

THE SCHUR CONE AND THE CONE OF LOG CONCAVITY

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ABSTRACT. Let $\{h_1, h_2, \dots\}$ be a set of algebraically independent variables. We ask which vectors are extreme in the cone generated by $h_i h_j - h_{i+1} h_{j-1}$ ($i \geq j > 0$) and h_i ($i > 0$). We call this cone the *cone of log concavity*. More generally, we ask which vectors are extreme in the cone generated by Schur functions of partitions with k or fewer parts. We give a conjecture characterizing which vectors are extreme in the cone of log concavity. We prove this characterization in one direction and give partial results in the other direction.

1. INTRODUCTION, PARTITIONS AND SYMMETRIC FUNCTIONS

Let $\{h_1, h_2, \dots\}$ be a set of algebraically independent variables. We ask which polynomials in these variables can be written as positive sums of products of polynomials of the form $h_i h_j - h_{i+1} h_{j-1}$ ($i \geq j > 0$) and h_i ($i > 0$). Such sums of products form a cone inside the algebra generated by these variables. We call this cone the *cone of log concavity*. It is natural to ask which of the generating vectors of this cone are extreme and which are not. That is, which can be written as positive linear combinations of the others and which are required to define the cone.

We can view the h_i as the homogenous symmetric functions in a set of indeterminates $\{x_1, x_2, \dots\}$. In this setting, the monomials of products of the h_i are a basis for the vector space of symmetric functions, and the monomials of homogenous degree N are a basis for the vector space of symmetric functions of that degree. Since the variables $\{h_1, h_2, \dots\}$ are algebraically independent, we will never have to refer to the underlying indeterminates $\{x_1, x_2, \dots\}$.

Having placed our problem in the context of symmetric functions, we need some preliminary definitions and results concerning partitions and symmetric functions. This material may be found in many other sources, most notably in [3] and in [6].

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ with integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$ and $N = \lambda_1 + \lambda_2 + \dots + \lambda_m$, then λ is called a *partition of N* , and we write $\lambda \vdash N$ and $|\lambda| = N$. The integers λ_i are called the *parts* of the partition and $m = l(\lambda)$ is the *number of parts*. Another common notation for partitions uses an exponential form. If the part k appears t_k times in the partition, we write k^{t_k} . Thus the partition of 18, $(4, 4, 2, 2, 2, 1, 1, 1, 1)$, can be written $4^2 2^3 1^4$.

Let \mathcal{P}_N be the set of partitions of N , \mathcal{P}_N^k be the set of partitions of N with k or fewer parts and \mathcal{P}^k be the set of partitions with k or fewer parts. Let $p(N)$ be the size of \mathcal{P}_N .

A partition λ is sometimes called a *shape*, especially when it is described by a *Ferrers diagram*, an array of left-justified cells with λ_1 cells in the first row, λ_2 cells in the second row, etc.

If $\lambda \vdash N$ and $\mu \vdash N$, we say λ *dominates* μ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i and we write $\lambda \supseteq \mu$ (and $\lambda \triangleright \mu$ if $\lambda \supseteq \mu$ and $\lambda \neq \mu$). In these partial sums, if one partition has more parts than the other, we pad with parts of size 0 as necessary. Dominance determines a partial order on \mathcal{P}_N .

If positive integers are placed in the cells of the shape λ , the resulting figure is called a *tableau*. The *content* of a tableau is a vector $\rho = (\rho_1, \rho_2, \dots)$ where ρ_i is the number of i 's in the tableau. Vectors such as ρ are called *compositions*.

If the entries of the tableau weakly increase across rows and strictly increase down columns, the tableau is called a *semistandard Young tableau*, or *SSYT*. The number of SSYT of shape λ and content ρ is $K_{\lambda, \rho}$, called the *Kostka number*. A well-known property of SSYT is that $K_{\lambda, \rho}$ does not depend upon the order of the entries in the vector ρ , so ρ is usually assumed to be a partition. In the next section, we shall describe a bijection on SSYT which proves this property.

We often encounter shapes more general than partitions. Given partitions λ and μ , we say $\mu \leq \lambda$ if each $\mu_i \leq \lambda_i$. Write λ/μ to denote the diagram obtained by removing the cells of the Ferrers diagram of μ from the cells of the Ferrers diagram of λ . This diagram is called a *skew shape*, and the idea of a SSYT extends naturally to skew shapes.

If T is a (possibly skew) SSYT, then $w(T)$, called the *word of T* , is the word obtained by reading the entries in T from right to left across the first (top) row, then right to left across the second row, etc. If α is a subset of the letters appearing in T , then $w_\alpha(T)$ is the subword of $w(T)$ which uses just letters in α .

A word, using the letters $t_1 < t_2 < \cdots < t_p$, is a *lattice word* if, at any point in the word (reading left to right), the number of t_i 's which have appeared is \geq the number of t_{i+1} 's which have appeared.

As mentioned earlier, the $\{h_1, h_2, \dots\}$ described above are usually defined to be the homogeneous symmetric functions in some set of indeterminates x_1, x_2, \dots . In this paper we will never need to refer to this underlying variable set. The fact that the h 's are algebraically independent gives us the freedom to move around among symmetric function bases without regard to the underlying set of indeterminates.

We write $h_\rho = h_{\rho_1} h_{\rho_2} \cdots$, where $\rho \vdash N$. The h_ρ , $\rho \vdash N$, form a basis of a vector space Λ^N of dimension $p(N)$.

We will use another basis, the Schur functions s_λ , extensively. We connect this basis with the h_ρ in two ways. The first is the equation

$$h_\rho = \sum_{\lambda \vdash N} K_{\lambda, \rho} s_\lambda,$$

where $\rho \vdash N$. The second is the Jacobi-Trudi identity,

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n}$$

where $n \geq l(\lambda)$.

Symmetric functions which can be written in the Schur function basis with integer coefficients are called *Schur-integral*; symmetric functions which can be written in the Schur function basis with non-negative coefficients are called *Schur-positive*.

2. THE LITTLEWOOD-RICHARDSON RULE

If two Schur functions are multiplied together, the resulting symmetric function can be expanded as a linear combination of Schur functions. More generally,

suppose $(\rho^1, \rho^2, \dots, \rho^k)$ is a vector of partitions of n_1, n_2, \dots, n_k respectively and $N = n_1 + n_2 + \dots + n_k$. Write

$$\prod_{i=1}^k s_{\rho^i} = \sum_{\lambda \vdash N} c_{\rho^1, \dots, \rho^k}^\lambda s_\lambda$$

The coefficients $c_{\rho^1, \dots, \rho^k}^\lambda$ are the well-known *Littlewood-Richardson coefficients*, whose computation is described below. They are non-negative integers, making the product of Schur functions both Schur-positive and Schur-integral.

The Littlewood-Richard coefficients are computed as follows. Let $\rho = \rho^1 \vee \dots \vee \rho^k$ be the composition formed by concatenating the parts of the partitions $\rho^1, \rho^2, \dots, \rho^k$. For example, if $\rho^1 = (2, 2, 1)$, $\rho^2 = (4, 2)$ and $\rho^3 = (3, 1)$, then $\rho = (2, 2, 1, 4, 2, 3, 1)$. We next form a SSYT, T , of shape $\lambda \vdash N$ and content ρ . We let α^i denote the subset of letters in T corresponding to the partition ρ^i . In our example, $\alpha^1 = \{1, 2, 3\}$, $\alpha^2 = \{4, 5\}$ and $\alpha^3 = \{6, 7\}$.

Finally, we say T is *LW* if, for each i , $w_{\alpha^i}(T)$ is a lattice word. For example, taking ρ as above and $\lambda = (5, 4, 4, 2)$, this tableau T is LW:

$$T = \begin{array}{cccccc} & 1 & 1 & 4 & 4 & 6 \\ & 2 & 2 & 5 & 6 & \\ & 3 & 4 & 6 & 7 & \\ & 4 & 5 & & & \end{array}$$

This is because the three words (11223) , (445454) and (6676) are each a lattice word.

Theorem 1. *The coefficient $c_{\rho^1, \dots, \rho^k}^\lambda$ is the number of SSYT of shape λ , content ρ which are LW.*

The Littlewood-Richardson rule has many proofs [6]. One ([2]) uses a well-known switching rule which can also be used to prove the Kostka numbers are independent of the order of the content. This rule, which is a rephrasing of the *jeu de taquin* of Schützenberger [6], swaps a letter from one alphabet through a tableau in a different alphabet. We make this more precise.

Suppose inside the (possibly skew) SSYT, T , the two letters $*$ and 0 appear, with $0 < *$ and no letter x such that $0 < x < *$. (We say such 0 and $*$ are *contiguous* or *appear contiguously*.) We swap the order of 0 and $*$ as follows. Whenever 0 and $*$ appear in a column, we call them *paired* and we swap the paired 0 and $*$. And in any row, we swap the unpaired 0 's with the unpaired $*$'s. The resulting tableau, $r(T)$, will have the 0 's and $*$'s occupying the same set of cells, with multiplicities unchanged, but with $* < 0$.

For example, if

$$T = \begin{array}{cccccc} & & & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & * & * \\ 0 & * & * & * & * & & & & \\ * & & & & & & & & \end{array}$$

then

$$r(T) = \begin{array}{cccccc} & & & & & * & 0 & 0 \\ & & & * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & & & & \\ 0 & & & & & & & & \end{array}$$

By iterating this process, two alphabets can be made to swap positions. That is, if T is a skew SSYT of shape λ/μ which uses two alphabets, α and β , with the α alphabet less than the β alphabet (written $\alpha < \beta$), then repeatedly passing letters from one alphabet through the other gives a second skew SSYT, S , of shape λ/μ , using the same alphabets, but with $\beta < \alpha$.

Furthermore, certain properties of these alphabets are maintained after this swapping. Write $S = r_{\beta < \alpha}(T)$ and $T = r_{\alpha < \beta}(S)$ to represent this swapping, and let T_α denote the skew subtableau of T which uses only the alphabet α . We have the following theorem, which appears in [1]:

Theorem 2. *If $S = r_{\beta < \alpha}(T)$, then T_α is LR if and only if S_α is LR and T_β is LR if and only if S_β is LR.*

3. THE CONE OF LOG-CONCAVITY

If A is a multiset from \mathcal{P}^k , define

$$wt(A) = \sum_{\lambda \in A} |\lambda|$$

and

$$s_A = \prod_{\lambda \in A} s_\lambda.$$

The homogeneous degree of s_A (as a polynomial in the h 's) is $wt(A)$. Define

$$\mathcal{SP}_N^k = \{A \mid wt(A) = N\}.$$

The (N, k) -Schur cone is

$$\mathcal{C}_N^k = \left\{ \sum_{A \in \mathcal{SP}_N^k} c_A s_A \mid c_A \geq 0 \right\}$$

A function s_A , $A \in \mathcal{SP}_N^k$ is *extreme* in \mathcal{C}_N^k if it cannot be written as a positive linear combination of other s_B , $B \in \mathcal{SP}_N^k$. We ask, for a given k , which elements $A \in \mathcal{SP}_N^k$ yield s_A which are extreme in this cone.

We distinguish two obvious special cases. When $k = 1$, $s_A = h_\lambda$ where λ is the partition whose parts are the 1-row partitions of A . Since the h_λ form a basis of Λ^N and are the only vectors defining \mathcal{C}_N^1 , they are the extreme vectors.

When $k \geq N$, then since the product of Schur functions is Schur-positive, the Schur functions s_λ are the extreme vectors.

It follows from the Jacobi-Trudi identity that the cone \mathcal{C}_N^2 consists of positive linear combinations of products of factors of the form

$$h_i h_j - h_{i+1} h_{j-1} \quad \text{and} \quad h_i \quad i \geq j \geq 1.$$

Thus, we call \mathcal{C}_N^2 the *cone of log concavity*.

There are many elements $A \in \mathcal{SP}_N^2$ which are not extreme in \mathcal{C}_N^2 . For example,

$$s_{(3,1)} s_{(2)} = s_{(3,2)} s_{(1)} + s_{(1,1)} s_{(4)}.$$

In fact, the extreme set of \mathcal{C}_6^2 is just these 13 elements:

$$\begin{array}{lll} s_{(6)} & s_{(4)}s_{(1,1)} & s_{(3)}s_{(2,1)} \\ s_{(5,1)} & s_{(3,1)}s_{(1,1)} & s_{(2,1)}^2 \\ s_{(4,2)} & s_{(2,2)}s_{(2)} & s_{(2)}s_{(1,1)}^2 \\ s_{(3,3)} & s_{(2,2)}s_{(1,1)} & s_{(1,1)}^3 \\ s_{(3,2)}s_{(1)} & & \end{array}$$

In this paper we conjecture a simple characterization of the extreme elements of \mathcal{SP}_N^2 . We give a proof of this conjecture in one direction and we prove an important special case in the other direction.

4. THE EXTREME SET

The conjectured characterization of the extreme elements of \mathcal{C}_N^2 is the following.

Conjecture 3. *The collection of pairs $A \in \mathcal{SP}_N^2$ is in the extreme set of \mathcal{C}_N^2 if and only if no pair of partitions $\{\lambda, \mu\}$ in A satisfies any one of the following conditions:*

i. $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, with

$$\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2;$$

ii. $\lambda = (\lambda_1 > \lambda_2 > 0)$, $\mu = (\mu_1 > 0)$, with

$$\lambda_1 \geq \mu_1 \geq \lambda_2;$$

iii. $\lambda = (\lambda_1 > 0)$, $\mu = (\mu_1 > 0)$.

If no pair of partitions in A satisfies any of these conditions, we say A is *nested*. The proof of one direction is easy.

Theorem 4. *If A is not nested then A is not in the extreme set of \mathcal{C}_N^2 .*

Proof. Suppose a pair $\{\lambda, \mu\}$ satisfies the first condition. This implies $\lambda_1 \geq \mu_1 + 1$ and $\lambda_2 - 1 \geq \mu_2$. Therefore, by Jacobi-Trudi,

$$(1) \quad s_\lambda s_\mu = s_{(\lambda_1, \mu_2)} s_{(\mu_1, \lambda_2)} + s_{(\lambda_1, \mu_1 + 1)} s_{(\lambda_2 - 1, \mu_2)}.$$

Suppose a pair $\{\lambda, \mu\}$ satisfies the second condition. If $\lambda_1 > \mu_1$ then by Jacobi-Trudi,

$$(2) \quad s_\lambda s_\mu = s_{(\lambda_1)} s_{(\mu_1, \lambda_2)} + s_{(\lambda_2 - 1)} s_{(\lambda_1, \mu_1 + 1)}.$$

If $\mu_1 > \lambda_2$, by Jacobi-Trudi

$$(3) \quad s_\lambda s_\mu = s_{(\lambda_2)} s_{(\lambda_1, \mu_1)} + s_{(\lambda_1 + 1)} s_{(\mu_1 - 1, \lambda_2)}.$$

Finally, suppose a pair $\{\lambda, \mu\}$ satisfies the third condition. Then

$$(4) \quad s_\lambda s_\mu = s_{(\lambda_1, \mu_1)} + s_{(\lambda_1 + 1)} s_{(\mu_1 - 1)}.$$

□

Let \mathcal{SSP}_N denote the nested sets $A \in \mathcal{SP}_N^2$. Thus, the extreme set of \mathcal{C}_N^2 is contained in \mathcal{SSP}_N .

For $A \in \mathcal{SP}_N^2$, let $\phi(A)$ be the partition defined by the parts of the partitions in A . For example, if $A = \{(4, 2), (3, 1), (3, 2), (2)\}$, then $\phi(A) = 43^22^31$.

Several $A \in \mathcal{SP}_N^2$ might have the same $\phi(A)$. For $\lambda \vdash N$, let $\mathcal{SP}_\lambda^2 = \{A \in \mathcal{SP}_N^2 \mid \phi(A) = \lambda\}$. For example, if $\lambda = 42^21$, then $\{(4, 2), (2, 1)\}$ and $\{(4, 1), (2, 2)\}$ are

both elements of \mathcal{SP}_λ . Similarly, define $\mathcal{SSP}_\lambda = \mathcal{SP}_\lambda^2 \cap \mathcal{SSP}_N$. Note that for every λ , $\mathcal{SSP}_\lambda \neq \emptyset$.

Remark 5. For $A \in \mathcal{SSP}_\lambda$, if λ has an even number of parts, then all the partitions of A have two parts, while if λ has an odd number of parts, then exactly one partition of A will have one part (and the remaining partitions in A will have two parts).

Note that when s_A is expanded in Schur functions, its support lies above $\phi(A)$ in dominance order, and its coefficients are non-negative. This is a consequence of the Littlewood-Richardson rule.

Proposition 6. If $A \in \mathcal{SP}_N^2$, then

$$s_A = \sum_{\mu \triangleright \phi(A)} c_A^\mu s_\mu,$$

with $c_A^{\phi(A)} = 1$ and $c_A^\mu \geq 0$.

Our primary tool in proving elements in \mathcal{SSP}_N are extreme in \mathcal{C}_N^2 is the well-known Farkas' Lemma (see [4]). Farkas' Lemma states that a vector \mathbf{v} is extreme in a cone if and only if there is a separating hyperplane, i.e., a hyperplane P such that \mathbf{v} lies on one side of P and all other generating vectors lie on the other side of P .

Since we are working in Λ^N and using the Schur functions as our basis, it is natural to determine separating hyperplanes by using the standard symmetric function inner product $\langle \cdot, \cdot \rangle$ for which the Schur functions are orthonormal.

Suppose $A \in \mathcal{SSP}_N$ and let f be a symmetric function such that $\langle f, s_B \rangle \leq 0$ for $B \in \mathcal{SSP}_N$, $B \neq A$, and $\langle f, s_A \rangle > 0$. Then we say f separates A . Restating Farkas' Lemma in our context:

Theorem 7. There is a symmetric function f which separates A for $A \in \mathcal{SSP}_N$ if and only if A is extreme in \mathcal{C}_N^2 .

To prove Conjecture 3, we seek therefore a set of separating functions, one for each A . We now use Proposition 6 to reduce the amount of work we must do in finding separating functions. In effect, Proposition 6 states that we need only work above $\phi(A)$ in dominance order. To formalize this idea, we introduce this definition. Let $\lambda = \phi(A)$, $A \in \mathcal{SSP}_N$. We say the symmetric function f separates A from above if

- i. f is Schur integral;
- ii. $\langle f, s_A \rangle > 0$;
- iii. $\langle f, s_B \rangle \leq 0$ for all B such that $\phi(B) \triangleright \lambda$, $B \neq A$.

For example, for $N = 6$ and $A = \{(2, 1), (2, 1)\}$, we have $\lambda = 2^2 1^2$. If we take $f = s_{2^2 1^2} + s_{2^3} + s_{31^3} - s_{321}$, then f separates A from above.

Lemma 8. If f separates A from above, then there is a symmetric function g which separates A .

Proof. Let I be a dual order ideal in the dominance poset (see [5] for definitions) with $\lambda = \phi(A) \in I$. Let u be a symmetric function with the following properties:

$$(5) \quad \begin{aligned} \langle u, s_A \rangle &> 0; \\ \langle u, s_B \rangle &\leq 0 \quad \text{for all } B \neq A \text{ such that } \phi(B) \in I. \end{aligned}$$

We show how to grow I and u . Let μ be a partition which lies “just below” I , that is, $\mu \notin I$ and $J = I \cup \{\mu\}$ is a dual order ideal. Let

$$m = \max_{B \in \mathcal{SSP}_\mu} \{\langle u, s_B \rangle\}.$$

If $m \leq 0$, then u satisfies (5) for J . Otherwise, let $u^* = u - ms_\mu$. We show that u^* satisfies (5) for J .

For any B such that $\phi(B) \in I$, we have $\mu \not\preceq \phi(B)$, since $\mu \notin I$. Therefore, by Proposition 6, $\langle s_\mu, s_B \rangle = 0$, and we have

$$\langle u^*, s_B \rangle = \langle u, s_B \rangle \begin{cases} \leq 0 & \text{if } B \neq A \\ > 0 & \text{if } B = A \end{cases}.$$

For $B \in \mathcal{SSP}_\mu$, by Proposition 6, $\langle s_\mu, s_B \rangle = 1$, so we have

$$\langle u^*, s_B \rangle = \langle u, s_B \rangle - m \leq 0.$$

The proof now proceeds by iterating this construction. If f separates A from above, then f satisfies (5) for the dual order ideal generated by λ . By iterating the construction above, we eventually arrive at a function g which satisfies (5) for I equal to the entire dominance poset. This is the same as saying g separates A . \square

An important special case is the following corollary.

Corollary 9. *If $|\mathcal{SSP}_\lambda| = 1$, that is, $\mathcal{SSP}_\lambda = \{A\}$, then A is extreme in \mathcal{C}_N^2 .*

Proof. The function s_λ separates A from above. \square

We can limit our search for separating functions even further by restricting to an interval in the dominance poset. Suppose $\phi(A) = \lambda$ and $\rho \succeq \lambda$. We will say the symmetric function f separates A on $[\lambda, \rho]$ if

- i. f is Schur integral;
- ii. $\langle f, s_\mu \rangle = 0$ whenever $\mu \notin [\lambda, \rho]$. That is, the support of f lies on $[\lambda, \rho]$;
- iii. $\langle f, s_A \rangle > 0$;
- iv. $\langle f, s_B \rangle \leq 0$ for all B such that $\phi(B) \in [\lambda, \rho]$, $B \neq A$.

Lemma 10. *If f separates A on $[\lambda, \rho]$, then f separates A from above.*

Proof. We show that for B such that $\phi(B) \triangleright \lambda$ but $\rho \not\preceq \phi(B)$, we have $\langle f, s_B \rangle = 0$. The support of s_B is $\succeq \phi(B)$ (Proposition 6). But then the support of s_B cannot be below ρ , so the support of s_B does not intersect the interval $[\lambda, \rho]$. \square

Again suppose $\phi(A) = \lambda$, where $A \in \mathcal{SSP}_N$. Two intervals above λ in dominance will be of particular interest to us. First, if $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0)$, define

$$\lambda^+ = (\lambda_1 + 1, \lambda_2, \lambda_3, \dots, \lambda_{m-1}, \lambda_m - 1).$$

For example, if $\lambda = (4, 3, 3, 2, 2, 2, 1)$, then $\lambda^+ = (5, 3, 3, 2, 2, 2)$. In the next section, we will find a symmetric function f which separates A on $[\lambda, \lambda^+]$ when λ has distinct parts. However, this interval is not sufficient when λ has repeated parts. For example, if $\lambda = 2^3 1^3$, then no such f separates $A = \{(2, 1), (2, 1), (2, 1)\}$ on this interval.

Now define

$$\lambda^{++} = \begin{cases} (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_k + 1, \lambda_{k+1} - 1, \dots, \lambda_m - 1) & \text{if } m = 2k \\ (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_k + 1, \lambda_{k+1}, \lambda_{k+2} - 1, \dots, \lambda_m - 1) & \text{if } m = 2k + 1. \end{cases}$$

In the previous example, $\lambda^{++} = (5, 4, 4, 2, 1, 1)$.

Conjecture 11. *For every $A \in \mathcal{SSP}_N$ with $\phi(A) = \lambda$ there is a symmetric function f such that f separates A on $[\lambda, \lambda^{++}]$.*

We have verified Conjecture 11 for $N \leq 20$.

5. DISTINCT PARTITIONS

In this section we show that if A is nested and if $\phi(A)$ has distinct parts, then s_A is extreme.

Theorem 12. *If λ has distinct parts and $A \in \mathcal{SSP}_\lambda$, then there is a symmetric function f which separates A on the interval $[\lambda, \lambda^+]$.*

Our strategy for proving Theorem 12 is to find a chain of subsets in \mathcal{SSP}_λ , starting with \mathcal{SSP}_λ itself and ending with $\{A\}$, such that there is a vector which separates the $(i+1)$ subset in the chain from the i subset. Putting these separating vectors together produces a separating vector for A .

To make this strategy precise, we need some technical definitions and lemmas. Suppose $Y \subseteq \mathcal{SSP}_\lambda$. We say symmetric function f separates Y on $[\lambda, \mu]$ if

- i. f is Schur-integral.
- ii. $\langle f, s_B \rangle = 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \notin [\lambda, \mu]$.
- iii. $\langle f, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \in [\lambda, \mu]$, $\nu \neq \lambda$.
- iv. $\langle f, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\lambda - Y$.
- v. $\langle f, s_A \rangle = k > 0$ for all $A \in Y$, where integer k does not depend on A .

Now suppose $X \subseteq Y \subseteq \mathcal{SSP}_\lambda$. We say symmetric function g partially separates (X, Y) on $[\lambda, \mu]$ if

- i. g is Schur-integral.
- ii. $\langle g, s_B \rangle = 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \notin [\lambda, \mu]$.
- iii. $\langle g, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \in [\lambda, \mu]$, $\nu \neq \lambda$.
- iv. $\langle g, s_B \rangle \leq 0$ for all $B \in Y - X$.
- v. $\langle g, s_A \rangle = l > 0$ for all $A \in X$, where integer l does not depend on A .

Note that the sign of $\langle g, s_B \rangle$ is not specified for $B \in \mathcal{SSP}_\lambda - Y$.

Lemma 13. *Suppose f separates Y on $[\lambda, \mu]$ and g partially separates (X, Y) on $[\lambda, \mu]$. Then there exists an h which separates X on $[\lambda, \mu]$.*

Proof. Let

$$m = \max_{B \in \mathcal{SSP}_\lambda - Y} \langle g, s_B \rangle.$$

Pick a non-negative integer $b \geq m/k$. Let

$$h = g + bf - bks_\lambda.$$

We now verify that h has the required properties. Clearly, h is Schur-integral and its support lies on $[\lambda, \mu]$.

Now suppose $B \in \mathcal{SSP}_\nu$, $\nu \in [\lambda, \mu]$, $\nu \neq \lambda$. Then $\langle g, s_B \rangle \leq 0$, $\langle f, s_B \rangle \leq 0$, and $\langle s_\lambda, s_B \rangle = 0$ (by Proposition 6). Thus

$$\langle h, s_B \rangle \leq 0,$$

since $b \geq 0$.

Next, suppose $B \in \mathcal{SSP}_\lambda - Y$. Then $\langle g, s_B \rangle \leq m$, $\langle f, s_B \rangle \leq 0$, and $\langle s_\lambda, s_B \rangle = 1$ (again by Proposition 6). Then

$$\langle h, s_B \rangle \leq m - bk \leq 0,$$

since $b \geq 0$ and $b \geq m/k$.

Next, suppose $B \in Y - X$. Then $\langle g, s_B \rangle \leq 0$, $\langle f, s_B \rangle = k$, and $\langle s_\lambda, s_B \rangle = 1$. Thus

$$\langle h, s_B \rangle \leq bk - bk = 0.$$

Finally, suppose $A \in X$. Then $\langle g, s_A \rangle = l$, $\langle f, s_A \rangle = k$, and $\langle s_\lambda, s_A \rangle = 1$, so

$$\langle h, s_A \rangle = l + bk - bk = l > 0,$$

and l is independent of the choice of A . \square

Corollary 14. *If $\lambda = \phi(A)$ and*

$$\mathcal{SSP}_\lambda = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_t = \{A\}$$

and for each $i = 0, 1, \dots, t-1$ there is an f_i which partially separates (X_{i+1}, X_i) on $[\lambda, \mu]$, then there is a symmetric function g which separates A on $[\lambda, \mu]$.

Proof. Clearly s_λ separates all of \mathcal{SSP}_λ on any interval. Iteratively applying Lemma 13 in this chain of subsets yields g which separates $\{A\}$ on $[\lambda, \rho]$. But that is the same as g separates A on $[\lambda, \rho]$. \square

Note that up to this point, we have not used the fact that λ is distinct. From now on, we assume λ is distinct.

Suppose $A, B \in \mathcal{SSP}_\lambda$ and $\rho = (\lambda_i, \lambda_j)$, but ρ not necessarily in either A or B . We say A and B agree within ρ if

- i. if $(\lambda_u, \lambda_v) \in A$ (resp. B) and either u or v is between i and j , then $i < u < v < j$;
- ii. if $i < u < v < j$, then $(\lambda_u, \lambda_v) \in A$ if and only if $(\lambda_u, \lambda_v) \in B$

In addition, we say A and B agree on ρ if they agree within ρ and $\rho \in A, B$.

Note that these two definitions allow a part of size 1 to be within ρ . That part must be in both A and B . Also define

$$\lambda[\rho] = (\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \dots, \lambda_m).$$

Note that $\lambda[\rho] \in [\lambda, \lambda^+]$.

We illustrate these definitions with an example. Let

$$\lambda = (17, 16, 15, 13, 12, 11, 9, 8, 7, 5, 4, 2),$$

and

$$A = (17, 2), (16, 7), (15, 11), (13, 12), (9, 8), (5, 4)$$

$$B = (17, 5), (16, 7), (15, 11), (13, 12), (9, 8), (4, 2)$$

$$C = (17, 16), (15, 11), (13, 12), (9, 8), (7, 2), (5, 4)$$

Let $\rho = (16, 7)$. Then A, B and C all agree within ρ , and A and B agree on ρ . Finally,

$$\lambda[\rho] = (17, 17, 15, 13, 12, 11, 9, 8, 6, 5, 4, 2).$$

The following lemma is crucial to our proof of Theorem 12.

Lemma 15. *Suppose $A, B \in \mathcal{SSP}_\lambda$, λ distinct. Suppose $\rho = (\lambda_i, \lambda_j)$, with $\rho \in A$, $\rho \notin B$, and A and B agree within ρ . Then the Littlewood-Richardson coefficients satisfy the following identity:*

$$c_A^{\lambda[\rho]} + 1 = c_B^{\lambda[\rho]}.$$

Furthermore, if $j = i + 1$, then $c_A^{\lambda[\rho]} = 0$ and $c_B^{\lambda[\rho]} = 1$.

We defer the proof of Lemma 15 for the moment and show how it leads directly to a proof of Theorem 12.

Proof of Theorem 12. Suppose $A \in \mathcal{SSP}_\lambda$ where $\lambda = (\lambda_1, \dots, \lambda_n)$. If $n = 2m$ is even, then all the partitions in A are 2-part partitions. If $n = 2m - 1$ is odd, then exactly one partition in A has one part.

Order all the partitions in A with two parts from the “inside out.” That is, list the partitions in A as ρ^1, ρ^2, \dots , where all the partitions within ρ^j appear among $\rho^1, \rho^2, \dots, \rho^{j-1}$. If λ is odd, put the 1-part partition last in the above list.

Let

$$X_i = \{B \in \mathcal{SSP}_\lambda \mid B \text{ and } A \text{ agree on } \rho^1, \rho^2, \dots, \rho^i\}.$$

We then have this chain of subsets:

$$\mathcal{SSP}_\lambda = X_0 \supseteq X_1 \supseteq \dots \supseteq X_m = \{A\}.$$

We wish to apply Corollary 14 to this chain, so we need to construct f_i which partially separates (X_{i+1}, X_i) on $[\lambda, \lambda^+]$. Note that X_i consists of all elements of \mathcal{SSP}_λ which agree with A on $\{\rho^1, \dots, \rho^i\}$ and $X_i - X_{i+1}$ are those which do not contain $\rho = \rho^{i+1}$. That is, $B \in X_i$ means B and A agree within ρ , but for such B , $B \in X_{i+1}$ if and only if $\rho \in B$. These are exactly the conditions needed to apply Lemma 15.

Let

$$f_i = (c_A^{\lambda[\rho]} + 1)s_\lambda - s_{\lambda[\rho]}.$$

We now verify that f_i partially separates (X_{i+1}, X_i) on $[\lambda, \lambda^+]$, thus completing the proof.

Clearly f_i is Schur-integral and its support lies on $[\lambda, \lambda^+]$.

For $B \in \mathcal{SSP}_\nu$ with $\nu \in [\lambda, \lambda^+]$ and $\nu \neq \lambda$, we have $\langle s_\lambda, s_B \rangle = 0$ (by Proposition 6) and $\langle s_{\lambda[\rho]}, s_B \rangle \geq 0$ (since s_B is Schur-positive). Thus $\langle f_i, s_B \rangle \leq 0$.

For $B \in Y - X$, $\langle s_\lambda, s_B \rangle = 1$ (by Proposition 6) and $\langle s_{\lambda[\rho]}, s_B \rangle = c_B^{\lambda[\rho]}$. So by Lemma 15, $\langle f_i, s_B \rangle = c_A^{\lambda[\rho]} + 1 - c_B^{\lambda[\rho]} = 0$.

For $B \in X$, $\langle s_\lambda, s_B \rangle = 1$ (by Proposition 6) and $\langle s_{\lambda[\rho]}, s_B \rangle = c_A^{\lambda[\rho]}$. Then $\langle f_i, s_B \rangle = c_A^{\lambda[\rho]} + 1 - c_A^{\lambda[\rho]} = 1$.

Therefore f_i partially separates (X_{i+1}, X_i) as required. □

We make a couple of observations about \mathcal{SSP}_λ when λ is distinct. If λ is even ($n = 2m$), then the elements of \mathcal{SSP}_λ are clearly counted by the Catalan numbers C_m . If λ is odd ($n = 2m - 1$) then the elements of \mathcal{SSP}_λ are counted again by the Catalan numbers C_m . Furthermore, for our ordering of the ρ^i 's, we may use the natural Catalan recursion induced by our realization of the partitions in \mathcal{SSP}_λ as nestings. See Exercise 6.19, part (o) in [6].

The awkward distinction between even n and odd n can be resolved in several ways. Our proof above placed the singleton part at the end so that all the work

had been accomplished before encountering it. Since a 1-part is incorporated in Lemma 15, this was not technically necessary. A heuristic of the general (non-distinct) problem seems to be that the odd case is easier than the even case.

6. A LITTLEWOOD-RICHARDSON IDENTITY

Our goal in this section is to prove Lemma 15. This lemma will be a corollary of a stronger theorem which we now describe.

We generalize somewhat the notation from Section 4. Suppose A is a multiset in \mathcal{P}_N (not necessarily 1 or 2 row partitions). As with 2-part partitions, let $\phi(A)$ be the partition defined by the parts of the partitions in A . Let $n = l(\phi(A))$, the number of parts of $\phi(A)$.

If $\phi(A)$ is distinct, the location in $\phi(A)$ of each part of each partition in A defines a set partition of $\{1, \dots, n\}$ with m blocks, where m is the number of partitions in A . If ρ^i is a partition in A , let α^i denote the corresponding block of the set partition.

For example, if $\rho^1 = (8, 3, 1)$ and $\rho^2 = (6, 4)$, and $A = \{\rho^1, \rho^2\}$ (with $n = 5$, $m = 2$ and $N = 22$), then $\phi(A) = (8, 6, 4, 3, 1)$, and $\alpha^1 = \{1, 4, 5\}$ and $\alpha^2 = \{2, 3\}$.

We will be forming tableaux with content equal to various rearrangements of $\phi(A)$. Since the elements of the blocks α^i will correspond to letters in such tableaux, we view these blocks as alphabets. When switching order between alphabets, we will keep the letters within alphabets intact, rather than relabel. Also, note that since $\phi(A)$ is distinct, there is no ambiguity in the definition of the set partition $\{\alpha^1, \dots, \alpha^m\}$.

Now for $\lambda \vdash N$ define d_A^λ to be the number of SSYT T of shape λ , content $\phi(A)$, such that each word $w_{\alpha^i}(T)$ is LW.

Here is an example of such a SSYT for the above A and $\lambda = (9, 9, 3, 1)$:

1	1	1	1	1	1	1	1	1	2
2	2	2	2	2	3	4	4	4	
3	3	3							
5									

The words

$$w_{\alpha^1}(T) = (1111111114445)$$

and

$$w_{\alpha^2}(T) = (2322222333)$$

are both lattice words.

The calculation of d_A^λ is similar to the Littlewood-Richardson calculation, except the content has been sorted into decreasing order and the subwords corresponding to each $\rho \in A$ are scattered throughout the tableau.

In general $d_A^\lambda \neq c_A^\lambda$. We shall give examples of this inequality later in this section. However, in one particular case, these two numbers are equal, and their equality allows us to calculate c_A^λ exactly.

Specifically, suppose $\lambda \vdash N$, λ with distinct parts, and $\nu = (\lambda_i, \lambda_j)$ is a two-part partition using two parts of λ . We consider SSYT of shape $\lambda[\nu]$ and content λ .

We first characterize these tableaux. Let T be such a tableau. Suppose $u < i$ or $u \geq j$. Then all the entries in row u of T are u 's. Furthermore, if $i \leq u < j$, then all the entries in row u of T are u 's, except for possibly the last one. We call the last cell in each of these rows *special*. A special cell does not have a cell below

it, since the only possible repeated rows in $\lambda[\nu]$ are rows $i - 1$ and i or rows j and $j + 1$.

The entries in the special cells (reading from row $j - 1$ up to row i) will form a permutation of $\{(i + 1), \dots, j\}$. However, for purposes which will soon become clear, we write this as a permutation of $\{i, \dots, j\}$ by inserting an i in the next-to-last position. We call this permutation $\pi = \pi_1, \dots, \pi_{j-i+1}$. Note that since π_1 is the entry in row $j - 1$, π_1 can only be j or $j - 1$. Similarly, π_2 can only be $j - 2$ or the value not used for π_1 . Continuing in this fashion, there are only two choices for each entry in π , except the last two. This description completely characterizes these SSYT.

Proposition 16. *The number of SSYT of shape $\lambda[\nu]$ and content λ is 2^{j-i-1} (that is, the Kostka number, $K_{\lambda[\nu], \lambda} = 2^{j-i-1}$).*

Suppose $i \leq u < v \leq j$. If $u \neq i$, it follows from the discussion above that $w_{\{u,v\}}(T)$ will be LW if and only if u follows v in π . Our insertion of i into π extends this to the $u = i$ case.

For example, let $\lambda = (7, 5, 3, 2, 1)$ and $\nu = (5, 1)$. There are 4 SSYT of shape $\lambda[\nu]$ and content λ . We list them with their corresponding π :

$$\begin{array}{l}
 A_1 = \begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 2 & 2 & 2 & 2 & 2 & 5 \\
 3 & 3 & 3 & & & & \\
 4 & 4 & & & & &
 \end{array} \quad \pi = 4325 \\
 \\
 A_2 = \begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 2 & 2 & 2 & 2 & 2 & 3 \\
 3 & 3 & 5 & & & & \\
 4 & 4 & & & & &
 \end{array} \quad \pi = 4523 \\
 \\
 A_3 = \begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 2 & 2 & 2 & 2 & 2 & 4 \\
 3 & 3 & 3 & & & & \\
 4 & 5 & & & & &
 \end{array} \quad \pi = 5324 \\
 \\
 A_4 = \begin{array}{cccccccc}
 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 2 & 2 & 2 & 2 & 2 & 3 \\
 3 & 3 & 4 & & & & \\
 4 & 5 & & & & &
 \end{array} \quad \pi = 5423
 \end{array}$$

From this example it is easy to verify, for instance, that $w_{\{2,3\}}(A_1)$ and $w_{\{2,3\}}(A_3)$ have the LW property but $w_{\{2,3\}}(A_2)$ and $w_{\{2,3\}}(A_4)$ do not. Also $w_{\{3,4\}}(A_1)$, $w_{\{3,4\}}(A_2)$ and $w_{\{3,4\}}(A_4)$ have the LW property, but $w_{\{3,4\}}(A_3)$ does not. This is reflected in the corresponding π 's.

Theorem 17. *Suppose $\lambda \vdash N$ is distinct, $\nu = (\lambda_i, \lambda_j)$, and A is a multiset of partitions, with $\phi(A) = \lambda$. Then*

$$c_A^{\lambda[\nu]} = d_A^{\lambda[\nu]}.$$

Furthermore, $c_A^{\lambda[\nu]}$ is the number of permutations $\pi = \pi_1, \dots, \pi_{j-i+1}$ of $\{i, \dots, j\}$ such that

i.

$$\pi_t \geq j - t \text{ for } t = 1, 2, \dots, j - i - 1$$

$$\pi_{j-i} = i$$

π_{j-i+1} is the only remaining value.

ii. if $u < v$ are in α^i , then v appears before u in π .

Proof. The characterization of the permutations follows from the discussion above.

Our strategy in proving the equality is to swap alphabets using Theorem 2. Starting with a SSYT T counted by $d_A^{\lambda[\nu]}$, as noted above, we view the blocks of the set partition $\{\alpha^1, \dots, \alpha^m\}$ as alphabets. Each alphabet is scattered throughout the tableau, not contiguously, so Theorem 2 does not directly apply. To circumvent this, we use Theorem 2 on a portion of the tableau where the alphabets do appear contiguously.

Suppose we have constructed a new tableau S from T , with special row k . Suppose S and k have the following properties.

- i. The shape of S is $\lambda[\nu]$.
- ii. The content of S is a rearrangement of λ . As noted above, we keep the same alphabets as T , but reordered without relabeling.
- iii. Each subword corresponding to each alphabet α^u is a lattice word.
- iv. At and above row k , S is identical to T . We shall say S is *pristine above k* .
- v. The letters $> k$ in each alphabet α^u appear contiguously in S . We shall say S is *clustered below k* . Note that “below” here means “greater than”. Letters $> k$ can appear in the pristine portion of S .

We will call such S *k -partially LR*.

Let $n = l(\lambda[\rho])$. (Note that n may or may not equal $l(\lambda)$, depending on whether ν includes the last part of λ and that part is 1.) Initially, T is n -partially LR. Also note that if S is 0-partially LR, then it is counted by $c_A^{\lambda[\nu]}$.

Now suppose S is k -partially LR. We show how to construct a new tableau S' which is $(k-1)$ -partially LR. Iterating will give a 0-partially LR tableau. Each step in this process will be seen to be reversible, thus proving the result.

Suppose k is in block α^{u_0} . If k is the largest in its block, then S will be clustered below $k-1$ and obviously pristine above $k-1$, so let $S' = S$.

Now suppose k is not the largest in its block. Since S is pristine above k , except possibly at the special cells, no letter greater than k appears above row k . Any swaps that take place between letters greater than k will not affect the special cells, since there is nothing below them, and so the tableau above row k will remain pristine.

Define for each u , $\beta^u = \alpha^u \cap \{k+1, \dots, n\}$. Since k is not largest in its block, β^{u_0} is non-empty. Since S is clustered below k , define S^u to be the skew subtableau of S containing the letters in β^u . We then apply Theorem 2 to move S^{u_0} through the S^u until the smallest letter in β^{u_0} and k are contiguous. This new tableau is S' . As noted above, these moves will leave the rows above k unaffected, so that S' is pristine above k . Furthermore, each of these moves will preserve the LW property within each S^u (by Theorem 2), even if some of the letters appear in special cells in the pristine region.

It remains to verify that the LW property holds in S' for each α^u . We say $x, y \in \alpha^u$ are *sequential* if there is no $z \in \alpha^u$ between x and y . Note that this is different

than contiguous: contiguous means there is no z between two entries in the tableau, while sequential means there is no z between two entries in the alphabet. Two letters could be sequential (same alphabet) but not contiguous (intervening letters from different alphabets). And two letters could be contiguous (no intervening letter within the tableau) but not sequential (from different alphabets).

It suffices to show the LR property holds in S' for all sequential pairs in α^u .

We consider three possible cases for sequential pair x, y . First, suppose $x, y > k$. Then x and y will have the LW property in α^u in S if and only if they have the LW property in S' by Theorem 2.

Second, suppose $x < k$ and $y \leq k$. Then neither x nor y is in β^u , so their relative positions will be unchanged by swapping alphabets.

Third, suppose $x \leq k$ and $y > k$. All the x 's appear in the pristine portion of S and so remain unchanged. All the y 's (except for possibly one in a special cell) are below row k . Therefore if x and y had the LW property before a swap, it would have it afterwards, and conversely. \square

We illustrate with some examples. First, suppose

$$\begin{aligned}\alpha^1 &= \{4, 8, 9, 13, 14, 16\} \\ \alpha^2 &= \{2, 5, 6, 7, 11, 15\} \\ \alpha^3 &= \{1, 3, 10, 12\}.\end{aligned}$$

Suppose S is 7-partially LR. Then

$$\begin{aligned}\beta^1 &= \{8, 9, 13, 14, 16\} \\ \beta^2 &= \{11, 15\} \\ \beta^3 &= \{10, 12\}.\end{aligned}$$

Furthermore, suppose $\beta^1 < \beta^3 < \beta^2$. That is, the order on the entries in S due to previous switches is

$$8 < 9 < 13 < 14 < 16 < 10 < 12 < 11 < 15.$$

Since 7 belongs to α^2 , β^2 will switch with β^3 then β^1 . The new order will be

$$7 < 11 < 15 < 8 < 9 < 13 < 14 < 16 < 10 < 12.$$

In this new order, 15 and 8 are contiguous but not sequential (different alphabets). And 4 and 8 are sequential but not contiguous (separated by 5, 6, 7, 11, 15).

Our second example illustrates the switching within a tableau. Let

$$\lambda = (9, 8, 7, 6, 5, 4, 3, 2, 1)$$

and $\mu = (9, 1)$. Then

$$\lambda[\mu] = (10, 8, 7, 6, 5, 4, 3, 2).$$

Let $A = \{\rho^1, \rho^2, \rho^3\}$, with $\rho^1 = (9, 5, 3)$, $\rho^2 = (8, 7, 4, 2)$ and $\rho^3 = (6, 1)$. Note that $\phi(A) = \lambda$. Also, since 9, 5 and 3 are in the 1, 5 and 7 positions of λ , $\alpha^1 = \{1, 5, 7\}$.

Similarly, $\alpha^2 = \{2, 3, 6, 8\}$ and $\alpha^3 = \{4, 9\}$. Initially let

$$T = \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & & & \\ 4 & 4 & 4 & 4 & 4 & 5 & & & & \\ 5 & 5 & 5 & 5 & 9 & & & & & \\ 6 & 6 & 6 & 6 & & & & & & \\ 7 & 7 & 7 & & & & & & & \\ 8 & 8 & & & & & & & & \end{array}$$

Note that $i = 1$, $j = 9$, and $\pi = 876953214$. Now suppose we have constructed the corresponding 3-partially LR tableau S . In fact, suppose

$$S = \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & & & \\ \hline 4 & 4 & 4 & 4 & 4 & 5 & & & & \\ 9 & 5 & 5 & 5 & 5 & & & & & \\ 7 & 7 & 7 & 6 & & & & & & \\ 6 & 6 & 6 & & & & & & & \\ 8 & 8 & & & & & & & & \end{array}$$

Then $\beta^1 = \{5, 7\}$, $\beta^2 = \{6, 8\}$ and $\beta^3 = \{4, 9\}$. Note that at this stage, $\beta^3 < \beta^1 < \beta^2$. Since 3 belongs to α^2 , S^2 will swap with S^1 , then S^3 , to form

$$S' = \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & & \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & & & \\ \hline 6 & 6 & 6 & 6 & 5 & 5 & & & & \\ 8 & 8 & 5 & 5 & 4 & & & & & \\ 5 & 7 & 7 & 4 & & & & & & \\ 7 & 4 & 4 & & & & & & & \\ 4 & 9 & & & & & & & & \end{array}$$

Note that T , S and S' all have the appropriate LW property. The new tableau S' will be pristine above $k - 1$ and clustered below $k - 1$. Lemma 15 now follows as a corollary

Proof of Lemma 15. By Theorem 17, $c_A^{\lambda[\rho]}$ (resp. $c_B^{\lambda[\rho]}$) counts permutations π of $\{i, \dots, j\}$ such that $\pi_i \geq j - i$ for $i = 1, 2, \dots, j - i - 1$ and $\pi_{j-i} = i$, and if $(\lambda_r, \lambda_s) \in A$ (resp. $\in B$), then s appears before r in π . It is clear that exactly one such permutation has i appearing before j , namely $j - 1, j - 2, \dots, i + 1, i, j$. This permutation is counted in $c_B^{\lambda[\rho]}$ but not in $c_A^{\lambda[\rho]}$.

If $j = i + 1$, then there is exactly one SSYT, T , and $\pi = (i, i + 1)$, so this T is LW for B but not for A . \square

We conclude with two examples which illustrate how special is the case of Theorem 17. First, to demonstrate that the content must be in decreasing order, let $\mu = (4, 4, 1, 1)$ with $A = \{(4, 1), (3, 2)\}$. Then $\lambda = \phi(A) = (4, 3, 2, 1)$. Note that $\mu = \lambda[(3, 2)]$. From Theorem 17, we know $c_A^\mu = d_A^\mu = 0$ and $K_{\mu, \lambda} = 1$. However, if

we use a different order, $(3, 4, 1, 2)$, for the content, we have

$$T = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 3 & & & \\ 4 & & & \end{array}$$

The corresponding words in this tableau are LW.

The second example illustrates how important it is that the shape be only slightly different than the content. Take $\lambda = (6, 5, 4, 3, 2, 1)$, $A = \{(6, 1), (5, 4), (3, 2)\}$ and $\mu = (7, 7, 5, 2)$. Note that λ is distinct and A is nested, but μ is not $\lambda[\rho]$ for any possible ρ . Now let

$$T = \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 2 & 2 & 4 & 6 \\ 3 & 3 & 3 & 3 & 4 & & \\ 5 & 5 & & & & & \end{array}$$

Clearly T has content λ and the words corresponding to A are LW. However, repeated switches yields this tableau (with content $(6, 1, 5, 4, 3, 2)$):

$$S = \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 6 \\ 2 & 2 & 2 & 2 & 2 & 4 & 4 \\ 3 & 3 & 3 & 3 & 4 & & \\ 5 & 5 & & & & & \end{array}$$

Note that S is not LW. In fact, $d_A^\mu = 15$ while $c_A^\mu = 13$.

7. NESTINGS AND PLANE PARTITIONS

There is an interesting connection between the nested sets of partitions described in Conjecture 3 and and plane partitions. See [5] for the relevant definitions associated with plane partitions.

Proposition 18. *If $\lambda \vdash N$ has $2m$ parts, then there is a one-to-one correspondence ψ between elements of \mathcal{SSP}_λ and plane partitions of N with shape (m, m) and parts λ_i .*

Proof. For each ρ in A , place ρ_1 in the first row of $\psi(A)$ and ρ_2 in the second row. Then write the rows in decreasing order. \square

We illustrate this bijection with the following table, when $\lambda = 12^334^25$. There are four elements of \mathcal{SSP}_λ .

$A \in \mathcal{SSP}_\lambda$	$\psi(A)$
$\{(5, 1), (4, 2), (4, 2), (3, 2)\}$	$\begin{array}{cccc} 5 & 4 & 4 & 3 \\ 2 & 2 & 2 & 1 \end{array}$
$\{(5, 1), (4, 2), (4, 3), (2, 2)\}$	$\begin{array}{cccc} 5 & 4 & 4 & 2 \\ 3 & 2 & 2 & 1 \end{array}$
$\{(5, 1), (4, 4), (3, 2), (2, 2)\}$	$\begin{array}{cccc} 5 & 4 & 3 & 2 \\ 4 & 2 & 2 & 1 \end{array}$
$\{(5, 3), (4, 4), (2, 1), (2, 2)\}$	$\begin{array}{cccc} 5 & 4 & 2 & 2 \\ 4 & 3 & 2 & 1 \end{array}$

When the number of parts of λ is odd, the bijection ψ becomes an injection. For example, suppose $\lambda = 1^3 2^3 3 4^2$.

$A \in \mathcal{SSP}_\lambda$	$\psi(A)$
$\{(4, 2), (4, 2), (3, 2), (1), (1, 1)\}$	$\begin{matrix} 4 & 4 & 3 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{matrix}$
$\{(4, 2), (4, 3), (2, 2), (1), (1, 1)\}$	$\begin{matrix} 4 & 4 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 0 \end{matrix}$
$\{(4, 4), (3, 2), (2, 2), (1), (1, 1)\}$	$\begin{matrix} 4 & 3 & 2 & 1 & 1 \\ 4 & 2 & 2 & 1 & 0 \end{matrix}$
$\{(4, 4), (3), (2, 1), (2, 1), (2, 1)\}$	$\begin{matrix} 4 & 3 & 2 & 2 & 2 \\ 4 & 1 & 1 & 1 & 0 \end{matrix}$
$\{(4, 4), (3), (2, 1), (2, 2), (1, 1)\}$	$\begin{matrix} 4 & 3 & 2 & 2 & 1 \\ 4 & 2 & 1 & 1 & 0 \end{matrix}$

To illustrate that the mapping to plane partitions is not a bijection, the plane partition

$$\begin{matrix} 4 & 4 & 3 & 2 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{matrix}$$

does not appear in this list.

8. REMARKS AND ACKNOWLEDGEMENTS

It is easy to describe the extreme vectors of \mathcal{C}_N^{N-1} . These vectors are all the Schur functions except s_{1^n} , which is replaced by $s_{1^{n-1}1}$. In a similar (but more complicated) fashion, it is possible to describe the extreme vectors of \mathcal{C}_N^{N-2} and \mathcal{C}_N^{N-3} .

However, we have been unable to replace the conditions in Conjecture 3 with general conditions for the cone \mathcal{C}_N^k . Even the case $k = 3$ seems difficult, requiring that the syzygies in Equations (1) to (4) be replaced with appropriate syzygies for 3-row partitions.

Lemma 14 gives a general recipe for constructing separating vectors. Unfortunately, if λ has repeated parts, the chain of subsets of \mathcal{SSP}_λ and the required partially separating vectors seem much more elusive than in the case of distinct parts.

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