Canonical forms and integrability of bi-Hamiltonian systems ☆

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Turiel's complete list of canonical forms for finite-dimensional, nondegenerate, compatible pairs of Hamiltonian structures is used to determine the precise local integrability of bi-Hamiltonian systems of ordinary differential equations. Also, classification of incompatible Hamiltonian pairs in four dimensions and the relationship between compatibility and integrability are discussed.

1. Introduction

A system of differential equations is called *bi-Hamiltonian* [1,2], if it can be written in Hamiltonian form in two distinct ways:

$$\dot{x}=J_1\nabla H_1=J_2\nabla H_0\,,\quad x\in \mathrm{M}\,. \tag{1}$$

Here M is a real or complex n-dimensional manifold, $H_0(x)$, $H_1(x)$ are the two Hamiltonian functions, and $J_1(x)$, $J_2(x)$ are skew symmetric $n\times n$ Hamiltonian matrices, not constant multiples of each other, determining Poisson brackets on M:

$$\{F,G\}_\nu=\nabla F^{\mathrm{T}}J_\nu(x)\nabla G\,,\quad \nu=1,2\,. \tag{2}$$

The Jacobi identity requires that each $J_\nu(x)$ satisfy a quadratically nonlinear system of partial differential equations (cf. ref. [2; proposition 6.8]). We call the structure defined by J_1, J_2 a *Hamiltonian pair.* The Hamiltonian pair is *compatible* if the sum $J_1(x)+J_2(x)$ also determines a Poisson bracket, which, owing to the quadratic nature of the Jacobi identity, implies that any constant coefficient linear combination of J_1 and J_2 is also a valid Poisson bracket. In the *symplectic* case, each $J_\nu(x)$ is nonsingular (so n is necessarily even), and the nonlinear Jacobi conditions can be replaced by the linear condition that the symplectic two-forms

$$\Omega_\nu=\tfrac{1}{2}\,\mathrm{d}x^{\mathrm{T}}\wedge K_\nu(x)\,\mathrm{d}x\,,\quad K_\nu(x)=J_\nu(x)^{-1} \tag{3}$$

are closed, i.e. $\mathrm{d}\Omega_\nu=0$. Compatibility is equivalent to the closure of the two-form

$$\tfrac{1}{2}\,\mathrm{d}x^{\mathrm{T}}\wedge[K_1(x)^{-1}+K_2(x)^{-1}]^{-1}\,\mathrm{d}x\,.$$

According to the fundamental theorem of Magri [1], [2; theorem 7.24], provided certain technical hypotheses hold, bi-Hamiltonian systems for compatible Hamiltonian pairs are completely integrable.

Theorem 1. Suppose J_1, J_2 form a compatible Hamiltonian pair, with J_1 symplectic. For each associated bi-Hamiltonian system (1), there exists a hierarchy of Hamiltonian functions $H_0, H_1, H_2, H_3, \ldots$, all in involution with respect to either Poisson bracket, $\{H_j, H_k\}_\nu=0$, generating mutually commuting bi-Hamiltonian flows

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$$\dot{x}=J_1\nabla H_k=J_2\nabla H_{k-1}\,. \tag{4}$$

Thus such bi-Hamiltonian systems are completely integrable in the classical sense provided enough of the Hamiltonians in the associated hierarchy are functionally independent. The goals of the present paper are two: first to investigate in more detail the integrability of particular bi-Hamiltonian systems; second, to determine the proper role that the compatibility condition plays in the integrability of such systems. Both investigations will rest on the determination of canonical forms for such Hamiltonian pairs, and then explicitly determining all associated bi-Hamiltonian systems. Note that for a given pair, the corresponding bi-Hamiltonian systems are found by solving the linear system of partial differential equations

$$\nabla H_0=M\nabla H_1\,,\quad M=J_2^{-1}J_1=K_2K_1^{-1}\,, \tag{5}$$

where M is the transpose of the recursion operator [2]. We remark here that the simple system of differential equations (5), which arises in a surprising number of different contexts, is not well understood, except in the particular case when the matrix M is constant, in which case the general solution can be found in ref. [3].

A Hamiltonian pair is called *nondegenerate* at the point x if the skew-symmetric matrix pencil

$$J_\lambda(x)\equiv J_2(x)-\lambda J_1(x)\,,\quad \lambda\in\mathbb{C}\cup\{\infty\}\,, \tag{6}$$

is nonsingular for at least one (and hence for all but a finite number of) λ. (For $\lambda=\infty$, set $J_\lambda(x)=J_1(x)$.) In this case, the λ's for which $J_\lambda(x)$ is singular are called the *eigenvalues* of the pair. If the pair is compatible, and nondegenerate, we can assume without loss of generality that J_1 is symplectic, i.e. $\lambda=\infty$ is not an eigenvalue. The complete algebraic type of a nondegenerate pair of skew-symmetric matrices is given by the elementary divisors and Segre characteristic of the matrix pencil $J_\lambda(x)$, cf. ref. [4]. (Degenerate pairs of skew-symmetric matrices are handled by the more detailed Kronecker theory.) A pair is called *elementary* at x if it has just one complex eigenvalue. A nondegenerate pair is called *irreducible* at x if it has Segre characteristic [(nm)] (analogous to a single Jordan block), so every elementary pair is the direct sum of irreducible pairs all having the same eigenvalue. Every nondegenerate complex skew-symmetric matrix pencil is algebraically the direct sum of irreducible skew-symmetric matrix pencils.

Theorem 2. The eigenvalues, elementary divisors and Segre characteristic of a Hamiltonian pair are invariant under the flow of any associated bi-Hamiltonian system.

Definition 3. A (nondegenerate) Hamiltonian pair is *generic* on a domain M if it has constant Segre characteristic over all of M, i.e. the algebraic type of the pair does not vary from point to point, and also the number of functionally independent eigenvalues does not vary, i.e. the dimension of the subspace of $\mathrm{T^*M}|_x$ spanned by their differentials $\{\mathrm{d}\lambda_1, ..., \mathrm{d}\lambda_l\}$ is independent of x. (In particular, each eigenvalue λ_m is either constant or has no critical points on M.)

Nongeneric points are singularities of the pair, and must be handled by more sophisticated techniques; see ref. [5] for the case of a single Poisson structure in the plane. According to theorem 2, as far as the flow of any bi-Hamiltonian system is concerned, we can safely restrict our attention to a domain where the pair is generic.

According to the results of Turiel [6,7], any complex-analytic generic nondegenerate compatible Hamiltonian pair can be locally expressed as the Cartesian product of elementary pairs, each having just one eigenvalue. (Turiel's classification results extend to real Hamiltonian pairs, as do our classifications of bi-Hamiltonian systems, but, for simplicity, we will restrict our attention here to complex analytic systems.) In the case of constant eigenvalue, any elementary pair is in turn the Cartesian product of irreducible pairs; however, this does *not* hold in the case of nonconstant eigenvalue. We will present the details of the Turiel classification and the structure of associated bi-Hamiltonian systems in five stages, corresponding to

(I) irreducible, constant eigenvalue pairs;
(II) elementary, constant eigenvalue pairs;
(III) irreducible, nonconstant eigenvalue pairs;
(IV) elementary, nonconstant eigenvalue pairs;
(V) general generic, compatible, nondegenerate Hamiltonian pairs.

We assume that neither 0 nor ∞ is an eigenvalue, so the Hamiltonian pair is determined by two compatible symplectic forms. Darboux' theorem [2; theorem 6.22], implies that we can write the first symplectic form in canonical form,

$$\Omega_1 = \mathrm{d}p_0 \wedge \mathrm{d}q_0 + \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \ldots + \mathrm{d}p_n \wedge \mathrm{d}q_n \,. \tag{7}$$

Therefore, the problem reduces to how to place the second form Ω_2 into a canonical form using canonical transformations of the phase space.

2. (I) Irreducible, constant eigenvalue pairs

Theorem 4. Any irreducible compatible Hamiltonian pair with constant nonzero eigenvalue λ has the canonical form

$$\Omega_2 = \mu\Omega_1 + \mathrm{d}p_0 \wedge \mathrm{d}q_1 + \mathrm{d}p_1 \wedge \mathrm{d}q_2 + \ldots \mathrm{d}p_{n-1} \wedge \mathrm{d}q_n \,, \tag{8}$$

where Ω_1 is given by (7), and $\mu = \lambda^{-1}$.

The associated symplectic matrices for the pair (7), (8) are

$$K_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \mu I + U \\ -\mu I - U^{\mathrm{T}} & 0 \end{pmatrix}, \tag{9}$$

which is the canonical algebraic form of Weierstrass for a nondegenerate pair of skew-symmetric complex matrices [4]. Here I is the $(n+1) \times (n+1)$ identity matrix, and U is the upper triangular matrix of the same size with 1's on the super-diagonal and 0's elsewhere. Inverting the matrices (9) will give a canonical form for the Hamiltonian matrices J_ν, although these are somewhat complicated to work with directly. However, this pair assumes the simpler canonical form

$$J_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \lambda I + U \\ -\lambda I - U^{\mathrm{T}} & 0 \end{pmatrix} \tag{10}$$

under the canonical transformation $(p, q) \to (A^{\mathrm{T}}p, A^{-1}q)$, where the matrix A is such that $A(\mu I + U)^{-1}A^{-1} = \lambda I + U$ is in Jordan canonical form. The general form for any associated bi-Hamiltonian system follows from the solution to (5) using the methods of ref. [3].

Theorem 5. Suppose $H_1(\boldsymbol{x})$, $H_0(\boldsymbol{x})$, $\boldsymbol{x} = (\boldsymbol{p}, \boldsymbol{q})$, are analytic functions which define a bi-Hamiltonian system (1) relative to the canonical irreducible constant eigenvalue Hamiltonian pair (7), (8) on a convex open subset. Then there exist scalar-valued functions $F_k(\xi, \eta)$, $k=0, ..., n$, such that

$$H_0(\boldsymbol{x}) = H_0^{(0)}(\boldsymbol{x}) - H_0^{(1)}(\boldsymbol{x}) + \ldots + H_0^{(n)}(\boldsymbol{x}) \,, \quad H_1(\boldsymbol{x}) = H_1^{(0)}(\boldsymbol{x}) + H_1^{(1)}(\boldsymbol{x}) + \ldots + H_1^{(n)}(\boldsymbol{x}) \,,$$

with

$$H_0^{(k)}(\boldsymbol{x}) = \mu \frac{\partial^k}{\partial s^k} F_k(\pi(s), \varpi(s)) \bigg|_{s=0} + k \frac{\partial^{k-1}}{\partial s^{k-1}} F_k(\pi(s), \varpi(s)) \bigg|_{s=0} \,,$$

$$H_1^{(k)}(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}F_k(\pi(s),\varpi(s))\bigg|_{s=0}\,, \tag{11}$$

where

$$\pi(s)=p_0+sp_1+s^2p_2+\ldots+s^np_n\,,\quad \varpi(s)=q_n+sq_{n-1}+s^2q_{n-2}+\ldots+s^nq_0\,. \tag{12}$$

Example 6. Consider the case $n=2$. Then

$$\pi(s)=p_0+sp_1+s^2p_2\,,\quad \varpi(s)=q_2+sq_1+s^2q_0\,.$$

Substituting into (11), we find that the general solution to (5) in this case is a sum of three particular types of solution:

$$H_0^{(0)}=\mu F_0(p_0,q_2)\,,\quad H_1^{(0)}=F_0(p_0,q_2)\,,$$

$$H_0^{(1)}=\mu H_1^{(1)}+F_1(p_0,q_2)\,,\quad H_1^{(1)}=\frac{\partial F_1}{\partial p_0}p_1+\frac{\partial F_1}{\partial q_2}q_1\,,$$

$$H_0^{(2)}=\mu H_1^{(2)}+2\frac{\partial F_2}{\partial p_0}p_2+2\frac{\partial F_2}{\partial q_2}q_0\,,$$

$$H_1^{(2)}=\frac{\partial^2F_2}{\partial p_0^2}p_1^2+2\frac{\partial^2F_2}{\partial p_0\partial q_2}p_1q_1+\frac{\partial^2F_2}{\partial q_2^2}q_1^2+2\frac{\partial F_2}{\partial p_0}p_2+2\frac{\partial F_2}{\partial q_2}q_0\,,$$

where F_0, F_1, F_2 are arbitrary functions of the variables p_0, q_2.

Similarly, it can be shown that the general Hamiltonians determined by (11) are polynomials in the *minor variables* $p_1, \ldots, p_n, q_0, \ldots, q_{n-1}$, whose coefficients are certain derivatives of the arbitrary smooth functions $F_k(p_0, q_n)$ of the remaining two *major variables*, p_0, q_n. In outline, the proof of this and similar subsequent results proceeds in two steps. First one verifies by direct, elementary computation that (11) really do define solutions to the system (5). Then, to show that these are the only solutions, we cross-differentiate to deduce that the general solution must be an affine function of the top order minor variables p_n, q_0, and, moreover, the coefficients of these variables can be matched by a suitable choice of the solution (11) for $k=n$. The linearity of (5) and an easy induction will complete the argument.

Theorem 5 demonstrates the complete integrability of any bi-Hamiltonian system corresponding to an irreducible, constant eigenvalue Hamiltonian pair. Indeed, the subsystem governing the time evolution of the major variables is the autonomous two-dimensional (one degree of freedom) Hamiltonian system with Hamiltonian $n!F_n(p_0, q_n)$,

$$\frac{\mathrm{d}p_0}{\mathrm{d}t}=-\frac{\partial H_1}{\partial q_0}=-n!\frac{\partial F_n}{\partial q_n}\,,\qquad \frac{\mathrm{d}q_n}{\mathrm{d}t}=\frac{\partial H_1}{\partial p_n}=n!\frac{\partial F_n}{\partial p_0}\,, \tag{13}$$

and is easily integrated by quadrature. (Curiously, the canonically conjugate variables p_0, q_n for the reduced system (13) are *not* canonically conjugate for the standard symplectic structure given by Ω_1, nor are they conjugate for Ω_2.) The time evolution of the minor variables is then determined by successively solving a hierarchy of two-dimensional forces linear Hamiltonian systems in the variables p_k, q_{n-k}, each of the form

$$\frac{\mathrm{d}p_k}{\mathrm{d}t}=-n!\left(\frac{\partial^2F_n}{\partial p_0\partial q_n}p_k+\frac{\partial^2F_n}{\partial q_n^2}q_{n-k}\right)-G_k(t)\,,\qquad \frac{\mathrm{d}q_{n-k}}{\mathrm{d}t}=n!\left(\frac{\partial^2F_n}{\partial p_0^2}p_k+\frac{\partial^2F_n}{\partial p_0\partial q_n}q_{n-k}\right)+\tilde{G}_k(t)\,, \tag{14}$$

where G_k, $\tilde{G}_k$ are certain explicit functions of $(p_0, \ldots, p_{k-1}, q_n, \ldots, q_{n-k+1})$, whose time evolution has thus already been determined. We conclude that any bi-Hamiltonian system for an irreducible, constant eigenvalue Hamiltonian pair can be integrated by solving a single two-dimensional autonomous Hamiltonian system, followed by a sequence of forced linear, nonautonomous two-dimensional Hamiltonian systems.

3. (II) Elementary, constant eigenvalue pairs

Theorem 7. Every compatible Hamiltonian pair with constant eigenvalue(s) is (locally) the Cartesian product of irreducible pairs.

We can therefore introduce local coordinates

$$\boldsymbol{x}=(\boldsymbol{p},\boldsymbol{q})=(\boldsymbol{p}^1,\ldots,\boldsymbol{p}^m,\boldsymbol{q}^1,\ldots,\boldsymbol{q}^m)\,,\quad \boldsymbol{p}^i=(p^i_0,\ldots,p^i_{n_i})\,,\quad \boldsymbol{q}^i=(q^i_0,\ldots,q^i_{n_i})\,, \tag{15}$$

such that $\Omega_\nu=\Omega^1_\nu+\ldots+\Omega^m_\nu$, where each $(2n_i+2)$-dimensional sub-pair Ω^i_1, Ω^i_2 is irreducible, compatible, taking the canonical form (7), (8) in its coordinates $(\boldsymbol{p}^i,\boldsymbol{q}^i)$. Restricting attention to the case of just one eigenvalue, assume that the sub-pairs are arranged in order of size: $n_1\geqslant n_2\geqslant\ldots\geqslant n_m$. For $k\geqslant 0$, let m_k denote the number of irreducible sub-pairs of dimension greater than $2k+1$, which, according to our ordering, will be the first m_k sub-pairs. In particular $m_0=m$.

Theorem 8. The most general bi-Hamiltonian system for a canonical elementary Hamiltonian pair with constant eigenvalue $\lambda=\mu^{-1}$ has the form (1) with

$$H_0(\boldsymbol{x})=H^0_0(\boldsymbol{x})+H^{(1)}_0(\boldsymbol{x})+\ldots+H^{(n_1)}_0\,,\quad H_1(\boldsymbol{x})=H^{(0)}_1(\boldsymbol{x})+H^{(1)}_1(\boldsymbol{x})+\ldots+H^{(n_1)}_1\,,$$

where

$$H^{(k)}_0(\boldsymbol{x})=\mu\frac{\partial^k}{\partial s^k}F_k(\pi^1(s),\varpi^1(s)\,,\ldots,\pi^{m_k}(s),\varpi^{m_k}(s))\Big|_{s=0}$$

$$+k\frac{\partial^{k-1}}{\partial s^{k-1}}F_k(\pi^1(s),\varpi^1(s),\ldots,\pi^{m_k}(s),\varpi^{m_k}(s))\Big|_{s=0}\,,$$

$$H^{(k)}_1(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}F_k(\pi_1(s),\varpi_1(s),\ldots,\pi_{m_k}(s),\varpi_{m_k}(s))\Big|_{s=0}\,. \tag{16}$$

Here $\pi^i(s)$, $\varpi^i(s)$ are the parametrized variables for the ith sub-pair, given by expressions (12), with $n=n_i$, and the additional index i on all the variables p^i_j, q^i_j.

As in the irreducible case, we find that these Hamiltonians are polynomials in the minor variables p^i_j, $q^i_{n_i-j}$, $j\geqslant 1$, whose coefficients are certain derivatives of arbitrary functions of the major variables p^i_0, $q^i_{n_i}$. Therefore, to solve such a bi-Hamiltonian system, we must integrate an autonomous m-dimensional Hamiltonian system in the major variables, followed by a sequence of linear non-autonomous Hamiltonian systems in the appropriate minor variables p^i_k, $q^i_{n_i-k}$, for all i with $n_i\geqslant k\geqslant 1$.

4. (III) Irreducible, nonconstant eigenvalue pairs

Theorem 9. A generic irreducible compatible Hamiltonian pair with nonconstant eigenvalue $\lambda=1/p_0$ can be put into the canonical form

$$\Omega_1=\sum_{i=0}^{n}\mathrm{d}p_i\wedge\mathrm{d}q_i\,,\qquad \Omega_2=\sum_{j+k=l}p_j\,\mathrm{d}p_k\wedge\mathrm{d}q_l\,. \tag{17}$$

The sum in Ω_2 is over all j, k, l from 0 to n satisfying $j+k=l$.

The coordinates employed here are different from those in ref. [6], although the transformation between the two is not hard. This particular bi-Hamiltonian structure has several remarkable properties. Define the $(n+1)\times(n+1)$ banded upper triangular matrix

$$P_n(\boldsymbol{p})=P(\boldsymbol{p})=\begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots & p_n \\ & p_0 & p_1 & p_2 & & p_{n-1} \\ & & p_0 & p_1 & & \vdots \\ & & & p_0 & & \\ & & & & \ddots & \\ & & & & & p_0 \end{pmatrix}. \tag{18}$$

Then the skew-symmetric matrix giving the symplectic two-form Ω_2 is

$$K_2(p)=\begin{pmatrix} 0 & P(p) \\ -P(p)^{\mathrm{T}} & 0 \end{pmatrix}.$$

The Hamiltonian matrix $J_2(p)=K_2(p)^{-1}$ is much more complicated, involving the inverse $R(\boldsymbol{p})=P(\boldsymbol{p})^{-1}$, which has the same banded upper triangular form as P with entries

$$r_0(p)=\frac{1}{p_0}, \qquad r_k(p)=\sum_{\substack{i_1+i_2+\ldots+i_\tau=k \\ i_\nu\geqslant 1}} (-1)^{\tau+1}\frac{p_{i_1}p_{i_2}\cdots p_{i_\tau}}{p_0^{\tau+1}}, \quad k\geqslant 1 . \tag{19}$$

However, the explicit change of coordinates

$$\tilde{\boldsymbol{p}}=\boldsymbol{r}(\boldsymbol{p}), \quad \tilde{\boldsymbol{q}}=P(\boldsymbol{p})^2\boldsymbol{q}, \tag{20}$$

is a canonical involution, which maps P to $P(\tilde{\boldsymbol{p}})=P(\boldsymbol{p})^{-1}$, and so changes the second Hamiltonian matrix into its inverse. Thus the pair has the alternative canonical form

$$J_1=\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad J_2=\begin{pmatrix} 0 & P(\tilde{\boldsymbol{p}}) \\ -P(\tilde{\boldsymbol{p}})^{\mathrm{T}} & 0 \end{pmatrix}, \tag{21}$$

with eigenvalue $\lambda=\tilde{p}_0$. Therefore, as in the constant eigenvalue case, we conclude that both $P(\boldsymbol{p})$ and its inverse determine isomorphic Hamiltonian matrices!

Theorem 10. Suppose $H_1(\boldsymbol{x})$, $H_0(\boldsymbol{x})$, $\boldsymbol{x}=(\boldsymbol{p},\boldsymbol{q})$, are analytic functions which define a bi-Hamiltonian system relative to the canonical irreducible nonconstant eigenvalue Hamiltonian pair (17) on a convex open subset. Then there exist scalar-valued functions $h(\xi)$, $\tilde{h}(\xi)$, $F_k(\xi,\eta)$, $k=0, ..., n-1$, such that

$$H_0(\boldsymbol{x})=\tilde{h}(p_1)+H_0^{(0)}(\boldsymbol{x})+\ldots+H_0^{(n-1)}(\boldsymbol{x}), \quad H_1(\boldsymbol{x})=h(p_1)+H_1^{(0)}(\boldsymbol{x})+\ldots+H_1^{(n-1)}(\boldsymbol{x}),$$

where $\tilde{h}'(\xi)=\xi h'(\xi)$, and

$$H_0^{(k)}(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}[\pi(s)\pi'(s)F_k(\pi(s),\varpi(s))]\bigg|_{s=0}, \qquad H_1^{(k)}(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}[\pi'(s)F_k(\pi(s),\varpi(s))]\bigg|_{s=0}. \tag{22}$$

Here $\pi(s)$, $\varpi(s)$ are given by (12), and $\pi'(s)$ is the derivative of π with respect to s.

Example 11. If $n=2$, the general bi-Hamiltonian system is given by a sum of the following three particular types of Hamiltonians:

$$H_0^{\{0)} = \tilde{h}(p_0)\,, \quad H_1^{\{0)} = h(p_0)\,, \quad \tilde{h}'(p_0) = p_0 h'(p_0)\,,$$

$$H_0^{\{1)} = p_0 p_1 F_0(p_0, q_2)\,, \quad H_1^{\{1)} = p_1 F_0(p_0, q_2)\,,$$

$$H_0^{\{2)} = (2p_0 p_2 + p_1^2) F_1(p_0, q_2) + p_0 p_1^2 \frac{\partial F_1}{\partial p_0} + p_0 p_1 q_1 \frac{\partial F_1}{\partial q_2}\,,$$

$$H_1^{\{2)} = 2p_2 F_1(p_0, q_2) + p_1^2 \frac{\partial F_1}{\partial p_0} + p_1 q_1 \frac{\partial F_1}{\partial q_2}\,.$$

In general, note that by theorem 2, the eigenvalue λ is a constant, hence p_0 is a first integral for such Hamiltonian systems. Once its value is fixed, the other minor variable, q_2, is found by solving a single autonomous ordinary differential equation. The minor variables satisfy a sequence of forced, linear planar Hamiltonian systems.

5. (IV) Elementary, nonconstant eigenvalue pairs

Theorem 12. Any generic elementary compatible Hamiltonian pair with nonconstant eigenvalue $\lambda = 1/p_0$ can be written in the canonical form

$$\Omega_1 = \mathrm{d}p_0 \wedge \mathrm{d}q_0 + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \mathrm{d}p_j^i \wedge \mathrm{d}q_j^i\,, \qquad \Omega_2 = p_0\, \mathrm{d}p_0 \wedge \mathrm{d}q_0 + \sum_{i=1}^{m} \sum_{\substack{j+k=l \\ j+k+l\neq 0}} p_j^i\, \mathrm{d}p_k^i \wedge \mathrm{d}q_l^i\,. \tag{23}$$

Beyond the eigenvalue coordinate p_0 and its canonical conjugate q_0, the coordinates come in conjugate sets $\boldsymbol{p}^i = (p_1^i, ..., p_{n_i}^i)$, $\boldsymbol{q}^i = (q_1^i, ..., q_{n_i}^i)$, $i = 1, ..., m$. We also set $p_0^i = p_0$ for all i by convention. The interior sum in Ω_2 is over all j, k, $l = 0, ..., n_i$ except the case $j = k = l = 0$, which already appears outside the double summation.

Note that this particular pair is algebraically reducible, but not reducible by canonical transformations. The corresponding symplectic matrix for Ω_2 is

$$K_2 = \begin{pmatrix} 0 & \mathbf{P}^*(\boldsymbol{p}) \\ -\mathbf{P}^*(\boldsymbol{p})^{\mathrm{T}} & 0 \end{pmatrix},$$

where

$$\mathbf{P}^*(\boldsymbol{p}) = \begin{pmatrix} p_0 & \boldsymbol{p}^1 & \boldsymbol{p}^2 & \cdots & \boldsymbol{p}^m \\ & P_{n_1}(\hat{\boldsymbol{p}}^1) & 0 & & 0 \\ & & P_{n_2}(\hat{\boldsymbol{p}}^2) & & \vdots \\ & & & \ddots & \\ & & & & P_{n_m}(\hat{\boldsymbol{p}}^m) \end{pmatrix}. \tag{24}$$

Here $\hat{\boldsymbol{p}}^m = (p_0, p_1^m, ..., p_{n_m}^m)$ (recall $p_0^m = p_0$) and the P_{n_m} are the corresponding upper triangular matrices of size $n_m \times n_m$ as given in (18). Note that the algebraically irreducible sub-blocks corresponding to just the variables $\hat{\boldsymbol{p}}^m$, $\hat{\boldsymbol{q}}^m$ are isomorphic to the irreducible pair (17), but these sub-blocks are all entangled since the eigenvalue $\lambda = 1/p_0$ is the same in all cases.

To describe the general form for any associated bi-Hamiltonian system, we let

$$\pi^i(s) = p_0 + s p_1^i + s^2 p_2^i + ... + s^{n_i} p_{n_i}^i\,,$$

$$\zeta^i(s) = p_1^i + s p_2^i + ... + s^{n_i - 1} p_{n_i}^i = s^{-1}[\pi^i(s) - p_0]\,,$$

$$\omega^i(s)=q^i_{n_i}+sq^i_{n_i-1}+\ldots+s^{n_i-1}q^i_1\,, \tag{25}$$

$$\mu^i(s)=\sum_{n=0}^{n_i-1}\frac{s^n[\zeta^1(s)]^{n+1}}{(n+1)!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\frac{1}{[\zeta^i(t)]^{n+1}}\bigg|_{t=0}\,,\quad i=2,\ldots,m\,,$$

$$\sigma^i(s)=q^i_{n_i}+\sum_{n=0}^{n_i-1}\frac{s^n[\zeta^1(s)]^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\frac{1}{[\zeta^i(t)]^n}\frac{\mathrm{d}\omega^i(t)}{\mathrm{d}t}\bigg|_{t=0}\,,\quad i=2,\ldots,m\,.$$

Remark. The expansions of μ^i and σ^i involve truncations of the remarkable nonlinear series differential operator

$$\mathscr{D}=\mathrm{D}^{-1}{:}\mathrm{e}^{s\mathrm{D}u}{:}\mathrm{D}=1+\sum_{n=1}^{\infty}\frac{s^n}{n!}\mathrm{D}^{n-1}u^n\mathrm{D}\,,\quad \mathrm{D}=\frac{\mathrm{d}}{\mathrm{d}t}\,, \tag{26}$$

where $u=u(t)$. The colons a *normal ordering* of the noncommuting operators D and u analogous to the Wick ordering in quantum mechanics. This operator has the surprising property that it commutes with *any* analytic function $\Phi(u)$:

$$\mathscr{D}\Phi(u)=\Phi(\mathscr{D}u)\,.$$

See ref. [8] for details and applications of this operator to deriving new derivative identities.

Theorem 13. The most general bi-Hamiltonian system for an elementary, nonconstant eigenvalue generic Hamiltonian pair (24) is given by

$$H_0(\boldsymbol{x})=\tilde{h}(p_0)+H_0^{0}(\boldsymbol{x})+\ldots+H_0^{(n_1-1)}(\boldsymbol{x})\,,\quad H_1(\boldsymbol{x})=h(p_0)+H_1^{(0)}(\boldsymbol{x})+\ldots+H_1^{(n_1-1)}(\boldsymbol{x})\,,$$

where

$$H_0^{(k)}(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}\left(s\zeta^1(s)\frac{\mathrm{d}\pi^1}{\mathrm{d}s}F_k(\pi^1(s),\mu^2(s),\ldots,\mu^{m_k}(s),\omega^1(s),\sigma^2(s),\ldots,\sigma^{m_k}(s))\right)\bigg|_{s=0}\,,$$

$$H_1^{(k)}(\boldsymbol{x})=\frac{\partial^k}{\partial s^k}\left(\frac{\mathrm{d}\pi^1}{\mathrm{d}s}F_k(\pi^1(s),\mu^2(s),\ldots,\mu^{m_k}(s),\omega^1(s),\sigma^2(s),\ldots,\sigma^{m_k}(s))\right)\bigg|_{s=0}\,, \tag{27}$$

where $h(\xi)$, $\tilde{h}(\xi)$, $F_k(\xi_1,\ldots,\xi_{m_k},\eta_1,\ldots,\eta_{m_k})$, $k=0,\ldots,n-2$, are scalar-valued functions, $\tilde{h}'(\xi)=\xi h'(\xi)$. We assume as before that the blocks are arranged in decreasing order $n_1\geqslant n_2\geqslant\ldots\geqslant n_m$, and m_k denotes the number of blocks of size $n_i\geqslant k$.

Example 14. Consider the case of two sub-blocks of size $n_1=n_2=2$, with coordinates $(p_0, p_1, p_2, p'_1, p'_2, q_0, q_1, q_2, q'_1, q'_2)$. The important variables for (27) are

$$\pi^1(s)=p_0+sp_1+s^2p_2\,,\quad \omega^1(s)=q_2+sq_1+s^2q_0\,,$$

$$\mu(s)=\frac{p_1}{p'_1}+s\left(\frac{p_2}{p'_1}-\frac{p_1^2p'_2}{p_1'^3}\right),\quad \sigma(s)=q'_2+s\frac{p_1q'_1}{p'_1}\,.$$

The general pair of Hamiltonians is a sum of the following two:

$$H_0^{(0)}=p_0p_1F_0(p_0,r,q_2,q'_2)\,,\quad H_0^{(1)}=p_1F_0(p_0,r,q_2,q'_2)\,,$$

$$H_0^{(2)}=p_0H_1^{(2)}+p_1^2F_1(p_0,r,q_2,q'_2)\,,$$

$$H_1^{(2)}=2p_2F_1(p_0,r,q_2,q'_2)+p_1^2\frac{\partial F_1}{\partial p_0}+p_1\left(\frac{p_2}{p'_1}-\frac{p_1^2p'_2}{p_1'^3}\right)\frac{\partial F_1}{\partial r}+p_1q_2\frac{\partial F_1}{\partial q_2}+\frac{p_1^2q'_1}{p'_1}\frac{\partial F_1}{\partial q'_2}\,,$$

where F_0, F_1 are arbitrary smooth functions of the variables p_0, $r=p_1/p'_1$, q_2, q'_2. Note that the corresponding bi-Hamiltonian system reduces to a system of three equations for the two top order positions q_2, q'_2, and the homogeneous momentum coordinate r; the remaining variables can be then determined by quadratures and solving linear systems.

In general, as in the example, such bi-Hamiltonian systems reduce to the integration of a $(2m-2)$-dimensional Hamiltonian system for the coordinates p_1^i, $q_{n_i}^i$, $i=1, ..., m$, followed by a sequence of forced linear Hamiltonian systems. The final coordinate q_0 is determined by quadrature. Actually, the initial Hamiltonian system can be reduced in order to $2m-3$ since it only involves the homogeneous ratios of momenta $r^i=p_1^i/p_1^1$, $i\geqslant 2$, as can be seen from the formula for μ^i.

6. (V) The general case

Theorem 15. Every generic nondegenerate, compatible Hamiltonian pair can be locally expressed as a Cartesian product of elementary (constant and nonconstant eigenvalue) Hamiltonian pairs. Every associated bi-Hamiltonian system decomposes into independent subsystems corresponding to each elementary subpair. Therefore the solution of the bi-Hamiltonian system reduces to a collection of autonomous Hamiltonian systems whose dimensions are determined by the number of irreducible sub-pairs in each distinct elementary pair, along with a sequence of linear, nonautonomous Hamiltonian systems.

As demonstrated by Turiel [6,7], all these results have real counterparts, obtained by taking real and imaginary parts of the complex pairs corresponding to complex eigenvalues. In particular, as in ref. [3], complex eigenvalue bi-Hamiltonian systems are necessarily analytic Hamiltonians. We note that theorem 15 gives a refined version of the integrability results appearing in refs. [9,10]. Moreover, this result can be applied to give a more complete version of the result in ref. [11], which demonstrates that the existence of enough bi-Hamiltonian vector fields implies that the two Hamiltonian structures must be constant multiples of each other.

This completes out classification of compatible nondegenerate bi-Hamiltonian systems. A significant open problem which has not been addressed is to extend these results to degenerate compatible Hamiltonian pairs.

7. Hamiltonian pairs in four dimensions

In 1946, Debever [12] used Cartan's equivalence method, cf. ref. [13], to classify pairs of symplectic two-forms on a four-dimensional complex manifold up to local (analytic) diffeomorphism. Debever does not impose any compatibility condition (which was not known at the time), but does impose an algebraic restriction, cf. (28) below; consequently the classification of Hamiltonian pairs in four dimensions is not complete. Nevertheless, Debever's result does produce new and interesting explicit examples of incompatible Hamiltonian pairs, and is worth reviewing in the context of Magri's theorem.

Theorem 16. Let Ω_1, Ω_2 be analytic, symplectic two-forms on a four-dimensional complex manifold M satisfying the algebraic condition

$$\Omega_1 \wedge \Omega_2 = 0 . \qquad (28)$$

Then there exist local coordinates (p_1, p_2, q_1, q_2) such that

$$\Omega_1 = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 , \qquad (29)$$

and, up to constant multiple, Ω_2 is equivalent to one of the three canonical forms

$$\Omega_2^{(1)} = dp_1 \wedge dq_1 - dp_2 \wedge dq_2 ,$$

$$\Omega_2^{(2)} = e^{p_1}(dp_1 \wedge dq_1 - dp_2 \wedge dq_2 - p_2 dp_1 \wedge dq_2) ,$$

$$\Omega_2^{(3)} = e^{p_1+p_2}(dp_1 \wedge dq_1 - dp_2 \wedge dq_2 + (q_1+q_2) dp_1 \wedge dp_2) . \quad (30)$$

Moreover, the canonical symplectic two-form Ω_1 is compatible with the symplectic two-form $\Omega_2^{(1)}$, but is *not* compatible with either $\Omega_2^{(2)}$ or $\Omega_2^{(3)}$.

Determining the general structure and integrability of bi-Hamiltonian systems associated with these Hamiltonian pairs is not difficult. In the compatible case 1, the pair has two distinct constant eigenvalues, $\pm\lambda$. According to theorem 15, any bi-Hamiltonian system for this pair decouples into a pair of autonomous planar Hamiltonian systems, and is thereby integrable, as can also be verified directly.

In case 2, the general bi-Hamiltonian system has Hamiltonians of the general form

$$H_1 = F(p_2 e^{p_1/2}, q_2) e^{-p_1/2} + g(p_1) , \quad H_0 = -F(p_2 e^{p_1/2}, q_2) e^{p_1/2} + \hat{g}(p_1) , \quad (31)$$

where $\hat{g}'(s) = e^{-s} g'(s)$. The corresponding bi-Hamiltonian system is

$$\frac{dp_1}{dt} = 0 , \quad \frac{dq_1}{dt} = \tfrac{1}{2} F(p_2 e^{p_1/2}, q_2) e^{-p_1/2} - \tfrac{1}{2} p_2 F_1(p_2 e^{p_1/2}, q_2) + g'(p_1) ,$$

$$\frac{dp_2}{dt} = -F_2(p_2 e^{p_1/2}, q_2) e^{-p_1/2} , \quad \frac{dq_2}{dt} = F_1(p_2 e^{p_1/2}, q_2) , \quad (32)$$

the subscripts on F indicating partial derivatives. Once the constant value of p_1 is fixed, the integration of this system reduces to solving a single autonomous planar Hamiltonian system for p_2, q_2, with Hamiltonian function $f(p_2 e^{p_1/2}, q_2) e^{-p_1/2}$. The remaining function q_1 can then be determined by a single quadrature.

In case 3, the Hamiltonians have the general form

$$H_0 = c(q_1 - q_2) + \tilde{f}(p_1, p_2) , \quad H_1 = c(q_1 - q_2) + f(p_1, p_2) ,$$

where c is a constant, and where

$$\frac{\partial \tilde{f}}{\partial p_1} = e^{p_1+p_2} \frac{\partial f}{\partial p_1} , \quad \frac{\partial \tilde{f}}{\partial p_2} = -e^{p_1+p_2} \frac{\partial f}{\partial p_2} . \quad (33)$$

Note that the integrability condition for (33) implies that f satisfies the partial differential equation $2f_{12} + f_1 + f_2 = 0$. The associated bi-Hamiltonian system is

$$\frac{dp_1}{dt} = -c , \quad \frac{dp_2}{dt} = c , \quad \frac{dq_1}{dt} = \frac{\partial \tilde{f}}{\partial p_1} , \quad \frac{dq_2}{dt} = \frac{\partial \tilde{f}}{\partial p_2} , \quad (34)$$

whose integration is trivial, reducing to just two quadratures.

Thus, by direct analysis, we are led to a strengthened version of Magri's theorem for this particular case: *Any* four-dimensional bi-Hamiltonian system satisfying the algebraic condition (28), compatible or not, is completely integrable. Indeed, the compatible bi-Hamiltonian structure leads to systems which reduce to the two decoupled planar autonomous Hamiltonian systems, whereas for the incompatible pairs given by $\Omega_2^{(2)}$ and $\Omega_2^{(3)}$, the systems reduce to one planar Hamiltonian system and one quadrature, or just two quadratures, respectively. It would be extremely interesting to extend Debever's classification to all possible nondegenerate pairs of two-forms on $\mathbb{C}^4$; this would go a long way to elucidating the precise role of the compatibility condition in the integrability of bi-Hamiltonian systems [#1]. Indeed, based on the evidence so far (including results on

[#1] This problem also has significant applications in Anderson's recent work on the Jesse Douglas inverse problem for second order systems [14].

quasi-linear hyperbolic systems [15]) it would appear that incompatible bi-Hamiltonian systems are, in a sense, even more integrable than compatible ones!

The proofs and further details on these results will appear in an expanded version to be published elsewhere.

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