 = U$ be a composition series of U. This is a series with semisimple factors and the radical series is the fastest descending such series, so $\operatorname{Rad}^r U \subseteq U_r$ for all r. Assuming inductively that $\operatorname{Rad}^r U = U_r$ the fact that $\operatorname{Rad}^{r+1} U \subseteq U_{r+1}$ and both $\operatorname{Rad}^r U/\operatorname{Rad}^{r+1} U$ and U_r/U_{r+1} are simple forces $\operatorname{Rad}^{r+1} U = U_{r+1}$. Thus the composition series is the same as the radical series and it is uniquely specified.

4. Let A be a finite dimensional algebra over a field. Show that A is semisimple if and only if all finite dimensional A-modules are projective.

Solution. If A is semisimple then the regular representation ${}_AA$ is a direct sum of simple modules, which are thus projective, and every simple A-module appears in the direct sum. Thus every finite dimensional module is projective because it is a direct sum of simple modules.
Conversely, if all finite dimensional A-modules are projective then every finite dimensional A-module is semisimple by induction on the composition length of such a module. Modules of length 1 are simple, and if all modules of length $r-1$ are semisimple and U is a module of length r, it has a simple homomorphic image $U \to S \to 0$, which must split by projectivity of S, so $U \cong S \oplus T$ where T has composition length $r-1$ and is thus semisimple. Thus U is semisimple. In particular A is semisimple as a module, so it is semisimple as a ring.

5. (a) Show that $\mathbb{F}_3 S_3$ has two isomorphism classes of simple modules.
(b) Let $e_1 \in \mathbb{F}_3 S_3$ be the idempotent $e_1 = \frac{1}{2}(() + (1,2))$, let $e_{-1} = \frac{1}{2}(() - (1,2))$, and consider the direct sum decomposition of left $\mathbb{F}_3 S_3$-modules $\mathbb{F}_3 S_3 = \mathbb{F}_3 S_3 e_1 \oplus \mathbb{F}_3 S_3 e_{-1}$. Show that, on restriction to the cyclic subgroup $\langle (1,2,3) \rangle$, each of the two modules in this direct sum is a copy of $\mathbb{F}_3\langle (1,2,3)\rangle$. Deduce that each module is indecomposable and uniserial as an $\mathbb{F}_3 S_3$-module.
(c) By considering a basis of each of these two indecomposable modules compatible with the action of $(1,2,3) - ()$ (or otherwise) and the action of (12) on this basis, identify the isomorphism types of the composition factors of these indecomposable modules, showing that the Cartan matrix of $\mathbb{F}_3 S_3$ is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Solution. (a) $O_3(S_3) = \langle(1,2,3)\rangle$ acts trivially on simple $\mathbb{F}_3S_3$-modules, so they are representations of $S_3/O_3(S_3) \cong C_2$, which has two simple modules 1 and ϵ (the sign).
(b) $\mathbb{F}_3S_3e_1$ has basis

$$\{e_1, (1,2,3)e_1, (1,3,2)e_1\}$$

and $\mathbb{F}_3S_3e_{-1}$ has basis

$$\{e_{-1}, (1,2,3)e_{-1}, (1,3,2)e_{-1}\},$$

because these sets of vectors are closed under the action of S_3, lie in the submodules generated by e_1 and e_{-1}, respectively, and are independent. Each set is permuted regularly by $\langle(1,2,3)\rangle$ so on restriction to $\langle(1,2,3)\rangle$ these modules are copies of the regular representation. As representations of $\langle(1,2,3)\rangle$ these modules are indecomposable and uniserial. It follows that they are also indecomposable and uniserial as representations of S_3 (because a decomposition over S_3 gives a decomposition over $\langle(1,2,3)\rangle$, and S_3-submodules are also $\langle(1,2,3)\rangle$-submodules, so are totally ordered).
(c) Consider the vectors obtained by applying powers of $(1,2,3) - ()$ to e_1, noting that $((1,2,3) - ())^2 = () + (1,2,3) + (1,3,2)$:

$$\{e_1, (1,2,3)e_1 - e_1, e_1 + (1,2,3)e_1 + (1,3,2)e_1\}$$

This basis is compatible with the radical series of $\mathbb{F}_3S_3e_1$ as a $\mathbb{F}_3\langle(1,2,3)\rangle$-module: the last two span the radical and the third spans the square of the radical. The third is fixed by $(1,2)$. The second, on multiplication by $(1,2)$ is

$$(1,2)((1,2,3)e_1 - e_1) = (1,3,2)e_1 - e_1 = -[(1,2,3)e_1 - e_1] + [e_1 + (1,2,3)e_1 + (1,3,2)e_1],$$

so $(1,2)$ acts as -1 modulo the radical square. Also $(1,2)$ fixes e_1. This means the radical series of $\mathbb{F}_3S_3e_1$ as a $\mathbb{F}_3\langle(1,2,3)\rangle$-module are also submodules for $\mathbb{F}_3S_3$, with simple factors $1, \epsilon, 1$, so this is also the radical series for $\mathbb{F}_3S_3$, and the projective is uniserial. A similar analysis of $\mathbb{F}_3S_3e_{-1}$ shows that it is uniserial with composition factors $\epsilon, 1, \epsilon$. We deduce the Cartan matrix from this.

6. The setup in this question is that U, V are (finite dimensional) kG-modules where k is a field. We write $U^* = \mathrm{Hom}_k(U, k)$ for the dual kG-module to U. We suppose we are given a non-degenerate bilinear pairing

$$\langle \quad , \quad \rangle : U \times V \to k$$

which has the property $\langle u, v\rangle = \langle gu, gv\rangle$ for all $u \in U$, $v \in V$, $g \in G$. (A *pairing* is like a bilinear form, except the spaces U and V may be different spaces. Non-degenerate means that the matrix of the pairing is non-degenerate, just like with bilinear forms, and there are other ways to express this, such as the left and right kernels are zero.) If U_1 is a subspace of U let $U_1^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U_1\}$ and if V_1 is a subspace of V let $V_1^\perp = \{u \in U \mid \langle u, v\rangle = 0 \text{ for all } v \in V_1\}$.

(a) Show that if U_1 and V_1 are kG-submodules, then so are $U_1^\perp$ and $V_1^\perp$.
(b) Show that the mapping $v \mapsto (u \mapsto \langle u, v\rangle)$ is an isomorphism $V \cong U^*$ as kG-modules.
(c) Show that if U_1 and U_2 are kG-submodules of U then $U_1 \subseteq U_2$ if and only if $U_1^\perp \supseteq U_2^\perp$. Show further in this case that

$$U_1^\perp / U_2^\perp \cong (U_2/U_1)^*$$

as kG-modules. Notice (but do not write anything about it) that the lattice of submodules of U is the opposite of the lattice of submodules of U^*.
(d) Let G permute a set Ω and let $U = V = k\Omega$ be the permutation module. Define $\langle \quad , \quad \rangle$ on basis elements $u, v \in \Omega$ by $\langle u, v\rangle = \delta_{u,v}$ (the Kronecker delta). Show that this pairing satisfies the condition $\langle u, v\rangle = \langle gu, gv\rangle$ always. Deduce that $k\Omega \cong (k\Omega)^*$. Deduce that if all indecomposable summands of $k\Omega$ have simple radical quotients, then they also all have simple socles.

Solution. (a) Let U_1 be a kG-submodule, $v \in U_1^\perp$ and $g \in G$. We verify that $gv \in U_1^\perp$ by computing $\langle u, gv\rangle = \langle g^{-1}u, v\rangle = 0$ for all $u \in U_!$. Thus $U_1^\perp$ is invariant under G and is a kG-submodule. The argument for $V_1^\perp$ is similar.
(b) The linear mapping shown is one-to-one because the right kernel of the form is zero, and it follows that it is invertible because U and V have finite dimension, so $\dim U = \dim V = \dim U^*$. The main thing now is to show that the linear map is a morphism of kG-modules. If $g \in G$ then gv is sent to the mapping $(u \mapsto \langle u, gv\rangle)$ but this equals $(u \mapsto \langle g^{-1}u, v\rangle) = g(u \mapsto \langle u, v\rangle)$ showing that we have a morphism of kG-modules.
(c) If $U_1 \subseteq U_2$ and $v \in U_2^\perp$ then $\langle u, v\rangle = 0$ for all $u \in U_2$, so this also holds for all $u \in U_1$ and thus $v \in U_1^\perp$. This shows that $U_1^\perp \supseteq U_2^\perp$ and the converse implication follows similarly, using the fact that $U_i^{\perp\perp} = U_i$.
Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & U_2^\perp & \to & V & \to & U_2^* & \to & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & \| & & \big\downarrow{\scriptstyle\beta} & & \\
0 & \to & U_1^\perp & \to & V & \to & U_1^* & \to & 0
\end{array}$$

in which the rows are exact. Here the isomorphism $V \to U^*$ composed with the surjective 'restriction of homomorphisms' map $U^* \to U_i^*$ gives a surjection $V \to U_i^*$, whose kernel is $U_i^\perp$. The map β is one of the maps obtained by applying $(-)^*$ to the short exact sequence $0 \to U_1 \to U_2 \to U_2/U_1 \to 0$, and because this duality is exact we get that $\operatorname{Ker} \beta \cong (U_2/U_1)^*$. The Snake Lemma now shows that $U_1^\perp / U_2^\perp \cong (U_2/U_1)^*$.