MAHONIAN Z STATISTICS

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ABSTRACT. The Z statistic of Zeilberger and Bressoud is computed by summing the major index of the 2-letter subwords. We generalize this idea to other splittable Mahonian statistics. We call splittable Mahonian statistics which produce other splittable Mahonian statistics in this fashion *Z-Mahonian*. We characterize Z-Mahonian statistics and include several examples.

1. Introduction

The major index statistic on words was first studied by Major P.A. MacMahon in the early part of the 20th century [16]. As MacMahon showed, the distribution of the major index on all rearrangements of a given word is the same as that of the inversion number. The significance of MacMahon's work has been recognized by giving the name *Mahonian* to any statistic whose distribution is the same as that of the inversion number (or the major index). In recent years, several other Mahonian statistics on words have emerged. These include the interpolating statistics of Rawlings [17], Kadell [13] and White [22], and an important new statistic due to Denert [7]. Also, Simion and Stanton [20] introduced new Mahonian statistics on binary words.

Of special interest is the Z statistic introduced by Zeilberger and Bressoud [23], obtained by summing the major index for each of the 2-letter subwords of a given word. Zeilberger and Bressoud showed that the Z statistic was Mahonian, a key step in their proof of the q-Dyson conjecture.

At this juncture, we emphasize that the Z statistic and the statistic from which it was manufactured have the same distribution. It is, therefore, natural to look for other statistics which have this "Z-ing" property. We restrict our attention to statistics on binary words which are *splittable*, an idea introduced in earlier work [9], [10]. In this paper we give precise conditions under which a family of splittable Mahonian statistics on binary words can be "Z-ed" to give a splittable Mahonian statistic on words. We call statistics with this property Z-Mahonian.

The next section defines words and statistics on words, and includes a more extended discussion of Mahonian statistics. In Section 3 we discuss the notion of a splittable Mahonian statistic and restate a key result from [10]. The definition of the Z operator that extends the ideas of Zeilberger and Bressoud and our main theorems appear in Section 4; illustrative examples are provided in Section 5.

In Section 6 we return to the topic of splittable statistics with several counting theorems. The paper concludes with a few remarks about the general applicability of our Z-operator results.

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2. Mahonian statistics

In this section we first define words and statistics on words. We will introduce the important class of statistics on words, Mahonian statistics.

Let K be a positive integer. A *letter* is an element of $[K] \equiv \{1, 2, \ldots, K\}$. A *word* is a sequence of letters. If $\mathbf{w}$ is a word, then the i-th letter is w_i. The *length* of $\mathbf{w}$ is the number of letters in $\mathbf{w}$. A letter x *appears* in $\mathbf{w}$ if $w_i = x$ for some i. If x and y are letters, then $\mathbf{w}[x, y]$ is the subword obtained by removing all letters except x and y from $\mathbf{w}$.

The *type* of $\mathbf{w}$, $\rho = (\rho_1, \rho_2, \ldots, \rho_K)$, is a vector representing the number of each kind of letter in $\mathbf{w}$. That is, ρ_1 is the number of 1's in $\mathbf{w}$. Thus, if the type of $\mathbf{w}$ is ρ, then x appears in $\mathbf{w}$ if and only if $\rho_x > 0$. Let $a(\rho)$ denote the number of letters appearing in words of type ρ and let $|\rho|$ denote the length of any word of type ρ.

The set of all words of type ρ is denoted W_ρ. If x appears in words of type ρ, we write $x \in \rho$. The new type $\rho - x$ is $(\rho_1, \ldots, \rho_x - 1, \ldots, \rho_K)$. That is, one x has been removed from the type.

An alternative description of type is an exponential form. If $\mathbf{w}$ has type ρ, then we write $1^{\rho_1} 2^{\rho_2} \ldots K^{\rho_K}$ to represent the type. This notation is useful because if x does not appear in $\mathbf{w}$, then the term x^0 won't appear in the exponential form.

If ρ and μ are two types, then $\mu < \rho$ if $\mu_i \leq \rho_i$ for all $1 \leq i \leq K$, with a strict inequality at least once. If $\mu < \rho$, define $\rho - \mu = (\rho_1 - \mu_1, \ldots, \rho_K - \mu_K)$.

Two special kinds of types are important. First, *binary words* are words with only two letters appearing. Second, *permutations* are words such that each letter which appears, appears only once.

A *statistic* on words in W_ρ is a function s from W_ρ to the non-negative integers. If the type is not specified, then the function is from all words to the non-negative integers. If s is a statistic on words of type ρ, then the corresponding *statistic generating function* is the polynomial

$$\sum_{\mathbf{w} \in W_\rho} q^{\mathrm{s}(\mathbf{w})} .$$

A statistic s on words of type ρ is *Mahonian* if its statistic generating function is

$$\begin{bmatrix} |\rho| \\ \rho \end{bmatrix}_q \equiv \begin{bmatrix} |\rho| \\ \rho_1, \rho_2, \ldots, \rho_K \end{bmatrix}_q ,$$

called the *q-multinomial coefficient*, and defined by

$$\begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q = \frac{[n]_q!}{[n_1]_q! [n_2]_q! \ldots [n_k]_q!} ,$$

$$[n]_q! = [n]_q [n-1]_q \ldots [1]_q ,$$

and

$$[m]_q = 1 + q + q^2 + \cdots + q^{m-1} .$$

In the case of binary words, the q-multinomial coefficient becomes the q-binomial coefficient. In the case of permutations, the q-multinomial coefficient becomes the q-analogue of $n!$.

Mahonian statistics on words have a long and important history. The interested reader is referred to [10] for a summary of this history. Over the years, many Mahonian statistics have been discovered and rediscovered. We list here some of the important ones, and give some of their definitions.

Define

$$\text{INV}(\mathbf{w}) = \#\{(i,j) \mid i < j, w_i > w_j\}$$

and

$$\text{MAJ}(\mathbf{w}) = \sum_{\{i \mid w_i > w_{i+1}\}} i\,.$$

The statistics INV and MAJ are classical Mahonian statistics, dating back to MacMahon [16].

In their proof of the q-Dyson conjecture, Zeilberger and Bressoud were led to another Mahonian statistic, which we call ZLB. (In the literature this statistic is referred to as Z.) This statistic interpolates between INV on permutations and MAJ on binary words. It is defined by

$$\text{ZLB}(\mathbf{w}) = \sum_{x<y} \text{MAJ}(\mathbf{w}[x,y])\,.$$

In a later paper [3], Bressoud describes "tournamented" statistics. His idea is extended in Section 4. Other Mahonian interpolating statistics include RAW [17], KAD [13] and WHT [22].

Another important Mahonian statistic first appeared in the work of Denert [7]. This statistic, DEN, was first proved Mahonian on permutations by Foata and Zeilberger [8], then on words by Han [11].

All the above statistics are defined on all words. However, many Mahonian statistics are defined only on permutations. These include some recent statistics due to Babson and Steingrimsson [2].

Since our plan is to bootstrap a Mahonian statistic on binary words up to all words, our discussion will not include Mahonian permutation statistics.

3. Splittable statistics

In this section, we introduce the idea of a splittable Mahonian statistic, and we review a couple of the results in [10].

The idea of a splittable statistic is described in detail in [10]. Roughly speaking, a statistic on words is splittable if it can be written as a sum of two pieces. One piece corresponds to the rightmost letter. The second piece corresponds to the subword obtained by removing the rightmost letter. The second piece must be (recursively) splittable.

More precisely, a Mahonian statistic S on W_ρ is *splittable* (or *splits*) if, for $\mathbf{v} = \mathbf{w}x \in W_\rho$,

$$\text{S}(\mathbf{w}x) = \alpha_x + \text{T}(\mathbf{w})\,, \tag{1}$$

where T is a Mahonian statistic on $W_{\rho-x}$ which splits, and α_x is an integer.

The following Theorem is a consequence of Theorem 6.2 in [10].

Theorem 1. *A Mahonian statistic* S *is splittable on words of type ρ if and only if for every $\mathbf{v} \in W_\mu$, $\emptyset \leq \mu < \rho$, there is a permutation of the letters in $\rho - \mu$, $\pi_{\mathbf{v}}$, such that if*

$$\mathbf{w} = x_1 x_2 \ldots x_n \in W_\rho$$

with

$$\mathbf{u}_i = x_1 \ldots x_i$$

and

$$\mathbf{v}_i = x_{i+1} \ldots x_n\,,$$

then

$$\mathrm{s} = \sum_{i=1}^{n} \alpha_i \,,$$

where

$$\alpha_i = \#\{y \text{ } in \text{ } \mathbf{u}_i \mid \pi_{\mathbf{v}_i} y > \pi_{\mathbf{v}_i} x_i\} \,.$$

For example, if $\rho = (1, 2, 1)$, then s defined below is splittable Mahonian.

$$\begin{array}{llll} \mathrm{s}(1223) = 3 & \mathrm{s}(1232) = 1 & \mathrm{s}(3122) = 1 & \mathrm{s}(2231) = 5 \\ \mathrm{s}(2123) = 2 & \mathrm{s}(2132) = 2 & \mathrm{s}(2312) = 3 & \mathrm{s}(2321) = 3 \\ \mathrm{s}(2213) = 4 & \mathrm{s}(1322) = 0 & \mathrm{s}(3212) = 2 & \mathrm{s}(3221) = 4 \end{array}$$

Thus, $\mathrm{s}(2312) = 3$, since $\pi_\emptyset = (123)$, $\pi_2 = (1)(23)$, $\pi_{12} = (23)$, and $\pi_{312} = (2)$, so that $\alpha_4 = 0$, $\alpha_3 = 2$, $\alpha_2 = 1$ and $\alpha_1 = 0$.

In what follows, we will identify the permutation $\pi_{\mathbf{v}}$ with the corresponding transitive tournament on the letters in the permutation.

Many Mahonian statistics on words appearing in the literature are splittable. These include MAJ, INV, DEN, RAW, KAD, WHT, and ZLB. These examples all appear in [10].

The fact that MAJ splits requires special attention. It splits because of an "encoding" due to Han [11] which gives exactly Equation (1). This encoding is as follows. For $x = w_j$ in the word $\mathbf{w}$, α_j counts the number of letters to the left of position j and cyclically between x and $y = w_{j+1}$ (with $w_{n+1} = k$), counting y's but not x's.

For example, if

$$\mathbf{w} = 2\,1\,1\,4\,3\,4\,2\,2\,3\,1\,3\,,$$

then the vector of α_j's is

$$(0, 0, 1, 3, 1, 3, 0, 1, 4, 5, 2)\,.$$

Thus, $\mathrm{MAJ}(\mathbf{w}) = 20$.

4. The Z operator

In this section we introduce the Z operator and we give two necessary and sufficient conditions for the Z operator to produce a splittable Mahonian statistic.

We now consider words of type ρ which use K letters. We call a collection of statistics, one for each type $x^{\rho_x} y^{\rho_y}$, a *binary family* of type ρ. If each statistic in a binary family of type ρ is splittable Mahonian, we say the binary family is splittable Mahonian. Suppose s is a binary family of type ρ. If $\mathbf{w}$ is a word of type $x^{\rho_x} y^{\rho_y}$, we abuse notation and write $\mathrm{s}(\mathbf{w})$ to denote the value of the statistic in the collection s on $\mathbf{w}$.

Now suppose s is a binary family of type ρ. Define $Z(\mathrm{s})$ on words of type ρ as follows:

$$Z(\mathrm{s}) = \sum_{\{x,y\}} \mathrm{s}(\mathbf{w}[x, y])\,.$$

It is obvious that if $\mathrm{s} = \mathrm{INV}$, then $Z(\mathrm{s}) = \mathrm{INV}$. In [23] it was shown that if $\mathrm{s} = \mathrm{MAJ}$, then $Z(\mathrm{s})$ is Mahonian. This is the statistic ZLB described in Section 2. A combinatorial proof that it is Mahonian was given in [12], and a proof that it is splittable appears in [10]. The statistic ZLB interpolates between INV and MAJ in the following sense. On binary words we have $\mathrm{ZLB} = \mathrm{MAJ}$. On permutations we have $\mathrm{ZLB} = \mathrm{INV}$.

The goal of the rest of this paper is to give general criteria under which $Z(\text{S})$ is a splittable Mahonian statistic. We call binary families with this property *Z-Mahonian.*

The following example will serve as illustration. Define EXS on 4 letters as follows. Let EXS = INV on words using 1's and 2's, on words using 1's and 3's, and on words using 1's and 4's. Also, let EXS = INV on words using 2's and 4's. Finally, on words using 2's and 3's or words using 3's and 4's, let EXS = MAJ. The following table describes EXS:

	2	3	4
1	INV	INV	INV
2		MAJ	INV
3			MAJ

Then for any type ρ, EXS is a binary family of splittable Mahonian statistics of type ρ. For instance, if

$$\mathbf{w} = 2\,1\,1\,4\,3\,4\,2\,2\,3\,1\,3\,,$$

then $Z(\text{EXS})(\mathbf{w}) = 19$, since

$$\text{EXS}(\mathbf{w}[1,2]) = \text{INV}(2\,1\,1\,2\,2\,1) = 5$$
$$\text{EXS}(\mathbf{w}[1,3]) = \text{INV}(1\,1\,3\,3\,1\,3) = 2$$
$$\text{EXS}(\mathbf{w}[1,4]) = \text{INV}(1\,1\,4\,4\,1) = 2$$
$$\text{EXS}(\mathbf{w}[2,3]) = \text{MAJ}(2\,3\,2\,2\,3\,3) = 2$$
$$\text{EXS}(\mathbf{w}[2,4]) = \text{INV}(2\,4\,4\,2\,2) = 4$$
$$\text{EXS}(\mathbf{w}[3,4]) = \text{MAJ}(4\,3\,4\,3\,3) = 4$$

For another example, take EXT to be the binary family given by this table:

	2	3	4
1	INV	INV	INV
2		MAJ if $\mathbf{w}[2,3]$ has even length, INV otherwise	INV if $\mathbf{w}[2,4]$ has odd length or if $\rho_2 \equiv \rho_4 \not\equiv \rho_3$ mod 2, MAJ otherwise
3			MAJ if $\mathbf{w}[3,4]$ has even length, INV otherwise

Again, EXT is a binary family of splittable Mahonian statistics of type ρ. If, for example,

$$\mathbf{w} = 2\,1\,1\,4\,3\,4\,2\,2\,3\,1\,3\,,$$

then $Z(\text{EXT})(\mathbf{w}) = 20$, since

$$\text{EXT}(\mathbf{w}[1,2]) = \text{INV}(2\,1\,1\,2\,2\,1) = 5$$
$$\text{EXT}(\mathbf{w}[1,3]) = \text{INV}(1\,1\,3\,3\,1\,3) = 2$$
$$\text{EXT}(\mathbf{w}[1,4]) = \text{INV}(1\,1\,4\,4\,1) = 2$$
$$\text{EXT}(\mathbf{w}[2,3]) = \text{MAJ}(2\,3\,2\,2\,3\,3) = 2$$
$$\text{EXT}(\mathbf{w}[2,4]) = \text{INV}(2\,4\,4\,2\,2) = 4$$
$$\text{EXT}(\mathbf{w}[3,4]) = \text{INV}(4\,3\,4\,3\,3) = 5$$

We shall find that EXS is not Z-Mahonian while EXT is Z-Mahonian.

Suppose S is a binary family of splittable Mahonian statistics of type ρ. Therefore, for each word $\mathbf{b}$ of type $x^i y^j$, $i \le \rho_x$ and $j \le \rho_y$, there is a "ranking" of x and

y. That is, for each word of the form $\mathbf{a}x\mathbf{b}$, the contribution of x, α_x, will depend upon $\mathbf{b}$, and will be either the number of y's in $\mathbf{a}$ (if y beats x) or 0 (if x beats y). Similarly, for each word of the form $\mathbf{a}y\mathbf{b}$, the contribution of y, α_y, will be either the number of x's in $\mathbf{a}$ (if x beats y) or 0 (if y beats x).

Now suppose $\mathbf{v}$ is a word of type $\mu \leq \rho$. Each $\mathbf{v}[x, y]$ determines a relative ranking of x and y, given by S. Together, these rankings form a tournament, which we call $T_{\mathbf{v}}$.

Theorem 2. *Suppose* S *is a binary family of splittable Mahonian statistics of type ρ. Then* S *is Z-Mahonian on type ρ if and only if for each $\mathbf{v}$ of type $\mu < \rho$, $T_{\mathbf{v}}$ is transitive on the letters in $\rho - \mu$.*

Proof. If $T_{\mathbf{v}}$ is transitive on the letters which appear in $\rho - \mu = \nu$, then it corresponds to a permutation $\pi_{\mathbf{v}}$ on these letters, and is exactly the permutation required by Theorem 1 for $Z(\text{S})$ to be a splittable Mahonian statistic.

Conversely, suppose $Z(\text{S})$ is splittable Mahonian. By Theorem 1, the tournament $T_{\mathbf{v}}$ must be transitive. □

Note that it is possible for S to be Z-Mahonian on some types, but not all types. Notable cases of splittable Mahonian binary families which are not Z-Mahonian for some types include DEN, RAW and KAD.

It is possible to describe how splittable statistics $Z(\text{S})$ can be built up. Suppose T is a transitive tournament. A transitive tournament S such that only team x changes positions relative to the remaining teams is called a *reassignment* of x in T.

Theorem 3. *Suppose* S *is a binary family of splittable Mahonian statistics of all types. Then each $T_{x\mathbf{v}}$ is a reassignment of x in $T_{\mathbf{v}}$ if and only if* S *is Z-Mahonian on all types.*

Proof. The "only if" part is immediate. For the "if" part, since $Z(\text{S})$ is splittable on all types, by Theorem 2, $T_{\mathbf{v}}$ is transitive for each $\mathbf{v}$. But all the 2-letter subwords of $x\mathbf{v}$ which do not involve x will be the same as the 2-letter subwords of $\mathbf{v}$, so the order of these letters in $T_{x\mathbf{v}}$ will be the same as the order of these letters in $T_{\mathbf{v}}$. Therefore, $T_{x\mathbf{v}}$ will be a reassignment of x in $T_{\mathbf{v}}$. □

5. Examples

In this section we will illustrate Theorem 2 and Theorem 3 with three basic examples. We can form other examples by combining these three basic examples in various ways.

Our first example is based on MAJ and the Zeilberger-Bressoud statistic ZLB. Using Han's encoding of MAJ [11], if the binary family is MAJ, then $T_{\mathbf{w}}$ is defined as follows:

x beats y if and only if the leftmost x in $\mathbf{w}$ appears to the left of the leftmost y.

To deal with cases where not all letters appear in $\mathbf{w}$, we concatenate a trailing $K\, K-1\, \ldots\, 1$ to $\mathbf{w}$. It is immediate from this description that $T_{\mathbf{w}}$ is transitive; in fact, π is the permutation of leftmost occurrences.

There are now several possible modifications. For example, instead of the leftmost x and y, we could choose the x which is p_x positions from the left, where p_x

is a nonnegative integer for each x (and similarly for y). Or we could choose the rightmost x and y.

The second example arises from a statistic related to work of Simion and Stanton [20]. Suppose $\mathbf{w}$ is a word using letters 1 and 2. An inversion in $\mathbf{w}$, $w_i = 2$, $w_j = 1$, with $i < j$, is of type A if there is a 1 to the left of w_i. It is of type B if, for every $i < k < j$, $w_k = 2$. Then $\text{SSS}(\mathbf{w})$ is the number of type A inversions plus the number of type B inversions. Note that some inversions get counted twice, while others not at all. For example, if

$$\mathbf{w} = 2\,2\,1\,2\,1\,1\,1\,2\,2\,1\,2\,1\,1,$$

then $\text{SSS}(\mathbf{w}) = 20$ since there are 14 type A inversions and 6 type B inversions.

This can be translated into an obvious splittable encoding as follows. For a word $\mathbf{w}$ using letters 1's and 2's, 1 beats 2 if and only if 1 appears in $\mathbf{w}$. In the above example, the rightmost 1 contributes 6 to the statistic (2 beats 1), but thereafter 1 beats 2, so the reverse inversions are counted (14 of them).

For a word $\mathbf{w}$ of arbitrary type ρ, the tournament $T_{\mathbf{w}}$ will have, for $x < y$, x beats y if and only if x appears in $\mathbf{w}$. This tournment is clearly transitive. The permutation π will first list all the letters appearing in $\mathbf{w}$ in increasing order, then all the letters not appearing in $\mathbf{w}$ in decreasing order.

Once again, many variations are possible. For example, instead of the first appearance of x, the p_x appearance could be used. Or, as letters appear in $\mathbf{w}$, they could become losers instead of winners.

The final example is the interpolating statistic hinted at in [22]. Call a subset $A \subseteq [K]$ *contiguous* if it has no "holes," that is, if $x, y \in A$ and $x < z < y$, then $z \in A$. Suppose $A_1, \ldots, A_m$ is a collection of disjoint contiguous subsets. Now define $\text{WHT}(\mathbf{w})$ on words using x's and y's as follows. If x and y are in the same A_i, then $\text{WHT}(\mathbf{w}) = \text{MAJ}(\mathbf{w})$. Otherwise, $\text{WHT}(\mathbf{w}) = \text{INV}(\mathbf{w})$.

Notice that if $m = 0$, then $\text{WHT} = \text{INV}$ and $Z(\text{WHT}) = \text{INV}$, while if $m = 1$ and $A_1 = [K]$, then $\text{WHT} = \text{MAJ}$ and $Z(\text{WHT}) = \text{ZLB}$.

Theorem 4. *The statistic* WHT *is Z-Mahonian.*

Proof. The tournament $T_{\mathbf{w}}$ is as follows. If x and y are both in the same A_i, then x beats y if and only if x appears to the left of y in $\mathbf{w}$. Otherwise, x beats y if and only if $x > y$. This tournament is clearly transitive. □

Once again, many variations are possible. Instead of MAJ, one of the statistics related to MAJ described above may be used. Or the statistic SSS may be used. Or MAJ on some of the A_i and SSS on other A_i.

We now return to the two examples given in Section 4, EXS and EXT. In the case of EXS, if $\mathbf{w} = 2\,3$, then in $T_{\mathbf{w}}$ 2 beats 3, 3 beats 4, and 4 beats 2.

In the case of EXT, the following table summarizes the situation. The first three columns are the possible mod 2 classes for ρ_2, ρ_3 and ρ_4. The next three columns give the statistic produced by EXT. The last column describes $Z(\text{EXT})$.

ρ_2	ρ_3	ρ_4	23	24	34	Z(EXT)
0	0	0	MAJ	MAJ	MAJ	WHT, $A_1 = \{2,3,4\}$
0	0	1	MAJ	INV	INV	WHT, $A_1 = \{2,3\}$
0	1	0	INV	INV	INV	INV
0	1	1	INV	INV	MAJ	WHT, $A_1 = \{3,4\}$
1	0	0	INV	INV	MAJ	WHT, $A_1 = \{3,4\}$
1	0	1	INV	INV	INV	INV
1	1	0	MAJ	INV	INV	WHT, $A_1 = \{2,3\}$
1	1	1	MAJ	MAJ	MAJ	WHT, $A_1 = \{2,3,4\}$

A few calculations will illustrate the frequency of splittable Mahonian statistics which are Z-Mahonian.

For type $(2,1,1)$, there are 32 binary families of splittable Mahonian statistics. Of these, Of these, 18 are Z-Mahonian.

For type $(2,2,1)$, there are 512 binary families of splittable Mahonian statistics. Of these, 138 are Z-Mahonian.

For type $(2,2,2)$, there are 32768 binary families, with 1122 Z-Mahonian.

6. Counting splittable statistics

In this section, we derive a generating function for the log of the number of splittable Mahonian statistics.

Given the simple recursive definition of a splittable Mahonian statistic, it is not surprising that we can find a recursion for the number of splittable statistics. Let f_ρ be the number of splittable Mahonian statistics on words of type ρ. We have $a(\rho)!$ different choices for the permutation π, and for each of these and for each possible letter x ending the word we have $f_{\rho-x}$ possible splittable Mahonian statistics on the shorter word. Putting this together, we get

Theorem 5.

$$f_\rho = (a(\rho)!) \prod_{x\in\rho} f_{\rho-x} \,.$$

Letting $g_\rho = \log f_\rho$, we have

$$g_\rho = \log(a(\rho)!) + \sum_{x\in\rho} g_{\rho-x} \,. \tag{2}$$

We can then find the generating function for the g_ρ. Let $\mathbf{x} = \{x_1, x_2, \dots\}$ be an infinite set of indeterminates and let t be another indeterminate. For a given type ρ, let

$$\mathbf{x}^\rho = \prod_{i\geq 1} x_i^{\rho_i} \,.$$

Now let

$$G(\mathbf{x};t) = \sum_\rho g_\rho \mathbf{x}^\rho t^{|\rho|} \,.$$

Theorem 6.

$$G(\mathbf{x};t) = \sum_{n\geq 0} H_n(\mathbf{x};t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n-m)!) \,,$$

where

$$H_n(\mathbf{x};t) = \frac{t^n e_n(\mathbf{x})}{1 - te_1(\mathbf{x})} \prod_{j\geq 1} \frac{1}{1 - x_j t},$$

and $e_n(\mathbf{x})$ is the n-th elementary symmetric function in the variables $\mathbf{x}$.

Two interesting special cases can be computed exactly. First, when the words have two letters, we have this corollary.

Corollary 7. *The coefficient of $x_1^i x_2^j t^{i+j}$ in $G(\mathbf{x};t)$ is*

$$(\binom{i+j}{i} - 1)\log 2\,.$$

Corollary 7 has a simple combinatorial proof. If the list of words using i 1's and j 2's is written in a natural recursive way, that is, with the reverse word ordered lexicographically, then between any pair of adjacent words, $\mathbf{v}$ and $\mathbf{w}$, in this list, there is a unique location k (from right to left) where they first differ, with $\mathbf{v}$ having a 1 and $\mathbf{w}$ having a 2. There are 2 possible tournament choices for this k. Since there are $\binom{i+j}{i}$ words in the list, there are $\binom{i+j}{i} - 1$ such adjacent pairs and therefore

$$2^{\binom{i+j}{i}-1}$$

possible statistics.

For permutations, we have this result.

Corollary 8. *The coefficient of $x_1 \dots x_n t^n$ in $G(\mathbf{x};t)$ is*

$$\sum_{k=2}^{n} \binom{n}{k} (n-k)! \log(k!)\,.$$

proof of Theorem 6. We first prove a finite version. Let

$$G_k(x_1,\dots,x_k;t) = G(\mathbf{x};t)|_{x_j=0,\ j>k}\,,$$

and

$$H_{k,n}(x_1,\dots,x_k;t) = H_n(\mathbf{x};t)|_{x_j=0,\ j>k}\,.$$

For a subset A of letters, let $\mathbf{x}[A]$ denote the list $\{x_i \mid i \in A\}$ and $x_A = \prod_{i\in A} x_i$. We will show

$$\tilde{G}_k(x_1,\dots,x_k;t) = \sum_{n=0}^{k} H_{k,n}(x_1,\dots,x_k;t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n-m)!) \tag{3}$$

satisfies the recursion (2). Since the initial conditions are trivially satisfied, we must have $\tilde{G}_k = G_k$. Letting $k \to \infty$ gives the theorem.

Recursion (2) and the definition of G translate into this identity:

$$\sum_{j=0}^{k} (-1)^j \sum_{\substack{A\subseteq[k] \\ |A|=k-j}} (1 - te_1(\mathbf{x}[A])) G_{k-j}(\mathbf{x}[A];t) = t^k e_k(\mathbf{x}) \prod_{i=1}^{k} \frac{1}{1 - x_i t}\,. \tag{4}$$

Replace G with $\tilde{G}$ into the left hand side of (4) and subtitute (3), giving

$$\sum_{j=0}^{k} (-1)^j \sum_{\substack{A\subseteq[k] \\ |A|=k-j}} (1 - te_1(\mathbf{x}[A])) \sum_{n=0}^{k-j} H_{k-j,n}(\mathbf{x}[A];t) \sum_{m=0}^{n} \binom{n}{m} (-1)^m \log((n-m)!)$$

If we now substitute for H and simplify, then the coefficient of

$$(-1)^{k+m}\log(m!)\prod_{i=1}^{k}\frac{1}{1-x_it}$$

is

$$\sum(-1)^{|A|+|B|+|D|}t^{|B|+|D|}x_Bx_D\ ,$$

where the sum is over subsets A, B, C, and D satsifying $C\subseteq B\subseteq A\subseteq[k]$, $D\subseteq A^c$, and $|C|=m$. It is easily seen that the coefficient of

$$t^{|U|}x_U$$

in this expression is 0 unless $C=B=A=[k]$, $D=\emptyset$, and $m=k$, in which case the right-hand side of Equation 3 emerges. □

We have not been able to derive a similar formula for all Z-Mahonian statistics of a given type. However, in the special case of permutations, we have this corollary of Theorem 2.

Corollary 9. *The number of Z-Mahonian statistics on permutations of length n is $n!$.*

Proof. We count the number of binary families which yield Z-Mahonian statistics under Theorem 2. For the empty word, we have $n!$ possible transitive tournaments. Let T be one such tournament. Now suppose $\mathbf{w}$ is a permutation ending in $\mathbf{v}$ and suppose that $\mathbf{v}$ is not the empty word. Then $T_{\mathbf{v}}$ is T with the letters appearing in $\mathbf{v}$ reassigned. But the letters remaining in $\mathbf{w}$ do not appear in $\mathbf{v}$, so these reassignments do not affect the original arrangement of the remaining letters. That arrangement is the same as in the initial T. □

7. Remarks

In [10] a new statistic, LP, was introduced. This statistic is notable for two reasons. First, it is splittable (though not Mahonian). Second, its distribution generating function is the same as the generating function for *charge* on words. Charge is the statistic discovered by Lascoux and Schützenberger [14] to resolve the Foulkes conjecture on the Kostka-Foulkes polynomials. For further details about charge, see [15].

Many of the results of this paper carry over to arbitrary splittable statistics with permuteable distribution generating functions (see [10] for definitions), as long as there is a generating statistic family (such as INV in the case of Mahonian statistics). In particular, it was conjectured in [10] that LP is such a generating statistic family, and this has since been proved by the second author. Therefore, not only can LP can be used to construct charge-distributed analogs of MAJ, DEN and other splittable mahonian statistics, but a theorem similar to Theorem 2 can be stated to describe when the Z-operator produces a charge-distributed statistic. In particular, there are charge distributed analogs of both MAJ and ZLB.

As a direction for further investigation, we note that several authors (see, for example, [1], [4], [5], [6], [18], [19], and [21]) have constructed statistics on signed permutations analogous to Mahonian statistics. These constructions invites the question: Can any of them be extended to "signed words?" If so, is there a concept corresponding to a splittable statistic and an analogous Z operator?

Finally, we suspect that, except for perhaps some sporadic cases, the only binary families which are splittable Mahonian are Z-Mahonian. However, we have no evidence and no theorems to this effect.

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