(*December 21, 2004*)

Weil-Schwartz envelopes for rapidly decreasing functions

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Definition: A complex-valued function f on $\mathbf{R}^n$ is **rapidly decreasing** if for every $m \geq 0$

$$\sup_{x \in \mathbf{R}^n} |x|^m \, |f(x)| < \infty$$

where the first $| * |$ is the usual norm on $\mathbf{R}^n$ and the second is the absolute value on $\mathbf{C}$.

Theorem:
• Given a rapidly-decreasing function f on $\mathbf{R}^n$ there is a Schwartz function φ such that $|\varphi| \geq |f|$.
• Given a countable collection $\{f_i\}$ of rapidly decreasing functions, there is a positive monotone decreasing Schwartz function φ such that f_i/φ is rapidly decreasing for every index i.
• Every Schwartz function on $\mathbf{R}^n$ is a product of two Schwartz functions.

Proof: Let

$$c_m = \sup_x \, (|x|^2 + 1)^m \cdot |f(x)| < \infty$$

For $0 \leq r \in \mathbf{R}$ define

$$F(r) = \inf_m \frac{c_m}{(1+r)^m}$$

Since for every m

$$|f(x)| \leq \frac{c_m}{(1+|x|^2)^m}$$

we have

$$|f(x)| \leq F(|x|^2)$$

For any collection of numbers c_m, and infimum such as F is rapidly decreasing, since the exponents in the denominators are unbounded. Since each $c_m/(1+r)^m$ is monotone decreasing as $|r| \to \infty$, so is F. Since F is the infimum of a countable collection of continuous functions, it is measurable.

Let β be a smooth non-negative function supported on $[-1, +1]$, with total mass 1. Take

$$\varphi(r) = (F * \beta)(r) = \int_{\mathbf{R}} F(r-t)\, \beta(t)\, dt$$

Certainly φ is positive. The monotonicity of F implies that

$$F(r) \leq \varphi(r-1) \quad (\text{for } r \geq 1)$$

We claim that φ is a Schwartz function on $\mathbf{R}$. It has derivatives

$$\varphi^{(n)}(r) = (F * \beta^{(n)})(r)$$

which are smooth (since β is) and rapidly decreasing (since F is). Let φ_2 be any non-negative Schwartz function on $\mathbf{R}^n$ such that for $|x| \leq 1$ we have $\varphi_2(x) \geq |f(x)|$. Then

$$|f(x)| \leq F(|x|^2) \leq \varphi(|x|^2 - 1) + \varphi_2(|x|^2)$$

Let

$$\Phi(x) = \varphi(|x|^2 - 1) + \varphi_2(|x|^2)$$

Then Φ is a Schwartz function and

$$|\Phi(y)| \leq |\Phi(x)| \quad (\text{for } 1 \leq |x| \leq |y|)$$

To ensure that Φ is positive and monotone, we may add to it a sufficiently large positive multiple of $e^{-|x|^2}$ without harming any of the other properties. This proves the first assertion.

Given a countable collection of rapidly decreasing functions f_i, we may suppose without loss of generality (by the first assertion of this theorem) that f_i is a positive monotone Schwartz function. By replacing f_i by

$$sup_{1 \leq j \leq i} f_j$$

we may supposed that $f_i \leq f_j$ for $i < j$. Inductively define $M_1 = 1$ and

$$M_n = \text{ smallest } t \geq 1 + M_{n-1} \text{ such that } |x|^n \, |f(x)| \leq 1 \text{ for } |x| \geq M_n$$

Define

$$f(x) = f_n(x) \quad (\text{for } M_n \leq |x| < M_{n+1})$$

Claim that f is rapidly decreasing. Indeed, given a positive integer m, for any $ngem$, for $M_n \leq |x| < M_{n+1}$,

$$|x|^m f(x) = |x|^m f_n(x) \leq |x|^n \, f_n(x) \leq 1$$

Again using the first part of the theorem, there is a positive monotone Schwartz function φ such that $\varphi \geq |f|$. Then for each index i the ratio f_i/φ is clearly bounded.

The square root $\varphi_2(x) = \sqrt{\varphi(x)}$ is still monotone, rapidly decreasing, and continuous. Since f_i/φ is bounded,

$$f_i/\varphi_2 = \varphi_2 \cdot f_i/\varphi$$

shows that each f_i/φ_2 is rapidly decreasing. Invoking the first assertion of the theorem yet again, there is a positive monotone Schwartz function ψ with $\psi \geq |\varphi_2|$. This yields the second assertion.

Given a Schwartz function f, consider the countable family of rapidly decreasing functions

$$\Phi = \{|\varphi(x)|^{1/m} : m = 1, 2, 3, \ldots, \ \varphi \text{ a derivative of } f\}$$

By the second assertion of the theorem, there is a positive monotone Schwartz function ψ such that φ/ψ is rapidly decreasing for every $\varphi \in \Phi$. To verify that $F = f/\psi$ is a Schwartz function requires the rapid decreas of all derivatives of F. Given a monomial D in the partial derivative operators $\partial/\partial x_i$, by induction on the order n of D one can show that DF is of the form

$$DF = \left(\sum_\alpha P\alpha(D^\alpha f) \right) / \psi^{n+1}$$

where P_α is a polynomial in ψ and its derivatives, D^α is a monomial in the operators D_i, and the sum is finite. Since each P_α is bounded, it would suffice to show that any ration of the form Df/ψ^n is of rapid decrease. By the defining property of ψ the function $|Df|^{1/n}/\psi$ is of rapid decrease, so its n^{th} power is, as well. That is, f/ψ is Schwartz, and $f = \psi \cdot f/\psi$. ///
