April 12, 2003; Due April 26, 2003.

Math 8652: Homework set #5 (Spring 2003)

1. Let X_n be a Markov processes (with respect to the filtration $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$). Let $A \in \mathcal{F}_n$, $B \in \sigma(X_n, X_{n+1}, \ldots)$. Prove that
$$P(A \cap B | X_n) = P(A|X_n)P(B|X_n)\,.$$

2. Let X_n be a (not necessary time homogeneous) Markov chain and suppose that
$$P(\cup_{m=n+1}^{\infty}\{X_m \in B_m\}|X_n) \geq \delta > 0 \quad \text{on}\ \{X_n \in A_n\}\,.$$
Let $\Gamma = \{X_n \in A_n\ i.o\}$, $\Lambda = \{X_n \in B_n\ i.o.\}$. Prove that
$$P(\Gamma \setminus \Lambda) = 0\,.$$
Remark: This is Theorem 2.3, Chapter 5 in Durrett's book. The proof there contains however a mistake!.

3. Let S be countable, let $A \subset S$, let X_n be a time-homogeneous Markov chain, and define
$$\tau_A = \inf\{n \geq 0 :\ X_n \in A\}, \quad T_A = \inf\{n \geq 1 :\ X_n \in A\}.$$
Prove that
$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_{\{y\}} = m)p^{n-m}(y,y)$$
where $p^n(x,y)$ is the transition probability matrix raised to the power n, more precisely $= P_x(X_n = y)$.

4. With the notations of #3, suppose that $D := S \setminus C$ is a finite set, and that for each $x \in D$, $P_x(\tau_C < \infty) > 0$. Prove that for some $N < \infty$ and $\epsilon > 0$ it holds that for all $y \in S$, $P_y(\tau_C > kN) \leq (1-\epsilon)^k$.
Remark: we say then that $[\tau_C/N]$ is dominated by a geometric random variable of parameter ϵ.

5. Still in the notations of #3, let B be such that $A \cap B = \emptyset$, $F = S \setminus (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$, for all $x \in F$. Define $h(x) = P_x(\tau_A < \tau_B)$. Prove that $h(\cdot)$ is *harmonic*, that is
$$h(x) = \sum_y p(x,y)h(y)\,, \quad \forall x \in F\,.$$

Show that any (necessarily bounded!) solution of this equation is such that $h(X_{n \wedge \tau_{A \cup B}})$ is a martingale, and prove that $P_x(\tau_A < \tau_B)$ is the *only* solution of this equation that is 1 on A and 0 on B. Finally, with X_n a simple random walk, use this result to compute

$$P(X.\,\text{hits}\,2000\,\text{before it hits}\,-4000)\,.$$

6. Consider the Markov chain on $S = \{0, 1, \ldots, N\}$ with transition probabilities

$$p(i,j) = \begin{cases} \frac{1}{2}, & |i-j| = 1, i \neq 0, i \neq N \\ 1, & |i-j| = 1, i = 0\,\text{or}\,i = N \\ 0, & \text{else.} \end{cases}$$

Find a stationary distribution for this Markov chain. Is it unique?

Assignment rules: Submitted work must be your own. You may, and in fact are encouraged to, collaborate on an assignment, provided that no more than 3 people are collaborating. In such case, you are requested to note the names of your collaborators on your submission. If collaboration is significant (more than two questions), you are requested to jointly submit your assignment.