(*May 7, 2015*)

Sporadic isogenies to orthogonal groups

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We will describe well-known 2-to-1 homomorphisms

$$\begin{cases} SL_2(\mathbb{C}) & \longrightarrow & SO(3,\mathbb{C}) \\ SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) & \longrightarrow & SO(4,\mathbb{C}) \\ Sp_2(\mathbb{C}) & \longrightarrow & SO(5,\mathbb{C}) \\ SL_4(\mathbb{C}) & \longrightarrow & SO(6,\mathbb{C}) \end{cases}$$

and well-known 2-to-1 homomorphisms to *real* special orthogonal groups $SO(p,q)$ with *signatures* (p,q):

$$SO(p,q) \;=\; \{g \in SL_{p+q}(\mathbb{R}) \;:\; g^\top Q g = Q\} \qquad (\text{where } Q = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix})$$

$$\begin{cases} SU(2) & \longrightarrow & SO(3) \\ SL_2(\mathbb{R}) & \longrightarrow & SO(2,1) \\ SU(2) \times SU(2) & \longrightarrow & SO(4) \\ SL_2(\mathbb{C}) & \longrightarrow & SO(3,1) \\ SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) & \longrightarrow & SO(2,2) \\ Sp^*(2,0) & \longrightarrow & SO(5) \\ Sp^*(1,1) & \longrightarrow & SO(4,1) \\ Sp_2(\mathbb{R}) & \longrightarrow & SO(3,2) \\ SU(4) & \longrightarrow & SO(6) \\ SL_2(\mathbb{H}) & \longrightarrow & SO(5,1) \\ SU(2,2) & \longrightarrow & SO(4,2) \\ SL_4(\mathbb{R}) & \longrightarrow & SO(3,3) \end{cases}$$

Thus, these are small examples of *spin groups*, two-fold covers of special orthogonal groups.

All these constructions are standard, in principle well-known, but often obscured or left as exercises in larger, systematic treatments of Lie theory or quadratic forms or Clifford algebras or Spin groups. [1]

[1] Thanks to Shaul Zemel for some corrections and suggestions, belatedly implemented.

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1. *Over* $\mathbb{C}$

[1.1] $SL_2(\mathbb{C}) \to SO(3,\mathbb{C})$ The space V of 2-by-2 complex matrices with trace 0, has symmetric bilinear form $\langle x,y\rangle = \mathrm{tr}(xy)$. The action of $SL_2(\mathbb{C})$ on V by $g\cdot x = gxg^{-1}$ preserves $\langle,\rangle$:

$$\langle g\cdot x, g\cdot y\rangle \;=\; \mathrm{tr}(gxg^{-1}\cdot gyg^{-1}) \;=\; \mathrm{tr}(g\cdot xy\cdot g^{-1}) \;=\; \mathrm{tr}(xy) \;=\; \langle x,y\rangle$$

An orthogonal basis is

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

with $\langle,\rangle$ values 2, 2, -2, demonstrating non-degeneracy. Thus, $SL_2(\mathbb{C})$ maps to a copy of $SO(3,\mathbb{C})$. The kernel is just $\{\pm 1\}$.

[1.2] $SL_2(\mathbb{C})\times SL_2(\mathbb{C}) \to SO(4,\mathbb{C})$ Let $V = M_2(\mathbb{C})$ be 2-by-2 complex matrices, with $(g,h)\in SL_2(\mathbb{C})\times SL_2(\mathbb{C})$ acting by $(g,h)\cdot x = gxh^{-1}$. Give V the bilinear form

$$\langle x,y\rangle \;=\; \mathrm{tr}(x\cdot wy^\top w^{-1}) \qquad\qquad (\text{where } w = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix})$$

It is *symmetric* because trace is invariant under transpose, and because $w^{-1} = -w$. For $g\in SL_2(\mathbb{C})$, $g^{-1} = wg^\top w^{-1}$, and the pairing is invariant under the group action:

$$\mathrm{tr}(gxh^{-1}\cdot w(gyh^{-1})^\top w^{-1}) \;=\; \mathrm{tr}(gxh^{-1}\cdot w(h^{-1})^\top w^{-1}\cdot wy^\top w^{-1}\cdot wg^\top w^{-1})$$
$$=\; \mathrm{tr}(gxh^{-1}\cdot h\cdot wy^\top w^{-1}\cdot g^{-1}) \;=\; \mathrm{tr}(g\cdot xwy^\top w^{-1}\cdot g^{-1}) \;=\; \mathrm{tr}(xwy^\top w^{-1})$$

Computing

$$\langle\begin{pmatrix}a & b\\ c & d\end{pmatrix}, \begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\rangle \;=\; \mathrm{tr}\Big(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}d' & -b'\\ -c' & a'\end{pmatrix}\Big) \;=\; \mathrm{tr}\begin{pmatrix}ad'-bc' & *\\ * & da'-cb'\end{pmatrix} \;=\; ad'-bc'-cb'+da'$$

an orthogonal basis is readily found: for example,[2]

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

with $\langle,\rangle$ values $2,-2,2,-2$, demonstrating non-degeneracy. Thus, $SL_2(\mathbb{C})\times SL_2(\mathbb{C})$ maps to a copy of $SO(4,\mathbb{C})$.

[1.3] $Sp_2(\mathbb{C}) \to SO(5,\mathbb{C})$ The symplectic group[3] is

$$Sp_2(\mathbb{C}) \;=\; \{g\in GL_4(\mathbb{C}) \;:\; g^\top Jg = J\} \qquad\qquad (\text{with } J = \begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix})$$

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[2] One can also observe from this expression that the bilinear form is a sum of two *hyperbolic planes*, thus giving signature $(2,2)$ without further computation.

[3] In some conventions, the subscript is made to be the *size*, so what we call Sp_2 here might be called Sp_4 elsewhere.

Write $g^\sigma = Jg^\top J^{-1}$, so the condition can be rewritten as $g^\sigma g = 1_2$. The $\mathbb{C}$-vectorspace V will be a subspace of the space $M_4(\mathbb{C})$ of 4-by-4 complex matrices. Let $\langle x,y\rangle = \mathrm{tr}(xy)$ on $M_4(\mathbb{C})$. Let $Sp_2(\mathbb{C})$ act on $M_4(\mathbb{C})$ by $g\cdot x = gxg^\sigma$. This action respects $\langle,\rangle$:

$$\langle g\cdot x, g\cdot y\rangle \;=\; \mathrm{tr}(gxg^\sigma\cdot gyg^\sigma) \;=\; \mathrm{tr}(g\cdot xy\cdot g^{-1}) \;=\; \mathrm{tr}(xy) \;=\; \langle x,y\rangle$$

Since $1_4 = g^\sigma g = g\cdot 1_4\cdot g^\sigma$, the action has fixed-point 1_4, and the subspace

$$V \;=\; \{x\in M_4(\mathbb{C}) \;:\; x^\sigma = x \text{ and } \langle x, 1_4\rangle = 0\}$$

is *stable* under the action. In 2-by-2 blocks, the condition $x^\sigma = x$ is

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}^\top\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} a^\top & c^\top\\ b^\top & d^\top\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix} d^\top & -b^\top\\ -c^\top & a^\top\end{pmatrix}$$

Thus, $d = a^\top$ and b,c are skew-symmetric. The condition $\langle x, 1_4\rangle = 0$ requires $\mathrm{tr}(a) = 0$. Thus, $\dim_{\mathbb{C}} V = 5$. To check that $\langle,\rangle$ is non-degenerate on V, identify an orthogonal basis, such as

$$\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0\\ & & 0 & -1\end{pmatrix}\;\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1\\ & & 1 & 0\end{pmatrix}\;\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1\\ & & 1 & 0\end{pmatrix}\;\begin{pmatrix} & & 0 & 1\\ & & -1 & 0\\ 0 & 1 & & \\ -1 & 0 & & \end{pmatrix}\;\begin{pmatrix} & & 0 & 1\\ & & -1 & 0\\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix}$$

where empty positions are 0.

[1.4] $SL_4(\mathbb{C}) \to SO(6,\mathbb{C})$

Let $SL_4(\mathbb{C})$ act in the natural way on the six-dimensional vectorspace $V = \bigwedge^2\mathbb{C}^4$, namely, $g\cdot(v\wedge w) = gv\wedge gw$. Let e_1,e_2,e_3,e_4 be the standard basis of $\mathbb{C}^4$, and define[4] $\langle,\rangle$ on V by

$$x\wedge y \;=\; \langle x,y\rangle\cdot e_1\wedge e_2\wedge e_3\wedge e_4 \qquad\qquad (\text{with } x,y\in \textstyle\bigwedge^2\mathbb{C}^4)$$

This form is *symmetric* because an even number of transpositions reverses the arguments:

$$(x\wedge y)\wedge(z\wedge w) \;=\; -x\wedge z\wedge y\wedge w \;=\; x\wedge z\wedge w\wedge y \;=\; -z\wedge x\wedge w\wedge y$$

$$=\; -z\wedge x\wedge w\wedge y \;=\; (z\wedge w)\wedge(x\wedge y) \qquad\qquad (\text{for } x,y,z,y\in\mathbb{C}^4)$$

The form is invariant under the action because

$$\big\langle g\cdot(x\wedge y), g\cdot(z\wedge w)\big\rangle\cdot e_1\wedge e_2\wedge e_3\wedge e_4 \;=\; gx\wedge gy\wedge gz\wedge gw \;=\; \det g\cdot x\wedge y\wedge z\wedge w$$

$$=\; \det g\cdot\big\langle x\wedge y, z\wedge w\big\rangle\cdot e_1\wedge e_2\wedge e_3\wedge e_4$$

To check non-degeneracy, observe

$$\big\langle e_1\wedge e_2, e_3\wedge e_4\big\rangle \;=\; 1 \qquad \big\langle e_1\wedge e_3, e_2\wedge e_4\big\rangle \;=\; -1 \qquad \big\langle e_1\wedge e_4, e_2\wedge e_3\big\rangle \;=\; 1$$

while $\langle e_i\wedge e_j, e_k\wedge e_\ell\rangle = 0$ when $\{i,j\}\cap\{k,\ell\}\neq\phi$. Thus, an orthogonal basis is

$$(e_1\wedge e_2)\pm(e_3\wedge e_4) \qquad (e_1\wedge e_3)\pm(e_2\wedge e_4) \qquad (e_1\wedge e_4)\pm(e_2\wedge e_3)$$

with $\langle,\rangle$ values $\pm 2, \mp 2, \pm 2$.

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[4] It is not necessary to choose a basis for $\mathbb{C}^4$, only to choose a basis for $\bigwedge^4\mathbb{C}^4$.

2. *Over* $\mathbb{R}$

Each homomorphism of complex groups gives rise to several homomorphisms of real groups.

[2.1] $SU(2) \to SO(3)$ The standard special unitary group $SU(2)$ is

$$SU(2) \;=\; \{g \in SL_2(\mathbb{C}) \;:\; g^* g = 1_2\} \qquad\qquad \text{(where } g^* \text{ is } g\text{-conjugate-transpose)}$$

The space V of 2-by-2 skew-hermitian complex matrices with trace 0 has symmetric real-valued real-bilinear form $\langle x, y\rangle = \mathrm{Re}(\mathrm{tr}(xy))$. An orthogonal basis is

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Each has value -2 for $\langle,\rangle$, so the *signature* of $\langle,\rangle$ on V is $(0,3)$. The action of $SU(2)$ on V by $g \cdot x = gxg^*$ preserves $\langle,\rangle$, because

$$\mathrm{tr}(gxg^* \cdot gyg^*) \;=\; \mathrm{tr}(g \cdot xy \cdot g^{-1}) \;=\; \mathrm{tr}(xy)$$

Thus, $SU(2)$ maps to a copy of $SO(3)$. The kernel is just $\{\pm 1\}$.

[2.2] $SL_2(\mathbb{R}) \to SO(2,1)$ The space V of 2-by-2 real matrices with trace 0, with symmetric bilinear form $\langle x, y\rangle = \mathrm{tr}(xy)$, has orthogonal basis

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The values of $\langle,\rangle$ are respectively 2, 2, -2, giving *signature* $(2,1)$. The action of $SL_2(\mathbb{R})$ on V by $g\cdot x = gxg^{-1}$ preserves $\langle,\rangle$:

$$\langle g\cdot x, g\cdot y\rangle \;=\; \mathrm{tr}(gxg^{-1}\cdot gyg^{-1}) \;=\; \mathrm{tr}(g\cdot xy\cdot g^{-1}) \;=\; \mathrm{tr}(xy) \;=\; \langle x, y\rangle$$

Thus, $SL_2(\mathbb{R})$ maps to a copy of $SO(2,1)$. The kernel is just $\{\pm 1\}$.

[2.3] $SU(2)\times SU(2) \to SO(4)$ Let [5]

$$V \;=\; \{\text{complex 2-by-2 matrices } x \;:\; x^* = wx^\top w^{-1}\} \qquad\qquad \Big(\text{with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Big)$$

$$=\; \{\text{2-by-2 complex matrices of the form } \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \text{ with } \alpha, \beta \in \mathbb{C}\}$$

Let $(g,h) \in SU(2)\times SU(2)$ act by $(g,h)\cdot x = gxh^*$. Give V the bilinear form

$$\langle x, y\rangle \;=\; \mathrm{Re}(\mathrm{tr}(xy^*))$$

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[5] It is not a coincidence that the vectorspace is a standard model of the Hamiltonian quaternions:

$$a + bi + cj + dk \;\longrightarrow\; \begin{pmatrix} a+bi & c+di \\ c-di & a-bi \end{pmatrix}$$

For $g \in SU(2) \subset SL_2(\mathbb{R})$, $g^{-1} = wg^\top w^{-1}$, giving the stabilization of V by the group action:

$$w(gxh^*)^\top w^{-1} \;=\; w(h^*)^\top w^{-1} \cdot wx^\top w^{-1} \cdot wg^\top w^{-1} \;=\; (h^*)^{-1} x^* g^{-1} \;=\; hx^*g^* \;=\; (gxh^*)^*$$

The pairing is invariant under the group action:

$$\mathrm{tr}(gxh^{-1} \cdot w(gyh^{-1})^\top w^{-1}) \;=\; \mathrm{tr}(gxh^{-1} \cdot w(h^{-1})^\top w^{-1} \cdot wy^\top w^{-1} \cdot wg^\top w^{-1})$$
$$=\; \mathrm{tr}(gxh^{-1} \cdot h \cdot wy^\top w^{-1} \cdot g^{-1}) \;=\; \mathrm{tr}(g \cdot xwy^\top w^{-1} \cdot g^{-1}) \;=\; \mathrm{tr}(xwy^\top w^{-1})$$

Computing

$$\langle \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \rangle \;=\; \mathrm{tr}\Big(\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} \Big) \;=\; \mathrm{tr} \begin{pmatrix} \alpha\overline{\alpha} + \beta\overline{\beta} & * \\ * & \alpha\overline{\alpha} + \beta\overline{\beta} \end{pmatrix}$$

an orthogonal basis is readily found: for example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with $\langle,\rangle$ values $2, 2, 2, 2$.

[2.4] $SL_2(\mathbb{C}) \to SO(3,1)$ With

$$V \;=\; \{\text{complex 2-by-2 matrices } x \;:\; x^* = wxw^{-1}\} \qquad (\text{with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$
$$=\; \{\text{2-by-2 complex matrices of the form } \begin{pmatrix} \alpha & ib \\ ic & \overline{\alpha} \end{pmatrix} \text{ with } \alpha \in \mathbb{C},\, b, c \in \mathbb{R}\}$$

use the $\mathbb{R}$-bilinear $\mathbb{R}$-valued form $\langle x, y\rangle \;=\; \mathrm{Re}\,(\mathrm{tr}(x\overline{y}))$, where the overline denotes entry-wise complex conjugation. An orthogonal basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

with $\langle,\rangle$ values $2, 2, 2, -2$. Thus, the signature of $\langle,\rangle$ is $3, 1$. The action $g \cdot x = gx\overline{g}^{-1}$ preserves the bilinear form $\langle x, y\rangle = \mathrm{Re}\,(\mathrm{tr}(x\overline{y}))$ on the larger $\mathbb{R}$-vectorspace of *all* complex 2-by-2 matrices, since

$$\mathrm{tr}(gx\overline{g}^{-1} \cdot \overline{gy\overline{g}^{-1}}) \;=\; \mathrm{tr}(gx\overline{g}^{-1} \cdot \overline{g}\overline{y}g^{-1}) \;=\; \mathrm{tr}(g \cdot x\overline{y} \cdot g^{-1}) \;=\; \mathrm{tr}(x\overline{y})$$

To check that $SL_2(\mathbb{C})$ stabilizes V, recall that $g^{-1} = wg^\top w^{-1}$ for $g \in SL_2(\mathbb{C})$. For $y \in V$, by design,

$$(gy\overline{g}^{-1})^* \;=\; (\overline{g}^{-1})^* y^* g^* \;=\; (g^\top)^{-1} \cdot wyw^{-1} \cdot \overline{g}^\top \;=\; w(g^\top)^\top w^{-1} \cdot wyw^{-1} \cdot w\overline{g}^{-1}w^{-1}$$
$$=\; wgw^{-1} \cdot wyw^{-1} \cdot w(\overline{g}^{-1})w^{-1} \;=\; w(gy\overline{g}^{-1})w^{-1}$$

so $SL_2(\mathbb{C})$ stabilizes V, and maps to a copy of $SO(3,1)$. The kernel is just $\{\pm 1\}$.

[2.5] $SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to SO(2,2)$ Let V be 2-by-2 real matrices, with $(g, h) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ acting by $(g, h) \cdot x = gxh^{-1}$. Give V the bilinear form

$$\langle x, y\rangle \;=\; \mathrm{tr}(x \cdot wy^\top w^{-1}) \qquad (\text{where } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$$

It is *symmetric* because trace is invariant under transpose, and because $w^{-1} = -w$. For $g \in SL_2(\mathbb{R})$, $g^{-1} = wg^\top w^{-1}$, and the pairing is invariant under the group action:

$$\mathrm{tr}(gxh^{-1} \cdot w(gyh^{-1})^\top w^{-1}) \;=\; \mathrm{tr}(gxh^{-1} \cdot w(h^{-1})^\top w^{-1} \cdot wy^\top w^{-1} \cdot wg^\top w^{-1})$$

$$=\; \mathrm{tr}(gxh^{-1} \cdot h \cdot wy^\top w^{-1} \cdot g^{-1}) \;=\; \mathrm{tr}(g \cdot xwy^\top w^{-1} \cdot g^{-1}) \;=\; \mathrm{tr}(xwy^\top w^{-1})$$

Computing

$$\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \rangle \;=\; \mathrm{tr}\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} \Big) \;=\; \mathrm{tr} \begin{pmatrix} ad' - bc' & * \\ * & da' - cb' \end{pmatrix} \;=\; ad' - bc' - cb' + da'$$

an orthogonal basis is readily found: for example,[6]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\langle , \rangle$ values $2, -2, 2, -2$, giving the desired signature.

[2.6] $Sp^*(2,0) \to SO(5)$ Let $\mathbb{H}$ be the Hamiltonian quaternions. One model of $G = Sp_2^* = Sp^*(2,0)$ is

$$Sp^*(2,0) \;=\; \{g \in GL_2(\mathbb{H}) \;:\; g^* g = 1_2\}$$

where $g^* = \overline{g}^\top$ with entry-wise quaternion conjugation. The $\mathbb{R}$-vectorspace V will be a subspace of the space $M_2(\mathbb{H})$ of 2-by-2 matrices with entries in $\mathbb{H}$. Let λ be the reduced trace

$$\lambda \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;=\; \tfrac{1}{2} \cdot (\alpha + \overline{\alpha} + \delta + \overline{\delta})$$

and on $M_2(\mathbb{H})$ let $\langle x, y \rangle = \lambda(xy)$. Let G act on $M_2(\mathbb{H})$ by $g \cdot x = gxg^*$. This action respects $\langle , \rangle$:

$$\langle g \cdot x, g \cdot y \rangle \;=\; \lambda(gxg^* \cdot gyg^*) \;=\; \lambda(g \cdot xy \cdot g^{-1}) \;=\; \lambda(xy)$$

Thus,

$$V \;=\; \{y \in M_2(\mathbb{H}) \;:\; y^* = y \text{ and } \langle y, 1_2 \rangle = 0\} \;=\; \{ \begin{pmatrix} a & \beta \\ \overline{\beta} & -a \end{pmatrix} \;:\; a \in \mathbb{R},\; \beta \in \mathbb{H}\}$$

is stable under this action, and $\dim_{\mathbb{R}} V = 5$. An orthogonal basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

with values $2, 2, 2, 2, 2$, giving the desired signature.

[2.7] $Sp^*(1,1) \to SO(4,1)$ One model of $G = Sp^*(1,1)$ is

$$Sp^*(1,1) \;=\; \{g \in GL_2(\mathbb{H}) \;:\; g^* S g = S\} \qquad\qquad (\text{with } S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

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[6] One can also observe from this expression that the bilinear form is a sum of two *hyperbolic planes*, thus giving signature $(2,2)$ without further computation.

where $g^* = \overline{g}^\top$ with entry-wise quaternion conjugation. Let $g^\sigma = Sg^*S^{-1}$, so the defining condition is $g^\sigma g = 1_2$. The $\mathbb{R}$-vectorspace V will be a subspace of the space $M_2(\mathbb{H})$ of 2-by-2 matrices with entries in $\mathbb{H}$. Let $\langle x, y\rangle = \lambda(xy)$. Let G act on $M_2(\mathbb{H})$ by $g \cdot x = gxg^\sigma$. This action respects $\langle , \rangle$:

$$\langle g\cdot x, g\cdot y\rangle \;=\; \lambda(gxg^\sigma \cdot gyg^\sigma) \;=\; \lambda(g\cdot xy\cdot g^\sigma) \;=\; \lambda(g\cdot xy\cdot g^{-1}) \;=\; \lambda(xy) \;=\; \langle x,y\rangle$$

The $\mathbb{R}$-vectorspace is

$$V \;=\; \{x \in M_2(\mathbb{H}) \;:\; x^\sigma = x \text{ and } \langle x, S\rangle = 0\} \;=\; \{\begin{pmatrix} \alpha & b \\ -b & \overline{\alpha}\end{pmatrix} \;:\; \alpha\in\mathbb{H},\; b\in\mathbb{R}\}$$

and is stable under the action, and $\dim_\mathbb{R} V = 5$. An orthogonal basis is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix} \qquad \begin{pmatrix} j & 0 \\ 0 & -j\end{pmatrix} \qquad \begin{pmatrix} k & 0 \\ 0 & -k\end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

with values $2, -2, -2, -2, -2$, giving the desired signature.

[2.8] $Sp_2(\mathbb{R}) \to SO(3,2)$ The symplectic group is

$$Sp_2(\mathbb{R}) \;=\; \{g \in GL_4(\mathbb{R}) \;:\; g^\top J g = J\} \qquad\qquad (\text{with } J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{pmatrix})$$

Write $g^\sigma = Jg^\top J^{-1}$, so the condition can be rewritten as $g^\sigma g = 1_2$. The $\mathbb{R}$-vectorspace V will be a subspace of the space $M_4(\mathbb{R})$ of 4-by-4 real matrices. Let $\langle x,y\rangle = \mathrm{tr}(xy)$. Let $Sp_2(\mathbb{R})$ act on $M_4(\mathbb{R})$ by $g\cdot x = gxg^\sigma$. This action respects $\langle , \rangle$:

$$\langle g\cdot x, g\cdot y\rangle \;=\; \mathrm{tr}(gxg^\sigma\cdot gyg^\sigma) \;=\; \mathrm{tr}(g\cdot xy\cdot g^{-1}) \;=\; \mathrm{tr}(xy) \;=\; \langle x,y\rangle$$

Since $1_4 = g^\sigma g = g\,1_4\,g^\sigma$, the action has fixed-point 1_4, and the subspace

$$V \;=\; \{x\in M_4(\mathbb{R}) \;:\; x^\sigma = x \text{ and } \langle x, 1_4\rangle = 0\}$$

is *stable* under the action. In 2-by-2 blocks, the condition $x^\sigma = x$ is

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}^\top\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a^\top & c^\top \\ b^\top & d^\top\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \;=\; \begin{pmatrix} d^\top & -b^\top \\ -c^\top & a^\top\end{pmatrix}$$

Thus, $d = a^\top$ and b, c are skew-symmetric. The condition $\langle x, 1_4\rangle = 0$ requires that $\mathrm{tr}(a) = 0$. Thus, $\dim_\mathbb{R} V = 5$. The easily observed orthogonal basis

$$\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & 1 & 0 \\ & & 0 & -1\end{pmatrix} \quad \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0\end{pmatrix} \quad \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0\end{pmatrix} \quad \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & & \\ -1 & 0 & & \end{pmatrix} \quad \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix}$$

has $\langle , \rangle$ values $4, 4, -4, -4, 4$, giving signature $3, 2$.

[2.9] $SU(4) \to SO(6)$ Let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{C}^4$. Give $\bigwedge^2\mathbb{C}^4$ the $\mathbb{C}$-valued $SL_4(\mathbb{C})$-invariant symmetric form

$$\langle x\wedge y, z\wedge w\rangle\cdot e_1\wedge e_2\wedge e_3\wedge e_4 \;=\; x\wedge y\wedge z\wedge w \qquad\qquad (\text{for } x,y,z,w\in\mathbb{C}^4)$$

A six-dimensional $\mathbb{R}$-subspace of $\bigwedge^2\mathbb{C}^4$ stable under $SU(4)$ will be identified as the fixed vectors of an $\mathbb{C}$-conjugate-linear isomorphism $J : \mathbb{C}^4 \to \mathbb{C}^4$ commuting with $SU(4)$, on which $\langle,\rangle$ takes real values.

To make such J, use the positive-definite hermitian form $(x, y) = y^*x$ on $\mathbb{C}^4$ invariant under $SU(4)$, giving a $\mathbb{C}$-conjugate-linear isomorphism $\mathbb{C}^4 \to (\mathbb{C}^4)^*$ by $x \to (y \to (y, x))$, which induces $\bigwedge^2\mathbb{C}^4 \to \bigwedge^2(\mathbb{C}^{4*}) \approx (\bigwedge^2\mathbb{C}^4)^*$. At the same time, the non-degenerate form $\langle,\rangle$ on $\bigwedge^2\mathbb{C}^4$ gives a $\mathbb{C}$-linear isomorphism $\bigwedge^2\mathbb{C}^4 \to \bigwedge^2\mathbb{C}^4$ by $v \to (w \to \langle w, v\rangle)$. Combining these,

$$\bigwedge^2\mathbb{C}^4 \xrightarrow{\langle,\rangle} (\bigwedge^2\mathbb{C}^4)^* \xrightarrow{\approx} \bigwedge^2(\mathbb{C}^{4*}) \xleftarrow{(,)\wedge(,)} \bigwedge^2\mathbb{C}^4 \qquad (J : \text{first} \to \text{last})$$

with the right-to-left arrow a $\mathbb{C}$-conjugate-linear isomorphism, gives a $\mathbb{C}$-conjugate-linear isomorphism J of $\bigwedge^2\mathbb{C}^4$ to itself. Since $SU(4)$ respects both $\langle,\rangle$ and $(,)$, the map J commutes with $SU(4)$. This is noted element-wise below.

We can track basis elements $e_k \wedge e_\ell$ under J. Since functionals $\langle -, e_1 \wedge e_2\rangle$ and $(-, e_3) \wedge (-, e_4)$ both compute the $e_3 \wedge e_4$ component of $\sum_{k<\ell} c_{k\ell} e_k \wedge e_\ell$, we have $J(e_1 \wedge e_2) = e_3 \wedge e_4$. That J commutes with the action of $g \in SU(4)$ can be made explicit:

$$g \cdot (e_1 \wedge e_2) \;\to\; \langle -, g(e_1 \wedge e_2)\rangle \;=\; \langle g^{-1}(-), e_1 \wedge e_2\rangle \;=\; g^{-1} \circ \langle -, e_1 \wedge e_2\rangle \;=\; g^{-1} \circ (-, e_3) \wedge (-, e_4)$$

$$=\; (g^{-1}(-), e_3) \wedge (g^{-1}(-), e_4) \;=\; (-, ge_3) \wedge (-, ge_4) \;\to\; ge_3 \wedge ge_4 \;=\; g \cdot (e_3 \wedge e_4) \;=\; g \cdot J(e_1 \wedge e_2)$$

A similar computation gives $J(e_3 \wedge e_4) = e_1 \wedge e_2$. Since $(,) \wedge (,)$ is conjugate-linear,

$$ie_1 \wedge e_2 \;\to\; i\langle -, e_1 \wedge e_2\rangle \;\to\; i(-, e_3) \wedge (-, e_4) \;=\; (-, (-i)e_3) \wedge (-, e_4) \;\to\; -ie_3 \wedge e_4$$

and $J(ie_3 \wedge e_4) = -ie_1 \wedge e_2$. Thus, on the real four-dimensional space with basis

$$e_1 \wedge e_2 \qquad e_3 \wedge e_4 \qquad ie_1 \wedge e_2 \qquad ie_3 \wedge e_4$$

the map J is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Thus, $J^2 = 1$ on this subspace, and this subspace has ± 1 eigenspaces of equal dimension. Similarly, functionals $(-1)\langle -, e_1 \wedge e_3\rangle$ and $(-, e_2) \wedge (-, e_4)$ both compute the $e_2 \wedge e_4$ component, and $(-1)\langle -, e_2 \wedge e_4\rangle$ and $(-, e_1) \wedge (-, e_3)$ both compute the $e_1 \wedge e_3$ component, so, noting the signs,

$$J(e_1 \wedge e_3) \;=\; -e_2 \wedge e_4 \qquad J(ie_1 \wedge e_3) \;=\; ie_2 \wedge e_4 \qquad J(e_2 \wedge e_4) \;=\; -e_1 \wedge e_3 \qquad J(ie_2 \wedge e_4) \;=\; ie_1 \wedge e_3$$

Thus, $J^2 = 1$ on this subspace, and this subspace has ± 1 eigenspaces of equal dimension. Functionals $\langle -, e_1 \wedge e_4\rangle$ and $(-, e_2) \wedge (-, e_3)$ both compute the $e_2 \wedge e_3$ component, and symmetrically, so

$$J(e_1 \wedge e_4) \;=\; e_2 \wedge e_3 \qquad J(ie_1 \wedge e_4) \;=\; -ie_2 \wedge e_3 \qquad J(e_2 \wedge e_3) \;=\; e_1 \wedge e_4 \qquad J(ie_2 \wedge e_3) \;=\; -ie_1 \wedge e_4$$

Again, $J^2 = 1$ on this subspace, and this subspace has ± 1 eigenspaces of equal dimension. An orthogonal basis for the $+1$-eigenspace for J is

$$e_1\wedge e_2+e_3\wedge e_4 \quad ie_1\wedge e_2-ie_3\wedge e_4 \quad e_1\wedge e_3-e_2\wedge e_3 \quad ie_1\wedge e_3+ie_2\wedge e_3 \quad e_1\wedge e_4+e_2\wedge e_3 \quad ie_1\wedge e_4-ie_2\wedge e_3$$

with $\langle,\rangle$ values $2, 2, 2, 2, 2, 2$.

[2.10] $SL_2(\mathbb{H}) \to SO(5,1)$ Imbed $\mathbb{H} \subset M_2(\mathbb{C})$ by

$$a + bi + cj + dk \longrightarrow \begin{pmatrix} a+bi & c+dj \\ -c+di & a-bi \end{pmatrix} \qquad \text{(with } a,b,c,d \in \mathbb{R}\text{)}$$

Note the characterization

$$\mathbb{H} \;=\; \{x \in M_2(\mathbb{C}) \;:\; \overline{x} \;=\; wxw^{-1}\} \qquad \text{(with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\text{)}$$

Thus, identify

$$SL_2(\mathbb{H}) \;=\; \{g \in SL_4(\mathbb{C}) \;:\; \overline{g} = WgW^{-1}\} \qquad \text{(where } W = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}\text{)}$$

where $g \to \overline{g}$ is entry-wise conjugation. Let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{C}^4$, and give $\bigwedge^2\mathbb{C}^4$ the $\mathbb{C}$-valued $SL_4(\mathbb{C})$-invariant symmetric form

$$\langle x \wedge y, z \wedge w\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \;=\; x \wedge y \wedge z \wedge w \qquad \text{(for } x,y,z,w \in \mathbb{C}^4\text{)}$$

A six-dimensional $\mathbb{R}$-subspace of $\bigwedge^2\mathbb{C}^4$ stable under $SU(4)$ will be identified as the fixed vectors of an $\mathbb{C}$-conjugate-linear isomorphism $J : \mathbb{C}^4 \to \mathbb{C}^4$ commuting with $SL_2(\mathbb{H})$, on which $\langle,\rangle$ takes real values.

Define conjugate-linear $J : \bigwedge^2\mathbb{C}^4 \to \bigwedge^2\mathbb{C}^4$ by

$$J(x \wedge y) \;=\; W\overline{x} \wedge W\overline{y}$$

By design, J commutes with the action of $g \in SL_2(\mathbb{H})$:

$$g \cdot J(x \wedge y) \;=\; gW\overline{x} \wedge gW\overline{y} \;=\; W\overline{W^{-1}\overline{g}Wx} \wedge W\overline{W^{-1}\overline{g}Wy} \;=\; W\overline{gx} \wedge W\overline{gy} \;=\; J(g \cdot x \wedge y)$$

The effect of J on $e_k \wedge e_\ell$ and $ie_k \wedge e_\ell$ is readily computed, since $We_1 = e_2$, $We_2 = -e_1$, $We_3 = e_4$, and $We_4 = -e_3$:

$$J(e_1 \wedge e_2) = -e_2 \wedge e_1 = e_1 \wedge e_2 \qquad J(e_3 \wedge e_4) = -e_4 \wedge e_3 = e_3 \wedge e_4$$

while

$$J(e_1 \wedge e_3) = e_2 \wedge e_4 \qquad J(e_2 \wedge e_4) = -e_1 \wedge -e_3 = e_1 \wedge e_3$$

and

$$J(e_1 \wedge e_4) = e_2 \wedge -e_3 = -e_2 \wedge e_3 \qquad J(e_2 \wedge e_3) = -e_1 \wedge e_4$$

Visibly, $J^2 = 1$ on these vectors. Since J is conjugate-linear, we have $J^2 = 1$. An orthogonal basis for $+1$ eigenvectors is

$$e_1 \wedge e_2 + e_3 \wedge e_4 \quad e_1 \wedge e_2 - e_3 \wedge e_4 \quad e_1 \wedge e_3 + e_2 \wedge e_4 \quad ie_1 \wedge e_3 - ie_2 \wedge e_4 \quad e_1 \wedge e_4 - e_2 \wedge e_3 \quad ie_1 \wedge e_4 + ie_2 \wedge e_3$$

with $\langle,\rangle$ values $2, -2, -2, -2, -2, -2$.

[2.11] $SU(2,2) \to SO(4,2)$ One model of $SU(2,2)$ is

$$SU(2,2) \;=\; \{g \in SL_4(\mathbb{C}) \;:\; g^*Sg = S\} \qquad \text{(where } S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\text{)}$$

Again, with e_1, e_2, e_3, e_4 the standard basis for $\mathbb{C}^4$, give $\bigwedge^2\mathbb{C}^4$ the $\mathbb{C}$-valued symmetric form

$$\langle x\wedge y, z\wedge w\rangle \cdot e_1\wedge e_2\wedge e_3\wedge e_4 \;=\; x\wedge y\wedge z\wedge w \qquad\qquad (\text{for } x,y,z,w\in\mathbb{C}^4)$$

A six-dimensional $\mathbb{R}$-subspace of $\bigwedge^2\mathbb{C}^4$ stable under $SU(2,2)$ will be identified as the fixed vectors of an $\mathbb{C}$-conjugate-linear isomorphism $J:\mathbb{C}^4\to\mathbb{C}^4$ commuting with $SU(2,2)$, and on which $\langle,\rangle$ takes real values.

Use the non-degenerate hermitian form

$$(x,y) = y^*Sx$$

on $\mathbb{C}^4$ invariant under $SU(2,2)$, giving $\mathbb{C}$-conjugate-linear isomorphism $\mathbb{C}^4\to(\mathbb{C}^4)^*$ by $x\to(y\to(y,x))$, which induces $\bigwedge^2\mathbb{C}^4\to\bigwedge^2(\mathbb{C}^{4*})\approx(\bigwedge^2\mathbb{C}^4)^*$. At the same time, the non-degenerate form $\langle,\rangle$ on $\bigwedge^2\mathbb{C}^4$ gives a $\mathbb{C}$-linear isomorphism $\bigwedge^2\mathbb{C}^4\to\bigwedge^2\mathbb{C}^4$ by $v\to(w\to\langle w,v\rangle)$. Combining these,

$$\bigwedge^2\mathbb{C}^4 \xrightarrow{\langle,\rangle} (\bigwedge^2\mathbb{C}^4)^* \xrightarrow{\approx} \bigwedge^2(\mathbb{C}^{4*}) \xleftarrow{(,)\wedge(,)} \bigwedge^2\mathbb{C}^4 \qquad (J : \text{first } \bigwedge^2\mathbb{C}^4 \to \text{last } \bigwedge^2\mathbb{C}^4)$$

with the right-to-left arrow a $\mathbb{C}$-conjugate-linear isomorphism, gives a $\mathbb{C}$-conjugate-linear isomorphism J of $\bigwedge^2\mathbb{C}^4$ to itself. Since $SU(2)$ respects both $\langle,\rangle$ and $(,)$, the map J commutes with $SU(2)$. This is noted element-wise below. It is important to check that $J^2=1$.

Tracking $e_k\wedge e_\ell$ and $ie_k\wedge e_\ell$ under J is nearly identical to that for $SU(4)$, with important sign flips.

Functionals $\langle -, e_1\wedge e_2\rangle$ and $(-,e_3)\wedge(-,e_4)$ both compute the $e_3\wedge e_4$ component of $\sum_{k<\ell} c_{k\ell}e_k\wedge e_\ell$. The two sign flips from $(e_3,e_3)=-1$ and $(e_4,e_4)=-1$ cancel. Thus, $J(e_1\wedge e_2)=e_3\wedge e_4$. A similar computation gives $J(e_3\wedge e_4)=e_1\wedge e_2$. Since $(,)\wedge(,)$ is conjugate-linear, $J(ie_1\wedge e_2)=-ie_3\wedge e_4$ and and $J(ie_3\wedge e_4)=-ie_1\wedge e_2$. Thus, on the real four-dimensional space with basis

$$e_1\wedge e_2 \qquad e_3\wedge e_4 \qquad ie_1\wedge e_2 \qquad ie_3\wedge e_4$$

the map J is

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\oplus\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

Thus, $J^2=1$ on this subspace, and this subspace has ±1 eigenspaces of equal dimension. This part is identical to that for $SU(2)$.

Functionals $(-1)\langle -, e_1\wedge e_3\rangle$ and $(-1)(-,e_2)\wedge(-,e_4)$ both compute the $e_2\wedge e_4$ component, with sign flip due to $(e_4,e_4)=-1$. Similarly, $(-1)\langle -, e_2\wedge e_4\rangle$ and $(-1)(-,e_1)\wedge(-,e_3)$ both compute the $e_1\wedge e_3$ component, with $(e_3,e_3)=-1$. Noting the signs,

$$J(e_1\wedge e_3) \;=\; e_2\wedge e_4 \qquad J(ie_1\wedge e_3) \;=\; -ie_2\wedge e_4 \qquad J(e_2\wedge e_4) \;=\; e_1\wedge e_3 \qquad J(ie_2\wedge e_4) \;=\; -ie_1\wedge e_3$$

Thus, $J^2=1$ on this subspace, with ±1 eigenspaces of equal dimension. Functionals $\langle -, e_1\wedge e_4\rangle$ and $(-1)(-,e_2)\wedge(-,e_3)$ both compute the $e_2\wedge e_3$ component, so

$$J(e_1\wedge e_4) \;=\; -e_2\wedge e_3 \qquad J(ie_1\wedge e_4) \;=\; -ie_2\wedge e_3$$

Functionals $\langle -, e_2\wedge e_3\rangle$ and $(-1)(-,e_1)\wedge(-,e_4)$ both compute the $e_1\wedge e_4$ component, so

$$J(e_2\wedge e_3) \;=\; -e_1\wedge e_4 \qquad J(ie_2\wedge e_3) \;=\; ie_1\wedge e_4$$

Again, $J^2=1$ on this subspace, with ±1 eigenspaces of equal dimension. An orthogonal basis for the $+1$-eigenspace for J is

$$e_1\wedge e_2+e_3\wedge e_4 \quad ie_1\wedge e_2-ie_3\wedge e_4 \quad e_1\wedge e_3+e_2\wedge e_3 \quad ie_1\wedge e_3-ie_2\wedge e_3 \quad e_1\wedge e_4-e_2\wedge e_3 \quad ie_1\wedge e_4+ie_2\wedge e_3$$

The last four have sign flips in comparison to the analogous basis for $SU(4)$, giving $\langle,\rangle$ values $2, 2, -2, -2, -2, -2$.

[2.12] $SL_4(\mathbb{R}) \to SO(3,3)$ This is just the obvious real form of the isogeny for $SL_4(\mathbb{C})$ above. Let $SL_4(\mathbb{R})$ act in the natural way on the six-dimensional vectorspace $V = \bigwedge^2 \mathbb{R}^4$, namely, $g \cdot (v \wedge w) = gv \wedge gw$. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$, and define $\langle,\rangle$ on V by

$$x \wedge y \;=\; \langle x, y\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \qquad \text{(with } x, y \in {\textstyle\bigwedge}^2 \mathbb{R}^4\text{)}$$

This form is *symmetric* because an even number of transpositions reverses the arguments:

$$(x \wedge y) \wedge (z \wedge w) \;=\; -x \wedge z \wedge y \wedge w \;=\; x \wedge z \wedge w \wedge y \;=\; -z \wedge x \wedge w \wedge y$$

$$=\; -z \wedge x \wedge w \wedge y \;=\; (z \wedge w) \wedge (x \wedge y) \qquad \text{(for } x, y, z, y \in \mathbb{R}^4\text{)}$$

The form is invariant under the action because

$$\big\langle g \cdot (x \wedge y), g \cdot (z \wedge w)\big\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \;=\; gx \wedge gy \wedge gz \wedge gw \;=\; \det g \cdot x \wedge y \wedge z \wedge w$$

$$=\; \det g \cdot \big\langle x \wedge y, z \wedge w\big\rangle \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4$$

To check non-degeneracy, observe

$$\big\langle e_1 \wedge e_2, e_3 \wedge e_4\big\rangle \;=\; 1 \qquad \big\langle e_1 \wedge e_3, e_2 \wedge e_4\big\rangle \;=\; -1 \qquad \big\langle e_1 \wedge e_4, e_2 \wedge e_3\big\rangle \;=\; 1$$

while $\langle e_i \wedge e_j, e_k \wedge e_\ell\rangle = 0$ when $\{i, j\} \cap \{k, \ell\} \neq \phi$. Thus, an orthogonal basis is

$$(e_1 \wedge e_2) \pm (e_3 \wedge e_4) \qquad (e_1 \wedge e_3) \pm (e_2 \wedge e_4) \qquad (e_1 \wedge e_4) \pm (e_2 \wedge e_3)$$

with $\langle,\rangle$ values $\pm 2, \mp 2, \pm 2$.

[2.13] Why not $SU(3,1)$? [7] One model of $SU(3,1)$ is

$$SU(3,1) \;=\; \{g \in SL_4(\mathbb{C}) \;:\; g^* S g = S\} \qquad \text{(where } S = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\text{)}$$

We could attempt the same procedure for $SU(3,1)$ as for $SU(4)$, $SL_2(\mathbb{H})$, and $SU(2,2)$, by arranging a conjugate-linear map J on $\bigwedge^2 \mathbb{C}^4$ and commuting with $SU(3,1)$, and hoping that the $SL_4(\mathbb{C})$-invariant $\mathbb{C}$-valued form $\langle,\rangle$ on $\bigwedge^2 \mathbb{C}^4$ is real-valued on J-eigenspaces. Indeed, the same diagrammatic description of J produces a conjugate-linear map J commuting with $SU(3,1)$, so $SU(3,1)$ stabilizes eigenspaces of J.

However, $J^2 = -1$, not $+1$, on $\mathbb{C}^4$:

The functionals $\langle -, e_1 \wedge e_2\rangle$ and $(-1)(-, e_3) \wedge (-, e_4)$ both compute the $e_3 \wedge e_4$ component, so

$$J(e_1 \wedge e_2) \;=\; -e_3 \wedge e_4$$

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[7] Also, as S. Zemel notes, the maximal compact $S(U(3) \times U(1))$ of $SU(3,1)$ has dimension 9, which is *not* the dimension $\frac{p(p-1}{2} + \frac{q(q-1)}{2}$ of the maximal compact $O(p) \times O(q)$ of $O(p,q)$ for any $p + q = 6$.

while $\langle -, e_3 \wedge e_4\rangle$ and $(-, e_1) \wedge (-, e_2)$ both compute the $e_1 \wedge e_2$ component, so

$$J(e_3 \wedge e_4) \;=\; e_1 \wedge e_2$$

Similarly, $(-1)\langle -, e_1 \wedge e_3\rangle$ and $(-1)(-, e_2) \wedge (-, e_4)$ both compute the $e_2 \wedge e_4$ component, so

$$J(e_1 \wedge e_3) \;=\; e_2 \wedge e_4$$

while $(-1)\langle -, e_2 \wedge e_4\rangle$ and $(-, e_1) \wedge (-, e_3)$ both compute the $e_1 \wedge e_3$ component, giving

$$J(e_2 \wedge e_4) \;=\; -e_1 \wedge e_3$$

Functionals $\langle -, e_1 \wedge e_4\rangle$ and $(-, e_2) \wedge (-, e_3)$ both compute the $e_2 \wedge e_3$ component, so

$$J(e_1 \wedge e_4) \;=\; e_3 \wedge e_3$$

while $\langle -, e_2 \wedge e_3\rangle$ and $(-1)(-, e_1) \wedge (-, e_4)$ both compute the $e_1 \wedge e_4$ component, so

$$J(e_2 \wedge e_3) \;=\; -e_1 \wedge e_4$$

Thus, $J^2 = -1$, not $+1$, on $\bigwedge^2 \mathbb{C}^4$. Thus, the only possible eigenvalues are $\pm i$.

Nevertheless, any J-eigenspace inside the $\mathbb{R}$-vectorspace $\bigwedge^2 \mathbb{C}^4$ is stabilized by $SU(3,1)$. But the conjugate-linearity of J shows that there cannot be $\pm i$-eigenvalues in $\bigwedge^2 \mathbb{C}^4$: if $Jv = iv$, then

$$-v \;=\; J^2 v \;=\; J(iv) \;=\; -iJv \;=\; (-i)iv \;=\; v$$

Thus, this device has failed to produce $SU(3,1)$-stable proper $\mathbb{R}$-subspaces of $\bigwedge^2 \mathbb{C}^4$.

3. *Appendix: isomorphism classes of forms over* $\mathbb{C}$ *and* $\mathbb{R}$

For convenience, we recall a classification over $\mathbb{C}$ and over $\mathbb{R}$: as elaborated below, *dimension* is the only invariant of non-degenerate symmetric bilinear forms over $\mathbb{C}$, and *signature* is the only invariant over $\mathbb{R}$.

A vector space V with a symmetric bilinear form over a field is *non-degenerate* when, for every $v \neq 0$ in V, there is $w \in V$ such that $\langle v, w\rangle \neq 0$.

The corresponding *orthogonal group* is the isometry group

$$\{g \in \mathrm{Aut}_k(V) : \langle gv, gw\rangle = \langle v, w\rangle, \text{ for all } v, w \in V\}$$

A basis $\{v_i\}$ is *orthogonal* when $\langle v_i, v_j\rangle = 0$ for $i \neq j$.

[3.1] Non-degenerate forms over $\mathbb{C}$ classified by dimension We claim that for a non-degenerate symmetric bilinear $\mathbb{C}$-valued form $\langle,\rangle$ on a finite-dimensional $\mathbb{C}$-vectorspace V, there is an orthogonal basis $v_1, \ldots, v_n$ such that $\langle v_i, v_i\rangle = 1$ for all i.

Given $v \neq 0$ in V, when $\langle v, v\rangle \neq 0$. Replace v by $v_1 = v/\sqrt{\langle v, v\rangle}$ with either square root, to arrange $\langle v_1, v_1\rangle = 1$. When $\langle v, v\rangle = 0$, use non-degeneracy to obtain w such that $\langle v, w\rangle \neq 0$. In case $\langle w, w\rangle \neq 0$, we are in the first case, and if $\langle w, w\rangle = 0$, then $\langle v + w, v + w\rangle = 2 \neq 0$, and again we are back to the first case.

That is, there is a vector with $\langle v, v\rangle = 1$.

To complete the induction argument, show that for $\langle v, v\rangle = 1$ the orthogonal complement

$$v^\perp = \{w \in V : \langle v, w\rangle = 0\}$$

is non-degenerate. Indeed, given $0 \neq v' \in v^\perp$, let $w \in V$ such that $\langle v', w\rangle \neq 0$. Retain this property while adjusting w to be in $v^\perp$ by replacing it by $w - \langle w, v\rangle$. ///

Thus, *dimension* is the only isomorphism-class invariant of non-degenerate symmetric bilinear forms over $\mathbb{C}$, or over any algebraically closed field of characteristic not 2. The standard model is

$$O(n, \mathbb{C}) = \{g \in GL_n(\mathbb{C}) : g^\top g = 1_n\}$$

[3.2] Non-degenerate forms over $\mathbb{R}$ classified by signature We claim that for non-degenerate $\mathbb{R}$-valued symmetric bilinear form $\langle,\rangle$ on a finite-dimensional $\mathbb{C}$-vectorspace V, there are non-negative integers p, q and an orthogonal basis $v_1, \ldots, v_p, w_1, \ldots w_q$ such that that $\langle v_i, v_i\rangle = 1$ for $1 \le i \le p$ and $\langle w_j, w_j\rangle = -1$ for $1 \le j \le q$.

This is Sylvester's *law of inertia.* The pair (p, q) is the *signature.* The standard model is

$$O(p, q) = \{g \in GL_{p+q}(\mathbb{R}) : g^\top Q g = Q\} \qquad (\text{where } Q = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix})$$

Given $v \neq 0$, when $\langle v, v\rangle \neq 0$, replacing v by $v/\sqrt{|\langle v, v\rangle|}$ gives $\langle v, v\rangle = \pm 1$. When $\langle v, v\rangle = 0$, there is w such that $\langle v, w\rangle \neq 0$. In case $\langle w, w\rangle \neq 0$, we are back to the first case. When $\langle w, w\rangle = 0$, $\langle v + w, v + w\rangle = 2 \neq 0$, and again we are back to the first case.

Thus, there is v with $\langle v, v\rangle = \pm 1$.

An argument nearly identical to the complex case shows that $v^\perp$ is non-degenerate, so and induction gives *existence* of a signature.

For *uniqueness*, let a *totally isotropic* subspace W of V be a subspace on which $\langle,\rangle = 0$, that is, $\langle w, w'\rangle = 0$ for all $w, w' \in W$. A *maximal* totally isotropic subspace is also called *Lagrangian*.

We claim that all Lagrangian subspaces W have the *same dimension*. Uniqueness of signature will follow from showing this common dimension is $\min(p, q)$.

A reformulation of the definition of *maximal* totally isotropic is that $W^\perp$ is just W itself. Thus, for W' another maximal totally isotropic subspace, the non-degenerate $\langle,\rangle$ gives a non-degenerate pairing of $W/(W\cap W')$ and $W'/(W\cap W')$. A non-degenerate pairing between finite-dimensional vectorspaces gives an isomorphism of each to the dual of the other, so the dimensions are equal.

Next, given a totally isotropic subspace W, there is another totally isotropic subspace W' such that $\langle,\rangle$ is non-degenerate on $W + W'$. Indeed, given $w_1 \in W$, find w_1' such that $\langle w_1, w_1'\rangle \neq 0$. Without loss of generality, $\langle w_1', w_1'\rangle = 0$, since otherwise replace w_1' by $w_1' - \frac{1}{2}\langle w_1', w_1'\rangle \cdot w_1$. As above, $(\mathbb{R}w_1 + \mathbb{R}w_1')^\perp$ is non-degenerate, and $W \cap (\mathbb{R}w_1 + \mathbb{R}w_1')^\perp$ is codimension 1 inside W. Thus, an induction chooses a basis $w_1', \ldots, w_m'$ for another totally isotropic subspace W, with $\langle w_i, w_i'\rangle = 1$ for all i, and $\langle w_i, w_j'\rangle = 0$ for $i \neq j$.

Thus, given a Lagrangian subspace W, there are corresponding $w_1, w_1', \ldots w_m, w_m'$, and the collection $w_i \pm w_i'$ gives an orthogonal basis for the span of $W + W'$ with m positive and m negative values. Thus, $\min(p, q) \geq m$.

On the other hand, taking $p \geq q$ and orthogonal basis $v_1, \ldots, v_p, w_1, \ldots, w_q$ as above, $v_1 + w_1, \ldots, v_q + w_q$ spans a totally isotropic subspace. This gives the opposite inequality, proving that $\min(p, q)$ is the (common) dimension of Lagrangian subspaces. ///
