Transvectants, Modular Forms, and the Heisenberg Algebra

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Abstract. We discuss the amazing interconnections between normal form theory, classical invariant theory and transvectants, modular forms and Rankin–Cohen brackets, representations of the Heisenberg algebra, differential invariants, solitons, Hirota operators, star products and Moyal brackets and coherent states.

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1. Introduction.

The transvectants are the most important computational tool in the classical invariant theory of binary forms, [**1**, **9**, **15**, **22**, **34**]. The transvection of two covariants yields a new covariant; moreover, starting with the ground form, every polynomial covariant and invariant can be constructed by successive transvection. They constituted the essential tool in Gordan's celebrated constructive proof of the finiteness theorem for the covariants of binary forms, [**15**]. Even Hilbert's monumental generalizations — the Hilbert Basis Theorem and Hilbert Syzygy Theorem — were firmly rooted in the classical transvection processes, [**22**]. In the symbolic calculus of classical invariant theory, the transvectants are based on a fundamental differential operator, known as Cayley's omega process; a key step in our analysis is establishing a formula for the omega and transvectant processes in the projective variable.

In the theory of modular forms, Rankin, [**36**], and H. Cohen, [**10**], discovered a set of bracket operations that map modular forms to modular forms. Zagier, [**48**], noticed the similarity between the Rankin–Cohen brackets of modular forms and transvectants of binary forms, and wondered if there was any direct connection. In [**33**, **34**], the first author noted that if one regards the degree of a binary form as minus the weight of a modular form, then, in fact, the two processes are identical! In particular, the invariance of the Rankin–Cohen brackets under discrete subgroups of the projective group is an immediate consequence of the invariance of transvectants under the full group $\mathrm{SL}(2,\mathbb{C})$. This observation serves to motivate the introduction of a "duality" between binary forms (homogeneous polynomials, or, better, their projective counterparts) and modular forms, where the degree n of the former is minus the weight w of the latter: $n = -w$. (We are using "duality" in a very loose sense here.)

The purpose of this paper is to develop this connection in some depth. The key result is that the two theories of modular and binary forms have a common limiting theory as $n = -w \to \infty$. The underlying transformation group of the limiting theory is a three-dimensional Heisenberg group. This limiting procedure is made precise on the Lie algebra (infinitesimal) level, realizing the solvable Heisenberg algebra as a contraction, [**47**], of the semisimple unimodular algebra $\mathfrak{sl}(2,\mathbb{C})$. Complicated identities in the transvectant and Rankin–Cohen bracket algebras reduce to much simpler identities in the Heisenberg limit. Moreover, an explicit procedure, in the form of a quantum deformation, for returning to the classical versions is presented.

Our constructions were originally motivated by the normal form theory for ordinary differential equations, [**37**, **39**, **40**]. Given a nilpotent matrix N, which represents the linear part of a dynamical system, we seek to embed it in an $\mathfrak{sl}(2,\mathbb{R})$ algebra, whose basis elements B, C, N satisfy the standard commutation relations

$$[\,B,N\,] = C, \qquad [\,N,C\,] = N, \qquad [\,B,C\,] = -B.$$

The normal forms for the nonlinear part of the dynamical system are identified as elements of $\ker B$. In the continuum limit, the nilpotent operator becomes the (non-nilpotent) total derivative operator $\mathcal{D} = D_x$. The infinitesimal action of the projective group on binary or modular forms of degree $n = -w$ leads to an embedding of the total derivative into an

$\mathfrak{sl}(2)$ algebra with commutation relations

$$[\,\mathcal{D},\mathcal{C}^{(n)}\,] = \frac{2}{n}\,\mathcal{D}, \qquad [\,\mathcal{D},\mathcal{B}^{(n)}\,] = -\,\mathcal{C}^{(n)}, \qquad [\,\mathcal{C}^{(n)},\mathcal{B}^{(n)}\,] = \frac{2}{n}\,\mathcal{B}^{(n)}.$$

The normal forms or elements of $\ker \mathcal{B}^{(n)}$ can be identified as classical covariants, or, in the modular forms picture, as elements of the algebra generated by iterated Rankin–Cohen brackets. In the $n \to \infty$ limit, the $\mathfrak{sl}(2)$ algebra reduces to a Heisenberg algebra:

$$[\,\mathcal{B},\mathcal{D}\,] = \mathcal{C}, \qquad [\,\mathcal{C},\mathcal{B}\,] = [\,\mathcal{C},\mathcal{D}\,] = 0.$$

The kernel of the limiting operator $\mathcal{B}$ is particularly easy to describe, and so connecting the classical theory with the limiting theory is of great interest.

The classical covariants and invariants of a binary form are all expressed as differential polynomials in the base form, and hence are (relative) differential invariants for the underlying projective transformation group, [**33**]. In accordance with a classical algebraic result due to Gordan, all of these can be constructed by successive transvection starting with the ground form. We prove that the space of differential invariants can be identified with the kernel of the operator $\mathcal{B}^{(n)}$. Moreover, we exhibit an explicit rational basis for this space consisting of the simplest quadratic and cubic transvectants, thereby generalizing a classical result of Stroh, [**43**], and Hilbert, [**22**], that these transvectants form a rational basis for the covariants of a binary form of arbitrary degree. (The striking simplicity of this result is in direct contrast with the intractable — at least in high degree — problem of finding an explicit polynomial basis for the invariants and/or covariants of a binary form.)

In the Heisenberg limit, the transvectants or Rankin–Cohen brackets reduce to the bilinear Hirota operators that originally arose in the study of integrable systems such as the Korteweg–de Vries and Kadomtsev–Petviashvili hierarchies, [**23**, **24**, **26**, **31**]. The limiting differential invariants are simply the logarithmic derivatives $D_x^k \log u$, for $k \geq 2$, of the ground form u. Indeed, this observation underlies Sato's approach to the solution of integrable systems based on the logarithmic derivatives of the tau function, which can itself be viewed as a modular form, [**45**]. Since the classical projective theories are identified as deformations of the simpler, Heisenberg theory, one can view the transvectants as (quantum?) deformations of the Hirota operators. Furthermore, the differential invariants of the Heisenberg limit can be interpreted as "perpetuants", [**44**, **43**, **29**], which, in the classical theory, are identified as the covariants of binary forms of "infinite" degree.

The symbolic form of the transvectant processes in terms of the Cayley omega process leads to the introduction of the associative star product on a two-dimensional phase space. The anti-symmetrization of the star product is known as the Moyal bracket, and was introduced as a quantum mechanical deformation of the classical Poisson bracket, [**30**, **46**, **3**]. The projective version of the star product and Moyal brackets introduces an associative algebra — and hence Lie algebra — structure on the spaces of classical covariants and of modular forms, where it is known as the Eholzer product, [**12**, **11**]. In the Heisenberg limit, the star and Eholzer products reduce to a very simple form, that has deep connections with the remarkable "exp–log formula" in the Hirota formalism, [**26**].

Since one can write down elementary explicit formulae, the limiting Heisenberg theory avoids many of the algebraic complications in the more classical polynomial/modular form theories. In the penultimate section, we provide an explicit procedure for returning from the Heisenberg theory to the classical level. This observation should have important implications, both theoretical and practical, for simplifying the complicated classical algebraic manipulations through the simpler Heisenberg theory.

In the final section, motivated by developments in quantum many-particle systems theory, [**17**], we introduce the notion of a coherent state for both the Heisenberg and projective differential invariants. This leads to a multilinear extension of the transvectant/Rankin–Cohen brackets that solves the problem of generalizing the Hirota operators to the multilinear case in a natural manner. We conclude with some open questions and interesting directions to pursue.

2. Transvectants.

The most important method for computing invariants and covariants of binary forms are the transvectants, discovered by Aronhold, [**1**], Clebsch, [**9**], and Gordan, [**15**]. See [**34**] for additional details.

Definition 2.1. The $m^{\rm th}$ order *transvectant* of a pair of analytic functions $Q(x,y)$, $R(x,y)$ is the function

$$(Q,\,R)^{(m)} = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \frac{\partial^m Q}{\partial x^{m-i}\partial y^i}\,\frac{\partial^m R}{\partial x^i \partial y^{m-i}}. \tag{2.1}$$

Particular examples are the product

$$(Q,\,R)^{(0)} = Q\,R,$$

the Jacobian determinant

$$(Q,\,R)^{(1)} = Q_x R_y - Q_y R_x,$$

and the polarized Hessian covariant

$$(Q,\,R)^{(2)} = Q_{xx}R_{yy} - 2Q_{xy}R_{xy} + Q_{yy}R_{xx}. \tag{2.2}$$

Remark: In the classical literature, additional degree-dependent numerical factors are often incorporated into the transvectants. These will be suppressed here.

The $m^{\rm th}$ transvectant $(Q,\,R)^{(m)}$ is symmetric or skew-symmetric under interchange of Q and R depending on whether m is even or odd:

$$(Q,\,R)^{(m)} = (-1)^m\,(R,\,Q)^{(m)}. \tag{2.3}$$

In particular, any odd transvectantof a form with itself automatically vanishes.

A function $Q(x,y)$ is *homogeneous* of degree $n = d(Q)$ if

$$Q(\lambda x, \lambda y) = \lambda^n\, Q(x,y).$$

In classical invariant theory, one restricts attention to homogeneous polynomials

$$Q(x,y) = \sum_{i=0}^{n} \binom{n}{i} a_i\, x^i y^{n-i}, \tag{2.4}$$

but our results apply equally well to general smooth/analytic functions. If Q and R are homogeneous of respective degrees $d(Q), d(R)$, then $(Q,\, R)^{(m)}$ is also homogeneous, of degree

$$d(Q,\, R)^{(m)} = d(Q) + d(R) - 2m. \tag{2.5}$$

A basic result is the covariance of transvectants under the linear action

$$\overline{x} = \alpha x + \beta y, \qquad \overline{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma = 1, \tag{2.6}$$

of $\mathrm{SL}(2) = \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$. The linear transformation maps the function $Q(x,y)$ to the function $\overline{Q}(\overline{x},\overline{y})$, defined so that

$$\overline{Q}(\overline{x},\overline{y}) = \overline{Q}(\alpha x + \beta y, \gamma x + \delta y) = Q(x,y). \tag{2.7}$$

Theorem 2.2. *If Q, R are mapped to $\overline{Q}, \overline{R}$ under a linear transformation* (2.6) *in* SL(2), *then their m^{th} transvectant $(Q,\, R)^{(m)}$ is mapped to $(\overline{Q},\, \overline{R})^{(m)}$.*

Remark: Allowing linear transformations of non-unit determinant introduces determinantal scaling factors into the transformation rules for the transvectants, cf. [**34**].

Given a homogeneous function of degree $n = d(Q)$, we let

$$u(x) = Q(x,1) \tag{2.8}$$

denote its inhomogeneous counterpart of degree $n = d(u)$. We can view $u\colon \mathbb{CP}^1 \to \mathbb{C}$ as a function on the associated projective space (or, more correctly, a section of the n^{th} power of universal line bundle over $\mathbb{CP}^1$, [**18**]). Note that one can reconstruct

$$Q(x,y) = y^n\, u\left(\frac{x}{y}\right), \qquad n = d(u). \tag{2.9}$$

The degree $d(u)$ of an inhomogeneous function $u(x)$ cannot be obtained from the local coordinate formula, but depends on the relation (2.9) to the homogeneous representative.

The induced action of a linear transformation (2.6) on the projective coordinate is governed by linear fractional transformations

$$\overline{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1. \tag{2.10}$$

The transformation rule corresponding to (2.7),

$$u(x) = (\gamma x + \delta)^n\, \overline{u}(\overline{x}) = (\gamma x + \delta)^n\, \overline{u}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right), \qquad n = d(u), \tag{2.11}$$

is a simple consequence of the basic correspondence (2.9). The factor $(\gamma p + \delta)^n$ is called the *multiplier*, and (2.11) defines the fundamental *multiplier representation* of SL(2) of degree n, cf. [**33**].

The planar transvectant formulae (2.1) will produce a projective transvectant formula for the inhomogeneous representatives of homogeneous forms, cf. [**19, 34, 35**].

Theorem 2.3. *The* m^{th} transvectant *of functions* $u(x)$ *and* $v(x)$ *is*

$$(u,\,v)^{(m)} = m! \sum_{r+s=m} (-1)^s \binom{d(u)-r}{s} \binom{d(v)-s}{r} D_x^r u \cdot D_x^s v\,. \tag{2.12}$$

The degree of $(u,\,v)^{(m)}$ *is*

$$d\,(u,\,v)^{(m)} = d(u) + d(v) - 2m. \tag{2.13}$$

Moreover, if u, v *are mapped to* $\overline{u}, \overline{v}$*, as given by* (2.11)*, under a linear fractional transformation, then their* m^{th} *transvectant* $(u,\,v)^{(m)}$ *is mapped to* $(\overline{u},\,\overline{v})^{(m)}$*.*

Remark: For more general, non-integral values of k, we define the binomial coefficient as

$$\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}\,.$$

Once the formula (2.12) is established, the other statements are immediate corollaries of Theorem 2.2.

3. Modular Forms and the Rankin–Cohen Bracket.

Let us recall the classical definition of automorphic and modular forms, [**27**, **42**]. Let $\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ denote the union of the upper half plane $\mathbb{H} \subset \mathbb{C}$ and the point at infinity.

Definition 3.1. A function $u(x)$ defined for $x \in \overline{\mathbb{H}}$ is called an *automorphic function* of *weight* $k = \omega(f)$ if it satisfies

$$u\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right) = (\gamma x+\delta)^k\, u(x), \tag{3.1}$$

for all $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, where $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ is a discrete subgroup. If $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, then u is called a *modular function* of weight k. An automorphic or modular function that is everywhere holomorphic on $\overline{\mathbb{H}}$ is called an *automorphic* or *modular form.* If in addition $u(\infty) = 0$, then u is called a *cusp form.*

A particular example is the *Eisenstein series*

$$G_{2k}(x) = \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{(mx+n\omega)^{2k}}, \qquad x \in \mathbb{H}, \tag{3.2}$$

which is a modular form of weight $2k$ for every integer $2 \le k \in \mathbb{Z}$. Its value at ∞ is given by

$$G_{2k}(\infty) = \frac{2\,\zeta(2k)}{\omega^{2k}}, \tag{3.3}$$

where $\zeta(s)$ is the Riemann zeta function. Here $\omega \in \mathbb{R}$ is a fixed real constant, usually set equal to $\omega = 1$, although the choice $\omega = \pi$ has the advantage of avoiding many factors of π in the resulting formulae. It is customary to let

$$g_2(x) = 60\,G_4(x), \qquad g_3(x) = 140\,G_6(x). \tag{3.4}$$

Comparing (3.1) with (2.11), we see that we can identify an automorphic form with a function of degree

$$d(u) = -\omega(u) = -k \tag{3.5}$$

which is *invariant* under the appropriate discrete subgroup $\Gamma \subset \mathrm{SL}(2)$. Thus, we should view modular and automorphic forms as homogeneous functions of *negative* degree, whereas binary forms are homogeneous functions of *positive* degree. This "duality" between modular and binary forms seems to be very important, and the two theories exhibit many parallelisms, not entirely understood. In this paper, we propose the Heisenberg representation as the connecting link between these two theories, bridging the gap from positive to negative degree via forms of "infinite degree".

The following definition originates in papers of Rankin, [**36**], and H. Cohen, [**10**]. See also [**11**, **48**], as well as [**7**, **12**] for extensions to functions of several variables.

Definition 3.2. The m^{th} *Rankin–Cohen bracket* between automorphic forms u and v is defined by the formula

$$[\,u,\,v\,]_m = \sum_{r+s=m} (-1)^r \binom{m+\omega(f)-1}{s} \binom{m+\omega(g)-1}{r} (D_x^r u)\,(D_x^s v)\,. \tag{3.6}$$

A basic result, first noted in [**33**; p. 102] and [**34**; p. 92], is that the Rankin–Cohen bracket is, in fact, just the classical transvectant, provided one identifies the weight of the form with minus its degree, cf. (3.5). This observation provides an elementary answer to a question of Zagier, [**48**], who states that the connection between transvectants and Rankin-Cohen brackets is an open problem. A more sophisticated treatment of this connection appears in a recent preprint by Choie, Mourrain, and Solé, [**8**].

Proposition 3.3. *If u, v are functions of respective degrees and weights $d(u) = -\omega(u)$, $d(v) = -\omega(v)$, then their m^{th} Rankin–Cohen bracket* (3.6) *is, up to a factor, equal to their m^{th} transvectant* (2.12)*:*

$$(u,\,v)^{(m)} = m!\,[\,u,\,v\,]_m.$$

Proof: This reduces to a simple binomial coefficient identity:

$$\binom{k-r}{s}\binom{l-s}{r} = (-1)^m \binom{m-k-1}{s}\binom{m-l-1}{r},$$

which is valid for $r+s=m$. *Q.E.D.*

Theorem 3.4. *If u, v are automorphic functions of respective weights $\omega(u), \omega(v)$, then their m^{th} Rankin–Cohen bracket $[\,u,\,v\,]_m$ is an automorphic function of weight*

$$\omega\big([\,u,\,v\,]_m\big) = \omega(u) + \omega(v) + 2m. \tag{3.7}$$

Proof: The invariance of the bracket under linear fractional transformations in the subgroup Γ is an immediate consequence of its transformation rules under the projective action of $\mathrm{SL}(2)$, as detailed in Theorem 2.3. The weight formula (3.7) is a consequence of (2.13), (3.5). *Q.E.D.*

Remark: This is a very simple proof, based on the omega processes and homogenization as discussed below, of the invariance properties of the Rankin–Cohen brackets. In [**11**; p. 26], the "easiest proof" relies on a much more sophisticated lifting from modular forms to "Jacobi–like forms" due to Kuznetsov and Cohen.

Remark: The connection with classical transvectants demonstrates that the result holds for *any* — not only discrete — subgroup of SL(2).

Theorem 3.5. *If u, v are automorphic forms, so is $[\,u,\,v\,]_m$. In particular, the even brackets $[\,u,\,u\,]_{2l}$ of a modular form with itself define cusp forms.*

Proof: The analyticity of $(u,\,v)^{(m)}$ on $\mathbb{H}$ is immediate; the proof of analyticity at ∞, and the vanishing result, can be found in [**36**, **10**, **48**]. *Q.E.D.*

For example, if

$$u(x) = g_2(x) = 60\, G_4(x)$$

is the Eisenstein series (3.4), then

$$[\,g_2,\,g_2\,]_2 = \frac{\pi^4}{3\;2^{12}\omega^4}\Delta, \qquad \text{where} \qquad \Delta = g_2^3 - 27\, g_3^2 \tag{3.8}$$

is the *modular discriminant*. Since g_2 and g_3 are modular forms of weight 4 and 6 respectively, Theorem 3.5 implies that Δ is a cusp form of weight 12. One can check the fact that $\Delta(\infty) = 0$ directly using the well-known values

$$g_2(\infty) = \frac{120}{\omega^4}\,\zeta(4) = \frac{4\omega^4}{3\pi^4}, \qquad g_3(\infty) = \frac{280}{\omega^6}\,\zeta(6) = \frac{8\omega^6}{27\pi^6}.$$

Similarly, one can prove that

$$[\,g_2,\,g_3\,]_2 = 0$$

and

$$[\,g_3,\,g_3\,]_2 = -\frac{25\pi^4}{64\omega^4}\,g_2\,\Delta.$$

4. Infinitesimal Generators.

The linear action of SL(2) on $\mathbb{C}^2$ induces the projective action

$$(x,u) \longmapsto \left(\,\frac{\alpha x+\beta}{\gamma x+\delta}\,,\;\frac{u}{(\gamma x+\delta)^n}\,\right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2), \tag{4.1}$$

on $\mathbb{C}^2$. The case of integral $n > 0$ governs the classical invariant theory of binary forms of degree n, while the case of integral $n = -w < 0$ (and restriction to a discrete subgroup) governs automorphic and modular forms of weight w. In this section, we discuss the infinitesimal version of this basic projective action, and show how it leads immediately to the limiting case $n \to \infty$ of a Heisenberg group action.

The infinitesimal generators of the projective action (4.1) are

$$\mathbf{v}_- = \partial_x, \qquad \mathbf{v}_0 = x\,\partial_x + \frac{n}{2}u\,\partial_u, \qquad \mathbf{v}_+ = x^2\,\partial_x + nxu\,\partial_u, \tag{4.2}$$

as described in [**33**; (5.15)] (although the signs are wrong there).

We are particularly interested in the limiting case $n \to \infty$, and so we replace the standard infinitesimal generators (4.2) by the alternative basis

$$\mathbf{w}_-^{(n)} = \mathbf{v}_- = \partial_x, \qquad \mathbf{w}_0^{(n)} = \frac{2}{n}\,\mathbf{v}_0 = \frac{2}{n} x\,\partial_x + u\,\partial_u, \qquad \mathbf{w}_+^{(n)} = \frac{1}{n}\,\mathbf{v}_+ = \frac{1}{n} x^2\,\partial_x + xu\,\partial_u. \tag{4.3}$$

These have the modified $\mathfrak{sl}(2)$ commutation relations

$$[\,\mathbf{w}_-^{(n)}, \mathbf{w}_0^{(n)}\,] = \frac{2}{n}\,\mathbf{w}_-^{(n)}, \qquad [\,\mathbf{w}_-^{(n)}, \mathbf{w}_+^{(n)}\,] = \mathbf{w}_0^{(n)}, \qquad [\,\mathbf{w}_+^{(n)}, \mathbf{w}_0^{(n)}\,] = -\frac{2}{n}\,\mathbf{w}_+^{(n)}. \tag{4.4}$$

In the limit $n \to \infty$, the $\mathfrak{sl}(2)$ Lie algebra generators (4.3) degenerate to

$$\mathbf{w}_- = \partial_x, \qquad \mathbf{w}_0 = u\,\partial_u, \qquad \mathbf{w}_+ = xu\,\partial_u. \tag{4.5}$$

(We drop the ∞ superscript to simplify the notation.) These vector fields form a Heisenberg algebra, with commutation relations

$$[\,\mathbf{w}_-, \mathbf{w}_0\,] = 0, \qquad [\,\mathbf{w}_-, \mathbf{w}_+\,] = \mathbf{w}_0, \qquad [\,\mathbf{w}_+, \mathbf{w}_0\,] = 0. \tag{4.6}$$

The corresponding Heisenberg group action is

$$(x,u) \longmapsto (x+\lambda, (\gamma x + \delta) u).$$

The unimodular Lie algebras (4.4) can thus be regarded as a Lie algebra deformation of the Heisenberg algebra (4.6).

Let pr $\mathbf{v}$ denote the usual prolongation, [**33**], of the vector field $\mathbf{v}$ to the (infinite) jet space J^∞, whose coordinates are the variables x, u and their derivatives

$$u_k = D_x^k u, \qquad k = 1, 2, \ldots .$$

The prolonged vector fields corresponding to (4.3) are easily computed:

$$\mathrm{pr}\,\mathbf{w}_-^{(n)} = \partial_x, \qquad \mathrm{pr}\,\mathbf{w}_0^{(n)} = \frac{2}{n} x\,\partial_x + \mathcal{C}^{(n)}, \qquad \mathrm{pr}\,\mathbf{w}_+^{(n)} = \frac{1}{n} x^2\,\partial_x + x\,\mathcal{C}^{(n)} + \mathcal{B}^{(n)}, \tag{4.7}$$

where

$$\mathcal{C}^{(n)} = \sum_{i=0}^{\infty} \left(1 - \frac{2i}{n}\right) u_i\,\frac{\partial}{\partial u_i}\,, \qquad \mathcal{B}^{(n)} = \sum_{i=1}^{\infty} \left(1 - \frac{(i-1)}{n}\right) i u_{i-1}\,\frac{\partial}{\partial u_i}\,. \tag{4.8}$$

In the limit $n \to \infty$, these reduce to

$$\mathrm{pr}\,\mathbf{w}_- = \partial_x, \qquad \mathrm{pr}\,\mathbf{w}_0 = \frac{2}{n}\,x\,\partial_x + \mathcal{C}, \qquad \mathrm{pr}\,\mathbf{w}_+ = \frac{1}{n}\,x^2\,\partial_x + x\,\mathcal{C} + \mathcal{B}, \tag{4.9}$$

where

$$\mathcal{C} = \sum_{i=0}^{\infty} u_i\,\frac{\partial}{\partial u_i}\,, \qquad \mathcal{B} = \sum_{i=1}^{\infty} i u_{i-1}\,\frac{\partial}{\partial u_i}\,, \tag{4.10}$$

are the limiting operators.

Remark: One can replace the infinitesimal generators (4.7) by their evolutionary forms

$$\mathbf{W}_-^{(n)} = -\mathcal{D}, \qquad \mathbf{W}_0^{(n)} = -\frac{2}{n}\, x\,\mathcal{D} + \mathcal{C}^{(n)}, \qquad \mathbf{W}_+^{(n)} = -\frac{1}{n}\, x^2\,\mathcal{D} + x\mathcal{C}^{(n)} + \mathcal{B}^{(n)}, \tag{4.11}$$

where

$$\mathcal{D} = \sum_{i=0}^{\infty} u_{i+1}\, \frac{\partial}{\partial u_i} \tag{4.12}$$

is the x-independent part of the total derivative

$$D_x = \partial_x + \mathcal{D}.$$

It is well known, [**33**], that the evolutionary forms satisfy the same commutation relations (4.4), so

$$[\,\mathbf{W}_-^{(n)}, \mathbf{W}_0^{(n)}\,] = \frac{2}{n}\,\mathbf{W}_-^{(n)}, \qquad [\,\mathbf{W}_-^{(n)}, \mathbf{W}_+^{(n)}\,] = \mathbf{W}_0^{(n)}, \qquad [\,\mathbf{W}_+^{(n)}, \mathbf{W}_0^{(n)}\,] = -\frac{2}{n}\,\mathbf{W}_+^{(n)}.$$

Substituting (4.11), and using the fact that $\mathcal{D}, \mathcal{C}^{(n)}, \mathcal{B}^{(n)}$ do not involve x differentiation, we conclude that

$$[\,\mathcal{D}, \mathcal{C}^{(n)}\,] = \frac{2}{n}\,\mathcal{D}, \qquad [\,\mathcal{D}, \mathcal{B}^{(n)}\,] = -\,\mathcal{C}^{(n)}, \qquad [\,\mathcal{C}^{(n)}, \mathcal{B}^{(n)}\,] = \frac{2}{n}\,\mathcal{B}^{(n)}. \tag{4.13}$$

Therefore, $\mathcal{D}, \mathcal{C}^{(n)}, \mathcal{B}^{(n)}$ form an $\mathfrak{sl}(2)$ algebra, while at $n = \infty$ they reduce to the Heisenberg commutation relations

$$[\,\mathcal{D}, \mathcal{C}\,] = 0, \qquad [\,\mathcal{D}, \mathcal{B}\,] = -\,\mathcal{C}, \qquad [\,\mathcal{C}, \mathcal{B}\,] = 0. \tag{4.14}$$

This provides a one-parameter family of realizations of the normal form construction that embeds the total derivative operator $\mathcal{D}$ in an $\mathfrak{sl}(2)$ algebra, or, in the limiting case, a Heisenberg algebra.

Remark: Since $\mathcal{C}^{(n)}, \mathcal{B}^{(n)}$ do not involve x, we can replace $\mathcal{D}$ by D_x without affecting the commutation relations (4.13) or (4.14). However, since we will only deal with constant coefficient differential polynomials, this will not make any difference in our analysis.

Remark: Truncating the operators (4.12) and (4.8) or (4.10) at some finite order m by setting $u_k = 0$ for $k > m$ is effectively the same as restricting their action to the space of polynomials of degree $\leq m$. This induces a family of finite-dimensional representations of $\mathfrak{sl}(2)$ on the $\mathcal{C}^{(n)}$ eigenspaces — the spaces of homogeneous differential polynomials $F(u, u_1, \ldots, u_m)$ of a fixed degree. In the limit, we obtain the corresponding finite-dimensional representations of the Heisenberg algebra. As we shall see, the transvectants provide natural maps between the tensor products of these eigenspace representations.

5. Differential Invariants.

As detailed in [**33**], transvectants can be characterized as relative differential invariants for the prolonged group action of SL(2) on J^∞. Let $u(x)$ be a function of degree $n = d(u)$, meaning that it transforms according to the degree n projective action (2.11) of SL(2). A *differential polynomial* is a smooth polynomial function $F[u] = F(x, u, u_1, \ldots, u_m)$ depending on finitely many derivatives of u. The *order* m of F is the highest order derivative it depends on.

Definition 5.1. A *relative differential invariant* of *weight* $k = \omega(R)$ for the transformation group (4.1) is a differential function $R[u]$ that satisfies the infinitesimal invariance conditions

$$\mathrm{pr}\,\mathbf{v}_-(R) = 0, \qquad \mathrm{pr}\,\mathbf{v}_0(R) = \tfrac{1}{2} knR, \qquad \mathrm{pr}\,\mathbf{v}_+(R) = knxR. \tag{5.1}$$

The extra factor of $n = d(u)$ is for later convenience. It does not appear in [**33**] and so our definition of weight is slightly different, but appropriate for taking the Heisenberg limit $n \to \infty$. In particular, $R = u$ is a relative differential invariant of weight $\omega(u) = 1$, independent of n. In general, the degree and weight of a relative differential invariant are related by

$$\omega(R) = n\, d(R) = d(u)\, d(R). \tag{5.2}$$

Since

$$\omega(R\,S) = \omega(R) + \omega(S),$$

absolute differential invariants, i.e., those of weight 0, can be found by taking the ratio of appropriate powers of relative differential invariants. In particular, if R is any relative differential invariant of weight $k = \omega(R)$, then $u^{-k}R$ is an absolute differential invariant.

Warning: The weight of a relative differential invariant has nothing to do with the weight of a modular form.

A key result is that the transvectant of two relative differential invariants is a relative differential invariant; see [**33**, **34**] for proofs.

Theorem 5.2. *If R and S are relative differential invariants of respective weights $k = \omega(R)$ and $l = \omega(S)$, then their m^{th} transvectant*

$$(R,\, S)^{(m)} = m! \sum_{r+s=m} (-1)^r \binom{kn - r}{s} \binom{ln - s}{r} (D_x^r R)\, (D_x^s S) \tag{5.3}$$

is a relative differential invariant of weight $k + l - \dfrac{2m}{n} = \omega(R) + \omega(S) - \frac{2m}{d(u)}$.

Using our rescaled generators (4.3), the conditions (5.1) for relative invariance become

$$\mathrm{pr}\,\mathbf{w}_-^{(n)}(R) = 0, \qquad \mathrm{pr}\,\mathbf{w}_0^{(n)}(R) = kR, \qquad \mathrm{pr}\,\mathbf{w}_+^{(n)}(R) = kxR,$$

where $k = \omega(R)$. This characterization of relative differential invariant carries over to the Heisenberg limit $n \to \infty$. Substituting the formulae (4.7), (4.9), we see that the infinitesimal invariance condition becomes

$$\partial_x R = 0, \qquad \mathcal{C}^{(n)}(R) = kR, \qquad \mathcal{B}^{(n)}(R) = 0.$$

Therefore, every relative differential invariant is independent of x and lies in the kernel of the operator $\mathcal{B}^{(n)}$. A key result is that the converse is valid: for any n, including $n = \infty$ the kernel of $\mathcal{B}^{(n)}$ is the space of relative differential invariants. Therefore, the differential invariants of the projective or Heisenberg groups can be identified as normal forms for differential polynomials with respect to the total derivative operator. To avoid technicalities, we restrict the action to the space of differential polynomials — the case of rational differential functions being an easy consequence.

Theorem 5.3. *Every differential polynomial $R[u]$ which satisfies $\mathcal{B}^{(n)}(R) = 0$ is a linear combination of relative differential invariants of various weights.*

To prepare for the proof of this result, and to produce a polynomial basis for the relative differential invariants and hence $\ker \mathcal{B}^{(n)}$, we first modify the transvectant formulae so as to be able to pass to the $n \to \infty$ limit. Given relative differential invariants R and S of respective weights k and l, let us define the *classical transvectant*

$$\begin{aligned} \tau_m^{(n)}(R,S) &= \frac{(-1)^m}{n^m}\,(R,\,S)^{(m)} \\ &= \sum_{r+s=m} (-1)^r \binom{m}{r} \prod_{i=s}^{m-1}\left(k-\frac{i}{n}\right) \prod_{j=r}^{m-1}\left(l-\frac{j}{n}\right) D_x^r R \cdot D_x^s S\,. \end{aligned} \tag{5.4}$$

In the case $n < 0$ these are rescaled versions of the Rankin–Cohen brackets. In the limit $n \to \infty$, this reduces to the limiting *Heisenberg transvectant*

$$\tau_m(R,S) = \sum_{r+s=m} (-1)^r \binom{m}{r} k^s l^r\, D_x^r R \cdot D_x^s S\,. \tag{5.5}$$

Since $\mathcal{C}(R) = k\,R$ and $\mathcal{C}(S) = l\,S$ are eigenfunctions of the scaling operator, we can rewrite (5.5) in the more suggestive form

$$\tau_m(R,S) = \sum_{r+s=m} (-1)^r \binom{m}{r} (\mathcal{D}^r\mathcal{C}^s R)\cdot(\mathcal{D}^s\mathcal{C}^r S) = (\mathcal{C}\wedge\mathcal{D})^m R\,S. \tag{5.6}$$

Assuming Theorem 5.3, we immediately conclude the following:

Corollary 5.4. *If $R, S \in \ker \mathcal{B}^{(n)}$, then $\tau_m^{(n)}(R,S) \in \ker \mathcal{B}^{(n)}$.*

Starting with u, which is a relative differential invariant of weight 1, we can produce the quadratic relative differential invariants

$$\gamma_{2k}^{(n)}[u] = \tfrac{1}{2}\tau_{2k}^{(n)}(u,u) = u u_{2k} + \cdots, \qquad\qquad k = 1,2,3,\ldots, \tag{5.7}$$

of even order $2k$ and weight $2 - 4k/n$. The simplest of these, when $2k = 2$, is the classical Hessian covariant, and the rest are higher order even transvectants of the form with itself. The fundamental odd order relative differential invariants are

$$\gamma_{2k+1}^{(n)}[u] = \tau_1^{(n)}(u, \gamma_{2k}^{(n)}) = u^2 u_{2k+1} + \cdots, \tag{5.8}$$

of odd order $2k+1$ and weight $3 - (4k+2)/n$. Since u is a relative differential invariant, Corollary 5.4 implies that all the $\gamma_k^{(n)}$ are relative differential invariants and hence belong to $\ker \mathcal{B}^{(n)}$. This result holds for any n, including the Heisenberg limit $n = \infty$.

In order to characterize the space $\ker \mathcal{B}^{(n)}$, we generalize an algebraic result due to Stroh, [**43**], and Hilbert, [**22**; p. 64], that establishes a rational basis for the covariants of binary forms; see also [**34**; p. 124]. We show that, modulo division by some power of u, every relative differential invariant can be written as a polynomial in u along with the fundamental quadratic and cubic relative differential invariants (5.7), (5.8). This provides a basis for the algebra of transvectants of a binary form, or, equivalently, the algebra of a modular form generated by its Rankin–Cohen brackets. Theorem 5.3 is then an immediate consequence of this basic result.

Theorem 5.5. *A differential polynomial P of order m belongs to* $\ker \mathcal{B}^{(n)}$ *if and only if, when multiplied by a power of u, it can be written as a polynomial in the fundamental relative differential invariants* (5.7), (5.8):

$$u^N P[u] = H(u, \gamma_2^{(n)}, \ldots, \gamma_m^{(n)}).$$

Each homogeneous summand of H is a relative differential invariant.

Proof: Each differential monomial

$$u^K = u^{k_0} u_1^{k_1} \cdots u_n^{k_n} \tag{5.9}$$

is uniquely specified by a multi-index

$$K = (k_0, k_1, \ldots, k_n) = (k_0, k_1, \ldots k_n, 0, 0, \ldots) \in \mathbb{N}^\infty,$$

where only finitely many terms are non-zero and we can suppress all trailing zeros. Let us introduce the reverse lexicographic ordering on multi-indices, so that $J < K$ if and only if

$$j_n = k_n \qquad \text{for all} \qquad n > i, \qquad \text{but} \qquad j_i < k_i.$$

This induces an ordering of differential monomials (5.9). The *leading term* of a differential polynomial is the last nonzero monomial in the reverse lexicographic ordering. In particular, the leading terms in our fundamental relative differential invariants are those indicated in the formulae (5.7), (5.8).

Lemma 5.6. *If $P \in \ker \mathcal{B}^{(n)}$, then the leading term in P does not contain u_1.*

Proof: Let u^K be the leading term in P. If $k_1 > 0$, then the leading term in $\mathcal{B}^{(n)}[P]$ is obtained from applying the first summand $u\,\partial_{u_1}$ in $\mathcal{B}^{(n)}$ to u^K, and is

$$\mathcal{B}^{(n)}(u^K) = k_1 u^{k_0+1} u_1^{k_1-1} u_2^{k_2} \cdots u_n^{k_n} + \cdots.$$

All other terms in $\mathcal{B}^{(n)}(u^K)$ come earlier in the lexicographic ordering. Thus, the resulting differential polynomial vanishes if and only if its leading term vanishes, which requires $k_1 = 0$. *Q.E.D.*

Lemma 5.6 implies that the leading term in $P \in \ker \mathcal{B}^{(n)}$ has the form

$$P = u^{k_0} u_2^{k_2} \cdots u_m^{k_m} + \cdots .$$

Consider the transvectant monomial

$$Q = (\gamma_2^{(n)})^{k_2} (\gamma_3^{(n)})^{k_3} \cdots (\gamma_m^{(n)})^{k_m} = u^{\widehat{k}} u_2^{k_2} u_3^{k_3} \cdots u_m^{k_m} + \cdots,$$

where

$$\widehat{k} = k_2 + 2k_3 + k_4 + 2k_5 + \cdots.$$

Let $k^\star = \min\{\, k_0,\ \widehat{k}\,\}$. Then the differential polynomial

$$u^{\widehat{k}-k^\star} P - u^{k_0-k^\star} Q \ \in\ \ker \mathcal{B}^{(n)}$$

and has a lower order leading term. A simple induction completes the proof. *Q.E.D.*

Corollary 5.7. *The space* $\ker \mathcal{B}^{(n)}$ *is a Poisson algebra with multiplication* $R \cdot S = \tau_0^{(n)}(R,S)$ *and Lie bracket* $[\,R,S\,] = \tau_1^{(n)}(R,S)$.

It is worth noting an alternative construction of a generating basis for the relative differential invariants, cf. [**33**; Theorem 5.19].

Proposition 5.8. *Every relative differential invariant of the projective action* (4.1) *is a homogeneous function of* u*, the Hessian covariant* $\psi_2^{(n)} = \tau_2^{(n)}(u,u)$*, and the successive Jacobians* $\psi_m^{(n)} = \tau_1^{(n)}(u, \psi_{m-1}^{(n)})$ *for* $m = 3, 4, 5, \ldots$.

In the limiting case, $n = \infty$, the simple change of variables

$$u = e^v$$

changes the infinitesimal generators (4.5) into

$$\mathbf{w}_- = \partial_x, \qquad \mathbf{w}_0 = \partial_v, \qquad \mathbf{w}_+ = x\,\partial_v, \tag{5.10}$$

which generate the elementary Heisenberg group action

$$(x,u) \longmapsto (x+\lambda, v+\gamma x+\delta).$$

The vector fields (5.10) have very simple prolongation:

$$\operatorname{pr}\mathbf{w}_- = \partial_x, \qquad \operatorname{pr}\mathbf{w}_0 = \partial_v, \qquad \operatorname{pr}\mathbf{w}_+ = x\,\partial_v + \partial_{v_1}, \tag{5.11}$$

and hence one can immediately write down all the differential invariants.

Proposition 5.9. *Every absolute differential invariant for the Heisenberg algebra* (5.10) *is given by a function* $F(v_2, v_3, \ldots v_n)$ *depending on the second and higher order derivatives of* v.

Therefore, in the new coordinates, the fundamental Heisenberg differential invariants are just the derivatives $v_2, v_3, \ldots$. In terms of the original variable $u = e^v$, these produce the fundamental absolute rational differential invariants

$$\lambda_k(u) = D_x^k \log u, \qquad k \geq 2. \tag{5.12}$$

The appearance of the second logarithmic derivative of u in $\lambda_2 = D_x^2 \log u$ is striking, and reminds one of the powerful τ function approach for finding explicit solutions to soliton equations, [**26**]. Indeed, Takhtajan, [**45**], shows how modular forms can be interpreted as τ functions.

Proposition 5.10. *Every relative differential invariant for the Heisenberg algebra* (4.5) *is given by a suitably homogeneous function of the basic differential polynomial invariants*

$$\psi_m = u^m\, D_x^m\, \log u, \qquad m = 2, 3, 4, \ldots. \tag{5.13}$$

Since u has weight 1, each ψ_m has weight m. Furthermore, since

$$D_x u^m = u^m D_x + m u^{m-1} u_x,$$

we find that the invariants (5.13) are obtained by successive transvection with u:

$$\psi_{m+1} = u D_x \psi_m - m u_x \psi_m = \tau_1(u, \psi_m). \tag{5.14}$$

Note that the ψ_m are the Heisenberg limits, $n \to \infty$, of the successive transvectants $\psi_m^{(n)}$, and so Proposition 5.10 is the limiting version of Proposition 5.8.

Combining Theorem 5.5 and Proposition 5.10 we see that there are universal formulae

$$\psi_m = P_m(u, \gamma_2, \ldots, \gamma_m) \tag{5.15}$$

relating the two sets of relative differential invariants. The first few of the *universal polynomials* are

$$\begin{aligned} &\psi_2 = \gamma_2, && \psi_3 = \gamma_3, \\ &\psi_4 = u^2\gamma_4 - 6(\gamma_2)^2, && \psi_5 = u^2\gamma_5 - 12\gamma_2\gamma_3, \\ &\psi_6 = u^4\gamma_6 - 30u^2\gamma_2\gamma_4 + 120(\gamma_2)^3, \qquad && \psi_7 = u^4\gamma_7 - 30u^2\gamma_2\gamma_5 - 30u^2\gamma_3\gamma_4 + 360(\gamma_2)^2\gamma_3. \end{aligned}$$

The following general formulae for the P_m will be proved below.

Theorem 5.11. *Let t be a formal parameter. Equating the powers of t in the formal series identity*

$$\sum_{m=1}^{\infty} \psi_{2m} \frac{t^{2m}}{(2m)!} = \frac{1}{2} \log \left(1 + 2 \sum_{m=1}^{\infty} u^{2m-2} \gamma_{2m} \frac{t^{2m}}{(2m)!} \right) \tag{5.16}$$

gives the even degree universal polynomials $\psi_{2m} = P_{2m}[u]$. The corresponding odd degree universal polynomials $\psi_{2m+1} = P_{2m+1}[u]$ are obtained from the series identity

$$\sum_{m=1}^{\infty} \psi_{2m+1} \frac{t^{2m}}{(2m)!} = \frac{\displaystyle\sum_{m=1}^{\infty} u^{2m-2} \gamma_{2m+1} \frac{t^{2m}}{(2m)!}}{\displaystyle 1 + 2 \sum_{m=1}^{\infty} u^{2m-2} \gamma_{2m} \frac{t^{2m}}{(2m)!}} \tag{5.17}$$

Remark: The formula for the odd order case is obtained by applying the derivation

$$F \longmapsto \tau_1^{(n)}(u, F)$$

to both sides of (5.16), and using (5.8), (5.14). One can obtain P_{2m+1} directly by applying the formal derivation

$$\Gamma = \sum_{n=1}^{\infty} \gamma_{2n+1} \frac{\partial}{\partial \gamma_{2n}} \tag{5.18}$$

to P_{2m}, so that $P_{2m+1} = \Gamma(P_{2m})$.

Remark: We may define the Heisenberg representation on the monomials generated by $\mathcal{D}$ and $\mathcal{C}$ as follows.

$$\mathcal{B} \cdot \mathcal{D}^k \mathcal{C}^m = k \mathcal{D}^{k-1} \mathcal{C}^{m+1}, \qquad \mathcal{C} \cdot \mathcal{D}^k \mathcal{C}^m = \mathcal{D}^k \mathcal{C}^{m+1}, \qquad \mathcal{D} \cdot \mathcal{D}^k \mathcal{C}^m = \mathcal{D}^{k+1} \mathcal{C}^m.$$

This allows us to induce a representation on the space of (pseudo-)differential operators

$$\sum_k F^{(k,m)} \mathcal{D}^k \mathcal{C}^m .$$

For instance,

$$\mathcal{D} \cdot F^{(k,0)} \mathcal{D}^k = F_1^{(k,0)} \mathcal{D}^k + F^{(k,0)} \mathcal{D}^{k+1}.$$

This allows us to extend the definition of the transvectants/Rankin–Cohen brackets τ_n to operators, cf. [**11**]. For example,

$$\tau_2(u, \mathcal{D}) = u\mathcal{D}^2 - 2u_1 \mathcal{D}\mathcal{C} + u_2 \mathcal{C}^2 \in \ker \mathcal{B}.$$

Application of the Poincaré–Birkhoff-Witt ordering allows us to extend this construction to the finite n representations of $\mathfrak{sl}(2)$.

6. The Omega Process.

The classical approach to transvectants is based on an important invariant differential operator originally introduced by Cayley, known as the omega process. The following summarizes basic constructions in the symbolic method from classical invariant theory, as detailed in [**34**].

We consider the joint action of $\mathrm{GL}(2,\mathbb{C})$ on Cartesian product spaces $\mathbb{C}^2 \times \cdots \times \mathbb{C}^2$, whose variables are labeled (symbolically) by Greek letters: $(x_\alpha, y_\alpha), (x_\beta, y_\beta), (x_\gamma, y_\gamma), \ldots$. Given a function $P(x,y)$, we define $P_\alpha = P(x_\alpha, y_\alpha)$. For example,

$$P_\alpha Q_\beta R_\gamma \qquad \text{represents the product} \qquad P(x_\alpha, y_\alpha)\, Q(x_\beta, y_\beta)\, R(x_\gamma, y_\gamma).$$

Equating the arguments in such a product will be viewed as a *trace* operation; for instance

$$\operatorname{tr}\,(P_\alpha Q_\beta R_\gamma) \equiv P(x_\alpha, y_\alpha)\, Q(x_\beta, y_\beta)\, R(x_\gamma, y_\gamma) \Big|_{\substack{x = x_\alpha = x_\beta = x_\gamma \\ y = y_\alpha = y_\beta = y_\gamma}} = P(x,y)\, Q(x,y)\, R(x,y). \tag{6.1}$$

Definition 6.1. The second order differential operator

$$\Omega_{\alpha\beta} = \det \mathbf{\Omega}_{\alpha\beta} = \det \begin{vmatrix} \dfrac{\partial}{\partial x_\alpha} & \dfrac{\partial}{\partial y_\alpha} \\ \dfrac{\partial}{\partial x_\beta} & \dfrac{\partial}{\partial y_\beta} \end{vmatrix} = \frac{\partial^2}{\partial x_\alpha \partial y_\beta} - \frac{\partial^2}{\partial x_\beta \partial y_\alpha} \tag{6.2}$$

is known as the *omega process* with respect to the variables (x_α, y_α) and (x_β, y_β).

The omega process is clearly invariant under the simultaneous transformation of the variables (x_α, y_α) and (x_β, y_β) by an element of $\mathrm{SL}(2)$. Therefore, the invariance of the transvectants comes from the following basic construction.

Lemma 6.2. *The* m^{th} *order* transvectant *of a pair of smooth functions* $Q(x,y)$, $R(x,y)$ *is the function*

$$(Q,\, R)^{(m)} = \operatorname{tr}\, (\Omega_{\alpha\beta})^m \big[\, Q(x_\alpha, y_\alpha)\, R(x_\beta, y_\beta) \,\big]. \tag{6.3}$$

The homogeneous transvectants (2.1) appear in the definition of the Moyal bracket, which arises in quantum mechanics as the essentially unique deformation of the classical Poisson bracket

$$\{\, P, Q \,\} = (P,\, Q)^{(1)} = P_x Q_y - P_y Q_x$$

on the two-dimensional phase space $X = \mathbb{R}^2$, [**30**, **46**, **3**, **34**].

Definition 6.3. Let t be a scalar parameter. The *star product* of the functions $Q(x,y)$ and $R(x,y)$ is the formal series

$$Q \star_t R = \operatorname{tr} \Big[\, (\exp t\, \Omega_{\alpha\beta}) Q_\alpha R_\beta \,\Big] = \sum_{m=0}^{\infty} \frac{t^m}{m!}\; (Q,\, R)^{(m)}. \tag{6.4}$$

The covariance properties of the transvectants imply that the star product is invariant under the projective group SL(2). The star product is the essentially unique deformation of the multiplicative product $(P,Q) \mapsto P \cdot Q$.

Proposition 6.4. *The star product is associative*:

$$P \star_t (Q \star_t R) = (P \star_t Q) \star_t R. \tag{6.5}$$

Proof: We use the fact that the omega process operators mutually commute:

$$P \star_t (Q \star_t R) = \operatorname{tr} \Big[\, \exp\big[\, t\, (\Omega_{\alpha\beta} + \Omega_{\alpha\gamma} + \Omega_{\beta\gamma}) \,\big]\; P_\alpha Q_\beta R_\gamma \,\Big],$$

which clearly equals $(P \star_t Q) \star_t R$, as well as $R \star_t (P \star_t Q)$, $Q \star_t (R \star_t P)$, etc. *Q.E.D.*

Remark: Associativity of the star product leads to many interesting transvectant identities, [**34**], which can alternatively be viewed as identities in the algebra of Rankin–Cohen brackets, cf. [**48**].

Since even transvectants are symmetric while odd ones are skew-symmetric, we have $P \star_t Q = Q \star_{(-t)} P$. The *Moyal bracket* is the "odd" part of the star product†

$$[\![\, P,\, Q \,]\!]_t = \frac{P \star_t Q - Q \star_t P}{2t} = \operatorname{tr}\, \left(\frac{\sinh t\, \Omega_{\alpha\beta}}{t} \right)\, P_\alpha Q_\beta, \tag{6.6}$$

which, by the associativity of the star product, automatically satisfies the Jacobi identity and provides a quantum deformation of the Poisson bracket:

$$[\![\, P,\, Q \,]\!]_t = \{\, P, Q \,\} + \mathrm{O}(t^2).$$

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† Some authors replace t by $\sqrt{-1}\, t$, converting the hyperbolic sinh to a trigonometric sine.

Let us now pass to the projective version. Recall Euler's formula

$$x\,\frac{\partial Q}{\partial x} + y\,\frac{\partial Q}{\partial y} = d\,Q \tag{6.7}$$

for a homogeneous function of degree $d = d(Q)$. The omega process reduces to

$$\begin{aligned}
\widetilde{\Omega}_{\alpha\beta} &= \frac{\partial}{\partial x_\alpha}\left(d_\beta - x_\beta\,\frac{\partial}{\partial x_\beta}\right) - \frac{\partial}{\partial x_\beta}\left(d_\alpha - x_\alpha\,\frac{\partial}{\partial x_\alpha}\right) \\
&= d_\beta\,\frac{\partial}{\partial x_\alpha} - d_\alpha\,\frac{\partial}{\partial x_\beta} + (x_\alpha - x_\beta)\,\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \\
&= d_\beta\,\partial_\alpha - d_\alpha\,\partial_\beta + (x_\alpha - x_\beta)\,\partial_{\alpha\beta}
\end{aligned} \tag{6.8}$$

when applied to the product $Q_\alpha\,R_\beta = Q(x_\alpha, y_\alpha)\,R(x_\beta, y_\beta)$ of homogeneous functions of respective degrees

$$d_\alpha = d(Q_\alpha) = d(Q), \qquad d_\beta = d(R_\beta) = d(R).$$

The projective omega process (6.8) can now be directly applied to the inhomogeneous representatives,

$$u_\alpha = u(x_\alpha) = Q(x_\alpha, 1), \qquad v_\beta = v(x_\beta) = R(x_\beta, 1).$$

However, it is important to note that $\Omega_{\alpha\beta}$ *decreases* the degree of each factor $Q_\alpha R_\beta$ by one. Therefore, powers of the omega process do not translate into powers of its projective version, since the degrees will vary from factor to factor. For example, in order to compute projective formula (2.12) for the m^{th} transvectant, we must use the noncommutative "falling factorial" or "Pochhammer product", [**34**; p. 101], of the omega operator (6.8),

$$\begin{aligned}
(u,\,v)^{(m)} = \operatorname{tr}\;&\big(\,[\,d_\beta - (m-1)\,]\,\partial_\alpha - [\,d_\alpha - (m-1)\,]\,\partial_\beta + (x_\alpha - x_\beta)\,\partial_{\alpha\beta}\,\big) \\
&\big(\,[\,d_\beta - (m-2)\,]\,\partial_\alpha - [\,d_\beta - (m-2)\,]\,\partial_\beta + (x_\alpha - x_\beta)\,\partial_{\alpha\beta}\,\big)\cdots \\
&\;\big(\,d_\beta\,\partial_\alpha - d_\alpha\,\partial_\beta + (x_\alpha - x_\beta)\,\partial_{\alpha\beta}\,\big)\;u_\alpha v_\beta.
\end{aligned} \tag{6.9}$$

The projective star product and Moyal bracket are constructed from these operations, and have the same properties. In the modular form case, $n < 0$, the projective star product for $t = 1$ is known as the Eholzer product, [**11, 12**], and induces a Lie algebra structure on the Rankin–Cohen bracket algebra. It would be interesting to see how the alternative associative products derived in [**11**; p. 29] fit into this picture.

In order to take the Heisenberg limit $n \to \infty$, we need to replace the degree by the weight, as in (5.2), and divide each omega process by the factor n. The resulting differential operator

$$\Omega^{(n)}_{\alpha\beta} = \omega_\beta\,\frac{\partial}{\partial x_\alpha} - \omega_\alpha\,\frac{\partial}{\partial x_\beta} + \frac{x_\alpha - x_\beta}{n}\,\frac{\partial^2}{\partial x_\alpha \partial x_\beta}, \tag{6.10}$$

is applied to a product $R_\alpha S_\beta$ of relative differential invariants of respective weights $\omega_\alpha = \omega(R_\alpha) = \omega(R)$, $\omega_\beta = \omega(S_\beta) = \omega(S)$. Again, owing to the change in weighting, one cannot use ordinary powers to compute transvectants. In particular, the m^{th} order transvectant (5.4) between two functions u, v of unit weight equals

$$\tau_m^{(n)}(u,v) = \operatorname{tr}\,\Big[\,\big(1 - \tfrac{m-1}{n}\big)\,(\partial_\alpha - \partial_\beta) + \tfrac{x_\alpha - x_\beta}{n}\,\partial_{\alpha\beta}\,\Big] \\ \Big[\,\big(1 - \tfrac{m-2}{n}\big)\,(\partial_\alpha - \partial_\beta) + \tfrac{x_\alpha - x_\beta}{n}\,\partial_{\alpha\beta}\,\Big]\cdots\Big[\,\partial_\alpha - \partial_\beta + \tfrac{x_\alpha - x_\beta}{n}\,\partial_{\alpha\beta}\,\Big]\,u_\alpha v_\beta. \tag{6.11}$$

In the Heisenberg limit $n \to \infty$, this reduces to an ordinary m^{th} power

$$\tau_m(u,v) = \operatorname{tr}\,\big(\partial_\alpha - \partial_\beta\big)^m\;u_\alpha\,v_\beta = \frac{\partial^m}{\partial t^m}\,u(x+t)\,v(x-t)\,\bigg|_{t=0} = \mathbb{D}_x^m\,(u\cdot v). \tag{6.12}$$

The symbol $\mathbb{D}_x$ denotes the Hirota bilinear operator, [**23**, **25**, **26**], that first arose in the classification of integrable systems. In this manner, we may interpret the Hirota operator $\mathbb{D}_x$ as the Heisenberg limit of the projective omega process (6.10). See also [**2**, **34**] for connections between transvectants and the Hirota formalism.

The star product (6.4) can be carried over to the projective version. In the limit $n \to \infty$ it reduces to a *Heisenberg star product*

$$R \star_t^{(\infty)} S = R(x + \omega(S)\,t)\,S(x - \omega(R)\,t) = \exp[\,t\,(\mathcal{C} \wedge \mathcal{D})\,]\,R\cdot S, \tag{6.13}$$

cf. (5.6). In particular,

$$u \star_t^{(\infty)} u = u(x+t)\,u(x-t). \tag{6.14}$$

(The *projective star product*, valid for finite n, involves a formal q-exponential type series.) The Hirota operators (6.12) naturally appear in the power series expansion of the star product (6.14). Using the "exp–log formula" of Jimbo and Miwa, [**26**; (3.5)], we can express the Heisenberg star product (6.14) in the remarkable form

$$\frac{u \star_t^{(\infty)} u}{u^2} = \frac{\exp(t\mathbb{D}_x)u\cdot u}{u^2} = \frac{u(x+t)\,u(x-t)}{u(x)^2} = \exp\left(2\sum_{m=1}^{\infty}\frac{t^{2m}}{(2m)!}\,D_x^{2m}\log u(x)\right). \tag{6.15}$$

We now apply (6.15) to prove Theorem 5.11. Consider the particular transvectants

$$\gamma_{2k}[u] = \tfrac{1}{2}\,\tau_{2k}(u,u) = \frac{1}{2}\operatorname{tr}\,\left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta}\right)^{2k} u(x_\alpha)\,u(x_\beta).$$

The right hand side in the series identity (5.16) can be rewritten in the form

$$\frac{1}{2}\operatorname{tr}\,\log\left(1 + \frac{1}{u(x)^2}\sum_{m=1}^{\infty}\frac{[t\,u(x)]^{2m}}{(2m)!}\left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta}\right)^{2m} u(x_\alpha)\,u(x_\beta)\right) = \\ = \frac{1}{2}\operatorname{tr}\,\log\left(\sum_{k=0}^{\infty}\frac{[t\,u(x)]^k}{k!}\left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta}\right)^{k} u(x_\alpha)\,u(x_\beta)\right) - \log u(x).$$

The summation is the Heisenberg star product (6.14) of u with itself but with t replaced by $tu(x)$, i.e.,

$$\operatorname{tr} \sum_{k=0}^{\infty} \frac{[t\,u(x)]^k}{k!} \left(\frac{\partial}{\partial x_\alpha} - \frac{\partial}{\partial x_\beta} \right)^k u(x_\alpha)\, u(x_\beta) = u \star^{(\infty)}_{tu(x)} u = u(x + t\,u(x))\, u(x - t\,u(x)), \tag{6.16}$$

the second equality following from (6.14). Therefore, (6.16) equals

$$\begin{aligned}\frac{1}{2} \log \frac{u(x + t\,u(x))\, u(x - t\,u(x))}{u(x)^2} &= \frac{1}{2} \left(\sum_{k=0}^{\infty} \frac{[t\,u(x)]^k + [-t\,u(x)]^k}{k!} \frac{\partial^k}{\partial x^k} \log u(x) \right) - \log u(x) \\ &= \sum_{m=1}^{\infty} \frac{t^{2m}}{(2m)!}\, u(x)^{2m} \frac{\partial^{2m}}{\partial x^{2m}} \log u(x) = \sum_{m=1}^{\infty} \frac{t^{2m}}{(2m)!}\, \psi_{2m}\,,\end{aligned} \tag{6.17}$$

is a consequence of the exp–log formula (6.15) with t replaced by $t\,u(x)$. *Q.E.D.*

7. The Heisenberg–Projective Connection.

We are now in a position to make precise our contention that the Heisenberg representation embodies the more complicated projective classical invariant theory and modular form theories, not just as a limiting procedure, but in a direct correspondence.

First, the First Fundamental Theorem of classical invariant theory implies that we can write every invariant as a linear combination of *partial transvectants*

$$\operatorname{tr} \Omega_{\mathbf{A}}\, (\, Q_\alpha Q_\beta \cdots Q_\varepsilon\,), \qquad \text{where} \qquad \mathbf{A} = ((\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)) \tag{7.1}$$

is an ordered collection of pairs chosen from the symbolic letters $\alpha, \beta, \ldots, \varepsilon$ appearing in the product in (7.1), and

$$\Omega_{\mathbf{A}} = \prod_{\nu=1}^{m} \Omega_{\alpha_\nu \beta_\nu} \tag{7.2}$$

is the corresponding product of omega processes. Note that $\Omega_{\mathbf{A}}$ is, in fact, symmetric in the pairs of indices, and anti-symmetric under a single interchange $(\alpha_\nu, \beta_\nu) \longmapsto (\beta_\nu, \alpha_\nu)$. In the projective version, we divide by n^m in order to take the Heisenberg limit. Given $\mathbf{A}$ as in (7.1), we define

$$a_k = \#\, \{\, \alpha_\nu = \alpha_k \mid \nu > k \,\}, \qquad b_k = \#\, \{\, \beta_\nu = \beta_k \mid \nu > k \,\}.$$

Then the projective counterpart of of the differential invariant (7.1) is

$$\operatorname{tr} \Omega^{(n)}_{\mathbf{A}}\, (\, u_\alpha u_\beta \cdots u_\varepsilon\,), \tag{7.3}$$

where

$$\Omega^{(n)}_{\mathbf{A}} = \left[\left(1 - \frac{b_1}{n}\right) \frac{\partial}{\partial x_{\alpha_1}} - \left(1 - \frac{a_1}{n}\right) \frac{\partial}{\partial x_{\beta_1}} + \frac{x_\alpha - x_\beta}{n} \, \frac{\partial^2}{\partial x_{\alpha_1} \partial x_{\beta_1}} \right]$$
$$\left[\left(1 - \frac{b_2}{n}\right) \frac{\partial}{\partial x_{\alpha_2}} - \left(1 - \frac{a_2}{n}\right) \frac{\partial}{\partial x_{\beta_2}} + \frac{x_\alpha - x_\beta}{n} \, \frac{\partial^2}{\partial x_{\alpha_2} \partial x_{\beta_2}} \right] \cdots \tag{7.4}$$
$$\left[\frac{\partial}{\partial x_{\alpha_m}} - \frac{\partial}{\partial x_{\beta_m}} + \frac{x_\alpha - x_\beta}{n} \, \frac{\partial^2}{\partial x_{\alpha_m} \partial x_{\beta_m}} \right].$$

Remarkably, this is still symmetric under interchanges of the indices in $\mathbf{A}$. Moreover, after we multiply out, any terms involving any $x_\beta - x_\gamma$ will vanish upon taking the trace, and so can be ignored. In the limit $n \to \infty$, this reduces to the product

$$\Omega^{(\infty)}_{\mathbf{A}} = \prod_{\nu=1}^{m} \left(\frac{\partial}{\partial x_{\alpha_\nu}} - \frac{\partial}{\partial x_{\beta_\nu}} \right) \tag{7.5}$$

of Hirota operators.

Thus, to change projective SL(2) invariants into Heisenberg invariants, we merely take the limit $n \to \infty$, after dividing through by the appropriate power of n. Conversely, given a Heisenberg invariant, we rewrite it as a sum of partial transvectants

$$R = \operatorname{tr} \Omega^{(\infty)}_{\mathbf{A}} \, (\, u_\alpha \cdots u_\varepsilon \,), \tag{7.6}$$

and then replace the Hirota product $\Omega^{(\infty)}_{\mathbf{A}}$ by the projective omega product $\Omega^{(n)}_{\mathbf{A}}$ to obtain the corresponding projective invariant.

Remark: To obtain the Hirota formula (7.6) for a Heisenberg invariant, first write each monomial as a trace. Then symmetrize over all permutations of the symbolic indices that leave the product monomial unchanged. Finally, replace each derivative ∂_γ by the corresponding Hirota operator $\partial_\gamma - \partial_\alpha$. For example, the Hessian invariant $u u_2 - u_1^2$ first becomes

$$\operatorname{tr} \; \left(\partial_\beta^2 - \partial_{\alpha\beta}\right) u_\alpha u_\beta$$

Since both factors are the same u, we symmetrize to produce

$$\tfrac{1}{2} \operatorname{tr} \; \left(\partial_\alpha^2 + \partial_\beta^2 - 2\,\partial_{\alpha\beta}\right) u_\alpha u_\beta = \tfrac{1}{2} \operatorname{tr} \; \left(\partial_\alpha - \partial_\beta\right)^2 u_\alpha u_\beta = \tfrac{1}{2}\, \mathbb{D}_x^2 (u \cdot u),$$

The final factorization being either done by inspection, or, more systematically, by replacing $\partial_\alpha \mapsto 0$, $\partial_\beta \mapsto \partial_\beta - \partial_\alpha$. To form the projective version, we replace

$$\begin{aligned} \left(\partial_\alpha - \partial_\beta\right)^2 \longmapsto & \left(\left(1 - \tfrac{1}{n}\right) (\partial_\alpha - \partial_\beta) + \tfrac{x_\alpha - x_\beta}{n} \, \partial_{\alpha\beta} \right) \left(\partial_\alpha - \partial_\beta + \tfrac{x_\alpha - x_\beta}{n} \, \partial_{\alpha\beta} \right) = \\ & = \left(1 - \tfrac{1}{n}\right) \left[(\partial_\alpha - \partial_\beta)^2 + \tfrac{2}{n} \, \partial_{\alpha\beta} \right] + (x_\alpha - x_\beta)\, Z \\ & = \left(1 - \tfrac{1}{n}\right) \partial_\alpha^2 - 2 \left(1 - \tfrac{1}{n}\right)^2 \partial_{\alpha\beta} + \left(1 - \tfrac{1}{n}\right) \partial_\beta^2 + (x_\alpha - x_\beta)\, Z. \end{aligned}$$

the remainder term $(x_\alpha - x_\beta)\, Z$ vanishes upon setting $x_\alpha = x_\beta$ and so can be ignored.

Remark: It would be very useful to have a general combinatorial formula for the terms in the Pochhammer product (7.4) which do not involve the differences $x_\beta - x_\gamma$ and so survive the trace operation.

8. Coherent States.

Suppose $\lambda \in \ker \mathcal{B}$ is a (linear combination of) relative differential invariants. In analogy with string theory, [**17**], we say that a differential function f is a *coherent state* if $\mathcal{B}f = \lambda f$. Clearly a coherent state cannot be a polynomial unless $\lambda = 0$. Denoting the space of coherent states with eigenvalue λ by V_λ, we see that $V_\lambda \cdot V_\mu = V_{\lambda+\mu}$. This enables us to construct elements in $\ker \mathcal{B}$ from coherent states, since $V_\lambda \cdot V_{-\lambda} \subset V_0 = \ker \mathcal{B}$. Observe that for $f \in V_\lambda$, $e^{\mu \mathcal{B}} f \in V_{e^{\mu}\lambda}$.

Coherent states can be constructed as series in a formal parameter t. Suppose λ is constant or a function of $\mathcal{C}$ (e.g. in the q-Heisenberg case take $\lambda = q^{\mathcal{C}}$), so that $[\mathcal{B}, \lambda] = [\mathcal{D}, \lambda] = 0$. Define the operator T^t_λ mapping im $\mathcal{C}$ into itself by

$$T^t_\lambda(g) = \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^n g. \tag{8.1}$$

Proposition 8.1. *Given λ as above, one has*

$$\mathcal{B}\, T^t_\lambda = T^t_\lambda\, \mathcal{B} + t\lambda\, T^t_\lambda. \tag{8.2}$$

Proof: We compute

$$\begin{aligned}
\mathcal{B}\, T^t_\lambda &= \mathcal{B} \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^n = \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} \mathcal{B} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^n \\
&= \sum_{n=1}^{\infty} \frac{t^n \lambda^n}{(n-1)!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^{n-1} + \sum_{n=1}^{\infty} \frac{t^n \lambda^n}{n!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^n \mathcal{B} \\
&= t\lambda \sum_{n=1}^{\infty} \frac{t^{n-1} \lambda^{n-1}}{(n-1)!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^{n-1} + \sum_{n=1}^{\infty} \frac{t^n \lambda^n}{n!} \left(\frac{\mathcal{D}}{\mathcal{C}} \right)^n \mathcal{B} \\
&= t\lambda\, T^t_\lambda + T^t_\lambda\, \mathcal{B}.
\end{aligned}$$

Q.E.D.

Corollary 8.2. *Given a positive integer $n \in \mathbb{N}$, let $\lambda_0, \ldots, \lambda_n$ be constants or functions of $\mathcal{C}$, such that $\sum_{i=0}^n \lambda_i = 0$. Then*

$$\nu^t_n(f_0 \otimes \cdots \otimes f_n) = \otimes_{i=0}^n\, T^t_{\lambda_i}(f_i), \qquad f_0, \ldots, f_n \in \operatorname{im} \mathcal{C}, \tag{8.3}$$

defines a multilinear map $\nu^t_n \colon \otimes_{i=0}^n \operatorname{im} \mathcal{C} \longrightarrow \otimes_{i=0}^n \operatorname{im} \mathcal{C}$, intertwining with the Heisenberg representation.

Corollary 8.3. *Given a positive integer $n \in \mathbb{N}$, let $\lambda_0, \ldots, \lambda_n$ be constants or functions of $\mathcal{C}$, such that $\sum_{i=0}^n \lambda_i = 0$. Then*

$$\mu^t_n(f_0, \cdots, f_n) = \prod_{i=0}^{n} T^t_{\lambda_i}(f_i), \qquad f_0, \ldots, f_n \in \ker \mathcal{B}, \tag{8.4}$$

defines a multilinear map $\mu^t_n \colon \otimes_{i=0}^n \ker \mathcal{B} \longrightarrow \ker \mathcal{B}$.

We now express the $n+1$ functions $\lambda_0, \ldots, \lambda_n$ in terms of n functions $c_2, \ldots, c_{n+1} \in \ker \mathcal{B}$ in such a way that the sum of the λ_i's is automatically zero. When this is done, one defines the transvectants $\tau_{i_2,\ldots,i_{n+1}}$ as the coefficients of the monomials in $c_2, \ldots c_{n+1}$, and labels them by the powers of these formal parameters. For example, the coefficient of $c_2^2 c_3$ will be called $\tau_{2,1}$. Due to the independence of the monomials in c_k, the generalized transvectants $\tau_{i_2,\ldots,i_{n+1}}$ map into $\ker \mathcal{B}$, where the indices i_j indicate the term with $c_2^{i_2} \cdots c_{n+1}^{i_{n+1}}$.

There are many ways to do this. In the symmetric case we make the following choice. We take

$$\lambda_i = \xi_i - \frac{1}{n+1} \sum_{i=0}^{n} \xi_i.$$

Let now $c_k = \sum_{i=0}^n \lambda_i^k, \qquad k = 2, \ldots, n+1$. One can now express the λ_i in terms of the c_k. These in turn can be considered as symbolic expressions which will give us elements in $\ker \mathcal{B}$ when whenever one computes a polynomial of the c_k. For example,

$$c_2 = \lambda_0^2 + \lambda_1^2 = \left(\xi_0 - \tfrac{1}{2}(\xi_0+\xi_1)\right)^2 + \left(\xi_1 - \tfrac{1}{2}(\xi_0+\xi_1)\right)^2 = \tfrac{1}{2}(\xi_0-\xi_1)^2,$$

and this is related to $\frac{1}{2}(u \otimes u_2 + u_2 \otimes u) - u_1 \otimes u_1$, which reduces to the Hessian $uu_2 - u_1^2$ in the symmetric case. In the cubic case,

$$\begin{aligned} c_3 &= \lambda_0^3 + \lambda_1^3 + \lambda_2^3 \\ &= \left(\xi_0 - \tfrac{1}{3}(\xi_0+\xi_1+\xi_2)\right)^3 + \left(\xi_1 - \tfrac{1}{3}(\xi_0+\xi_1+\xi_2)\right)^3 + \left(\xi_2 - \tfrac{1}{3}(\xi_0+\xi_1+\xi_2)\right)^3 \\ &= \tfrac{4}{3}\,\xi_0\xi_1\xi_2 - \tfrac{1}{3}\,\xi_0^2\xi_1 - \tfrac{1}{3}\,\xi_0^2\xi_2 - \tfrac{1}{3}\,\xi_0\xi_1^2 - \tfrac{1}{3}\,\xi_0\xi_2^2 - \tfrac{1}{3}\,\xi_1^2\xi_2 - \\ &\qquad - \tfrac{1}{3}\,\xi_1\xi_2^2 + \tfrac{2}{9}\,\xi_0^3 + \tfrac{2}{9}\,\xi_1^3 + \tfrac{2}{9}\,\xi_2^3. \end{aligned}$$

This reduces, when we desymbolize, to

$$c_3 u^3 = \tfrac{4}{3}\,u_1^3 - 2\,uu_1u_2 + \tfrac{2}{3}\,u^2u_3.$$

We now compute $\mu_3^1(u,u,u)$, ignoring all terms that are not cubic in λ.

$$\begin{aligned} \mu_3^1(u,u,u) = T_{\lambda_0}^1(u)T_{\lambda_1}^1(u)T_{\lambda_2}^1(u) &= \prod_{i=0}^{2} \left(u + \lambda_i u_1 + \frac{\lambda_i^2}{2}u_2 + \frac{\lambda_i^3}{6}u_3\right) \\ &= c_3\left(\tfrac{1}{6}u^2u_3 - \tfrac{1}{2}uu_1u_2 + \tfrac{1}{3}u_1^3\right) \end{aligned}$$

We find that $\tau_{0,1}(u,u,u) = \frac{1}{6}u^2u_3 - \frac{1}{2}uu_1u_2 + \frac{1}{3}u_1^3$. The fact that the symbolic expression c_3 gives rise to the same transvectant is not too surprising when one considers that the coherent state method carefully labels each differentiation with the appropriate symbol λ_i. The connection, however, is not quite straightforward since the condition that the sum of the λ_i's should vanish does not appear in the symbolic method. It might be worth while to pursue this further since it may lead to effective computation methods for covariants.

In the nonsymmetric case we proceed as follows. Since we have $n+1$ parameters and 1 relation among them, we would like to find n parameters in which things can be expressed, in order to insure that the monomials in these new parameters are linearly independent.

Let $\omega \in \mathbb{C}$ be a primitive $(n+1)^{\mathrm{st}}$ root of unity: $\omega^{n+1} = 1, \omega^p \neq 1$ for any $0 < p < n+1$. Put

$$\lambda_i = \sum_{j=2}^{n+1} \omega^{ij} c_j, \qquad i = 0, \ldots, n, \qquad \text{for} \qquad c_2, \ldots, c_{n+1} \in \ker \mathcal{B},$$

to obtain $T^t_{\lambda_i}(f_i) \in V_{t\lambda_i}$, $i = 0, \ldots, n$. Then $\mu^t_n(f_0, \ldots, f_n) \in \ker \mathcal{B}$. For $n = 1$ reduces to the star product $f_0 \star_{tc_2} f_1$. Note that

$$\sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} \sum_{j=2}^{n+1} \omega^{ij} c_j = \sum_{j=2}^{n+1} n\delta(j) c_j = 0$$

as it should be. We can view the λ_i as Discrete Fourier Transforms of the c_i, where we take $c_1 = 0$ from the start. In the symmetric case the reader may want to verify that when we now apply the Inverse Discrete Fourier Transform to the c_i, we obtain the formulae employed in our analysis of the symmetric case. Let us write out the formal expansion when $\omega = -1$, $f_0 = R, f_1 = S$:

$$\begin{aligned} R \star_{tc_2} S = T^t_{c_2}(R) \cdot T^t_{-c_2}(S) &= \left\{ \sum_{r=0}^{\infty} \frac{t^r c_2^r}{r!} \left(\frac{\mathcal{D}}{\mathcal{C}}\right)^r R \right\} \left\{ \sum_{s=0}^{\infty} \frac{(-1)^s t^s c_2^s}{s!} \left(\frac{\mathcal{D}}{\mathcal{C}}\right)^s S \right\} \\ &= \sum_{m=0}^{\infty} t^m \, c_2^m \, \hat{\tau}_m(R, S), \end{aligned} \tag{8.5}$$

where

$$\hat{\tau}_m(R, S) = \sum_{r+s=m} (-1)^s \frac{1}{r!s!} \left(\frac{\mathcal{D}}{\mathcal{C}}\right)^r R \cdot \left(\frac{\mathcal{D}}{\mathcal{C}}\right)^s S. \tag{8.6}$$

Note that if R, S are homogeneous, so $\mathcal{C}(R) = kR$, $\mathcal{C}(S) = lS$, then

$$\hat{\tau}_n(R, S) = \frac{1}{(kl)^n} \, \tau_m(R, S)$$

is simply a multiple of the Heisenberg transvectant (5.5). The fact that the coherent state procedure gives a multilinear generalization suggests that $\hat{\tau}_m$ is the more natural definition for the transvectant.

In the multilinear case one obtains analogous formulae which are labeled by monomials in $c_2, \ldots, c_{n+1}$. This procedure basically solves the problem of generalizing the Hirota operator, [**23**, **24**] to the multilinear case in a natural way, cf. [**16**, **20**, **21**]. It is rather surprising that the coherent state method, which relies on the fact that $\mathcal{C}$ commutes with the whole algebra, can be used to compute classical covariants. But it does provide a very nice illustration of the power of the methods covered in this paper. In [**38**] analogous results for the q-Heisenberg representation are derived.

9. Conclusions and Further Directions.

We have covered a number of subjects which are all related by two facts:

(a) The Heisenberg algebra plays a role.

(b) They are of importance in modern physical theories.

Although it is too early to claim any deep connection between these two facts, the thread seems to be interesting and leading to nontrivial results. As objects of possible further research we mention:

(1) Applications to integrable systems. These include further developments of modular forms and their brackets as tau functions for soliton equations, would be well worth pursuing. While it is perfectly possible to apply the multilinear Hirota operators to integrable equations, so far this does not seem to simplify matters in any way. In particular one would like a normal form result, in which integrability would be a divisibility condition in terms of the c_k.

(2) Transvectants. In [**32**], these were shown to be particular cases of general multilinear and multidimensional differential operators called "hyperjacobians", which are based on Cayley's old, pre-transvectant theory of hyperdeterminants, [**4**, **5**], and have interesting formulations as higher dimensional determinants, [**13**, **14**]. A detailed investigation into the connections with our multilinear generalizations of the Hirota operators would be worth pursuing.

(3) Application of multilinear generalizations of the Hirota operators, and their projective analogues.

(4) Connections with combinatorics. In [**41**], Schimming and Strampp connect differential polynomials arising in the Sato approach to soliton equations with the combinatorial Bell polynomials. The further development of these connections and their analogues for modular forms has significant potential.

(5) Intertwining operators. The coherent state method suggests that it might be natural to look at the tensor products and formulate the problem in terms of intertwining operators.

(6) Application of the q-Heisenberg analysis to quantum groups, [**6**]. A very interesting generalization of classical invariant theory and the transvectant calculus to quantum groups appears in the recent paper of [**28**]. Our methods, which in themselves realize the classical theory as a deformation of the Heisenberg theory, should be particularly relevant. The paper [**38**] extends results in this paper to the q-Heisenberg representation.

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