Introduction to Partial Differential Equations

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Corrections to Second Printing (2016)

Last updated: July 4, 2022

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Exercise 1.10(a): change $4t^2 - x^2$ to $4t^2 + x^2$.

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Exercise 2.2.31(b): insert $= 0$ in equation: $u_t + y\,u_x - x\,u_y = 0$.

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In the last two displayed formulas, the first term on the right hand side of the equals sign is missing a minus sign:

$$\frac{\partial v}{\partial \xi}(\xi,\eta) = -\frac{1}{2c}\frac{\partial u}{\partial t}\left(\frac{\eta-\xi}{2c},\frac{\eta+\xi}{2}\right) + \frac{1}{2}\frac{\partial u}{\partial x}\left(\frac{\eta-\xi}{2c},\frac{\eta+\xi}{2}\right),$$

and so, in particular,

$$\frac{\partial v}{\partial \xi}(\xi,\xi) = -\frac{1}{2c}\frac{\partial u}{\partial t}(0,\xi) + \frac{1}{2}\frac{\partial u}{\partial x}(0,\xi) = 0,$$

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Exercise 3.2.14: insert after first sentence:

(See (3.81) for the definition of the function $\operatorname{sign} x$.)

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Line -4: change "second derivative" to "first derivative"

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Line 2 before (3.106): remove square root from "… equal to $\dfrac{1}{2\pi}\displaystyle\int_a^b |\,\varphi(x)\,|^2\,dx$."

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In (4.37), change final $+$ to $-$:

$$u(t,x) \approx \tfrac{1}{2}a_0 + e^{-t}(a_1\cos x + b_1 \sin x) = \tfrac{1}{2}a_0 + r_1 e^{-t}\cos(x-\delta_1), \qquad (4.37)$$

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Line -6: change "… appear the context of boundary value problems." to "… appear in the context of boundary value problems."

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Line 5 after (4.86): change "heat flux out of a plate" to "heat flux into a plate"

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Last line of table: change $x^4 - 4x^2y^2 + y^4$ to $x^4 - 6x^2y^2 + y^4$

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Exercise 4.3.25(b): change $x^2 + y^2 = 1$; to $x^2 + y^2 = 2$;

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Exercise 4.4.12(a): switch t and x in the function: $u_n(t,x) = \dfrac{\cosh n\pi t \ \sin n\pi x}{n}$.

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Line 2 before (5.14): change
"… heat equation (5.14) …" to "… heat equation (5.7) …".

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Example 5.4: change rest of sentence after displayed formula to

"…used earlier in Example 4.1, along with homogeneous Dirichlet boundary conditions, so $u(t,0) = u(t,1) = 0$."

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Equation (5.28): change $\mathrm{O}\big((\Delta t)^2\big)$ to $\mathrm{O}(\Delta t)$:

$$\frac{\partial u}{\partial t}\,(t_j, x_m) \approx \frac{u(t_j, x_m) - u(t_{j-1}, x_m)}{\Delta t} \ + \ \mathrm{O}(\Delta t). \tag{5.28}$$

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Change sentence after equation (5.40): "We use step sizes $\Delta t = \Delta x = .005$, set $\ell = 1$, and try four different values of the wave speed."

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Equation (5.45): change denominator to $2\,\Delta x$:

$$\frac{\partial u}{\partial x}\,(t_j, x_m) \approx \frac{u_{j,m+1} - u_{j,m-1}}{2\,\Delta x} + \mathrm{O}\big((\Delta x)^2\big). \tag{5.45}$$

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Line 4 after (5.50): reverse the inequality: $\Delta x/\Delta t \geq |\,c_{j,m}\,|$

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Correct last displayed equation by switching indices on the $u_{i,j}$:

$$\begin{array}{lll} u_{1,1} = .1831, & u_{2,1} = .2589, & u_{3,1} = .1831, \\ u_{1,2} = .3643, & u_{2,2} = .5152, & u_{3,2} = .3643, \\ u_{1,3} = .5409, & u_{2,3} = .7649, & u_{3,3} = .5409, \end{array}$$

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Equation (5.78): the sub- and super-diagonal matrix elements should be -1, not $-\rho^2$:

$$B_\rho = \begin{pmatrix} 2(1+\rho^2) & -1 & & & & & \\ -1 & 2(1+\rho^2) & -1 & & & & \\ & -1 & 2(1+\rho^2) & -1 & & & \\ & & -1 & 2(1+\rho^2) & -1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2(1+\rho^2) & -1 \\ & & & & & -1 & 2(1+\rho^2) \end{pmatrix} \tag{5.78}$$

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Equation (5.82): replace $\mathbf{w}^{(n-1)}$ by $U_{n-1}\mathbf{w}^{(n-1)}$ and U_j by U_k:

$$\begin{array}{lll} \mathbf{z}^{(1)} = \widehat{\mathbf{f}}^{(1)}, & \mathbf{z}^{(j+1)} = \widehat{\mathbf{f}}^{(j+1)} - L_j \mathbf{z}^{(j)}, & j = 1, 2, \ldots, n-2, \\ U_{n-1}\mathbf{w}^{(n-1)} = \mathbf{z}^{(n-1)}, & U_k \mathbf{w}^{(k)} = \mathbf{z}^{(k)} - \rho^2\,\mathbf{w}^{(k+1)}, & k = n-2, n-3, \ldots, 1, \end{array} \tag{5.82}$$

Equation (5.83): replace L_j by L_k:

$$\mathbf{w}^{(k)} = L_k\big(\mathbf{w}^{(k+1)} - \rho^{-2}\,\mathbf{z}^{(k)}\big), \qquad k = n-2, n-3, \ldots, 1. \tag{5.83}$$

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Equation (6.40): delete initial fraction:

$$\int_{-\pi}^{\pi} s_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{\,\mathrm{i}\,k\,x}\,dx = 1, \tag{6.40}$$

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In the first integral in the displayed equation after (6.65) change $\sinh\omega y$ to $\sinh\omega\xi$:

$$u(x) = \int_0^x \frac{\sinh\omega(1-x)\sinh\omega\xi}{\omega\sinh\omega}\,d\xi \;+\; \int_x^1 \frac{\sinh\omega x\,\sinh\omega(1-\xi)}{\omega\sinh\omega}\,d\xi$$

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Displayed formula after Theorem 6.17: change $\mathbb{R}^2$ to Ω:

$$u(x,y) = -\iint_\Omega G_0(x,y;\xi,\eta)\,\Delta u(\xi,\eta)\,d\xi\,d\eta.$$

Equation (6.108): change $\mathbb{R}^2$ to Ω twice:

$$\iint_\Omega \delta(x-\xi)\,\delta(y-\eta)\,u(\xi,\eta)\,d\xi\,d\eta = \iint_\Omega -\Delta G_0(x,y;\xi,\eta)\,u(\xi,\eta)\,d\xi\,d\eta. \tag{6.108}$$

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Equation (6.135): correct left hand side:

$$\frac{\partial G}{\partial \rho}(r,\theta;1,\phi) = -\frac{1}{2\pi}\,\frac{1-r^2}{1+r^2-2\,r\cos(\theta-\phi)}\,, \tag{6.135}$$

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The proof of Proposition 7.10 is flawed. Since $k\,\delta(k)\equiv 0$, dividing equation (7.47) by $\mathrm{i}\,k$ could also introduce a multiple of the delta function, and it seems difficult to dismiss this term. A better proof uses the Convolution Theorem 7.13, as follows.

The first step is to note that we can write the integral of $f(x)$ as a convolution with the step function:

$$g(x) = \int_{-\infty}^{x} f(\xi)\,d\xi = \int_{-\infty}^{\infty} \sigma(x-\xi)\,f(\xi)\,d\xi,$$

Thus, according to the convolution formula (7.55),

$$\widehat{g}(k) = \sqrt{2\pi}\;\widehat{\sigma}(k)\,\widehat{f}(k).$$

Consulting our Table of Fourier transforms, we find

$$\widehat{g}(k) = \sqrt{2\pi}\left(\sqrt{\frac{\pi}{2}}\,\delta(k) - \frac{\mathrm{i}}{\sqrt{2\pi}\,k}\right)\widehat{f}(k) = -\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\widehat{f}(k)\,\delta(k) = -\frac{\mathrm{i}}{k}\,\widehat{f}(k) + \pi\widehat{f}(0)\,\delta(k),$$

which establishes the desired formula.

$\star\star$ Thanks to William Young for alerting me to this issue and for sharing the above convolution-based proof.

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In the displayed equation immediately above the Exercises, delete one factor of $1/k$ in the first term after the equals sign:

$$\widehat{f}(k) = \left(-\frac{\mathrm{i}}{k}\sqrt{\frac{\pi}{2}}\,e^{-\,|\,k\,|} + \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k)\right) - \frac{\pi^{3/2}}{\sqrt{2}}\,\delta(k) = -\,\mathrm{i}\sqrt{\frac{\pi}{2}}\,\frac{e^{-\,|\,k\,|}}{k}\,.$$

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Exercise 7.2.12. insert factor of $\sqrt{2\pi}$ in formula $\widehat{f}(k) = \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\,\delta(k-n)$.

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Second displayed formula after equation (8.63): insert missing factor of $\frac{1}{2}$:

$$u(t,x) = c_1 + c_2 \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right).$$

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Insert parenthetical comment at end of page:

(The case $q(x) \equiv 0$ can also be positive definite, when subject to suitable boundary conditions, but is treated differently, in accordance with the weighted inner product construction appearing in Example 9.23.)

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Change (9.131–132) to the following:

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{\infty} \big[\, c_k u_k(t,x) + d_k \widetilde{u}_k(t,x) \,\big] \\ &= \sum_{k=1}^{\infty} \big[\, c_k \cos(\omega_k t) + d_k \sin(\omega_k t) \,\big]\, v_k(x) = \sum_{k=1}^{\infty} r_k \cos(\omega_k t - \delta_k)\, v_k, \end{aligned} \tag{9.131}$$

where (r_k, δ_k) are the polar coordinates of (c_k, d_k):

$$c_k = r_k \cos\delta_k, \qquad d_k = r_k \sin\delta_k. \tag{9.132}$$

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Change (9.145) to the following:

$$0 = \langle\, h - 2\,a\,\omega_k v_k \,,v_k \,\rangle = \langle\, h \,, v_k \,\rangle - 2\,a\,\omega_k \,\|\, v_k \,\|^2, \qquad \text{and hence} \qquad a = \frac{\langle\, h \,, v_k \,\rangle}{2\,\omega_k \,\|\, v_k \,\|^2}\,, \tag{9.145}$$

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Correct sign errors in (9.149):

$$v_\star(x) = \frac{\sin k\pi x}{k^2\pi^2 c^2 - \omega^2}\,, \qquad \text{so that} \qquad u_\star(t,x) = \frac{\cos\omega t\ \sin k\pi x}{k^2\pi^2 c^2 - \omega^2}\,, \tag{9.149}$$

and in the last displayed equation:

$$z(0,x) = f(x) \;-\; \frac{\sin k\pi x}{k^2\pi^2 c^2 - \omega^2}\,, \qquad\qquad \frac{\partial z}{\partial t}\,(0,x) = g(x),$$

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Line 9 in paragraph beginning "The first ...": change "vertexvertices" to "vertices".

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Last equation in (10.32): change y_k to y_l:

$$\begin{aligned} \omega_l^\nu(x_i,y_i) &= \alpha_l^\nu + \beta_l^\nu\, x_i + \gamma_l^\nu\, y_i = 0, \\ \omega_l^\nu(x_j,y_j) &= \alpha_l^\nu + \beta_l^\nu\, x_j + \gamma_l^\nu\, y_j = 0, \\ \omega_l^\nu(x_l,y_l) &= \alpha_l^\nu + \beta_l^\nu\, x_l + \gamma_l^\nu\, y_l = 1. \end{aligned} \tag{10.32}$$

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The first term in the integral in (10.38) is the Euclidean norm of a vector:

$$\begin{aligned} Q[w] = Q\left[\sum_{i=1}^n c_i\,\varphi_i\right] &= \iint_\Omega \left[\left\|\sum_{i=1}^n c_i\,\nabla\varphi_i\right\|^2 - f(x,y)\left(\sum_{i=1}^n c_i\,\varphi_i\right)\right] dx\,dy \\ &= \frac{1}{2}\sum_{i,j=1}^n k_{ij}\,c_i\,c_j - \sum_{i=1}^n b_i\,c_i = \tfrac{1}{2}\,\mathbf{c}^T K\,\mathbf{c} - \mathbf{b}^T\mathbf{c}. \end{aligned} \tag{10.38}$$

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Correct last line in (10.45):

$$\begin{aligned} k_{ij}^\nu &= \frac{1}{2}\,\frac{(y_j-y_l)(y_l-y_i)+(x_l-x_j)(x_i-x_l)}{(\Delta_\nu)^2}\,|\,\Delta_\nu\,| = -\,\frac{(\mathbf{x}_i-\mathbf{x}_l)\cdot(\mathbf{x}_j-\mathbf{x}_l)}{2\,|\,\Delta_\nu\,|}\,, \quad i\neq j, \\ k_{ii}^\nu &= \frac{1}{2}\,\frac{(y_j-y_l)^2+(x_l-x_j)^2}{(\Delta_\nu)^2}\,|\,\Delta_\nu\,| = \frac{\|\,\mathbf{x}_j-\mathbf{x}_l\,\|^2}{2\,|\,\Delta_\nu\,|} \\ &= \frac{(\mathbf{x}_i-\mathbf{x}_l)\cdot(\mathbf{x}_j-\mathbf{x}_l)+(\mathbf{x}_l-\mathbf{x}_j)\cdot(\mathbf{x}_i-\mathbf{x}_j)}{2\,\Delta_\nu} = -\,k_{ij}^\nu - k_{il}^\nu\,. \end{aligned} \tag{10.45}$$

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Change first sentence in Example 10.7 to

A metal plate has the shape of an oval running track, consisting of a square, with side lengths 2 m, and two semi-circular disks glued onto opposite sides, as sketched in Figure 10.9.

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Exercise 10.3.16: change $n=2$ in part (*b*) to $n=3$, and change $n=3$ in part (*c*) to $n=4$.

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Exercise 10.4.3(c): change "… wave equation." to "… transport equation."

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Exercise 11.2.12: correct boundary conditions:

$$u(t,0,y) = u(t,\pi,y) = 0 = u(t,x,0), \quad u(t,x,\pi) = f(x), \quad 0 < x,y < \pi, \quad t > 0.$$

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In equation (11.91) and the subsequent displayed formula, change all s, t, r to a, b, c:

$$a_0\, r\,(r-1) + b_0\, r + c_0 = 0, \tag{11.91}$$

where, referring back to (11.71),

$$a_0 = a(x_0), \qquad b_0 = b(x_0), \qquad c_0 = c(x_0),$$

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Case (iii): change $r_2 = r_1 + k$ to $r_1 = r_2 + k$; change "smaller" to "larger", and change x^{r_2} to $(x-x_0)^{r_2}$ in equation (11.93):

(iii) Finally, if $r_1 = r_2 + k$, where $k > 0$ is a positive integer, then there is a nonzero solution $\widehat{u}(x)$ with a convergent Frobenius expansion corresponding to the larger index r_1. One can construct a second independent solution of the form

$$\widetilde{u}(x) = c\log(x-x_0)\,\widehat{u}(x) + v(x), \quad \text{where} \quad v(x) = (x-x_0)^{r_2} + \sum_{n=1}^{\infty} v_n (x-x_0)^{n+r_2} \tag{11.93}$$

is a convergent Frobenius series, and c is a constant, which may be 0, in which case the second solution $\widetilde{u}(x)$ is also of Frobenius form.

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Correct formulas after equation (11.96) as follows:

$$\begin{aligned} u'' + \left(\frac{1}{x} + \frac{x}{2}\right) u' + u &= v\left[\widehat{u}'' + \left(\frac{1}{x} + \frac{x}{2}\right)\widehat{u}' + \widehat{u}\right] + v'\left[2\,\widehat{u}' + \left(\frac{1}{x} + \frac{x}{2}\right)\widehat{u}\right] + v''\,\widehat{u} \\ &= e^{-x^2/4}\left[v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v'\right]. \end{aligned}$$

If u is to be a solution, v' must satisfy a linear first-order ordinary differential equation:

$$v'' + \left(\frac{1}{x} - \frac{x}{2}\right) v' = 0, \qquad \text{and hence} \qquad v' = \frac{c}{x}\, e^{x^2/4}, \qquad v = c\int \frac{e^{x^2/4}}{x} + d,$$

where c, d are arbitrary constants. We conclude that the general solution to the original differential equation is

$$\widetilde{u}(x) = v(x)\,\widehat{u}(x) = \left(c \int \frac{e^{x^2/4}}{x} + d \right) e^{-x^2/4}. \tag{11.97}$$

⋆⋆ Thanks to Manuel Mañas for alerting me to this error.

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In Figure 11.5, the graphs of $Y_1(x), Y_2(x), Y_3(x)$ are poorly reproduced:

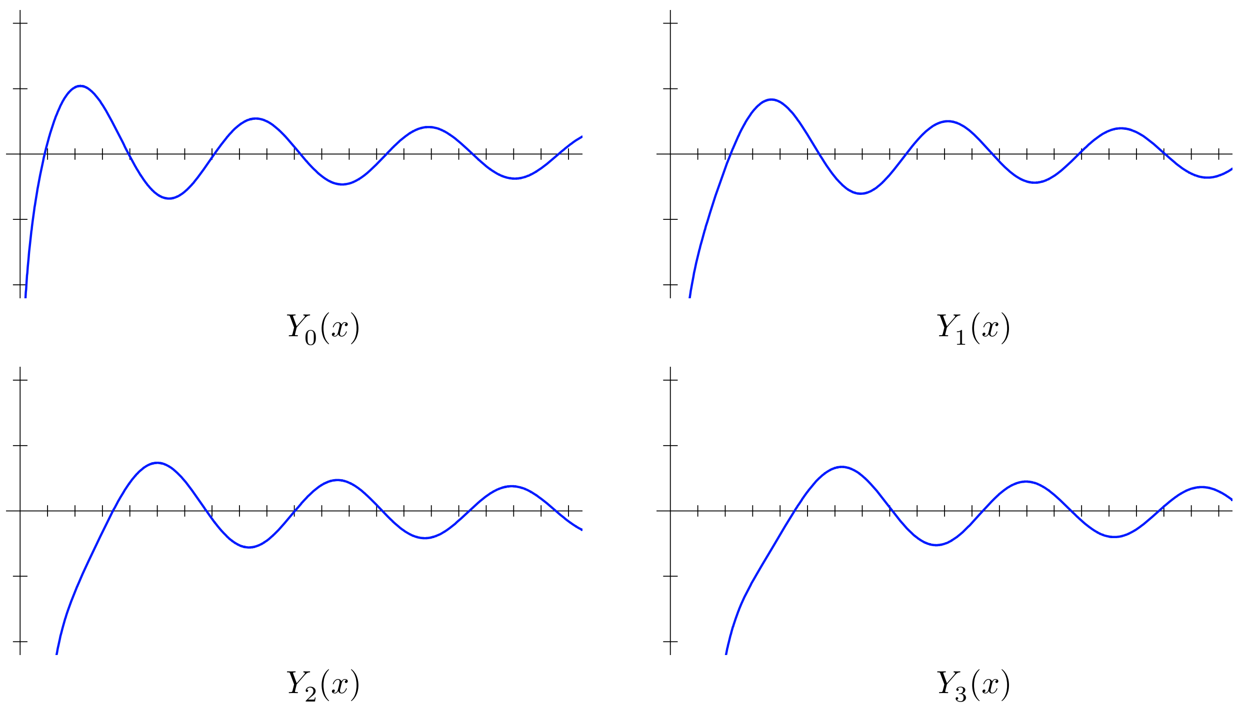

Figure 11.5. Bessel functions of the second kind.

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Add movie symbol ⊎ to Figure 11.10.

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Example 11.15: correct equation 2 lines from the end: $\zeta_{0,1}/\zeta_{0,2} \approx .43565$

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Exercise 11.6.41: switch indices on $\omega_{i,j}$:

(a) $\omega_{0,4}$, (b) $\omega_{2,4}$, (c) $\omega_{4,2}$, (d) $\omega_{3,3}$, (e) $\omega_{5,1}$.

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Line -3: change comma to semicolon in $v(\mathbf{x}; \boldsymbol{\xi})$

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Equation (12.131): add t dependence to $u_{0,0,n}$ and $\widehat{u}_{0,0,n}$, and correct denominators in final expressions:

$$\begin{aligned} u_{0,0,n}(t,r,\varphi,\theta) &= \cos(c\,n\,\pi\,t)\;S_0(n\,\pi\,r) = \frac{\cos c\,n\,\pi\,t\;\sin n\,\pi\,r}{n\,\pi\,r}\,, \\ \widehat{u}_{0,0,n}(t,r,\varphi,\theta) &= \sin(c\,n\,\pi\,t)\;S_0(n\,\pi\,r) = \frac{\sin c\,n\,\pi\,t\;\sin n\,\pi\,r}{n\,\pi\,r}\,, \end{aligned} \qquad n = 1,2,3,\ldots\,. \tag{12.131}$$

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In equation (12.145) and the displayed equation immediately after, the limit should be as $t \to 0$:

$$\lim_{t\to 0} \mathrm{M}_{ct}\,[\,f\,] = \mathrm{M}_0\,[\,f\,] = f(\mathbf{0}). \tag{12.145}$$

$$\lim_{t\to 0} \langle\, u(t,\cdot\,)\,,f\,\rangle = \langle\, u(0,\cdot\,)\,,f\,\rangle = 0 \qquad \text{for } \textit{all} \text{ functions} \quad f,$$

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Replace the period in equation (12.151) by a comma, and replace the following sentence by

where $\mathrm{M}^{\mathbf{x}}_{ct}\,[\,g\,]$ denotes the average of the initial velocity function g over the sphere $S^{\mathbf{x}}_{ct} = \{\,\|\,\boldsymbol{\xi} - \mathbf{x}\,\| = c\,t\}$ of radius $c\,t$ centered at the point $\mathbf{x}$. Thus, the value of our solution at position $\mathbf{x}$ and time $t > 0$ only depends upon the initial data a distance ct away from the point $\mathbf{x}$.

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At the end of the statement of Theorem B.15, add "; for the triangle equality, the scalar multiples must be nonnegative."

Acknowledgements: Many thanks to Lawrence Baker, Henry Boateng, Dario Valdebenito, Joseph Feneuil, Adam Kay, Manuel Mañas, Dani Rozenbroek, Mikhail Safonov, Radu Slobodeanu, and William Young for their comments and corrections.

