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The Equivalence Problem and Canonical Forms for Quadratic Lagrangians

PETER J. OLVER*

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

The general equivalence and canonical form problems for quadratic variational problems under arbitrary linear changes of variable are formulated, and the role of classical invariant theory in their general solution is made clear. A complete solution to both problems for planar, first order quadratic variational problems is provided, including a complete list of canonical forms for the Lagrangians and corresponding Euler–Lagrange equations. Algorithmic procedures for determining the equivalence class and the explicit canonical form of a given Lagrangian are provided. Applications to planar anisotropic elasticity are indicated.

1. INTRODUCTION

The basic problem of the calculus of variations is to determine the minima of a variational integral

$$\mathscr{L}[\mathbf{u}] = \int_\Omega L[\mathbf{u}]\,dx.$$

Here the integral is over a subdomain Ω of the space $\mathbb{R}^p$ of independent variables $\mathbf{x} = (x_1, \dots, x_p)$, and the solutions $\mathbf{u} = \mathbf{f}(\mathbf{x})$ are restricted to be in some appropriate space of vector-valued functions from $\mathbb{R}^p$ to $\mathbb{R}^q$, the space of dependent variables $\mathbf{u} = (u^1, \dots, u^q)$. The *Lagrangian* $L[\mathbf{u}]$ is a function of $\mathbf{x}, \mathbf{u}$, and the derivatives of $\mathbf{u}$, which, for simplicity, we take to be smooth in its arguments. The smooth minima of this variational problem are known to be solutions of the associated Euler–Lagrange equations $E(L) = 0$.

Since the process of minimization does not depend on any particular coordinate system in use, it makes eminent sense to try to simplify the Lagrangian, and hence the associated Euler–Lagrange equations, as much

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as possible through the introduction of "adapted" coordinates. More specifically, consider a change of variables

$$\mathbf{x} = \boldsymbol{\varphi}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}), \qquad \mathbf{u} = \boldsymbol{\psi}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}), \tag{1}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are smooth, invertible functions. (More restrictively, one might require that $\boldsymbol{\varphi}$ only depend on $\tilde{\mathbf{x}}$, so the new independent variables do not depend on the old dependent variables. although this would exclude interesting changes of variable like the hodograph transformation of gas dynamics.) In the new variables $\tilde{\mathbf{x}}, \tilde{\mathbf{u}}$, the variational problem has an analogous form

$$\tilde{\mathscr{L}}[\tilde{\mathbf{u}}] = \int_{\tilde{\Omega}} \tilde{L}[\tilde{\mathbf{u}}]\, d\tilde{x}.$$

The minima of the two variational problems $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are in one-to-one correspondence under the change of variables (1), so from a coordinate-free standpoint, they are essentially the same problem.

Once we allow ourselves the freedom of changing variables in the treatment of variational problems, we are immediately confronted with two problems of fundamental importance. The first is to determine when two variational problems are really the same under some change of variables:

EQUIVALENCE PROBLEM. Given two Lagrangian functions $L[\mathbf{u}]$ and $\tilde{L}[\tilde{\mathbf{u}}]$, when does there exist a change of variables (1) taking L to $\tilde{L}$? If so, how does one explicitly construct the change of variables?

Perhaps even more important from the point of view of analyzing the solutions to a given variational problem is the problem of determining a change of variables which has the effect of simplifying the integral as much as possible. Thus one is led to the problem of determining lists of simple "canonical forms" for variational problems, so that any other variational problem of a given type is equivalent to one of these canonical forms.

CANONICAL FORM PROBLEM. Find a complete list of canonical forms for Lagrangians with the property that any other Lagrangian of a given type (e.g., first order, planar, etc.) is equivalent to precisely one of the canonical forms on the list.

In the above general formulation, there is a powerful method due to Cartan [1] which will algorithmically solve the general equivalence problem. However, to date, it has only been implemented in the special cases of first-order Lagrangians on the line ($p = 1$) [2, Section 6] and scalar first-order Lagrangians in the plane ($p = 2$, $q = 1$) [3]. The extension to the truly vector-valued case ($p > 1$, $q > 1$), which includes the equations of nonlinear continuum mechanics, has yet to be done. Once Cartan's program has been implemented in this case, there will, I believe, be profound applications to the study of variational problems in elasticity and continuum mechanics. The canonical form problem, however, appears to be quite a bit more difficult, and Cartan's method is not as directly useful.

In the present paper, we tackle a much more modest version of the above problems, namely the equivalence and canonical form problems for homogeneous quadratic Lagrangians. Thus, we make the assumption that the Lagrangian L is a quadratic function of just the kth order derivatives of the dependent variables $\mathbf{u}$. The corresponding Euler–Lagrange equations then form a $2k$th order self-adjoint system of linear partial differential equations, the special case $k = 1$ being of especial interest as it includes the equations of linear elasticity, cf. [5]. Since we have restricted the Lagrangian to be quadratic, we will only allow linear changes of variable

$$\mathbf{x} = A\tilde{\mathbf{x}}, \qquad \mathbf{u} = B\tilde{\mathbf{u}}, \tag{2}$$

in which A and B are invertible $p \times p$ and $q \times q$ matrices, respectively. We thus have restricted versions of the equivalence problem to determine when two quadratic Lagrangians are equivalent under a linear change of variables (2), and the closely related problem of finding canonical forms for quadratic Lagrangians of a given order. The motivation for the study of these problems was the author's continuing studies on conservation laws in linear elasticity [8] and more specifically, attempts to extend the results on linear isotropic elasticity to the anisotropic case [9]. It was found that without some kind of elementary canonical form, the intervening computations for symmetries and conservation laws are just too complex to effectively analyze in the case of general linearly elastic materials.

In this paper, the equivalence and canonical form problems for first-order planar quadratic Lagrangians ($p = q = 2$), which include the case of planar anisotropic elasticity, are completely solved. Here the independent variables $\mathbf{x} = (x, y)$ and the dependent variables $\mathbf{u} = (u, v)$ are both in $\mathbb{R}^2$, and the Lagrangian is a constant-coefficient quadratic function of the four first-order derivatives u_x, u_y, v_x, v_y. A preliminary classification into 15 different canonical forms is provided by our first main result.

THEOREM 1. *Any planar first-order quadratic Lagrangian is equivalent to a "canonical" Lagrangian from precisely one of the following classes*:

(1) $\pm u_x^2 \pm v_y^2 + \alpha(u_y^2 \pm v_x^2) + 2\beta u_x v_y$, $\quad \alpha, \beta$ constants,

(2) $\pm u_x^2 \pm v_y^2 \pm u_y^2 + 2\beta u_x v_y$, $\quad \beta$ constant,

(3) $\pm u_x^2 \pm u_y^2 + u_x v_y$,

(4) $\pm u_x^2 \pm v_x^2 + u_x v_y$,

(5) $\pm u_x^2 + u_x v_y$,

(6) $u_x^2 - v_x^2 + u_x u_y + v_x v_y$,

(7) $u_x^2 - u_y^2 + u_x v_x + u_y v_y$,

(8) $\pm(u_x^2 + v_y^2) + 2u_x v_y + u_y v_y$,

(9) $\pm u_x^2 + u_y v_y$,

(10) $\pm u_x^2 + v_x v_y$,

(11) $u_x u_y + u_x v_x$,

(12) $\pm u_x^2 \pm u_y^2$,

(13) $\pm u_x^2 \pm v_x^2$,

(14) $\pm u_x^2$,

(15) 0.

In each of these canonical forms, one can take any desired combination of + or − signs. Also note that one has the extra freedom of adding in a multiple of the basic quadratic null Lagrangian $u_x v_y - u_y v_x$ without affecting the Euler–Lagrange equations. Consequently, there are fifteen canonical forms for a self-adjoint system of second-order linear Euler–Lagrange equations in the plane, each of which corresponds to one of the above canonical Lagrangians.

Included in this classification is the important case of anisotropic linear planar elasticity, so our analysis provides canonical forms for elastic moduli under general linear changes of variable. In fact, it is easy to see that of the above fifteen canonical forms, only one satisfies the Legendre–Hadamard strong ellipticity condition (see Eq. (6) below) required of an elastic problem.

THEOREM 2. *Let $L[\mathbf{u}]$ be a first-order planar quadratic Lagrangian which satisfies the Legendre–Hadamard strong ellipticity condition. Then L is equivalent to an* ***orthotropic Lagrangian***

$$u_x^2 + v_y^2 + \alpha\left(u_y^2 + v_x^2\right) + 2\beta u_x v_y,$$

where the "moduli" α and β are constants, satisfying the inequalities

$$\alpha > 0, \qquad |\beta| < \alpha + 1.$$

The corresponding Euler–Lagrange equations are thus equivalent to a "generalized" system of Navier's equations

$$u_{xx} + \alpha u_{yy} + \beta v_{xy} = 0, \qquad \beta u_{xy} + \alpha v_{xx} + v_{yy} = 0.$$

In other words, for planar elasticity, once we allow arbitrary linear changes of variable, there are in reality only two independent elastic moduli. The name "orthotropic" refers to the fact that such Lagrangians are rescaled versions of the stored energy function for an orthotropic linear elastic medium, meaning that in the given coordinates there are three orthogonal planes of reflectional symmetry, cf. [5, p. 207]. The Lagrangian for linear isotropic elasticity is a special case corresponding to the condition

$$\alpha + \beta = 1.$$

In this particular case, we can let

$$\alpha = \mu/(2\mu + \lambda), \qquad \beta = (\mu + \lambda)/(2\mu + \lambda),$$

where μ and λ are the classical Lamé moduli [5, p. 162]. In particular, two isotropic Lagrangians determine the same orthotropic Lagrangian if and only if their Lamé moduli are proportional: $\lambda/\mu = \tilde{\lambda}/\tilde{\mu}$, or, equivalently, they have the same value for Poisson's ratio $\nu = \lambda/(\mu + \lambda)$. In the more general anisotropic case, the α and β play the role of "canonical elastic moduli." Thus, while the general planar elastic problem in a general coordinate system has 6 elastic moduli c_{ijkl}, Theorem 2 shows that if we choose a special adapted coordinate system, there are in reality only *two* independent moduli. The implications of this result for the study of planar elasticity, including the determination of symmetries and conservation laws, and the direct application of complex variable methods into anisotropic elasticity, will be the subject of subsequent papers.

Although the above fifteen classes of canonical forms for planar Lagrangians are all inequivalent, meaning that a Lagrangian from one class cannot possibly be equivalent to a Lagrangian from a different class, it is still possible for two different Lagrangians within the same class to be equivalent. For the case of orthotropic Lagrangians, which is the most physically interesting case, the basic result is as follows.

THEOREM 3. *Let L and $\tilde{L}$ be different orthotropic Lagrangians with moduli α, β and $\tilde{\alpha}, \tilde{\beta}$, respectively. Then L is equivalent to $\tilde{L}$ if and only if their moduli are related by one of the following pairs of equations*:

(i) $\tilde{\alpha} = \alpha, \quad \tilde{\beta} = -\beta,$

(ii) $\tilde{\alpha} = 1/\alpha, \quad \tilde{\beta} = \beta/\alpha,$

(iii) $\tilde{\alpha} = 1/\alpha, \quad \tilde{\beta} = -\beta/\alpha,$

(iv) $\tilde{\alpha} = (1 + \alpha - \beta)/(1 + \alpha + \beta), \quad \tilde{\beta} = (2 - 2\alpha)/(1 + \alpha + \beta),$

(v) $\tilde{\alpha} = (1 + \alpha - \beta)/(1 + \alpha + \beta), \quad \tilde{\beta} = (2\alpha - 2)/(1 + \alpha + \beta),$

(vi) $\tilde{\alpha} = (1 + \alpha + \beta)/(1 + \alpha - \beta), \quad \tilde{\beta} = (2 - 2\alpha)/(1 + \alpha - \beta),$

(vii) $\tilde{\alpha} = (1 + \alpha + \beta)/(1 + \alpha - \beta), \quad \tilde{\beta} = (2\alpha - 2)/(1 + \alpha - \beta).$

Note that transformation (i), (iv) and (v) leave an isotropic Lagrangian unchanged, but (ii), (iii), (vi), and (vii) change it into a different orthotropic Lagrangian with

$$\tilde{\alpha} - \tilde{\beta} = 1.$$

In particular, excluding isotropic Lagrangians, in the strongly elliptic case, one can always use one of the above transformations to make the moduli α and β satisfy the additional restrictions

$$0 < \alpha \le 1, \qquad 0 \le \beta \le 1 - \alpha.$$

Once one eliminates the possible interrelationships between the canonical forms in Theorem 1 and takes into account the possible plus or minus signs, there is a detailed list of 60 completely inequivalent canonical forms for planar, first-order quadratic Lagrangians. The complete list is given in Section 9.

The methods utilized to prove these theorems come from the powerful techniques of classical invariant theory, cf. [4, 6, 7]. In essence, classical invariant theory is concerned with the direct analogs of the equivalence and canonical form problems, but in the case of ordinary polynomials. The solution of these problems relies on the introduction of certain functions, called invariants or, more generally, covariants, whose values do not change under the given linear changes of variable. In the paper [10], an extension of classical invariant theory to the case of symbols of quadratic Lagrangians has been developed, and the particular invariants and covariants constructed there are directly applied to the equivalence and canonical form problems here. The specifics of the method are presented in Section 3 below.

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2. Symbols

This paper is concerned with the study of homogeneous kth order quadratic variational problems

$$\mathscr{L}[\mathbf{u}] = \textstyle\int_\Omega L[\mathbf{u}]\, d\mathbf{x},$$

where the *Lagrangian* is a quadratic function of the kth order derivatives of the dependent variables $\mathbf{u}$:

$$L[\mathbf{u}] = \sum a_{IJ}^{\alpha\beta} u_I^\alpha u_J^\beta. \tag{3}$$

Here $\mathbf{u} = (u^1, \dots, u^q)$ are the unknown dependent variables, which are functions of the independent variables $\mathbf{x} = (x_1, \dots, x_p)$ defined over some domain $\Omega \subset \mathbb{R}^p$. Given $1 \le \alpha \le q$, and a kth order multi-index $I = (i_1, \dots, i_p)$, so $i_1 + \cdots + i_p = k$, we let $u_I^\alpha = \partial_I u^\alpha$ denote the corresponding kth order partial derivative of u^α:

$$u_I^\alpha = \partial_I u^\alpha = \partial^k u^\alpha / \partial x_1^{i_1} \cdots \partial x_p^{i_p}.$$

In (3) the coefficients $a_{IJ}^{\alpha\beta}$ are assumed to be constants, satisfying the symmetry condition $a_{IJ}^{\alpha\beta} = a_{JI}^{\beta\alpha}$, and the sum is over all α, $\beta = 1, \dots, q$, and all kth order multi-indices I and J.

The Euler–Lagrange equations for such a variational problem are the self-adjoint linear system of $2k$th order partial differential equations

$$E_\alpha(L) = 2\sum a_{IJ}^{\alpha\beta} u_{I,J}^{\beta} = 0, \qquad \alpha = 1, \dots, q, \tag{4}$$

the sum now over β, I, J, where $u_{I,J}^{\beta} = \partial_I u_J^{\beta}$. In most applications, $k = 1$, so we are dealing with a first-order Lagrangian, with second-order Euler–Lagrange equations.

Definition 4. The *symbol* of the quadratic variational problem (3) is the polynomial

$$Q(\mathbf{x}, \mathbf{u}) = \sum a_{IJ}^{\alpha\beta} x^I x^J u^\alpha u^\beta, \tag{5}$$

where $\mathbf{u} = (u^1, \dots, u^q) \in \mathbf{R}^q$, $\mathbf{x} = (x_1, \dots, x_p) \in \mathbf{R}^p$, and, for $I = (i_1, \dots, i_p)$, x^I denotes the product monomial $x_1^{i_1} x_2^{i_2} \cdots x_p^{i_p}$.

(Technically speaking, the symbol is defined on the cotangent space at each point, so we should be conforming to the usual practice of using letters other than the independent and dependent variables $\mathbf{x}$ and $\mathbf{u}$ for its arguments. However, for later puposes it will be convenient to adopt the present notation, which should not be a source of confusion.)

For a first-order variational problem, the symbol is just the polynomial appearing in the Legendre–Hadamard condition for strong ellipticity, which is the condition that Q be positive definite,

$$Q(\mathbf{x}, \mathbf{u}) > 0 \qquad \text{whenever } \mathbf{x} \neq 0 \text{ and } \mathbf{u} \neq 0, \tag{6}$$

cf. [5, p. 86]. From the definition, it appears that the symbol does not uniquely determine the Lagrangian. For example, the terms $u_x v_y$ and $u_y v_x$ both contribute the identical term $xyuv$ to the symbol; however, these two terms differ by a total divergence

$$u_x v_y - u_y v_x = D_x(u v_y) - D_y(u v_x),$$

and hence lead to the same Euler–Lagrange equations. It is not hard to prove that for quadratic Lagrangians this remark holds in general.

Proposition 5. *Two homogeneous, quadratic Lagrangians have the same symbol if and only if they differ by a total divergence. Consequently, two quadratic variational problems have the same symbol if and only if they have the same Euler–Lagrange equations.*

The symbol of a kth order quadratic Lagrangian is a special case of a *biform* of bidegree $(2k, 2)$, meaning that it is a homogeneous polynomial separately in each of the two variables $\mathbf{x}$ and $\mathbf{u}$, of degrees $2k$ and 2, respectively. In particular, for a first-order planar Lagrangian, so $p = q = 2$, $k = 1$, which includes the case of two-dimensional linear anisotropic elasticity, we have a symbol which is a "binary biquadratic biform," or biquadratic for short.

Proposition 5 also points out a key difference between quadratic Lagrangians and Lagrangians of higher degree, e.g., cubic Lagrangians. It is *not* possible to replace a higher degree Lagrangian by a well-defined biform. The net effect is that the equivalence and canonical form problems for higher degree Lagrangians are considerably more complicated, and the tools of classical invariant theory are not so readily applicable.

3. Changes of Variable

In our treatment of the equivalence problem for quadratic Lagrangians, we will only allow real linear changes of independent and dependent variables,

$$\mathbf{x} = A \cdot \tilde{\mathbf{x}}, \qquad \mathbf{u} = B \cdot \tilde{\mathbf{u}},$$

where A is a matrix in $\mathrm{GL}(p, \mathbb{R})$, the general linear group of real invertible $p \times p$ matrices, and, similarly, B is in $\mathrm{GL}(q, \mathbb{R})$. The variational problem in the new $\tilde{\mathbf{x}}, \tilde{\mathbf{u}}$ coordinates is found by substituting $A \cdot \tilde{\mathbf{x}}$ for $\mathbf{x}$ and $B \cdot \tilde{\mathbf{u}}$ for $\mathbf{u}$ wherever they occur in the integral. Since we will have cause to drop the tildes on the new variables, we will use the "substitutional notation"

$$\mathbf{x} \mapsto A \cdot \tilde{\mathbf{x}}, \qquad \mathbf{u} \mapsto B \cdot \tilde{\mathbf{u}}, \tag{7}$$

for the above change of variables. To distinguish the two general linear groups (especially in the case $p = q$), we will use the notation $G_{\mathbf{x}}$ to denote the action of $\mathrm{GL}(p, \mathbb{R})$ on the space $\mathbb{R}^p$ of independent variables, and, correspondingly, $G_{\mathbf{u}}$ to denote the action of $\mathrm{GL}(q, \mathbb{R})$ on the space $\mathbb{R}^q$ of dependent variables. (Occasionally, it will be convenient to allow the *real* general linear groups $G_{\mathbf{x}}$ and $G_{\mathbf{u}}$ to also act on the complex vector spaces $\mathbb{C}^p$ and $\mathbb{C}^q$.) In this notation, a change of variables will then correspond to an element of the Cartesian product Lie group,

$$\mathbf{A} = (A, B) \in G_{\mathbf{x}} \times G_{\mathbf{u}}.$$

When the variational integral is subjected to such a linear change of variables, it remains quadratic, but the coefficients $a_{IJ}^{\alpha\beta}$ will, of course, change. The explicit formulas for the new coefficients are not difficult to write down explicitly, but are not overly helpful when it comes to a detailed analysis, especially from the point of view of solving the fundamental equivalence and canonical form problems. As a first step towards addressing these problems, we note that the change of variables easily translates into the standard action of $G_{\mathbf{x}} \times G_{\mathbf{u}}$ on the symbol of the variational problem.

LEMMA 6. *Let $\mathscr{L}$ be a variational problem with symbol $Q(\mathbf{x}, \mathbf{u})$. If $\tilde{\mathscr{L}}$ is the transformed variational problem under the change of variables* (7) *induced by $\mathbf{A} = (A, B) \in G_{\mathbf{x}} \times G_{\mathbf{u}}$, then $\tilde{\mathscr{L}}$ has symbol*

$$\tilde{Q}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = |\det A| \cdot Q(A\tilde{\mathbf{x}}, B\tilde{\mathbf{u}}). \tag{8}$$

Thus, except for the inessential scaling factor $|\det A|$ coming from the $d\mathbf{x}$ in the integrand, the symbol is transformed by the obvious action of the group $G_{\mathbf{x}} \times G_{\mathbf{u}} = \mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})$ on the space of homogeneous polynomials (biforms) in the variables $\mathbf{x} \in \mathbb{R}^p$, $\mathbf{u} \in \mathbb{R}^q$.

4. INVARIANT THEORY

Classical invariant theory is concerned with those properties of homogeneous polynomial functions which do not change under linear changes of variable. The principal tool is the construction of particular functions, known as invariants, which depend on the coefficients of the given homogeneous polynomial or "form" $Q(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^p$ and with the property that they are unchanged (up to a factor) by the action of the general linear group $\mathrm{GL}(p, \mathbb{R})$. A simple example is the discriminant $\Delta = ac - b^2$ of a quadratic polynomial $ax^2 + 2bxy + cy^2$. See [4, 6, or 7] for a basic introduction to classical invariant theory. In the case of biforms $Q(\mathbf{x}, \mathbf{u})$, the only difference is the appearance of a second general linear group $\mathrm{GL}(q, \mathbb{R})$ corresponding to the changes in dependent variable. However, all the standard techniques of classical invariant theory readily generalize to analyze this more complicated problem. In this section, we summarize the basic results from the invariant theory of biforms, developed in detail in the paper [10], which will be required for our solution to the equivalence and canonical form problems.

Consider a general biform

$$Q(\mathbf{x}, \mathbf{u}) = \sum a_{IJ} x^I u^J,$$

of bidegree (m, n), defined for $\mathbf{x} \in \mathbb{R}^p$, $\mathbf{u} \in \mathbb{R}^q$. Under the linear change of variables

$$\mathbf{x} \mapsto A \cdot \tilde{\mathbf{x}} \qquad \mathbf{u} \mapsto B \cdot \tilde{\mathbf{u}},$$

determined by the group element $\mathbf{A} = (A, B) \in G_{\mathbf{x}} \times G_{\mathbf{u}}$, the coefficients a_{IJ} of Q get transformed into new coefficients $\tilde{a}_{IJ}$:

$$\tilde{Q}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = Q(A \cdot \tilde{\mathbf{x}}, B \cdot \tilde{\mathbf{u}}) = \sum \tilde{a}_{IJ} \tilde{x}^I \tilde{u}^J.$$

The explicit formulas for the $\tilde{a}_{IJ}$, while easy to write down, are not very useful. What are useful are particular functions of the coefficients a_{IJ} which are essentially unchanged by such a change of variables, i.e., have the same formula in terms of the new coefficients $\tilde{a}_{IJ}$.

DEFINITION 7. An *invariant* of *biweight* (g, h) of the biform Q is a polynomial function $I(\mathbf{a}) = I(\dots, a_{IJ}, \dots)$ of the coefficients of Q which, up to a determinantal factor, does not change under the action of the group $G_{\mathbf{x}} \times G_{\mathbf{u}}$:

$$I(\tilde{\mathbf{a}}) = (\det A)^g (\det B)^h I(\mathbf{a}), \qquad \mathbf{A} = (A, B) \in G_{\mathbf{x}} \times G_{\mathbf{u}}.$$

A *covariant* of *biweight* (g, h) of Q is a polynomial function $J(\mathbf{a}, \mathbf{x}, \mathbf{u})$ depending both on the coefficients a_{IJ} and the independent and dependent variables x^i, u^α, which, up to a factor, is similarly unchanged:

$$J(\tilde{\mathbf{a}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = (\det A)^g (\det B)^h J(\mathbf{a}, \mathbf{x}, \mathbf{u}), \qquad \mathbf{A} = (A, B) \in G_{\mathbf{x}} \times G_{\mathbf{u}}.$$

(Note that invariants are special types of covariants.)

As an immediate application, note that the vanishing of a covariant for a particular biform Q is independent of any particular choice of coordinates, and thus has intrinsic geometrical meaning. For example, the discriminant of a quadratic polynomial vanishes if and only if the polynomial is a perfect square. Similarly, for a positive or negative definite covariant of *even* biweight, meaning that both g and h are even integers, the sign of the covariant is also unaffected by real changes of variables and has intrinsic geometrical meaning. Again, the discriminant of a quadratic polynomial is positive if and only if the polynomial has complex conjugate roots, which will be the case in any real coordinate system. Note also that the extra determinantal factor in the transformation rule (8) for symbols of quadratic Lagrangians does not alter the definition of properties of covariants.

For the purposes of implementing our solution to the planar equivalence problem, we present some of the basic covariants associated with a first-order planar quadratic Lagrangian $L(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} = (x_1, x_2) = (x, y)$ and $\mathbf{u} = (u^1, u^2) = (u, v)$ are both in $\mathbb{R}^2$. After taking into account the ambiguity stemming from basic null Largangian $u_x v_y - u_y v_x$, we see that we can take L to be of the form

$$\begin{aligned} L(\mathbf{x}, \mathbf{u}) = {} & a_{11}^{11} u_x^2 + a_{12}^{11} u_x u_y + a_{22}^{11} u_y^2 + a_{11}^{12} u_x v_x + a_{12}^{12} u_x v_y \\ & + a_{22}^{12} u_y v_y + a_{11}^{22} v_x^2 + a_{12}^{22} v_x v_y + a_{22}^{22} v_y^2, \end{aligned} \tag{9}$$

where the coefficients a^{ij}_{kl} are constant. (They are closely related to, in fact linear combinations of, the components c_{ijkl} of the elasticity tensor in the elastic case.) In this case, the symbol is a biquadratic polynomial $Q(\mathbf{x}, \mathbf{u})$:

$$Q(\mathbf{x}, \mathbf{u}) = a^{11}_{11}x^2u^2 + a^{11}_{12}xyu^2 + a^{11}_{22}y^2u^2 + a^{12}_{11}x^2uv + a^{12}_{12}xyuv + a^{12}_{22}y^2uv + a^{22}_{11}x^2v^2 + a^{22}_{12}xyv^2 + a^{22}_{22}y^2v^2. \tag{10}$$

As Q is a quadratic polynomial in the $\mathbf{x}$-variables, the $\mathbf{x}$-*discriminant*

$$\Delta_{\mathbf{x}}(\mathbf{u}) = \tfrac{1}{4}\left\{ Q_{xx} \cdot Q_{yy} - Q^2_{xy} \right\}$$

will clearly be a covariant of biweight (2,0). (The subscripts on Q indicate partial derivatives.) The vanishing of $\Delta_{\mathbf{x}}(\mathbf{u})$ at a particular $\mathbf{u}_0$ indicates that at $\mathbf{u}_0$ the quadratic polynomial $Q(\mathbf{x}, \mathbf{u}_0)$ is a perfect square:

$$Q(\mathbf{x}, \mathbf{u}_0) = \pm(bx + cy)^2.$$

Similarly, the discriminant

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \tfrac{1}{4}\left\{ Q_{uu} \cdot Q_{vv} - Q^2_{uv} \right\}$$

of Q with respect to the variables $\mathbf{u}$ is also a covariant, of biweight (0, 2), with a similar interpretation. There is also a mixed covariant

$$C_2(\mathbf{x}, \mathbf{u}) = Q_{xu}Q_{yv} - Q_{xv}Q_{yu},$$

which is of biweight (1, 1), and is itself a biquadratic polynomial.

The simplest invariant of the symbol Q is the quadratic expression

$$I_2 = 8a^{11}_{11}a^{22}_{22} - 4a^{11}_{12}a^{22}_{12} + 8a^{11}_{22}a^{22}_{11} - 4a^{12}_{11}a^{12}_{22} + \left(a^{12}_{12}\right)^2.$$

It has even biweight (2, 2). There is also an important cubic invariant

$$I_3 = a^{11}_{11}a^{12}_{12}a^{22}_{22} - a^{11}_{11}a^{12}_{22}a^{22}_{12} - a^{11}_{12}a^{12}_{11}a^{22}_{22} + a^{11}_{11}a^{12}_{22}a^{22}_{12} + a^{11}_{22}a^{12}_{11}a^{22}_{12} - a^{11}_{22}a^{12}_{12}a^{22}_{11},$$

of biweight (3, 3). Both of these will be of use in our classification program.

The composition of covariants provides an easy method to compute further covariants of the biquadratic form Q. For example, one can construct the discriminant of the quartic from $\Delta_{\mathbf{x}}$, or, alternatively, the discriminant of $\Delta_{\mathbf{u}}$, both of which are invariants of degree 12. Remarkably, these two invariants are exactly the same! In particular, this implies that if $\Delta_{\mathbf{x}}$ has a repeated root, the same is true of $\Delta_{\mathbf{u}}$. In fact, more than this is true: the cross ratio of the four roots of $\Delta_{\mathbf{x}}$ is the same as that of $\Delta_{\mathbf{u}}$. See [10] for the proofs of these results, as well as more information on the construction of invariants and covariants of biforms using a generalization of the symbolic method of Aronhold.

In this section we begin the determination of the canonical forms for a first-order quadratic Lagrangian (9) in two independent and two dependent variables. As above, the standard action of $\mathrm{GL}(2, \mathbb{R})$ on the space of independent variables $\mathbf{x} = (x, y) \in \mathbb{R}^2$ is denoted by $G_{\mathbf{x}}$, while $G_{\mathbf{u}}$ denotes the action of $\mathrm{GL}(2, \mathbb{R})$ on the space of dependent variables $\mathbf{u} = (u, v) \in \mathbb{R}^2$. Occasionally, we will extend the actions of $G_{\mathbf{x}}$ and $G_{\mathbf{u}}$ to the complex space $\mathbb{C}^2$.

The symbol for such a variational problem will be a biquadratic function $Q(\mathbf{x}, \mathbf{u})$ of $\mathbf{x}$ and $\mathbf{u}$, cf. (10). We begin by distinguishing certain particular types of symbols corresponding to special types of Lagrangians.

DEFINITION 8. The symbol $Q(\mathbf{x}, \mathbf{u})$ is called *factorizable* if it can be written as a product $Q(\mathbf{x}, \mathbf{u}) = S(\mathbf{x}) \cdot T(\mathbf{u})$ of polynomials of $\mathbf{x}$ and $\mathbf{u}$ alone.

Notice that factorizability is independent of the particular coordinates used. Of the 15 canonical forms listed in Theorem 1, cases 12–15 are factorizable, as well as case 1 when $\beta = 0, \alpha = \pm 1$, and there are an even number (0, 2, or 4) of minus signs in the Lagrangian.

The determination of canonical forms for factorizable symbols obviously reduces to the determination of canonical forms for homogeneous polynomials over $\mathbb{R}$, a problem which has been dealt with in the classical literature, cf. [6, p. 252]. For real quadratic polynomials in $\mathbf{x}$, there are six inequivalent canonical forms:

$$x^2 + y^2, \qquad -x^2 - y^2, \qquad x^2 - y^2, \qquad x^2, \qquad -x^2, \qquad 0,$$

and every other quadratic polynomial can be reduced to exactly one of these six by a transformation in $G_{\mathbf{x}} = \mathrm{GL}(2, \mathbb{R})$. Clearly, then, if Q is factorizable, we can separately reduce each of its two factors S and T to canonical form. This proves that the above 5 cases constitute the complete set of inequivalent canonical forms for factorizable biquadratic symbols.

A wider class of symbols is provided by the "semi-diagonal symbols":

DEFINITION 9. The symbol $Q(\mathbf{x}, \mathbf{u})$ is called *semi-diagonal* if it has the form

$$Q = px^2u^2 + qy^2u^2 + 2rxyuv + sx^2v^2 + ty^2v^2, \tag{11}$$

where p, q, r, s, t are real constants. (The factor of 2 is just introduced for convenience.)

Most, but not all, biquadratic symbols are equivalent to a semi-diagonal symbol. For instance, our above analysis of factorizable symbols proves that every factorizable symbol is equivalent to a semi-diagonal symbol. Of the 15 canonical symbols listed in Theorem 1, cases 1–5 and the factorizable cases 12–15 are semi-diagonal. The remaining "special" canonical forms are *not* equivalent to a semi-diagonal symbol.

We begin our determination of canonical forms with a complete analysis of semi-diagonal symbols.

LEMMA 10. *Suppose the symbol for the quadratic Lagrangian is in the semi-diagonal form* (11). *Then it is equivalent to one of the canonical forms* 1–5 *or* 12–15 *of Theorem* 1.

Proof. First note that if all four coefficients $p = q = s = t = 0$ vanish, or if two or three vanish and also $pt = qs = 0$ and $r = 0$, then Q is factorizable. These cases have been already analyzed, so we concentrate on the other possibilities.

If all of the four coefficients, p, q, s, t are non-zero, we just rescale x, y and u:

$$x \mapsto \left(|q/pst|\right)^{1/4}x, \qquad y \mapsto y/\sqrt{|t|}, \qquad u \mapsto \left(|st/pq|\right)^{1/4}u.$$

This has the effect of changing Q into the symbol

$$Q = \pm x^2u^2 + \alpha y^2u^2 + 2\beta xyuv \pm \alpha x^2v^2 \pm y^2v^2,$$

where

$$\alpha = \operatorname{sign}(q)\sqrt{|qs/pt|} \qquad \text{and} \qquad \beta = r/\sqrt{|pt|}.$$

This is the symbol of the canonical Lagrangian of type 1 as listed in Theorem 1.

If precisely one of the coefficients p, q, s, t vanishes, then by possibly interchanging x and y, or u and v, we can assume without loss of generality that $s = 0$. A similar rescaling argument shows that such a symbol is equivalent to one of the form

$$Q = \pm x^2u^2 \pm y^2u^2 + 2\beta xyuv \pm y^2v^2,$$

which corresponds to a Lagrangian of type 2.

If precisely two of the coefficients p, q, s, t vanish, then there are various subcases. If $s = t = 0$, then we can rescale to a symbol of type 3, while if $p = q = 0$ we interchange u and v before rescaling to also get a symbol of type 3. If $q = t = 0$, or if $p = s = 0$, we similarly have a symbol of type 4. The only other cases not covered are when $q = 0 = s$, where a simple rescaling makes Q of type 1 with $\alpha = 0$, and when $p = 0 = t$, where we interchange x and y before rescaling to also reduce to type 1.

If three or four of the coefficients p, q, s, t vanish, then the symbol is automatically factorizable, and so has been already analyzed. It is worth remarking that the special factorizable case $p = q = s = t = 0$, $r \neq 0$, corresponding to the Lagrangian $2ru_xv_y$, is changed into a Lagrangian of type 1 by the special change of variables

$$(x, y) \mapsto (x - y, x + y), \qquad (u, v) \mapsto (u - v, u + v). \tag{12}$$

(Here, and below, we will consistently drop the tildes when we make a change of variables.) This particular transformation occurs many times in our classification procedure.

6. Root Structure

At the moment, we are not able to determine whether or not two different canonical forms are really the same under some as yet undetermined change of variables. There are several tools at our disposal for resolving this problem, including the invariants I_2 and I_3 constructed in Section 4. The most powerful tool, however, is the structure of the roots of the two discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$. Note that the action of $G_{\mathbf{x}}$ or $G_{\mathbf{u}}$ cannot change the basic multiplicities or geometric configuration of these roots. For example, if $\Delta_{\mathbf{x}}$ has two double roots in one coordinate system, then it always has two double roots; if two of the roots form a complex conjugate pair, then they are complex conjugates in every coordinate system. Thus the multiplicity of the roots of the discriminants is an easy way to distinguish inequivalent symbols. We will use the notation $1^i2^j \cdots$ for a polynomial with i simple roots, j double roots, etc. (including the roots at ∞). For example, $1^2 2$ denotes a quartic polynomial with two simple roots and one double root, while 4 denotes a quartic with a single quadruple root. The quartic which is identically zero will be denoted by the symbol ∞ (indicating a root of "infinite multiplicity"). It is then a simple matter to see that, by possibly interchanging the variables x and y or u and v, any semi-diagonal symbol can be cast into one of the 14 forms listed in Table I.

Comparing this table with the canonical Lagrangians in Theorem 1, we see that under rescaling, cases 1–4 correspond to Lagrangians of type 1. Cases 5–6 correspond to type 2, while 7, 8, 11 correspond to types 3, 4, 5, respectively. The remaining cases are all factorizable.

From the structure of the roots of the discriminants, the only cases which have any chance of being equivalent are 2, 3, 11, and 13, or 4, 12, and 14. Clearly case 14, where Q is identically 0, is distinct from 4 and 12. Furthermore, case 4 has a nonzero invariant I_3, whereas I_3 vanishes in case

TABLE I

Case	Conditions			$\Delta_{\mathbf{x}}$	$\Delta_{\mathbf{u}}$
1	$p, q, s, t \neq 0,$		$(qs + pt - r^2)^2 \neq 4pqst$	1^4	1^4
2	$p, q, s, t \neq 0,$		$(qs + pt - r^2)^2 = 4pqst$	2^2	2^2
3	$p, t \neq 0,$	$q = s = 0,$	$r^2 \neq pt$	2^2	2^2
4	$p, t \neq 0,$	$q = s = 0,$	$r^2 = pt$	∞	∞
5	$p, s, t \neq 0,$	$q = 0,$	$r^2 \neq pt$	$1^2 2$	$1^2 2$
6	$p, s, t \neq 0,$	$q = 0,$	$r^2 = pt$	4	4
7	$p, q \neq 0,$	$s = t = 0,$	$r \neq 0$	$1^2 2$	2^2
8	$p, s \neq 0,$	$q = t = 0,$	$r \neq 0$	2^2	$1^2 2$
9	$p, q \neq 0,$	$s = t = 0,$	$r = 0$	4	∞
10	$p, s \neq 0,$	$q = t = 0,$	$r = 0$	∞	4
11	$p \neq 0,$	$q = s = t = 0,$	$r \neq 0$	2^2	2^2
12	$p \neq 0,$	$q = s = t = 0,$	$r = 0$	∞	∞
13	$p = q = s = t = 0,$		$r \neq 0$	2^2	2^2
14	$p = q = s = t = 0,$		$r = 0$	∞	∞

12, so these cannot be equivalent. Case 13 can be reduced to a version of case 2 by the previously indicated change of variables (12). Similarly, if we are in case 3 with p and t of the same sign, we can rescale to make them equal and then use the same change of variables to change it into a version of Case 2. Except for these, all the other cases are inequivalent. (This will follow from our more sophisticated classification in Section 9.)

7. General Canonical Forms

The determination of canonical forms for more general biquadratic symbols rests on the properties of the roots of either one of the discriminants of Q. Clearly in the present situation $\mathbf{x}$ and $\mathbf{u}$ play interchangeable roles, so it does not really matter which discriminant is the primary object of interest. For definiteness we select $\Delta_{\mathbf{x}}$, which is a quartic polynomial in $\mathbf{u}$. Since $\Delta_{\mathbf{x}}(\mathbf{u})$ is a covariant, it is unaffected (except for a determinantal factor) by transformations A in $G_{\mathbf{u}}$. In other words, if we make the change of variables $\mathbf{u} \mapsto \tilde{\mathbf{u}} = A\mathbf{u}$, whereby Q changes into $\tilde{Q}$, then $\Delta_{\mathbf{x}}$ transforms into $\tilde{\Delta}_{\mathbf{x}}$, where

$$\tilde{\Delta}_{\mathbf{x}}(\tilde{\mathbf{u}}) = (\det A)^2 \Delta_{\mathbf{x}}(A\tilde{\mathbf{u}}).$$

In particular, the roots of $\Delta_{\mathbf{x}}$ get transformed according to the projective action of $G_{\mathbf{u}}$ on $\mathbb{C}$ via linear fractional transformations:

$$z \mapsto (az + b)/(cz + d), \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{u}} = \mathrm{GL}(2, \mathbb{R}).$$

(Here we are viewing $\mathbf{u} = (u, v)$ as projective coordinates for the roots z, so we can identify z with the ratio u/v, with $v = 0$ corresponding to $z = \infty$.)

We label the four (complex) roots of $\Delta_{\mathbf{x}}$ as z_1, z_2, z_3, z_4, with corresponding representative points $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, in $\mathbf{C}^2$. (Some of the roots can coincide; also the precise choice of projective coordinates $\mathbf{u}_j = (u_j, v_j)$ to represent a given $z_j = u_j/v_j$ is unimportant.) Given a root z_j of $\Delta_{\mathbf{x}}$, we note that, by the basic property of the discriminant of a quadratic polynomial, the complex quadratic polynomial $Q_j(\mathbf{x}) \equiv Q(\mathbf{x}, \mathbf{u}_j)$ must be a perfect square:

$$Q_j(x, y) = \pm\left(b_j x + c_j y\right)^2.$$

There are several different cases to analyze, depending on whether $\Delta_{\mathbf{x}}$ has only real roots or has at least one pair of complex conjugate roots. We begin by analyzing the latter.

Case I. Assume that $\Delta_{\mathbf{x}}(\mathbf{u})$ has a pair of complex conjugate roots z_1 and $z_2 = \bar{z}_1$. In this case, since Q is real, the corresponding polynomials Q_1 and Q_2 are complex conjugates

$$Q_2(\bar{x}) = \overline{Q_1(x)}.$$

Since the action of $G_{\mathbf{u}}$ on the complex upper half plane is transitive, we can move the two roots to $z_1 = i = \sqrt{-1}$, $z_2 = -i$, by a suitable element of $G_{\mathbf{u}}$.

Case Ia. The two perfect squares Q_1 and Q_2 are genuinely complex polynomials. In other words,

$$Q_1(x, y) = -4[(a + ib)x + (c + id)y]^2,$$

where $a + ib$ and $c + id$ are not real multiples of each other, and Q_2 is the complex conjugate square. By replacing (x, y) by $(ax + cy, bx + dy)$, we can transform Q_1 into the elementary square $-4(x + iy)^2$ and hence Q_2 to its conjugate $-4(x - iy)^2$. Thus,

$$\begin{aligned} Q = {} & (x - iy)^2(u - iv)^2 \\ & + \left\{\rho(x - iy)^2 + \sigma(x - iy)(x + iy) + \bar{\rho}(x + iy)^2\right\} \\ & \times(u - iv)(u + iv) + (x + iy)^2(u + iv)^2, \end{aligned}$$

where ρ is complex and σ is real. Now, simultaneously rotating (x, y) through an angle θ and (u, v) through angle $-\theta$ (which is the same as multiplying $x + iy$ by $e^{i\theta}$ and $u + iv$ by $e^{-i\theta}$) leaves Q in the same form, has the only effect of multiplying ρ by $e^{2i\theta}$. Thus we can choose the angle θ

so that ρ is real, and Q takes the semi-diagonal form (11), where

$$p = 2\rho + \sigma + 2, \qquad q = -2\rho + \sigma - 2, \qquad r = -8,$$
$$s = 2\rho + \sigma - 2 \qquad t = -2\rho + \sigma + 2.$$

Thus we are in the case covered by Lemma 10, and we know the canonical forms.

Case Ib. The two perfect squares Q_1 and Q_2 are complex conjugate multiples of the same real square. In other words,

$$Q_1 = 2(a + ib)^2(cx + dy)^2, \qquad Q_2 = 2(a - ib)^2(cx + dy)^2,$$

where a, b, c, d are real. If both Q_1 and Q_2 are zero, then Q is factorizable. Otherwise, we can transform them into multiples of the elementary square x^2 by an element of G_x. Thus Q has the form

$$Q = \tfrac{1}{2}x^2\{(a - ib)^2(u - iv)^2 + (a + ib)^2(u + iv)^2\}$$
$$+(\hat{\rho}x^2 + \hat{\sigma}xy + \hat{\tau}y^2)(u - iv)(u + iv),$$

for certain real constants $\hat{\rho}, \hat{\sigma}, \hat{\tau}$. If we replace (u, v) by $(au - bv, bu + av)$, then the symbol becomes

$$\begin{aligned} Q &= \tfrac{1}{2}x^2\{(u - iv)^2 + (u + iv)^2\} \\ &\quad +(\rho x^2 + \sigma xy + \tau y^2)(u - iv)(u + iv) \\ &= x^2(u^2 - v^2) + (\rho x^2 + \sigma xy + \tau y^2)(u^2 + v^2), \end{aligned}$$

where ρ, σ, τ are just equal to $\hat{\rho}, \hat{\sigma}, \hat{\tau}$ divided by $a^2 + b^2$. Keeping x fixed, we can translate and scale y so as to arrange that the polynomial $\rho x^2 + \sigma xy + \tau y^2$ takes one of two forms. If $\tau \neq 0$, or $\tau = \sigma = 0$, then we can arrange for the coefficient σ of the xy term to vanish, and hence the symbol to be semi-diagonal. In the other case $\tau = 0$, $\sigma \neq 0$, we can translate and scale to make it equal to xy, and so the symbol takes the form

$$Q = x^2(u^2 - v^2) + xy(u^2 + v^2).$$

This is not in semi-diagonal form, nor can it be changed into semi-diagonal form. Indeed, the discriminants Δ_x and Δ_u of this particular symbol have two double complex roots and a double and two simple real roots respectively (i.e., of types 2^2 and $1^2 2$ in our earlier notation). This could coincide with case 8 in the semi-diagonal classification of Table I, except for the fact that for case 8, Δ_x has two double *real* roots, and so cannot be changed into the current Δ_x. Thus we have our first example of a genuinely non-semi-diagonal symbol. This case corresponds to a canonical Lagrangian of type 6 in the list of Theorem 1.

Case II. In this case, the discriminant Δ_x has three distinct real roots, z_1, z_2, z_3. (The fourth root z_4 may or may not coincide with one of the other three.) We can then use a transformation in G_u to place the roots at $z_1 = 0$, $z_2 = \infty$, $z_3 = 1$. There are then four distinct subcases, depending on the relative forms of the corresponding perfect squares Q_1, Q_2, Q_3.

Case IIa. Assume further that no two of the corresponding three perfect squares Q_1, Q_2, and Q_3 are scalar multiples of each other, i.e., $Q_1 \neq \lambda Q_2$, etc. (In particular, no Q_i vanishes identically.) In this case, at least two of the squares Q_1, Q_2, and Q_3 have the same sign, which we can take to be Q_1 and Q_2, and use a transformation in G_x to make $Q_1 = \pm x^2$ and $Q_2 = \pm y^2$ (with the same sign). Therefore

$$Q = \pm\left[x^2u^2 + y^2v^2\right] + \left(\rho x^2 + \sigma xy + \tau y^2\right)uv$$

for certain real constants ρ, σ, τ. If ρ and τ are both positive, both zero, or both negative, then we can rescale

$$(x, y) \mapsto (\lambda x, \mu y), \qquad (u, v) \mapsto \left(\lambda^{-1}u, \mu^{-1}v\right)$$

to make $\tau = \rho$. We then use the change of variables (12) to change Q into the semi-diagonal form (11), with

$$\begin{aligned} p &= 2\rho + \sigma \pm 2, & q &= -2\rho + \sigma \pm 2, & r &= 8, \\ s &= 2\rho + \sigma \mp 2, & t &= -2\rho + \sigma \mp 2. \end{aligned}$$

The other cases when ρ and τ have opposite signs, or one or the other vanishes, are a bit more tricky. For definiteness, assume Q_1 and Q_2 are both positive (the opposite case being treated analogously), so

$$Q = x^2u^2 + y^2v^2 + \left(\rho x^2 + \sigma xy + \tau y^2\right)uv. \tag{13}$$

At the third root $z_3 = 1$, i.e., $(u_3, v_3) = (1, 1)$, the perfect square is

$$Q_3(x, y) = (\rho + 1)x^2 + \sigma xy + (\tau + 1)y^2 = \pm(bx + cy)^2,$$

where neither b nor c is 0 (otherwise Q_3 would be a multiple of either Q_1 or Q_2). The only way for ρ and τ to be of opposite signs or zero is when Q_3 is also a positive square. There are two possibilities. If $\rho \geq 0$, $\tau < 0$, then $|b| \geq 1$, $|c| < 1$. We perform the change of variables

$$(x, y) \mapsto \left(x, c^{-1}(y - bx)\right), \qquad (u, v) \mapsto (1 - u, v). \tag{14}$$

Then net effect is to permute the roots z_1, z_2, z_3 into $1, \infty, 0$, respectively, and to take the corresponding perfect squares Q_1, Q_2, Q_3 to the new forms $(\tilde{b}x + \tilde{c}y)^2, x^2, y^2$, where $\tilde{b} = -b/c$, $\tilde{c} = 1/c$. Thus Q has the same form (13), with a new $\tilde{\rho}, \tilde{\sigma}, \tilde{\tau}$. Note also that if $|b| \geq 1$, $|c| < 1$, then the corresponding inequalities are *not* both true for $\tilde{b}$ and $\tilde{c}$. Therefore $\tilde{\rho}$ and $\tilde{\tau}$ necessarily have the same sign, and we are back to the previous case. Similarly, in the case $\rho < 0$, $\tau \geq 0$, so $|b| < 1$, $|c| \geq 1$, we perform the same change of variables (14) to make ρ, τ the same sign. Thus all versions of this case are equivalent to semi-diagonal symbols.

Case IIb. Suppose Q_1 and Q_2 are nonzero multiples of each other, but Q_3 is a different nonzero square. Then we can arrange that $Q_1 = \pm x^2$, $Q_2 = \pm\sigma^2 x^2$, and $Q_3 = \pm y^2$, so that

$$Q = \pm x^2u^2 \pm \sigma^2x^2v^2 + \left\{-(\pm 1 \pm \sigma^2)x^2 \pm y^2\right\}uv.$$

Note that we can rescale v to make

$$Q = \pm x^2u^2 \pm x^2v^2 + \left(\rho x^2 + \tau y^2\right)uv,$$

for certain constants ρ, τ and where $\tau \neq 0$. There are then two subcases:

Case IIb1. If Q_1 and Q_2 have the same sign, then

$$Q = \pm x^2(u^2 + v^2) + \left(\rho x^2 + \tau y^2\right)uv.$$

Now, if we replace (u, v) by $(u + v, u - v)$, then $u^2 + v^2$ becomes $2(u^2 - v^2)$, while uv becomes $u^2 - v^2$, so Q is placed in semi-diagonal form.

Case IIb2. If Q_1 and Q_2 have opposite signs, then

$$Q = \pm x^2(u^2 - v^2) + \left(\rho x^2 + \tau y^2\right)uv.$$

In this case, the **u**-discriminant is

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \left(\rho x^2 + \tau y^2\right)^2 + 4x^4,$$

which only has complex roots. If we interchange the roles of (x, y) and (u, v), we thus have a symbol of the sort discussed in Case I. Therefore, we can employ our earlier reasoning to conclude that the only non-semi-diagonal form which Q could take is the analog of Case Ib3, namely

$$Q = u^2(x^2 - y^2) + uv(x^2 + y^2). \tag{15}$$

This corresponds to a canonical Lagrangian of type 7.

Case IIc. Suppose Q_2 is identically 0, and Q_1 and Q_3 are not scalar multiples of each other. Thus we can make $Q_1 = \pm x^2$, and $Q_3 = \pm y^2$, so that

$$Q = \pm x^2(u^2 - uv) \pm y^2 uv.$$

If the two signs are the same, then we make the change of variables

$$(x, y) \mapsto (x - y, x + y), \qquad (u, v) \mapsto \left(u, v + \tfrac{1}{2}u\right)$$

to place Q into semi-diagonal form. If, on the other hand, we have opposite signs, then we use the change of variables

$$(u, v) \mapsto \left(u, \tfrac{1}{2}u - v\right)$$

to change Q to

$$Q = \tfrac{1}{2}u^2\left(x^2 - y^2\right) + uv\left(x^2 + y^2\right),$$

which is easily rescaled to the symbol (15).

Case IId. If Q_1, Q_2, and Q_3 are all multiples of the same perfect square, then Q is factorizable.

Case III. In this case, the discriminant Δ_x has precisely two distinct real roots, which we can assume to be at $z_1 = 0$, $z_2 = \infty$.

Case IIIa. Assume that the two corresponding polynomials Q_1 and Q_2 are not scalar multiples of each other. Then, using a transformation in G_x, we can arrange that

$$Q = \pm x^2u^2 + \left(\rho x^2 + \sigma xy + \tau y^2\right)uv \pm y^2v^2 \tag{16}$$

for certain constants ρ, σ, τ. There are two possibilities:

Case IIIa1. Both roots of Δ_x are double roots, so Δ_x is a multiple of u^2. Computing Δ_x directly from (16), we conclude that this is possible only if $\rho = \tau = 0$, and so Q is semi-diagonal.

Case IIIa2. The root $z_1 = 0$ is a triple root and the root $z_2 = \infty$ is a simple root. In this case Δ_x is a multiple of u^3, which implies that the two $\pm$ signs in (16) are the same, $\rho = 0$ and $\sigma = \pm 2$, the sign of σ being the same as that of the terms x^2u^2 and y^2v^2 in Q. Furthermore, if $\tau \neq 0$ (otherwise Q is semi-diagonal), we can rescale y and v to convert τ to 1. There is still the freedom of replacing x by $-x$, so we conclude that there are just two possible forms

$$Q = \pm\left(x^2u^2 + y^2v^2\right) + \left(2xy + y^2\right)uv.$$

These are not equivalent to a semi-diagonal symbol, since both the discriminants have root structure 13 (i.e., one simple and one triple root) which does not appear among the 14 semi-diagonal forms. These correspond to canonical Lagrangians of type 8.

Case IIIb. Assume that the two polynomials Q_1 and Q_2 are nonzero scalar multiples of each other, so, transforming by $G_{\mathbf{x}}$, we have

$$Q = \pm x^2u^2 \pm x^2v^2 + (\rho x^2 + \sigma xy + \tau y^2)uv.$$

Note that we cannot have a triple root for $\Delta_{\mathbf{x}}$ in this case; thus $\Delta_{\mathbf{x}}$ must be a multiple of u^2, and this implies that $\tau = 0$. If $\sigma \neq 0$, we can replace y by $\rho x + \sigma y$ to place Q into semi-diagonal form. If $\sigma = 0$, then Q is factorizable.

Case IIIc. If only Q_1 is nonzero, then

$$Q = \pm x^2u^2 + \{\rho x^2 + \sigma xy + \tau y^2\} uv.$$

If $\Delta_{\mathbf{x}}$ has two double roots, then $\tau = 0$, and, as in Case IIIb, we either get a semi-diagonal or a factorizable symbol. If $\Delta_{\mathbf{x}}$ has a triple root, then $4\rho\tau = \sigma^2$, and so the polynomial in brackets is a perfect square. Thus

$$Q = \pm x^2u^2 \pm (\mu x + \nu y)^2 uv.$$

If $\nu = 0$, we are back to a factorizable symbol; otherwise we can replace y by $\mu x + \nu y$, and, possibly, v by $-v$ to give Q the elementary form

$$Q = \pm x^2u^2 + y^2uv.$$

This is not a semi-diagonal case, since the discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$ have root multiplicities 13 and 4, respectively, which does not appear in our table of semi-diagonal symbols. These correspond to canonical Lagrangians of type 9.

Case IIId. If both Q_1 and Q_2 vanish identically, then Q is factorizable.

Case IV. Finally, we look at the remaining case when the discriminant $\Delta_{\mathbf{x}}$ has a single quadruple root, which we place at $z_1 = \infty$.

Case IVa. The corresponding perfect square Q_1 is not identically zero, so we arrange that it be $\pm x^2$. Then the symbol takes the form

$$Q = \pm\{x^2u^2 + (\rho x^2 + \sigma xy + \tau y^2)uv + (\lambda x^2 + \mu xy + \nu y^2)v^2\}.$$

Since the discriminant $\Delta_{\mathbf{x}}$ must be a multiple of v^4, we find that

$$\tau = 0, \qquad \sigma^2 = 4\nu, \qquad \sigma\mu = 2\rho\nu,$$

If we have $\sigma \neq 0$, then

$$Q = \pm\left\{x^2u^2 + x(\rho x + \sigma y)uv + \left[\tilde{\lambda}x^2 + \tfrac{1}{4}(\rho x + \sigma y)^2\right]v^2\right\},$$

where $\tilde{\lambda} = \lambda - \frac{1}{4}\rho^2$. Thus we can replace y by $\rho x + \sigma y$ to convert Q into semi-diagonal form.

If, on the other hand, $\sigma = 0$, then $\nu = 0$ as well, and

$$Q = \pm\left\{x^2u^2 + \rho x^2uv + \left(\lambda x^2 + \mu xy\right)v^2\right\}.$$

If $\mu = 0$, then Q is factorizable. Otherwise, we replace y by $\lambda x + \mu y$ and rescale to make Q take either the form

$$Q = \pm\left\{x^2u^2 + x^2uv + xyv^2\right\},$$

(if $\rho \neq 0$), or the form

$$Q = \pm x^2u^2 + xyv^2,$$

(if $\rho = 0$). The first case can be reduced to the second by the transformation

$$(x, y) \mapsto \left(x, \tfrac{1}{4}x \pm y\right), \qquad (u, v) \mapsto \left(u - \tfrac{1}{2}v, v\right).$$

This is not equivalent to a semi-diagonal symbol, as the discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$ have root multiplicities 4 and 13, respectively. Note that this case corresponds to canonical Lagrangians of type 10, and is the counterpart of Case IIIc.)

Case IVb. If the perfect square Q_1 is identically zero, then the symbol takes the form

$$Q = \left(\rho x^2 + \sigma xy + \tau y^2\right)uv + \left(\lambda x^2 + \mu xy + \nu y^2\right)v^2.$$

The condition that the discriminant $\Delta_{\mathbf{x}}$ be a multiple of v^4 implies that $\sigma^2 = 4\rho\tau$, so the coefficient of uv in Q is a perfect square. If it vanishes identically, then Q is factorizable; otherwise, we can transform x, y (and possibly change the sign of u) so that

$$Q = x^2uv + \left(\tilde{\lambda}x^2 + \tilde{\mu}xy + \tilde{\nu}y^2\right)v^2.$$

Moreover, for $\Delta_{\mathbf{x}}$ be a multiple of v^4, we necessarily have $\tilde{\nu} = 0$. Provided $\tilde{\mu} \neq 0$ (otherwise we are back in a factorizable case) we can replace y by $\tilde{\lambda}x + \tilde{\mu}y$ to put Q into the simplified form

$$Q = x^2uv + xyu^2.$$

In this case, both discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$ have root multiplicities 4. However, this is not equivalent to semi-diagonal case 6 since the invariant I_2 vanishes here, but is nonvanishing in the semi-diagonal case. This case corresponds to canonical Lagrangians of type 11.

This completes the classification. Summarizing, we find that besides the fourteen types of semi-diagonal symbols, we have the following additional canonical forms:

Case	Symbol	Δ_x	Δ_u
15.	$y^2(u^2 - v^2) + xy(u^2 + v^2)$	2^2	$1^2 2$
16.	$u^2(x^2 - y^2) + uv(x^2 + y^2)$	$1^2 2$	2^2
17.	$\pm(x^2u^2 + y^2v^2) + (2xy + y^2)uv$	13	13
18.	$\pm x^2u^2 + y^2uv$	13	4
19.	$\pm x^2u^2 + xyv^2$	4	13
20.	$x^2uv + xyu^2$	4	4

The reader may be struck by the asymmetry of case 17, and may wonder why there is not another case corresponding to an interchange of the variables (x, y) and (u, v). It turns out that the resulting symbol is equivalent to the original one. For instance, the symbol

$$x^2u^2 + \left(2xy + y^2\right)uv + y^2v^2$$

is transformed into the "interchanged" symbol

$$x^2u^2 + \left(2uv + v^2\right)xy + y^2v^2$$

by the change of variables

$$(x, y) \mapsto \left(\tfrac{1}{4}x + y, -x\right), \qquad (u, v) \mapsto \left(-v, u + \tfrac{1}{4}v\right),$$

as the reader can verify. A similar result holds for the case with a minus sign.

8. Equivalence of Orthotropic Symbols

We have now determined the basic 15 classes of canonical forms for first-order planar quadratic Lagrangians, and shown that no two canonical forms from different equivalence classes can possibly be equivalent under a linear change of variables. There remains the possibility that two different canonical forms within the *same* equivalence class might be equivalent. For most equivalence classes, there is just a finite number of canonical forms corresponding to different choices of plus or minus signs, and it is not difficult to see that all such canonical forms are certainly inequivalent with respect to real changes of variables. That leaves only canonical forms of type 1, with three sign choices and two arbitrary constants, and type 2, with three sign choices and one arbitrary constant, to be investigated.

For simplicity, we write out the detailed solution to the problem of equivalence of canonical forms only for the case of orthotropic Lagrangians, meaning those of type 1 with all plus signs:

$$u_x^2 + v_y^2 + \alpha\left(u_y^2 + v_x^2\right) + 2\beta u_x v_y. \tag{17}$$

The methods used for determining the possible equivalences of orthotropic Lagrangians will readily extend to the other cases, but, as our primary interest is in the elastic case, we leave the details of these cases to the reader.

The symbol of an orthotropic Lagrangian is the biquadratic polynomial

$$Q(\mathbf{x}, \mathbf{u}) = x^2u^2 + \alpha y^2u^2 + 2\beta xyuv + \alpha x^2v^2 + y^2v^2.$$

We assume to begin with that $\alpha \neq 0$. (In terms of the canonical form for semi-diagonal symbols, this corresponds to not being in case 4. It was shown at the end of Section 6 how to convert case 4 to case 2, so we are not losing any generality by our assumption.) The $\mathbf{x}$-discriminant for Q is the quartic polynomial

$$\Delta_{\mathbf{x}}(\mathbf{u}) = \alpha(u^4 + 2\sigma u^2v^2 + v^4),$$

where

$$2\sigma = \left(\alpha^2 + 1 - \beta^2\right)/\alpha.$$

One important remark is that although the parameter σ plays a fundamental role in the structure of orthotropic symbols, it is *not* an invariant of the symbol! Note also that $\Delta_{\mathbf{u}}(\mathbf{x})$ has the exact same form with (x, y) replacing (u, v); in particular, the two discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$ have the same roots.

The quadratic formula gives the explicit formula for the roots of $\Delta_{\mathbf{x}}$, which depend only on the parameter σ. There are five geometrically distinct configurations that these roots can have:

(1) $\sigma > 1$ roots all on the imaginary axis at $\pm\tau i$, $\pm\tau^{-1}i$, $\tau > 1$.

(2) $\sigma = 1$ two double roots at $\pm i$.

(3) $|\sigma| < 1$ roots all on the unit circle at $\pm\exp(\pm i\theta)$, $0 < \theta < \frac{1}{2}\pi$.

(4) $\sigma = -1$ two double roots at ± 1.

(5) $\sigma < -1$ roots all on the real axis at $\pm\tau$, $\pm\tau^{-1}$, $\tau > 1$.

Note that case 2 includes the isotropic Lagrangians, i.e., $\alpha + \beta = 1$, as well as the complementary, but equivalent case $\alpha - \beta = 1$ mentioned in Theorem 3.

Suppose to begin with that $\alpha > 0$. The case $\alpha < 0$ will be dealt with subsequently. We can also assume without loss of generality that $\beta \geq 0$

(otherwise replace x by $-x$). In particular, Q satisfies the Legendre–Hadamard strong ellipticity condition (6) if and only if we are in cases 1–3, i.e., $\sigma > -1$.

We begin the analysis by looking at case 1, where all the roots are on the imaginary axis. Given $\tau > 1$, let S_τ denote the set of four points

$$S_\tau = \{\tau i, -\tau i, \tau^{-1} i, -\tau^{-1} i\}.$$

LEMMA 11. *Let* $\mathrm{GL}(2, \mathbb{R})$ *act on the complex plane via linear fractional transformations. Given* $\tau, \tau' > 1$, *a transformation* $A \in \mathrm{GL}(2, \mathbb{R})$ *maps the set* S_τ *to the set* $S_{\tau'}$ *if and only if* $\tau = \tau'$ *and* A *is a multiple of one of the four matrices* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *or* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Thus, the only changes of variable which can preserve the orthotropic root structure of Q for a given $\sigma > 1$ are interchanges of x and y or of u and v, and rescalings. If we are also requiring the symbol to be in orthotropic form, these are only three possibilities: We can change the sign of one of the variables x, y, u, or v; this changes β into $-\beta$. We can interchange x and y, and rescale $(x, y) \mapsto (y/\sqrt{\alpha}, x/\sqrt{\alpha})$; this changes α into $1/\alpha$ and β into β/α. Or we can compose these two transformations, changing α into $1/\alpha$ and β into $-\beta/\alpha$. All three of these transformations leave σ unchanged. These correspond to transformations (i)–(iii) in Theorem 3.

Next consider case 3, in which $|\sigma| < 1$, and the roots lie on the unit circle. The linear fractional transformation determined by the matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ will map the unit circle onto the imaginary axis, and so onto a root structure as in case 1. This can be realized in the $\mathbf{x}$ and $\mathbf{u}$ variables by our familiar transformation

$$(x, y) \mapsto (x - y, x + y), \qquad (u, v) \mapsto (u - v, u + v). \tag{18}$$

This has the net effect of changing Q into the symbol

$$\begin{aligned}\tilde{Q} = {} & (2 + 2\alpha + 2\beta)(x^2u^2 + y^2v^2) \\ & + (2 + 2\alpha - 2\beta)(y^2u^2 + x^2v^2) + (8 - 8\alpha)xyuv.\end{aligned}$$

Note that under our assumptions, $1 + \alpha + \beta > 0$, so simple rescaling will convert $\tilde{Q}$ into a orthotropic symbol, with moduli

$$\tilde{\alpha} = (1 + \alpha - \beta)/(1 + \alpha + \beta), \qquad \tilde{\beta} = (2 - 2\alpha)/(1 + \alpha + \beta).$$

A simple computation shows that the corresponding parameter $\tilde{\sigma}$ for the new orthotropic symbol is related to the original σ by the transformation

$$\tilde{\sigma} = -(\sigma - 3)/(\sigma + 1),$$

which has the desired effect of changing a symbol of case 3 into one of case 1. Thus we have reconstructed transformation (iv) of Theorem 3. Once we are in case 1, the only possible further transformations preserving the orthotropic form of the Lagrangian are the above ones, and this leads to the remaining transformations (v)–(vii).

If we are in the orthotropic case 2, then the transformation (18) has no effect. (Indeed, when coupled with the rescaling, it is just one of an entire one-parameter family of simultaneous rotations in $\mathbf{x}$ and $\mathbf{u}$, which preserve isotropic Lagrangians. This extra degree of symmetry is another feature that distinguishes the isotropic case from the more general orthotropic cases). If we are in case 4, then $1 + \alpha - \beta = 0$, in which case (18) produces an orthotropic Lagrangian with $\alpha = 0$.

If we are in case 5, with $\alpha > 0$, $\beta \geq 0$, and $\sigma < -1$, then $1 + \alpha - \beta < 0$, so the effect of (18) and the rescaling is the produce an orthotropic Lagrangian with $\tilde{\alpha} < 0$, and $\tilde{\sigma} < -1$. Conversely, if we begin with an orthotropic Lagrangian with $\alpha < 0$, $\sigma < -1$, then we can invert the transformation to produce an equivalent orthotropic Lagrangian with $\alpha > 0$, $\sigma < -1$. If, on the other hand, $\alpha < 0$, but $\sigma > -1$, then we can never produce an orthotropic Lagrangian with $\alpha > 0$, although we still have the same correspondence between the $\sigma > 1$ and $-1 < \sigma < 1$ cases.

The upshot is that for an orthotropic Lagrangian (17), for $\sigma \neq -1$ there are five different types of canonical forms, namely

(i) $0 < \alpha \leq 1, 0 \leq \beta < 1 - \alpha$ giving $\sigma > 1$,
(ii) $\beta > 1, 0 < \alpha < \beta - 1$, giving $\sigma < -1$,
(iii) $\alpha < 0$, $|\alpha + 1| < \beta < 1 - \alpha$, giving $-1 < \sigma < 1$,
(iv) $\alpha > 0$, $\beta \geq 0$, $\alpha + \beta = 1$, giving $\sigma = 1$,
(v) $\alpha < 0$, $\beta \geq 0$, $\alpha + \beta = 1$, giving $\sigma = 1$,
(vi) $\alpha = 0$, $\beta \geq 0$, giving $\sigma = -1$,
(vii) the special form $2xu$,

with the property that every other orthotropic Lagrangian is equivalent to precisely *one* of these forms. (In particular, no two of the orthotropic Lagrangians on this reduced list are equivalent.)

Similar types of reasoning can be applied to the other types of canonical forms, but we leave the details to the reader.

9. The Complete List of Canonical Forms

We are now able to give a complete list of inequivalent canonical forms for first-order planar quadratic Lagrangians (Table II) which will take care

TABLE II

Type	Canonical form	Restrictions	Root structure
Ia.	$x^4 + 2\sigma x^2y^2 + y^4$,	$-1 < \sigma \neq 1$	$1^4\mathbb{C}$
Ib.	$-(x^4 + 2\sigma x^2y^2 + y^4)$,	$-1 < \sigma \neq 1$	$1^4\mathbb{C}$
Ic.	$x^4 + 2\sigma x^2y^2 - y^4$,		$1^4\mathbb{R}$-$\mathbb{C}$
Id.	$x^4 + 2\sigma x^2y^2 + y^4$,	$\sigma < -1$	$1^4\mathbb{R}$
IIa.	$x^2y^2 + y^4$,		$1^2 2\mathbb{R}$-$\mathbb{C}$
IIb.	$-x^2y^2 - y^4$,		$1^2 2\mathbb{R}$-$\mathbb{C}$
IIc.	$-x^2y^2 + y^4$,		$1^2 2\mathbb{R}$
IId.	$x^2y^2 - y^4$,		$1^2 2\mathbb{R}$
IIIa.	$x^4 + 2x^2y^2 + y^4$,		$2^2\mathbb{C}$
IIIb.	$-(x^4 + 2x^2y^2 + y^4)$,		$2^2\mathbb{C}$
IIIc.	x^2y^2,		$2^2\mathbb{R}$
IIId.	$-x^2y^2$,		$2^2\mathbb{R}$
IV.	x^3y,		$13\mathbb{R}$
Va.	x^4,		$4\mathbb{R}$
Vb.	$-x^4$,		$4\mathbb{R}$
VI.	0,		∞

of the repetitions in our earlier list in Theorem 1. The primary feature that we will use to distinguish the different classes of canonical form will be the structure of the two discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$. Therefore, before stating the result, we recall the 16 canonical forms for a binary quartic over the reals. (See [6; Exercises 25.13, 25.14], although our numbering differs slightly from Gurevich's.)

In the root structure, the symbols $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{R}$-$\mathbb{C}$ indicate all complex roots, all real roots, and two real and two complex roots, respectively. (Clearly in cases IIa, b, the simple roots are the complex conjugate pair.)

We now list the canonical forms for first-order, planar quadratic Lagrangians, as distinguished by the canonical forms of their discriminants, in Table III. The verification that this list is exhaustive, so every first-order, planar quadratic Lagrangian is equivalent to precisely one of the following 60 types of canonical forms, has, for the most part been done, and the remaining details are not hard, but not very instructive.

For the most part, these canonical forms are distinguished by the root structure, i.e., canonical form, of the corresponding quartic discriminants $\Delta_{\mathbf{x}}$ and $\Delta_{\mathbf{u}}$. Occasionally, two different canonical forms will have the same canonical forms for both their discriminants, and they must be distinguished by some other invariants or covariants. The hardest cases to distinguish by purely invariant-theoretic means are when a Lagrangian L and its negative $-L$ are two *different* canonical forms. There are many such pairs, and they are always listed together with a $+$ and $-$ sign

TABLE III

Type	Canonical form of Lagrangian	Restrictions
1	$u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$0 < \alpha \leq 1, 0 \leq \beta < 1 - \alpha$
2	$-(u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y)$,	$0 < \alpha \leq 1, 0 \leq \beta < 1 - \alpha$
3	$u_x^2 - v_y^2 + \alpha(-u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha > 0, \beta \geq 0$,
4	$u_x^2 - v_y^2 + \alpha(u_y^2 - v_x^2) + 2\beta u_x v_y$,	$\alpha > 0, \beta \geq 0$,
5	$u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha < 0, \lvert\alpha + 1\rvert < \beta < 1 - \alpha$
6	$u_x^2 - v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha \neq 0, \beta \geq 0$,
7	$u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha < 0, 0 \leq \beta < \lvert\alpha + 1\rvert$
8	$u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y$,	$0 \leq \beta < 1$
9	$-(u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y)$,	$0 \leq \beta < 1$
10	$u_x^2 + v_y^2 - u_y^2 + 2\beta u_x v_y$,	$\beta > 1$
11	$-(u_x^2 + v_y^2 - u_y^2 + 2\beta u_x v_y)$,	$\beta > 1$
12	$-u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y$,	$\beta \geq 0$
13	$-(-u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y)$,	$\beta \geq 0$
14	$u_x^2 - v_y^2 + u_y^2 + 2\beta u_x v_y$,	$\beta \geq 0$
15	$-(u_x^2 - v_y^2 + u_y^2 + 2\beta u_x v_y)$,	$\beta \geq 0$
16	$u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y$,	$\beta > 1$
17	$-(u_x^2 + v_y^2 + u_y^2 + 2\beta u_x v_y)$,	$\beta > 1$
18	$u_x^2 + v_y^2 - u_y^2 + 2\beta u_x v_y$,	$0 \leq \beta < 1$
19	$-(u_x^2 + v_y^2 - u_y^2 + 2\beta u_x v_y)$,	$0 \leq \beta < 1$
20	$u_x^2 - u_y^2 + 2u_x v_y$,	
21	$u_x^2 + u_y^2 + 2u_x v_y$,	
22	$-(u_x^2 + u_y^2 + 2u_x v_y)$,	
23	$u_x^2 - u_y^2 + u_x v_x + u_y v_y$,	
24	$u_x^2 - v_x^2 + u_x u_y + v_x v_y$,	
25	$u_x^2 - v_x^2 + 2u_x v_y$,	
26	$u_x^2 + v_x^2 + 2u_x v_y$,	
27	$-(u_x^2 + v_x^2 + 2u_x v_y)$,	
28	$u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha > 0, \beta \geq 0, \alpha + \beta = 1$
29	$-(u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y)$,	$\alpha > 0, \beta \geq 0, \alpha + \beta = 1$
30	$u_x^2 + v_y^2 + \alpha(u_y^2 + v_x^2) + 2\beta u_x v_y$,	$\alpha < 0, \beta \geq 0, \alpha + \beta = 1$
31	$u_x^2 + v_y^2 + 2\beta u_x v_y$,	$0 \leq \beta < 1$
32	$-(u_x^2 + v_y^2 + 2\beta u_x v_y)$,	$0 \leq \beta < 1$
33	$u_x^2 + v_y^2 + 2\beta u_x v_y$,	$\beta > 1$
34	$-(u_x^2 + v_y^2 + 2\beta u_x v_y)$,	$\beta > 1$
35	$u_x^2 - v_y^2 + 2\beta u_x v_y$,	
36	$u_x^2 + 2u_x v_y$,	
37	$-(u_x^2 + 2u_x v_y)$,	
38	$u_x v_y$,	
39	$u_x^2 + v_y^2 + 2u_x v_y + u_y v_y$,	

TABLE III—*Continued*

Type	Canonical form of Lagrangian	Restrictions
40	$-(u_x^2 + v_y^2 + 2u_xv_y + u_yv_y)$,	
41	$u_x^2 + u_yv_y$,	
42	$-(u_x^2 + u_yv_y)$,	
43	$u_x^2 + v_xv_y$,	
44	$-(u_x^2 + v_xv_y)$,	
45	$u_x^2 + v_y^2 + u_y^2 + 2u_xv_y$,	
46	$-(u_x^2 + v_y^2 + u_y^2 + 2u_xv_y)$,	
47	$u_x^2 + v_y^2 - u_y^2 + 2u_xv_y$,	
48	$-(u_x^2 + v_y^2 - u_y^2 + 2u_xv_y)$,	
49	$u_xu_y + u_xv_x$,	
50	$u_x^2 + u_y^2$,	
51	$-(u_x^2 + u_y^2)$,	
52	$u_x^2 - u_y^2$,	
53	$u_x^2 + v_x^2$,	
54	$-(u_x^2 + v_x^2)$,	
55	$u_x^2 - v_x^2$,	
56	$u_x^2 + v_y^2 + 2u_xv_y$,	
57	$-(u_x^2 + v_y^2 + 2u_xv_y)$,	
58	u_x^2,	
59	$-u_x^2$,	
60	0,	

following in the last column of Table IV, e.g., types 10 and 11. The fact that each pair constitutes two inequivalent canonical forms is easily verified by considering which changes of variables preserve the actual roots of the two discriminants, and then checking that no such change of variables will take L to $-L$. The only way to distinguish L and $-L$ by use of covariants is by looking at those of odd degree and even biweight, since covariants of even degree are the same for L and $-L$, while those of odd degree and one or the other weight odd can be simply changed in sign either by $L \mapsto -L$, or by the change of variables $\mathbf{x} \mapsto -\mathbf{x}$, or $\mathbf{u} \mapsto -\mathbf{u}$.

The simplest covariant of odd degree and even biweight is the symbol Q itself. If Q is positive (semi-)definite, then it itself can be used to distinguish L from $-L$; this occurs with a few of these pairs. However, if Q is not of one sign, then more subtle covariants are required. For example, to distinguish types 41 and 42, we can proceed as follows. The covariant C_2 is of even degree, and biweight (1, 1), so the covariant

$$C_3 = C_{2xu}Q_{yv} - C_{2xv}Q_{yu} - C_{2yu}Q_{xv} + C_{2yv}Q_{xu},$$

TABLE IV

Type	Δ_x	Δ_x	Invariants	Sign
1	Ia	Ia	$Q \geq 0$	+
2	Ia	Ia	$Q \leq 0$	−
3	Ia	Ib		
4	Ib	Ia		
5	Ib	Ib		
6	Ic	Ic		
7	Id	Id		
8	IIa	IIa	$Q \geq 0$	+
9	IIa	IIa	$Q \leq 0$	−
10	IIb	IIb		+
11	IIb	IIb		−
12	IIb	IIc		+
13	IIb	IIc		−
14	IIc	IIb		+
15	IIc	IIb		−
16	IIc	IIc		+
17	IIc	IIc		−
18	IId	IId		+
19	IId	IId		−
20	IIb	IIId		
21	IIc	IIId		+
22	IIc	IIId		−
23	IId	IIIb		
24	IIIb	IId		
25	IIId	IIb		
26	IIId	IIc		+
27	IIId	IIc		−
28	IIIa	IIIa		+
29	IIIa	IIIa		−
30	IIIb	IIIb		
31	IIIc	IIIc	$Q \geq 0$	+
32	IIIc	IIIc	$Q \leq 0$	−
33	IIId	IIId	$I_3 \neq 0$, C_2 type 31	+
34	IIId	IIId	$I_3 \neq 0$, C_2 type 31	−
35	IIId	IIId	C_2 type 35 or 38	
36	IIId	IIId	$I_3 = 0$, C_2 type 58	+
37	IIId	IIId	$I_3 = 0$, C_2 type 58	−
38	IIId	IIId	$C_2 = 0$ (type 60)	
39	IV	IV		+
40	IV	IV		−
41	IV	Vb		+
42	IV	Vb		−
43	Vb	IV		+
44	Vb	IV		−
45	Va	Va	$Q \geq 0$	+
46	Va	Va	$Q \leq 0$	−
47	Vb	Vb	$I_2 < 0$	+

TABLE IV—*Continued*

Type	Δ_x	Δ_x	Invariants	Sign
48	Vb	Vb	$I_2 < 0$	$-$
49	Vb	Vb	$I_2 = 0$	
50	Va	VI		$+$
51	Va	VI		$-$
52	Vb	VI		
53	VI	Va		$+$
54	VI	Va		$-$
55	VI	Vb		
56	VI	VI	$Q \geq 0, I_2 > 0$	$+$
57	VI	VI	$Q \leq 0, I_2 > 0$	$-$
58	VI	VI	$Q \geq 0, I_2 = 0$	$+$
59	VI	VI	$Q \leq 0, I_2 = 0$	$-$
60	VI	VI	$Q = 0$	

which is also a biquadratic, is of odd degree and even biweight (2, 2), cf. [10]. For case 41

$$Q = x^2u^2 + y^2uv,$$

so

$$C_2 = Q_{xu}Q_{yv} - Q_{yu}Q_{xv} = 8xyu^2.$$

Therefore

$$C_3 = 32y^2u^2,$$

which is positive semi-definite. For case 42, $C_3 = -32y^2u^2$ is negative semi-definite, and hence the covariant C_3 serves to distinguish the two cases. (In the other $\pm$ pairs, the required covariants can be most easily constructed using hyperjacobian combinations, cf. [10].)

I have not completely determined all the covariants of odd degree and even biweight which can serve to completely distinguish all the $\pm L$ pairs of canonical forms. This is because (a) the calculations are rather tedious, (b) the results are not particularly enlightening, and, most importantly, (c) the proof of the general theorem gives the algorithm for constructing the canonical form of any given Lagrangian, so one can determine which case is applicable by direct construction anyway. However, I certainly believe that, with enough effort, each pair can ultimately be distinguished by an appropriate covariant.

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