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Equivalence Problems for First Order Lagrangians on the Line

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Complete solutions to and applications of the equivalence problems for first order particle Lagrangians under the pseudo-groups of contact, point, and fiber-preserving transformations, both with and without the addition of divergence terms, are presented. © 1989 Academic Press, Inc

1. INTRODUCTION

A basic problem in the calculus of variations is to recognize when two variational problems are actually manifestations of the same problem, but expressed in different coordinate systems. The solution of such equivalence problems is potent tool in the analysis and simplification of complicated variational problems, and it is essential if one is to attempt to solve the more difficult problem of determination of canonical forms for Lagrangians. Applications to particle dynamics, elasticity, symmetry groups and conservation laws, and classical invariant theory are but a few of the benefits of such a solution. Elie Cartan (cf. [6]) developed a powerful construction for completely resolving equivalence problems which can be recast into the framework of exterior differential systems. In this paper, we apply the Cartan method to study the case of a first order Lagrangian on the line. Although the simplest of the possible equivalence problems arising in the calculus of variations, nevertheless this problem already embodies many of

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the complications inherent in the application of Cartan's equivalence method, and it serves as a good testing ground for more complicated variational problems of interest in field theory and elasticity. It also serves as an excellent prototype equivalence problem, which can be analyzed in detail. Many subtle phenomena associated with the equivalence method, which are not commonly acknowledged in the literature, already manifest themselves in one or more of the versions of the Lagrangian equivalence problem. Furthermore, a new application of this problem to classical invariant theory has recently appeared [20, 21].

We begin by looking at the possible equivalence problems associated with a general variational problem

$$\mathscr{L}[u] = \int_{\Omega} L(x, u^{(n)})\, dx. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^p$, and the Lagrangian L is a smooth function of the independent variables $x \in \Omega$, the dependent variables $u \in \mathbb{R}^q$, and their derivatives up to some order n, denoted $u^{(n)}$. This paper will deal exclusively with the first order case of particle Lagrangians, $p = q = n = 1$, but it is still useful to present the equivalence problems in full generality. There are at least six different versions of the notion of "equivalence of variational problems." First, there are three possible choices of coordinate changes or pseudogroups which can be used to transform the problem. These are:

(1) The *fiber-preserving transformations*, in which the new independent variables depend only on the old independent variables, so that the transformations have the form

$$\bar{x} = \varphi(x), \qquad \bar{u} = \psi(x, u).$$

(2) The general *point transformations*, in which an arbitrary change of independent and dependent variables is allowed, and the transformations have the form

$$\bar{x} = \varphi(x, u), \qquad \bar{u} = \psi(x, u).$$

(3) *Contact transformations*, where the transformations also depend on first order derivatives

$$\bar{x} = \varphi(x, u^{(1)}), \qquad \bar{u} = \psi(x, u^{(1)})$$

in such a way that the ideal generated by the contact forms

$$\omega^\alpha = du^\alpha - \sum p_i^\alpha\, dx^i, \qquad \alpha = 1, \ldots, q,$$

is preserved. Of course, by Bäcklund's Theorem [1] these generalize case 2 only in the case of one dependent variable.

The other choice to be made is how one wishes to view the equivalence of the Lagrangians themselves. In our treatment of the problem, it helps to take the differential-geometric viewpoint that the variational integral (1.1) is an *oriented integral*, which means that the integrand is a p-form $L\,dx$, not just the function L. In other words, under an orientation-reversing transformation on the base Ω, the Lagrangian L will change sign. (It is not hard to extend our results to the unoriented case—it just involves fiddling with $\pm$ signs.) With this agreed, there are two choices for deciding when two Lagrangians are equivalent:

(a) The first is to require the two variational problems to agree on all possible functions $u = f(x)$. This implies that the two Lagrangians are related by the change of variables formula for p-forms:

$$\bar{L}(\bar{x}, \bar{u}^{(n)}) = L(x, u^{(n)}) \frac{1}{\det J}, \tag{1.2}$$

where J is the Jacobian matrix of total derivatives $D_i \varphi^j$. We shall call this the *standard equivalence problem.* (In the unoriented case, we would use the change of variables formula for multiple integrals, so there would be an absolute value on the factor $\det J$.)

(b) For the *divergence equivalence problem*, one only requires that the variational problems agree on extremals, or, equivalently, that the two sets of Euler–Lagrange exressions are mapped into each other by the change of variables. A standard result [18] says that two Lagrangians have the same Euler–Lagrange equations if and only if they differ by a divergence. Thus, L and $\bar{L}$ are divergence-equivalent Lagrangians if they are related by the formula

$$\bar{L}(\bar{x}, \bar{u}^{(n)}) = \frac{L(x, u^{(n)}) + \operatorname{Div} F}{\det J}, \tag{1.3}$$

by the prescribed change of variables. Here J is as before, $F(x, u^{(m)})$ is an arbitrary p-tuple of functions of x, u, and derivatives of u, and Div denotes the total divergence.

The contact transformations occupy a somewhat anomalous position. They are relevant only in the case of one dependent variable. Moreover, in the case to be discussed here, first order one independent variable, it is not hard to see that (a) any two Lagrangians are divergence equivalent under a contact transformation, while (b) standard equivalence of Lagrangians under contact transformations reduces to equivalence under point transformations (cf. [2]). Thus we can eliminate contact transformations from consideration without any appreciable loss of generality. Thus, there are four distinct equivalence problems for Lagrangians of current interest. To distinguish them concisely, we make the following definition:

DEFINITION 1.1. Let m be an integer from 1 to 4. Two Lagrangians L and $\bar{L}$ are said to be *m-equivalent* if they are mapped into each other by

$m=1$, a fiber-preserving transformation,

$m=2$, a point transformation,

$m=3$, a fiber-preserving transformation up to a divergence,

$m=4$, a point transformation up to a divergence.

Some of these are subclasses of others; for example, if two Lagrangians are equivalent under fiber-preserving transformations, they are certainly equivalent under point transformations, and hence also divergence equivalent. The basic relations are illustrated in Fig. 1, where the arrows point from the easier equivalence problem, which is a subcase of the more complicated equivalence problem.

Some work has already been done on special versions of the Lagrangian equivalence problem. Cartan [8], in the course of solving a more general equivalence problem, completely solved the case of first order Lagrangians on the line, $n=p=q=1$, under point transformations. In a subsequent paper [7], he also treats the case of second order Lagrangians, $n=2$, $p=q=1$, under contact transformations. Gardner [10] sets up the general first order particle Lagrangian equivalence problem under point transformations, $n=p=1$, $q \geqslant 1$. The intrinsic solution has been effected by Bryant

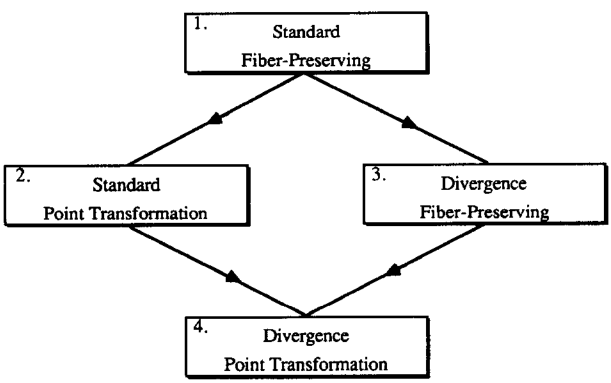

FIGURE 1

and Gardner [11]. Gardner and Shadwick [12] have considered the case of first order Lagrangians in the plane, $p=2$, $n=q=1$, under point transformations. Shadwick [22] sets up the fiber-preserving equivalence problem, $p=q=n=1$, as an overdetermined equivalence problem, finds the adapted coframe, and the simplest of the three invariants. Bryant [2] formulates the standard fiber-preserving, standard point transformation, and contact transformation with divergence equivalence problems (which he names diffeomorphism equivalence, simple equivalence, and divergence equivalence, respectively) for Lagrangians on the line, $p=1$, $n \geqslant 1$, $q \geqslant 1$. Bryant's approach to the divergence equivalence problem differs from that adopted here and has the advantage of allowing the Lagrangian to have a zero locus; however, he treats only the case of contact transformations, which constitutes a trivial equivalence problem for first order Lagrangians on the line $(q=1)$ since all such Lagrangians are divergence equivalent under a contact transformation, and his method does not easily generalize to several dimensions. Indeed, it is a nontrivial problem to cast the general divergence equivalence problem into a form amenable to the Cartan algorithm; in the case $n=1$, $p=q=2$ or 3, a solution to this equivalence problem would have significant applications to the study of nonlinear elasticity (cf. [19]).

As stressed by Gardner [10], there are two principal approaches to using the Cartan method to solve a given equivalence problem. The first is the *parametric method* in which the calculation of exterior derivatives, absorption of torsion, group normalization, etc., are all done explicitly. The advantages of this method are the following: (i) The explicit, albeit complicated, expressions for the invariants appear directly at the end of the computation. (ii) The possible branches are determined explicitly at each step of the procedure. The great disadvantages of the parametric approach are twofold: (i) The Lie group must be parametrized explicitly, something one can achieve globally only in the simplest cases, and even then the explicit parametrization might not be amenable to relatively simple symbolic manipulation. (ii) The computations are *extremely* long, and the intermediate expressions very complicated, as illustrated by the fiber-preserving equivalence problem, the apparently simplest case in this paper. However, with the advent of powerful symbolic manipulation programs, many of these purely computational difficulties can be done automatically, eliminating much of the drudgery from the method. Indeed, all the computations in this paper were performed symbolically on two different programs, one written in MAPLE on the University of Waterloo VAX 785 computer, and the second in SMP on an Apollo workstation at the University of Minnesota. The answers were compared for accuracy and also, to the extent possible, checked by hand. One should however emphasize that the algebraic tools, such as classical invariant theory, needed to identify and discuss the various cases which arise in the normalization of torsion when the dimension of the underlying manifold is large have not yet been incorporated in any symbolic manipulation program. Also, the computations can even grow too complicated for present-day symbolic manipulation programs to handle.

The second approach to the equivalence method, inspired by the theory of principal components in Cartan's method of moving frames [4], is to work *intrinsically* and use the closure of the exterior derivative operation to determine the group action which will normalize the unabsorbable torsion coefficients. The advantages of this method are the following: (i) It is relatively easy to implement by hand and leads as quickly as possible to the appropriate structure equations for the problem. (ii) It requires only a knowledge of the (linear) defining relations of the Lie algebra of the structure group, as opposed to an explicit parametrization of the group itself. The principal disadvantages are that (i) it does not give the explicit formulas for the invariants, which must be recomputed parametrically, and (ii) occasionally, the explicit forms of certain torsion coefficients must be determined parametrically in order to rule out inapplicable branching. Nevertheless, we find that the intrinsic method is a very powerful tool to unravel the full structure of some very hard problems. (A good example is the equivalence problem for an overdetermined system of two second order partial differential equations in two independent and one dependent variable which is in involution (cf. Cartan [5].))

In this paper, the intrinsic method serves as an extremely useful guide to map out the more complicated parametric approach and to thereby maintain accuracy in the ensuing parametric calculations. Actually, in the simplest case ($m=1$), we indicate how parametric expressions for invariants can sometimes be directly determined from primarily intrinsic calculations. Our point of view, then, is that *both* parametric and intrinsic approaches are useful for the solution of all but the most elementary equivalence problems. We therefore indicate both in our outline of the solutions to the Lagrangian equivalence problems.

The paper is organized as follows: Section 2 is devoted to setting up the various equivalence problems for first order Lagrangians on the line. As required by Cartan's algorithm, each problem is recast in the language of differential forms. In Section 3, we provide a detailed solution to the simplest of our four equivalence problems—standard equivalence under fiber-preserving transformations. The presentation includes a brief review of how the invariants and derived invariants for an equivalence problem provide necessary and sufficient conditions for equivalence. Section 4 introduces a new "inductive" approach to the solution of equivalence problems where the structure group contains a subgroup whose associated equivalence problem has a simpler solution, which can be used to "induce" a solution to the more complicated problem, thereby providing explicit formulae expressing the invariants for the more complicated problem in terms of those of the simpler problem. This procedure is illustrated in Section 5 by solving the remaining three versions of the Lagrangian equivalence problem and thereby determining how they are related. Section 6 is devoted to applications of our solutions to some variational problems of physical interest. In the final section, we comment on the connections between the Lagrangian equivalence problems and the Tresse equivalence problem for second order ordinary differential equations, although we do not attempt the complete solution of the latter problem using the induction approach.

2. Equivalence of First Order Lagrangians on the Line

We begin by specializing our initial considerations to the case of first order particle Lagrangians on the line. Consider a variational problem

$$\mathscr{L}[u] = \int_a^b L(x, u, p)\, dx, \tag{2.1}$$

where the Lagrangian L is an analytic, real-valued function defined on an open subdomain Ω of the first jet space $J^1 = J^1(\mathbb{R}, \mathbb{R})$, with coordinates x, u, and p. The associated Euler–Lagrange equation is a second order ordinary differential equation

$$\frac{\partial^2 L}{\partial p^2}\frac{d^2 u}{dx^2} = \frac{\partial L}{\partial u} - \left(\frac{\partial^2 L}{\partial x\, \partial p} + \frac{du}{dx}\frac{\partial^2 L}{\partial u\, \partial p}\right), \tag{2.2}$$

where the derivatives of the Lagrangian L are all evaluated at $x, u, u' = du/dx$. If L is an affine function of p, i.e., $L = a(x, u)p + b(x, u)$, then the Euler–Lagrange equation (2.2) reduces to an algebraic equation for x, u. We will disregard this case throughout the discussion as (a) it is trivial from the point of view of the calculus of variations and (b) it leads to infinite pseudo-groups in the equivalence procedure and must be treated separately throughout the discussion. Otherwise, the points where the second derivative L_{pp} vanishes are singular points for the ordinary differential equation (2.2). By shrinking the domain Ω, we can assume that the Lagrangian satisfies the condition $L_{pp} \neq 0$ for $(x, u, p) \in \Omega$. This assumption will be maintained throughout the discussion.

To conserve space, we introduce some notation for important differential operators arising in our problem. Let

$$\tilde{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} \tag{2.3}$$

denote the J^1-component of the total derivative, and

$$\tilde{E}(L) = \frac{\partial L}{\partial u} - \tilde{D}_x \frac{\partial L}{\partial p} \tag{2.4}$$

the J^1-component of the Euler operator or variational derivative. Avoiding singular points where $L_{pp} = 0$, we can write the Euler–Lagrange equation (2.2) in the concise form

$$u'' = Q(x, u, u'), \qquad \text{where} \quad Q = \frac{\tilde{E}(L)}{L_{pp}}. \tag{2.5}$$

Finally, we let

$$\hat{D}_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial u} + Q(x, u, p)\frac{\partial}{\partial p} \tag{2.6}$$

denote the total derivative operator determined by the solutions to (2.5).

The *contact form* on the first jet space J^1 is the one-form

$$\omega_1 = du - p\, dx.$$

A general diffeomorphism $\Phi: J^1 \to J^1$, which is given by

$$\bar{x} = \varphi(x, u, p), \qquad \bar{u} = \psi(x, u, p), \qquad \bar{p} = \chi(x, u, p), \tag{2.7}$$

determines a contact transformation if and only if it preserves the contact form up to multiple

$$\Phi^*(\omega_1) = \lambda\omega_1, \tag{2.8}$$

where λ is a function of x, u, p.

Next consider how the general diffeomorphism (2.7) acts on the variational integral (2.1). In this case, the basic invariance condition (1.3) becomes

$$\Phi^*(\bar{L}) = \frac{L + \tilde{D}_x f}{\tilde{D}_x \varphi}. \tag{2.9}$$

Now L and $\bar{L}$ can depend only on first order derivatives. If we are allowed arbitrary contact transformations (2.7), and the divergence term f can depend on x, u, p, then it is easily seen that any Lagrangian L can be mapped to any other Lagrangian $\bar{L}$, so the equivalence problem is trivial. However, if we require the divergence f to depend on just x and u, then (2.9) and the restriction to first order Lagrangians imply that φ can depend only on x and u, and so the diffeomorphism (2.7) comes from a point transformation; conversely, if we start by requiring the diffeomorphism to come from a point transformation, then, for the same reason, f can depend only on x and u. In other words, except in the trivial case when everything is equivalent, equivalence under contact transformations automatically reduces to equivalence under point transformations, with the divergence term f depending only on x and u.

Using the elementary identity

$$\Phi^*(d\bar{x}) = \varphi_x\, dx + \varphi_u\, du = \tilde{D}_x \varphi\, dx + \varphi_u \omega_1$$

(cf. (2.3)), we see that the equivalence condition (2.9) can be recast in the form

$$\Phi^*(\bar{L}\, d\bar{x}) = \{L + \tilde{D}_x f\}\, dx + \tilde{\mu}\, \omega_1,$$

where $\tilde{\mu}$ is a function on J^1, whose precise form is unimportant. Furthermore, since

$$df = f_x\, dx + f_u\, du = \tilde{D}_x f\, dx + f_u \omega_1,$$

we make the important deduction that two Lagrangians L and $\bar{L}$ will be divergence equivalent under a point transformation (i.e., 4-equivalent in the terminology of definition 1.1) if and only if there is a diffeomorphism $\Phi: J^1 \to J^1$ such that the contact condition (2.8) is maintained, and

$$\Phi^*(\bar{L}\, d\bar{x}) = L\, dx + \mu\, \omega_1 + df, \tag{2.10}$$

where $f(x, u)$ is a real-valued function on J^0, and $\mu(x, u, p)$ a real-valued function on J^1. In the standard equivalence problem, we have the same equivalence conditions (2.8), (2.10), but where the divergence component f is set to 0.

The equivalence condition (2.10) and the condition (2.8) that Φ define a contact transformation define a proper sub-pseudo-group of the pseudo-group of contact transformations of J^1. In order to apply Cartan's method of equivalence to this problem, we need to reformulate these defining relations in terms of invariance conditions on a coframe or basis for the cotangent space T^*X, where X is the appropriate "base space" for the problem [6, 10]. In our case, the first element of the coframe will be the contact form, the second being the integrand $\omega_2 = L\, dx$, provided we restrict to the domain where the Lagrangian itself does not vanish, i.e., $L \neq 0$. (Note that for the divergence equivalence problem, we can always arrange this on bounded domains by adding in a suitable divergence term.) We complete these two one-forms to a coframe by including $\omega_3 = dp$. In the standard equivalence problem, $\{\omega_1, \omega_2, \omega_3\}$ will constitute the required coframe on the base $X = J^1$.

More generally, to incorporate the extra divergence term f, we need to introduce an auxiliary real $w \in \mathbb{R}$, whose transformation rule will be

$$\bar{w} = w + f(x, u). \tag{2.11}$$

In other words, we let the base space be $X = J^1 \times \mathbb{R}$, with coordinates (x, u, p, w), and let $\tilde{\Phi}: J^1 \times \mathbb{R} \to J^1 \times \mathbb{R}$, as determined by (2.7), (2.11), denote the *lift* of $\Phi: J^1 \to J^1$. The action of $\tilde{\Phi}$ on the differential dw is given by

$$\tilde{\Phi}^*(d\bar{w}) = dw + df = dw + \tilde{D}_x f\, dx + f_u \omega_1. \tag{2.12}$$

Therefore, provided the Lagrangian L does not vanish on the domain Ω, the basic coframe on $\Omega \times \mathbb{R} \subset J^1 \times \mathbb{R}$ will be given by the one-forms

$$\omega_1 = du - p\, dx, \qquad \omega_2 = L\, dx, \qquad \omega_3 = dp, \qquad \omega_4 = dw. \tag{2.13}$$

To encode the various equivalences, we require appropriate structure groups. To this end, we introduce certain Lie subgroups of $GL(3, \mathbb{R})$ or $GL(4, \mathbb{R})$:

$$G_1 = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 1 & 0 \\ a_4 & a_5 & a_6 \end{pmatrix} : a_i \in \mathbb{R},\ a_1 \cdot a_6 \neq 0 \right\},$$

$$G_2 = \left\{ \begin{pmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_4 & b_5 & b_6 \end{pmatrix} : b_i \in \mathbb{R},\ b_1 \cdot b_6 \neq 0 \right\},$$

$$G_3 = \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_3 & 0 & 0 \\ c_4 & c_5 & c_6 & 0 \\ c_7 & c_3 - 1 & 0 & 1 \end{pmatrix} : c_i \in \mathbb{R},\ c_1 \cdot c_3 \cdot c_6 \neq 0 \right\},$$

$$G_4 = \left\{ \begin{pmatrix} d_1 & 0 & 0 & 0 \\ d_2 & d_3 & 0 & 0 \\ d_4 & d_5 & d_6 & 0 \\ d_7 & d_3 - 1 & 0 & 1 \end{pmatrix} : d_i \in \mathbb{R},\ d_1 \cdot d_3 \cdot d_6 \neq 0 \right\}.$$

Since the standard equivalence problem can be embedded in the divergence equivalence problem by setting the divergence term to zero (and hence the extra variable w will experience no change), we will often have cause to view $GL(3, \mathbb{R})$ as a subgroup $GL(4, \mathbb{R})$ according to the embedding

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \qquad A \in GL(3, \mathbb{R}).$$

This will be used in the sequel without further comment. Under it, some of the Lie groups G_m can be viewed as subgroups of others, according to the diagram

$$\begin{array}{ccccc} & & G_2 & & \\ & \rotatebox{45}{$\subset$} & & \rotatebox{-45}{$\supset$} & \\ G_1 & & & & G_4 \\ & \rotatebox{-45}{$\subset$} & & \rotatebox{45}{$\supset$} & \\ & & G_3 & & \end{array}$$

The next theorem shows how this diagram of subgroup structure reflects the relationships among the various equivalence problems expressed in Fig. 1.

PROPOSITION 2.1. *Let $1 \leqslant m \leqslant 4$ be an integer. Two Lagrangians L and $\bar{L}$ are m-equivalent if and only if there is a diffeomorphism $\tilde{\Phi}: J^1 \times \mathbb{R} \to J^1 \times \mathbb{R}$ which satisfies*

$$\tilde{\Phi}^*(\bar{\omega}_i) = \sum_{j=1}^{4} g_{ij}\omega_j, \qquad i = 1, ..., 4, \tag{2.14}$$

where $g = (g_{ij})$ is a G_m-valued function on $J^1 \times \mathbb{R}$. (For the two cases of standard equivalence, i.e., $m = 1$ or 2, the one-form ω_4 can be omitted, and the sum in (2.14) *only runs from* 1 *to* 3.)

Proof. We just do the most complicated version, $m = 4$, leaving the other three to the reader. Suppose the two Lagrangians L and $\bar{L}$ are divergence equivalent under a point transformation $\varphi: J^0 \to J^0$. Let $\Phi: J^1 \to J^1$ be the first prolongation of φ, and let $\tilde{\Phi}: J^1 \times \mathbb{R} \to J^1 \times \mathbb{R}$ be defined so that $\tilde{\Phi}|_{J^1} = \Phi$, and

$$\bar{w} \circ \tilde{\Phi} = w + f(x, u). \tag{2.15}$$

A straightforward calculation shows that Eqs. (2.8), (2.10), and (2.12) imply equation (2.14) holds for some (base-dependent) group element $g = g(x, u, p) \in G_4$.

To prove the converse statement, note first that since G_4 consists of lower triangular matrices, it immediately follows that there exists a diffeomorphism $\Phi: J^1 \to J^1$ such that $\tilde{\Phi}|_{J^1} = \Phi$. If we examine the last row of the matrix determined by the group element g, we see that there exists a function $f: J^0 \to \mathbb{R}$ such that (2.15) holds, and hence the third group parameter must be given by $d_3 = 1 + \tilde{D}_x f/L$. Now the condition

$$\Phi^*(\bar{L}\, d\bar{x}) = d_3 L\, dx + d_2 \omega_1$$

is precisely the equivalence condition (2.10). This proves the proposition.

To apply Cartan's algorithm for each equivalence problem defined by the coframe $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ on $J^1 \times \mathbb{R}$ and one of the groups G_m, $m = 1, ..., 4$, we must "lift" the coframe to the space $J^1 \times \mathbb{R} \times G_m$. Each of these *lifted coframes* takes the form

$$\theta_i = \sum_{j=1}^{4} g_{ij} \omega_j, \qquad i = 1, ..., 4, \tag{2.16}$$

where $g = (g_{ij}) \in G_m$. In the sequel, we will often abbreviate the relation (2.16) simply as $\theta = g \cdot \omega$. The equivalence condition (2.14) can then be stated symmetrically in terms of the lifted coframes:

PROPOSITION 2.2. *Let L and $\bar{L}$ be two Lagrangians. Then L and $\bar{L}$ are m-equivalent if and only if there is a diffeomorphism*

$$\Psi: J^1 \times \mathbb{R} \times G_m \to J^1 \times \mathbb{R} \times G_m,$$

which commutes with the natural left action of G_m on $J^1 \times \mathbb{R} \times G_m$, and maps the appropriate lifted coframe elements to each other:

$$\Psi^*(\bar{\theta}_i) = \theta_i, \qquad i = 1, ..., 4. \tag{2.17}$$

The proof is straightforward.

3. THE STANDARD FIBER-PRESERVING EQUIVALENCE PROBLEM

We begin with a detailed discussion of the most elementary of the four problems, the equivalence under fiber-preserving transformations without any divergence term, i.e., the case $m = 1$. To keep our presentation as short as possible, we will assume that the reader has a basic familiarity with the mechanics of the equivalence method of Cartan, as discussed in Refs. [6, 10]. Let $\{\omega_1, \omega_2, \omega_3\}$, as given by (2.13), denote the base coframe. Using the first structure group G_1, we introduce the lifted coframe

$$\xi_1 = a_1 \omega_1, \qquad \xi_2 = \omega_2, \qquad \xi_3 = a_4 \omega_1 + a_5 \omega_2 + a_6 \omega_3, \tag{3.1}$$

which corresponds to (2.16) in this particular case. The basic tool in Cartan's method is the invariance of the exterior derivative operation under smooth maps, so we begin by computing the differentials $d\xi_i$. They are found to have the form

$$d\xi_1 = \alpha_1 \wedge \xi_1 + \tau_{112}\xi_1 \wedge \xi_2 + \tau_{123}\xi_2 \wedge \xi_3,$$
$$d\xi_2 = \tau_{212}\xi_1 \wedge \xi_2 + \tau_{223}\xi_2 \wedge \xi_3,$$
$$d\xi_3 = \alpha_4 \wedge \xi_1 + \alpha_5 \wedge \xi_2 + \alpha_6 \wedge \xi_3 + \tau_{312}\xi_1 \wedge \xi_2 + \tau_{323}\xi_2 \wedge \xi_3.$$

Here $\alpha_1, \alpha_4, \alpha_5, \alpha_6$ form a basis for the right-invariant one-forms on the Lie group G_1. In the absorption part of Cartan's process, we are allowed to replace each one-form α_j by an expression of the form $\alpha_j + \sum z_{jk}\xi_k$, where the functions z_{jk} are chosen so as to make as many of the torsion coefficients τ_{ijk} vanish as possible. Here we can readily "absorb" all the torsion components except

$$\tau_{123} = \frac{a_1}{a_6 L},$$
$$\tau_{212} = \frac{a_6 L_u - a_4 L_p}{a_1 a_6 L},$$
$$\tau_{223} = -\frac{L_p}{a_6 L}.$$

These components must be invariants of the problem. Since they depend on the group parameters, the next step is to normalize them to as simple a form as possible through a suitable choice of the group parameters. Assuming $L_p \not\equiv 0$ (otherwise the variational problem is trivial), we can normalize these torsion components to $1, 0, -1$, respectively, by setting

$$a_1 = L_p, \qquad a_4 = \frac{L_u}{L}, \qquad a_6 = \frac{L_p}{L}. \tag{3.2}$$

The normalizations (3.2) have the effect of reducing the original Lie group G_1 to a one-parameter subgroup, with a_5 the only remaining undetermined parameter.

In the second loop through the equivalence procedure, we substitute expressions (3.2) into the formulas for the lifted coframe (3.1), and recompute the differentials. We find the new structure equations have the form

$$d\xi_1 = \tau_{112}\xi_1 \wedge \xi_2 + \tau_{113}\xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3,$$
$$d\xi_2 = -\xi_2 \wedge \xi_3,$$
$$d\xi_3 = \alpha_5 \wedge \xi_2 + \tau_{312}\xi_1 \wedge \xi_2 + \tau_{323}\xi_2 \wedge \xi_3.$$

We can still absorb all the torsion in the expression for $d\xi_3$. so we have two unabsorbable pieces of torsion. The first is

$$\tau_{112} = -\frac{L_p(L_u - L_{xp} - pL_{up}) + a_5 L^2 L_{pp}}{LL_p^2} = -\frac{L_p \tilde{E}(L) + a_5 L^2 L_{pp}}{LL_p^2}.$$

Disregarding the trivial case when the Lagrangian L is an affine function of p, we can normalize this torsion component to 0 by setting

$$a_5 = -\frac{L_p \tilde{E}(L)}{L^2 L_{pp}} = -\frac{L_p Q}{L^2} \tag{3.3}$$

(cf. (2.5)). The other unabsorbable torsion component is

$$\tau_{113} = -\frac{LL_{pp}}{L_p^2},$$

which is thus the first fundamental invariant of the problem.

We have now eliminated all the group parameters, reducing the equivalence problem to the case of an $\{e\}$-structure or local parallelism. The equivalence problem for $\{e\}$-structures has a simple solution, which will be described below, and then applied to the Lagrangian equivalence problem. The invariant coframe is determined from (3.1), where the group parameters now have the prescribed values (3.2), (3.3); we therefore have shown that the one-forms

$$\begin{aligned}
\xi_1 &= L_p(du - p\,dx), \\
\xi_2 &= L\,dx, \\
\xi_3 &= \frac{L_u}{L}(du - p\,dx) + \frac{L_p}{L}(dp - Q\,dx), \\
&= d(\log L) - \hat{D}_x(\log L)\,dx,
\end{aligned} \tag{3.4}$$

form an invariant coframe on J^1, meaning that their expressions do not change under the action of the pseudo-group of fiber-preserving transformations (cf. [22]). (See (2.5), (2.6) for the definitions of Q and $\hat{D}_x$.) The structure equations take the form

$$\begin{aligned}
d\xi_1 &= -I_1 \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3, \\
d\xi_2 &= -\xi_2 \wedge \xi_3, \\
d\xi_3 &= I_2 \xi_1 \wedge \xi_2 + I_3 \xi_2 \wedge \xi_3,
\end{aligned} \tag{3.5}$$

where I_1, I_2, I_3 are the fundamental invariants of the problem, one of which we already know from the formula for τ_{113}.

Before writing out the explicit formulae for these invariants, which turn out to be quite complicated rational combinations of the partial derivatives of the Lagrangian L, we recall that if $F(x, u, p)$ is any smooth function on Ω, then its *covariant derivatives* with respect to the coframe (3.4) are the functions $F_{,\xi_j}$, $j = 1, 2, 3$, defined by the formula

$$dF = F_{,\xi_1}\xi_1 + F_{,\xi_2}\xi_2 + F_{,\xi_3}\xi_3. \tag{3.6}$$

Explicitly,

$$\begin{aligned} F_{,\xi_1} &= \frac{1}{L_p^2}\frac{\partial(L, F)}{\partial(p, u)}, \\ F_{,\xi_2} &= \frac{1}{L}\hat{D}_x F, \\ F_{,\xi_3} &= \frac{L}{L_p}\frac{\partial F}{\partial p}. \end{aligned} \tag{3.7}$$

Employing this notation, we find that the invariants have a remarkably simple explicit form:

$$\begin{aligned} I_1 &= \frac{LL_{pp}}{L_p^2} = \frac{L_{p,\xi_3}}{L_p}, \\ I_2 &= -\frac{L_{,\xi_2,\xi_1}}{L}, \\ I_3 &= \frac{L_{,\xi_2,\xi_3}}{L}. \end{aligned} \tag{3.8}$$

Although these expressions can be deduced directly from the parametric calculations, it is not an easy matter to recognize that the resulting complicated explicit combinations of partial derivatives of L can be so easily expressed using the covariant derivatives. A simpler approach is to find the formulae by comparison with the intrinsic calculations. Applying (3.6) when $F = L$, we can compute

$$\begin{aligned} d\xi_2 = d(L\,dx) = dL \wedge dx &= \frac{1}{L}\,dL \wedge \xi_2 \\ &= \frac{1}{L}\{L_{,\xi_1}\xi_1 \wedge \xi_2 - L_{,\xi_3}\xi_2 \wedge \xi_3\}. \end{aligned}$$

This will agree with the second structure equation in (3.5) if and only if

$$L_{,\xi_1} = 0, \qquad L_{,\xi_3} = L,$$

formulae that can also be verified directly from (3.7). Therefore

$$dL = L_{,\xi_2}\xi_2 + L\xi_3.$$

Furthermore, using the structure equations (3.5) themselves, we compute

$$\begin{aligned} 0 = d^2L &= d(L_{,\xi_2}) \wedge \xi_2 + dL \wedge \xi_3 + L_{,\xi_2}\, d\xi_2 + L\, d\xi_3 \\ &= \{L_{,\xi_2,\xi_1} + LI_2\}\, \xi_1 \wedge \xi_2 + \{-L_{,\xi_2,\xi_3} + LI_3\}\, \xi_2 \wedge \xi_3. \end{aligned}$$

This will vanish if and only if the second and third identities in (3.8) hold.

Similarly, another way to derive the formula for I_1 is to look at

$$\begin{aligned} d\xi_1 &= \frac{1}{L_p}\, dL_p \wedge \xi_1 - L_p\, dp \wedge dx \\ &= \frac{L_u - LL_{p,\xi_2}}{LL_p}\, \xi_1 \wedge \xi_2 - \frac{L_{p,\xi_3}}{L_p}\, \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_3. \end{aligned}$$

Comparing with (3.5), we conclude that I_1 must be given by the first equation in (3.8), and, moreover,

$$L_{p,\xi_2} = \frac{L_u}{L}.$$

Using (3.7), we see that this equation reduces to the identity

$$\hat{D}_x L_p = L_u, \tag{3.9}$$

which we recognize to be the Euler–Lagrange equation (2.5)! (One can play the same game with the third invariant form ξ_3, leading to further, more complicated expressions for the invariants in terms of covariant derivatives; however, we have not found these expressions to be of much use.)

The covariant derivatives of any the fundamental invariants (3.8), which are called the *derived invariants*, are also invariants. Not all of these are independent. In general, if we apply the exterior derivative d to the structure equations (3.5), we can expect to derive certain relations among the invariants, which Cartan refers to as generalized "Bianchi identities." In our case, an easy calculation shows that $d^2\xi_2 = 0$ automatically, while the identities $d^2\xi_1 = d^2\xi_3 = 0$ imply the following relations among the invariants,

$$I_3 = -\frac{I_{1,\xi_2}}{I_1}, \qquad I_{2,\xi_3} + I_{3,\xi_1} + (I_1 + 1)I_2 = 0, \tag{3.10}$$

which will prove to be of use later. Alternatively, these identities can be deduced directly from the "curvature" identities among "mixed partials," which are found by evaluating $d^2F=0$ using (3.6) and the structure equations (3.5). (Here, we are ascribing to Cartan's point of view (cf. [8]) that the structure equations define a "generalized geometry.") In our case, they take the explicit form

$$\begin{aligned} F_{,\xi_1,\xi_2} &= F_{,\xi_2,\xi_1} + I_2 F_{,\xi_3}, \\ F_{,\xi_1,\xi_3} &= F_{,\xi_3,\xi_1} - I_1 F_{,\xi_1}, \\ F_{,\xi_2,\xi_3} &= F_{,\xi_3,\xi_2} + F_{,\xi_1} - F_{,\xi_2} + I_3 F_{,\xi_3}. \end{aligned} \tag{3.11}$$

In essence, the collection of all the invariants and their derived invariants will completely solve our equivalence problem, providing explicit necessary and sufficient conditions for two Lagrangians to be equivalent under a fiber-preserving transformation. To precisely formulate the theorem which gives necessary and sufficient conditions for equivalence, it is useful to recall the general result on the equivalence of $\{e\}$-structures, and to obtain the equivalence condition for Lagrangians as a special case (cf. Sternberg [23; Sect. 7.4]). In general, an analytic $\{e\}$-structure on a domain $\Omega \subset \mathbb{R}^n$ is given by a coframe $\{\theta_1, ..., \theta_n\}$, with structure equations

$$d\theta_i = \sum_{j,k} C^i_{jk} \theta_j \wedge \theta_k, \qquad i = 1, ..., n.$$

The *torsion coefficients* C^i_{jk} (which are analytic functions, some of which may be constant) are invariants for the $\{e\}$-structure over Ω. We use the notation

$$C_J = C^i_{jk,\theta_{m_1}, ..., \theta_{m_s}}, \qquad \text{where} \quad J = (i, j, k, m_1, ..., m_s),$$

to denote the derived invariant of *order* $s = \text{order } J$, labelled by the multi-index J. Let

$$\mathscr{F}_s = \{C_J : \text{order } J \leqslant s\}, \qquad s = 0, 1, ...,$$

denote the family of functions consisting of the derived invariants up to order s. Assume (for simplicity) that each family $\mathscr{F}_s$ is regular on Ω, which means that its rank r_s is constant on all of Ω. (By the *rank* of a collection of functions at a point $x \in \Omega$, we mean the dimension of the subspace of $T^*\Omega|_x$ spanned by their differentials. For a regular family, the rank equals the number of functionally independent functions in it.) By the *order* s and *rank* k of a regular $\{e\}$-structure, we mean the smallest $s \geqslant 0$ at which the ranks stabilize, meaning $r_s = r_{s+1} = k$, which implies that $r_t = k$, for all $t \geqslant s$. If our regular $\{e\}$-structure has order s and rank k, it follows that locally there exist k functionally independent invariants $I_1 = C_{J_1}, ..., I_k = C_{J_k} \in \mathscr{F}_s$, with the property that any other invariant or derived invariant appearing in any $\mathscr{F}_t$, $t \geqslant 0$, can be expressed as a function of the *fundamental invariants* $I_1, ..., I_k$:

$$C_J = F_J(I_1, ..., I_k). \tag{3.12}$$

The functions F_J corresponding to multi-indices J of order $\leqslant s+1$ are called the *determining functions* of the $\{e\}$-structure, since they provide the complete solution to the equivalence problem. The fundamental theorem underlying the solution to the equivalence problem for $\{e\}$-structures is as follows.

THEOREM 3.1. *Let Ω and $\bar{\Omega}$ be open subsets of $\mathbb{R}^n$, and let the coframes $\{\theta_1, ..., \theta_n\}$, $\{\bar{\theta}_1, ..., \bar{\theta}_n\}$ determine regular analytic $\{e\}$-structures on Ω and $\bar{\Omega}$ of ranks k and $\bar{k}$ and orders s and $\bar{s}$, respectively. Then there exists a map $\Phi: \Omega \to \bar{\Omega}$ which maps the coframes to each other, $\Phi^*(\bar{\theta}_i) = \theta_i$, $i = 1, ..., n$, if and only if the following conditions hold*:

(i) *the orders and ranks are the same*: $k = \bar{k}$ *and* $s = \bar{s}$,

(ii) *the fundamental invariants can be chosen to have the same labels*:

$$I_\nu = C_{J_\nu}, \qquad \bar{I}_\nu = \bar{C}_{J_\nu}, \qquad \nu = 1, ..., k,$$

(iii) *the determining functions are identical*:

$$F_J(t_1, ..., t_k) = \bar{F}_J(t_1, ..., t_k), \qquad \textit{order } J \leqslant s+1, \tag{3.13}$$

(iv) *the invariant equations*

$$I_j(x) = \bar{I}_j(\bar{x}), \qquad j = 1, ..., k, \tag{3.14}$$

have a common real solution $\bar{x} = h(x) \in \bar{\Omega}$, for $x \in \Omega$.

Note that (3.13) implies that if one invariant C_J is constant, then the corresponding invariant $\bar{C}_J$ must have the same constant value in order that their (constant) determining functions are the same. The local solvability condition (3.14) is often glossed over in treatments of the equivalence of $\{e\}$-structures, but it is essential to the validity of the theorem. As we shall see, there are examples in which all the other hypotheses of the theorem are met, but there is no real solution to (3.14), and the coframes are not equivalent. For $\{e\}$-structures on complex manifolds, analytic continuation implies that this is no longer an issue (except possibly at isolated points), and so Theorem 3.1 holds for complex $\{e\}$-structures without the local solvability condition. This remark is relevant to the equivalence problem for complex-analytic Lagrangians under the analogous pseudo-groups of complex-analytic transformations. Although not perhaps of such immediate physical interest, the complex Lagrangian equivalence problem has applications to classical invariant theory [21].

If the rank equals the dimension, $k = n$, then the equations (3.14) implicitly determine the change of variables $\bar{x} = \Phi(x)$ transforming the coframe $\{\theta_1, ..., \theta_n\}$ into $\{\bar{\theta}_1, ..., \bar{\theta}_n\}$. However, one needs to exercise some care at this point; if the determining functions are multiply-valued, as is often the case in practice, a naïve solution to (3.14) might not actually transform one coframe to another, since the corresponding derived invariants might correspond to different branches of the multiply-valued determining function F_J, and so (3.13) would not hold. Really, one should regard the derived invariants $C_J(x)$, order $J \leqslant s+1$, as parametrizing a k-dimensional submanifold of some Euclidean space, the *determining manifold* for the $\{e\}$-structure. In this language, condition (iii) would say that the two determining manifolds must be identical, while (3.14) would require the change of variables $\bar{x} = \Phi(x)$ to be a map taking x and $\bar{x}$ to the *same* point of the determining manifold. (See [21] for more details.) If the rank is less than the dimension, $k < n$, then any transformation between coframes must still satisfy (3.14) (suitably interpreted), but now the set of transformations leaving one of the coframes fixed forms an $(n-k)$-parameter Lie group—the symmetry group of the coframe. Thus, solutions to the equivalence problem can be used to determine symmetry groups; see [16] for applications of this remark to the study of symmetry groups of ordinary differential equations.

The simplest case is when there are already n independent invariants $I_1, ..., I_n$ among the original torsion coefficients C^i_{jk}. In this case, the rank equals the dimension, and the order is 0, so the determining functions in this case arise from expressing C^i_{jk} and their first covariant derivatives in terms of the fundamental invariants:

$$C^i_{jk} = F^i_{jk}(I_1, ..., I_n), \qquad C^i_{jk,m} = F^i_{jkm}(I_1, ..., I_n). \tag{3.15}$$

In our Lagrangian equivalence problem, this occurs when the three invariants I_1, I_2, I_3 are independent, i.e.,

$$\frac{\partial(I_1, I_2, I_3)}{\partial(x, u, p)} \neq 0. \tag{3.16}$$

Since all the other coefficients in the structure equations (3.5) are constant, the determining functions come just from rewriting the derived invariants I_{j,ξ_k} in terms of the fundamental invariants:

$$I_{j,\xi_k} = F_{jk}(I_1, I_2, I_3). \tag{3.17}$$

Of course, some of the determining functions F_{jk} are "universal," since they follow directly from the Bianchi identities (3.10); others, however, will be different, depending on the equivalence class of the Lagrangian in question. According to Theorem 3.1, if the invariants for the two Lagrangians $L(x, u, p)$ and $\bar{L}(\bar{x}, \bar{u}, \bar{p})$ satisfy (3.16), then L and $\bar{L}$ are equivalent if and only if

(a) the determining functions relating these invariants are *identical*, $F_{jk} = \bar{F}_{jk}$, and

(b) the invariant equations

$$I_j(x, u, p) = \bar{I}_j(\bar{x}, \bar{u}, \bar{p}), \qquad j = 1, 2, 3, \tag{3.18}$$

have a common real solution.

Complications of higher order and/or lower rank arise when only one or two of the invariants I_1, I_2, I_3 are functionally independent. If there are no further functionally independent invariants among the derived invariants I_{j,ξ_k}, then the ranks stabilize at order 0, and the same determining functions (3.17) will completely solve the equivalence problem. Otherwise one needs to look at second order derived invariants I_{j,ξ_k,ξ_m}, etc. The "worst" case is when the order is 2, so there is only one independent function among the basic invariants I_1, I_2, I_3, one additional independent function appears among the first derived invariants, and yet another independent function appears among the second order derived invariants; in this case, to derive equivalence conditions like (a) and (b) above, one needs to look at the determining functions for all the covariant derivatives up to order 3. For reasons of space, we will not explicitly catalogue all the possibilities, since the basic underlying philosophy should be clear.

EXAMPLE 3.2. The local solvability requirement (3.18) is essential for the problem of equivalence under real analytic transformations, as the following example indicates. Consider the Lagrangians

$$L(x, u, p) = e^{1/(2p^2)}, \qquad \bar{L}(\bar{x}, \bar{u}, \bar{p}) = e^{-1/(2\bar{p}^2)}.$$

It is not hard to see that there is no real transformation taking L to $\bar{L}$. (There is, of course, an obvious complex transformation, namely $(x, u) \to (x, \sqrt{-1}u)$, mapping L to $\bar{L}$; indeed, this fact is really the source of the problem.) Nevertheless, the determining functions for L and $\bar{L}$ are identical. Evaluating (3.8), we deduce that the three invariants for L are

$$I_1 = 3p^2 + 1, \qquad I_2 = 0, \qquad I_3 = 0.$$

According to (3.7), the only nonzero derived invariant is

$$I_{1,\xi_3} = -6p^4 = -\tfrac{2}{3}I_1^2 + \tfrac{4}{3}I_1 - \tfrac{2}{3}.$$

Therefore the order is 0, and there is a single independent invariant, I_1, so the problem has rank 1. (This reflects the two parameter symmetry group of L consisting of translations in x and u.) Of the various determining functions, the only nonzero function is that relating I_{1,ξ_3} to I_1; we find $I_{1,\xi_3} = F(I_1)$, where

$$F(t) = -\tfrac{2}{3}t^2 + \tfrac{4}{3}t - \tfrac{2}{3}. \tag{3.19}$$

Similarly, in the case of $\bar{L}$ we find

$$\bar{I}_1 = -3\bar{p}^2 + 1, \qquad \bar{I}_2 = 0, \qquad \bar{I}_3 = 0,$$

and the only nontrivial derived invariant is again

$$\bar{I}_{1,\xi_3} = -6\bar{p}^4 = F(\bar{I}_1),$$

with precisely the same determining function (3.19). Therefore, the $\{e\}$-structure for $\bar{L}$ has the same order and rank as that for L, and the same determining function(s). The coframes (3.4) for these two Lagrangians satisfy all the conditions of Theorem 3.1 except for the local solvability criterion (3.14), but they are nevertheless not real-equivalent Lagrangians since there is no real solution branch to the equation

$$I_1(p) = \bar{I}_1(\bar{p}).$$

4. An Inductive Approach to Equivalence Problems

Before proceeding to the more complicated versions of the Lagrangian equivalence problem, we make an elementary but powerful observation on the use of subgroups to "induce" solutions to equivalence problems. Suppose we are given two equivalence problems with the same coframe $\omega = \{\omega_i\}$ on the base space X, but different structure groups G and H, the first of which is a subgroup of the second: $G \subset H$. The two problems therefore lead to different lifted coframes $\theta = g \cdot \omega$ on $X \times G$ and $\zeta = h \cdot \omega$ on $X \times H$, respectively (cf. (2.16)). For example, the standard fiber-preserving and point transformation equivalence problems are of this form, with $G = G_1$ and $H = G_2$. In some sense this means that the G-equivalence problem is "simpler" than the H-equivalence problem, and that the solution of the simpler equivalence problem should therefore aid us in the solution to the more complicated problem.

Indeed, suppose that we have solved the G-equivalence problem and hence have determined the G-adapted coframe

$$\theta = g_0 \cdot \omega, \tag{4.1}$$

for some $g_0 \in G$, that normalizes all the torsion. (For simplicity, we assume that the G-equivalence problem reduces directly to an $\{e\}$-structure, i.e., we do not need to prolong, and there are no infinite pseudo-groups to worry about.) For the Lagrangian example, g_0 would denote the group element of G_1 determined by (3.2) and (3.3) in our solution to the fiber-preserving equivalence problem. The idea is then instead of merely reverting to the original base coframe, we use the *G-adapted coframe* to reformulate the H-equivalence problem, and thereby effect an easier calculation for the more complicated problem. In other words, to solve the H-equivalence problem, rather than using the direct lifted coframe

$$\zeta = \hat{h} \cdot \omega, \qquad \hat{h} \in H,$$

we work with the *adapted coframe*

$$\zeta = h \cdot \theta = h \cdot g_0 \cdot \omega, \qquad h \in H. \tag{4.2}$$

The group elements h and $\hat{h}$ must be related by the elementary formula

$$\hat{h} = h \cdot g_0.$$

Since the structure equations for the G-equivalence problem give the differentials of θ in terms of the G-invariants, we can use these expressions in the absorption and normalization process to obtain structure equations for ζ with the normalized group parameters, and, ultimately, obtain expressions for the H-invariants written in terms of the G-invariants and their derived invariants. Thus, we will automatically derive expressions for the invariants of the more complicated problem in terms of the invariants and their derived invariants of the simpler problem, and thereby explicitly expose the relationship between the two problems. Use of the inductive method will become clearer in the examples to be treated in the following section.

5. Solution of Lagrangian Equivalence Problems

In this section we outline our complete solutions to the other versions of the Lagrangian equivalence problem, based on the inductive approach outlined in Section 4. For reasons of space, we will not present all the details, but only the most basic steps in the reduction and absorption procedure.

(a) *Standard Point Transformation Equivalence*

We begin with the derivation of the invariants for the point transformation equivalence problem, $m=2$, in terms of those of the fiber-preserving equivalence problem $m=1$, found in Section 3, and thereby recover results of Cartan [8] and Gardner [10]. However, our presentation is simplified by using the inductive approach based on the inclusion $G_1 \subset G_2$. The adapted invariant coframe for the G_1-equivalence problem, corresponding to (4.1), is given by the one-forms (3.4). For the G_2-equivalence problem, the lifted coframe, as given by (4.2), has the form

$$\begin{aligned} \eta_1 &= b_1 \xi_1, \\ \eta_2 &= b_2 \xi_1 + \xi_2, \\ \eta_3 &= b_4 \xi_1 + b_5 \xi_2 + b_6 \xi_3. \end{aligned} \tag{5.1}$$

We now apply the Cartan algorithm to the equivalence problem determined by the lifted one-forms (5.1). In terms of the right-invariant one-forms β_j on G_2, the differentials are

$$\begin{aligned} d\eta_1 &= \beta_1 \wedge \eta_1 + \sigma_1, \\ d\eta_2 &= \beta_2 \wedge \eta_1 + \sigma_2, \\ d\eta_3 &= \beta_4 \wedge \eta_1 + \beta_5 \wedge \eta_2 + \beta_6 \wedge \eta_3 + \sigma_3, \end{aligned}$$

where the torsion components have the general form

$$\sigma_i = \sum_{j,k} \tau_{ijk} \eta_j \wedge \eta_k, \qquad i = 1, 2, 3. \tag{5.2}$$

We now outline the basic features in the application of the equivalence method. At each stage, we indicate all the unabsorbable torsion components, as labelled in (5.2), the chosen normalizations, and the group reductions resulting from the normalizations. Before we begin, however, note that since L is not an affine function of p, the invariant I_1 (cf. (3.8)), cannot vanish identically. By possibly shrinking the domain Ω, we can assume that I_1 does not vanish anywhere on Ω.

Phase 1.

$$\begin{aligned} \tau_{123} &= 1, & b_1 &= b_6, \\ \tau_{223} &= 0, & b_2 &= 1. \end{aligned}$$

Note that at this stage, the one-form

$$\eta_2 = \xi_1 + \xi_2 = (L - pL_p)\, dx + L_p\, du$$

is invariant. This is the well-known Cartan form from the calculus of variations [13] (also known as Hilbert's invariant integral).

Phase 2.

$$\tau_{212}=0, \qquad b_5=0,$$
$$\tau_{213}=-\varepsilon, \qquad b_6=\kappa\sqrt{|I_1|},$$

where $\varepsilon, \kappa=\pm 1$. Here there are two distinct branches to the real-equivalence problem, determined by $\varepsilon=\operatorname{sign}(I_1)$, which lead to different normalizations for the torsion coefficient τ_{213}. There is also, even in the complex case, an unavoidable ambiguity in the sign κ of the group parameter b_6 which cannot be resolved at this point. We will discuss κ in more detail after we complete the equivalence procedure.

Phase 3.

$$\tau_{112}=\tau_{323}=0, \qquad b_4=\frac{\varepsilon\kappa I_{1,\xi_2}}{2\sqrt{|I_1|}}=-\frac{1}{2}\kappa I_3\sqrt{|I_1|},$$

where we have used the Bianchi identities (3.10). We have now normalized all the group parameters, leading to the structure equations

$$\begin{aligned} d\eta_1&=-\varepsilon\kappa J_1\eta_1\wedge\eta_3+\eta_2\wedge\eta_3,\\ d\eta_2&=-\varepsilon\,\eta_1\wedge\eta_3,\\ d\eta_3&=J_2\eta_1\wedge\eta_2+\varepsilon\kappa J_3\eta_1\wedge\eta_3,\end{aligned} \tag{5.3}$$

where

$$\begin{aligned} J_1&=\frac{\frac{1}{2}I_{1,\xi_3}+I_1^2+I_1}{|I_1|^{3/2}},\\ J_2&=\tfrac{1}{2}I_{3,\xi_2}+I_2-\tfrac{1}{4}I_3^2,\\ J_3&=\frac{\frac{1}{2}I_3I_{1,\xi_3}+I_{1,\xi_1}+I_1I_{3,\xi_3}+I_1^2I_3}{2\,|I_1|^{3/2}}.\end{aligned} \tag{5.4}$$

The basic invariants are κJ_1, J_2, κJ_3. The explicit formula for the simplest of these functions is

$$J_1=\frac{LL_{ppp}+3L_pL_{pp}}{2\,|L|^{1/2}\,|L_{pp}|^{3/2}}. \tag{5.5}$$

Note that the function J_1, or even εJ_1, is *not* an invariant of the problem, and the ambiguity in the sign of the invariant κJ_1 is unavoidable. For instance, under the elementary transformation

$$\bar{x} = x, \qquad \bar{u} = -u, \tag{5.6}$$

the function $J_1(p)$ gets mapped to $-\bar{J}_1(-\bar{p})$, so the sign *can* change. However, by composing any change of variables with the orientation-reversing map (5.6), we can always change the sign of J_1 if required, so the ambiguity is of an inessential kind. One way to avoid this ambiguity is to use $\hat{J}_1(p) = J_1(p)^2$ as the fundamental invariant, which is convenient when discussing the applications to classical invariant theory (cf. [21]). However, we will not do this here.

The derived invariants for this problem have the form

$$\begin{aligned} J_{,\eta_1} &= \kappa \frac{J_{,\xi_1} - J_{,\xi_2} + \frac{1}{2} I_3 J_{,\xi_3}}{\sqrt{|I_1|}}, \\ J_{,\eta_2} &= J_{,\xi_2}, \\ J_{,\eta_3} &= \kappa \frac{J_{,\xi_3}}{\sqrt{|I_1|}}. \end{aligned} \tag{5.7}$$

We note the "Bianchi identities"

$$\begin{aligned} J_3 &= -J_{1,\eta_2}, \\ J_{2,\eta_3} - J_{3,\eta_2} + J_1 J_2 &= 0, \end{aligned} \tag{5.8}$$

stemming from the identities $d^2\eta_1 = d^2\eta_3 = 0$.

(b) *Fiber-Preserving Divergence Equivalence*

For the case $m = 3$, we start with the fiber-preserving coframe $\{\xi_1, \xi_2, \xi_3\}$, supplemented by the additional form $\xi_4 = dw$. We proceed to an induced equivalence problem based on the groups $G_1 \subset G_3$. The lifted coframe for the G_3 problem is

$$\begin{aligned} \zeta_1 &= c_1 \xi_1, \\ \zeta_2 &= c_3 \xi_2, \\ \zeta_3 &= c_4 \xi_1 + c_5 \xi_2 + c_6 \xi_3, \\ \zeta_4 &= c_7 \xi_1 + (c_3 - 1)\, \xi_2 + \xi_4. \end{aligned}$$

We compute the differentials

$$d\zeta_1 = \gamma_1 \wedge \zeta_1 + \sigma_1,$$
$$d\zeta_2 = \gamma_3 \wedge \zeta_2 + \sigma_2,$$
$$d\zeta_3 = \gamma_4 \wedge \zeta_1 + \gamma_5 \wedge \zeta_2 + \gamma_6 \wedge \zeta_3 + \sigma_3,$$
$$d\zeta_4 = \gamma_7 \wedge \zeta_1 + \gamma_3 \wedge \zeta_2 + \sigma_4,$$

where, as before, the torsion components have the form

$$\sigma_i = \sum_{j,k} \tau_{ijk} \zeta_j \wedge \zeta_k, \qquad i = 1, ..., 4.$$

We now indicate how the absorption and reduction process runs:

Phase 1.

$$\tau_{123} = 1, \qquad c_1 = c_3 c_6,$$
$$\tau_{223} = \tau_{423}, \qquad c_7 = -1,$$

Phase 2.

$$\tau_{212} = \tau_{412}, \qquad c_5 = 0,$$
$$\tau_{413} = 1, \qquad c_3 = \frac{I_1}{c_6^2},$$

(as above, we can assume that the invariant I_1 does not vanish),

Phase 3.

$$\tau_{112} = \tau_{323}, \qquad (*)$$
$$\tau_{223} = 2\tau_{113} + 1, \qquad c_6 = I_4,$$

where (*) means that the equation is satisfied identically, and I_4 denotes the invariant

$$I_4 \equiv \frac{I_{1,\xi_3}}{I_1} + 2I_1 - 1 = \frac{LL_{ppp}}{L_p L_{pp}}. \tag{5.9}$$

There are two cases to be considered according to whether I_4 does or does not vanish, i.e., whether or not L is a quadratic function of p. If L is a quadratic function of p, we cannot obtain any further group reduction at this stage, and so need to prolong the system, which we shall discuss at the end of this subsection. Otherwise, $I_4 \not\equiv 0$, and, by restricting to a subdomain where I_4 does not vanish, we can reduce further:

Phase 4.

$$\tau_{323}=0, \qquad c_4=I_5,$$

where I_5 denotes the invariant

$$I_5=-I_3I_4-I_{4,\xi_2}, \tag{5.10}$$

and we have used the Bianchi identities (3.10) to simplify this formula. We have now normalized all the group parameters, and therefore obtain the structure equations

$$\begin{aligned}
d\zeta_1&=-K_1\zeta_1\wedge\zeta_3+\zeta_2\wedge\zeta_3,\\
d\zeta_2&=K_2\zeta_1\wedge\zeta_2+(1-2K_1)\,\zeta_2\wedge\zeta_3,\\
d\zeta_3&=K_3\zeta_1\wedge\zeta_2+K_4\zeta_1\wedge\zeta_3,\\
d\zeta_4&=K_2\zeta_1\wedge\zeta_2+\zeta_1\wedge\zeta_3+(1-2K_1)\zeta_2\wedge\zeta_3.
\end{aligned}\tag{5.11}$$

The basic invariants are given by

$$\begin{aligned}
K_1&=\frac{I_1}{I_4}+\frac{1}{I_1}\left(\frac{I_1}{I_4}\right)_{,\xi_3},\\
K_2&=\frac{I_4I_{1,\xi_1}-2I_1I_{4,\xi_1}+I_4^2I_5}{I_1^2},\\
K_3&=\frac{I_4^4I_2+I_4^3(I_3I_5-I_{5,\xi_2})+I_4^2(I_5^2+I_5I_{4,\xi_2})}{I_1^2},\\
K_4&=\frac{I_{4,\xi_1}-I_{5,\xi_3}}{I_1}-I_5.
\end{aligned}\tag{5.12}$$

The simplest of these is

$$K_1=2-\frac{L_{pp}L_{pppp}}{L_{ppp}^2}. \tag{5.13}$$

Interestingly, even though the Lagrangian $L(x, u, p)$ is defined over a three-dimensional domain, there are four invariants arising in the structure equations. There will always be a one-parameter symmetry group provided by the translations in the auxiliary variable w, so at most three of these invariants can be independent functions on the base space J^1. Thus there is at least one functional relation, which can usually be taken to be of the form

$$K_4=H(K_1, K_2, K_3).$$

The function H appearing in this relation is one of the determining functions for the Lagrangian, as in (3.12), the others coming from the first (and, if the order is >0, higher) derived invariants. However, there is no *universal* functional relation among the invariants K_i common to all (non-quadratic) Lagrangians because the invariants are independent functions of the derivatives of L; in other words, H will inevitably depend on the equivalence class of L. In fact, we see that the highest order derivatives of L occurring in the invariants I_1, I_2, I_3, are

$$L_{pp}, \qquad L_{pp*}, \qquad L_{ppp},$$

respectively, where by L_{pp*} we mean all derivatives of L_{pp}, i.e., L_{ppx}, L_{ppu}, L_{ppp}. Similarly, the highest order derivatives in I_4, I_5 (cf. (5.9), (5.10)) are

$$L_{ppp}, \qquad L_{ppp*},$$

respectively. Thus, using (5.12), the highest order derivatives of L occurring in the invariants K_1, K_2, K_3, K_4, are

$$L_{pppp}, \qquad L_{ppp*}, \qquad L_{ppp**}, \qquad L_{pppp*}.$$

Thus the four invariants are clearly independent differential functions.

The explicit formulas for the derived invariants are

$$\begin{aligned} K_{,\zeta_1} &= \frac{I_4}{I_1} K_{,\xi_1} - \frac{I_5}{I_1} K_{,\xi_3} + \frac{I_4}{I_1} K_{,\xi_4}, \\ K_{,\zeta_2} &= \frac{I_4^2}{I_1} K_{,\xi_2} + \left(\frac{I_4^2}{I_1} - I_4\right) K_{,\xi_4}, \\ K_{,\zeta_3} &= \frac{K_{,\xi_3}}{I_4}, \\ K_{,\zeta_4} &= K_{,\xi_4}. \end{aligned} \tag{5.14}$$

The Bianchi identities are

$$\begin{aligned} K_{1,\zeta_2} + K_2 + K_4 &= 0, \\ K_{2,\zeta_3} + 2K_{1,\zeta_1} + K_1(K_2 + 2K_4) &= 0, \\ K_{3,\zeta_3} - K_{4,\zeta_1} - K_1 K_3 &= 0. \end{aligned} \tag{5.15}$$

The other case to consider is when the invariant (5.9) vanishes identically, so that L is a quadratic function of p. Now, after phase 3 of the absorption procedure, we can absorb all the nonconstant torision, and we have reduced the structure group to a two-parameter subgroup $\tilde{G}_3 \subset G_3$, parametrized by c_4, c_6. The resulting structure equations are

$$\begin{aligned} d\zeta_1 &= -\beta \wedge \zeta_1 + \zeta_2 \wedge \zeta_3, \\ d\zeta_2 &= -2\beta \wedge \zeta_2, \\ d\zeta_3 &= \alpha \wedge \zeta_1 + \beta \wedge \zeta_3, \\ d\zeta_4 &= -2\beta \wedge \zeta_2 + \zeta_1 \wedge \zeta_3. \end{aligned} \tag{5.16}$$

Here α and β are equivalent, modulo the base coframe, to the right-invariant one-forms γ_4 and γ_6 on $\tilde{G}_3$ (see below). The action of the reduced group $\tilde{G}_3$ on the torsion is trivial since there is no nonconstant torsion left in (5.16). There is thus no further possible group reduction, and the question arises of as to whether (5.16) are the structure equations for a transitive infinite Lie pseudo-group, or, equivalently, whether the system of partial differential equations

$$\Phi^*(\bar{\zeta}_i) = \zeta_i, \qquad i = 1, 2, 3, 4, \tag{5.17}$$

is *in involution*. There is an arithmetic test due to Cartan which provides necessary and sufficient conditions for a system of the form (5.16) to be involutive (cf. [3]). One computes the reduced characters s_i', and, in this case, finds that $s_1' = 2$, while $s_i' = 0$ for $i > 1$. On the other hand, writing (5.16) as the system of polar equations for an admissible integral element, we find that it has a solution space of dimension 1. Since $1 < \sum is_i' = 2$, the system is *not* involutive, and, according to the Cartan–Kuranishi theorem, one has to prolong, that is, add the differential consequences of (5.16) to the system. (Another way to detect this is to prove that a nonaffine first order particle Lagrangian can never have an infinite-dimensional symmetry group.)

Explicitly, the forms α, β in (5.16) are given by

$$\begin{aligned} \alpha &= \gamma_4 - m\zeta_1 - \tau_{312}\zeta_2 - (\tfrac{1}{2}\tau_{212} + \tau_{313})\zeta_3, \\ \beta &= \gamma_6 - \tfrac{1}{2}\tau_{212}\zeta_1 + \tau_{112}\zeta_2 + \tau_{113}\zeta_3, \end{aligned} \tag{5.18}$$

where the only remaining freedom is reflected by the undetermined parameter m, and it serves to define the structure group $G^{(1)}$ for the prolonged equivalence problem. The intrinsic forms for the structure equations for the prolonged problem are found by expressing the fact that the right-hand sides of (5.16) are closed under the exterior derivative d, and then using Cartan's lemma to solve for $d\alpha$ and $d\beta$. After absorption, we find, in addition to (5.16), the equations

$$d\alpha = \mu \wedge \zeta_1 + 2\alpha \wedge \beta + T\zeta_2 \wedge \zeta_3,$$
$$d\beta = -\alpha \wedge \zeta_2 + T\zeta_1 \wedge \zeta_2, \tag{5.19}$$

where μ is congruent modulo ζ_i, α, β to a right-invariant one-form on the one-parameter group $G^{(1)}$. The $G^{(1)}$ action on the nonconstant torsion component T is intrinsically determined in infinitesimal form by using the closure of the right-hand sides of (5.19), and then solving for dT. We find

$$dT + 3T\beta + \mu \equiv 0 \bmod \zeta_1, \zeta_2,$$

which implies that $G^{(1)}$ acts on T by translation. Indeed, we find, after a long parametric computation using the Bianchi identities (3.10) to effect some simplification, that

$$T = m - I_6 c_6^3 - I_7 c_4 c_6^2 - I_8 c_4^2 c_6,$$

where

$$I_6 = \frac{I_1(I_{2,\xi_3} + 2I_2) + I_2 - I_3 I_{1,\xi_1}}{2I_1^3},$$
$$I_7 = \frac{I_1(I_{3,\xi_3} + 4I_3) + I_{1,\xi_1}}{2I_1^3}, \tag{5.20}$$
$$I_8 = \frac{2I_1 - 1}{I_1^2}.$$

We can thus translate T to zero by setting

$$m = I_6 c_6^3 + I_7 c_4 c_6^2 + I_8 c_4^2 c_6,$$

which reduces our prolonged problem to an $\{e\}$-structure on $J^1 \times \tilde{G}_3$, with structure equations

$$d\zeta_1 = -\beta \wedge \zeta_1 + \zeta_2 \wedge \zeta_3,$$
$$d\zeta_2 = -2\beta \wedge \zeta_2,$$
$$d\zeta_3 = \alpha \wedge \zeta_1 + \beta \wedge \zeta_3,$$
$$d\zeta_4 = -2\beta \wedge \zeta_2 + \zeta_1 \wedge \zeta_3, \tag{5.21}$$
$$d\alpha = B\zeta_1 \wedge \zeta_2 + 2\alpha \wedge \beta.$$
$$d\beta = -\alpha \wedge \zeta_2.$$

The action of $\tilde{G}_3$ on the invariant B is found intrinsically by looking at the integrability condition $d^2\alpha = 0$. A simple calculation shows that

$$dB - 5B\beta \equiv 0 \quad \mathrm{mod}\, \zeta_1, \zeta_2. \tag{5.22}$$

Indeed, the parametric expression for the invariant B is given by

$$B = c_6^5 \left(\frac{I_1^2 I_{6,\xi_2} - I_{2,\xi_1}}{I_1^3} \right)$$

which is indeed scaled according to the intrinsic result (5.22).

If $B \equiv 0$, i.e.,

$$I_1^2 I_{6,\xi_2} = I_{2,\xi_1}, \tag{5.23}$$

then (5.21) constitute the Maurer–Cartan equations for the symmetry algebra of the Lagrangian with the largest symmetry group, namely $L = \frac{1}{2} p^2$, with generators

$$\begin{aligned} &\mathbf{v}_1 = \frac{\partial}{\partial x}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial u}, \qquad \mathbf{v}_3 = 2x \frac{\partial}{\partial x} + 2u \frac{\partial}{\partial u}, \\ &\mathbf{v}_4 = x \frac{\partial}{\partial u}, \qquad \mathbf{v}_5 = x^2 \frac{\partial}{\partial x} + xu \frac{\partial}{\partial u}, \qquad \mathbf{v}_6 = \frac{\partial}{\partial w}. \end{aligned} \tag{5.24}$$

Otherwise, we can scale B to 1 by setting

$$c_6 = I_9 \equiv \left(\frac{I_1^2 I_{6,\xi_2} - I_{2,\xi_1}}{I_1^3} \right)^{-1/5}. \tag{5.25}$$

This ensures that

$$\beta = A_1 \zeta_1 + A_2 \zeta_2, \tag{5.26}$$

where, according to the parametric calculations,

$$A_1 = \frac{I_1 I_{9,\xi_1} - I_9 I_{1,\xi_1}}{2 I_1^2}, \tag{5.27}$$

whereas

$$A_2 = \frac{I_9 I_{9,\xi_2} - I_9^2 I_3 - c_4 I_9}{I_1}.$$

(The expression for A_1 has been simplified using the intrinsically determined fact that the coefficient of ζ_3 in (5.26) has to be zero, which implies that

$$I_{9,\zeta_3} = I_9(1 - I_1),$$

an identity which also follows, albeit much more tediously, from (3.10), (5.20), (5.25).)

Substituting (5.26) into the structure equations (5.21), we see that A_1 and A_2 are both new invariants. The former contains no further group parameters, while the latter can be translated to zero by normalizing the remaining group parameter

$$c_4 = I_{10} \equiv I_{9,\zeta_2} - I_9 I_3.$$

This implies that

$$\alpha = B_1\zeta_1 + B_2\zeta_2 + B_3\zeta_3,$$

where $A_1 = B_3$, which is a consequence of the relation $d^2\zeta_1 = 0$. Thus, the final structure equations for the reduced $\{e\}$-structure on $J^1 \times \mathbb{R}$ are

$$\begin{aligned} d\zeta_1 &= \zeta_2 \wedge \zeta_3, \\ d\zeta_2 &= 2A_1\zeta_1 \wedge \zeta_2, \\ d\zeta_3 &= -B_2\zeta_1 \wedge \zeta_2 - 2A_1\zeta_1 \wedge \zeta_3, \\ d\zeta_4 &= 2A_1\zeta_1 \wedge \zeta_2 + \zeta_1 \wedge \zeta_3. \end{aligned} \tag{5.28}$$

Thus there are two fundamental invariants of the prolonged problem, namely (5.27) and

$$B_2 = \frac{I_9^3 I_{10,\zeta_2} - I_9^2 I_{10} I_{9,\zeta_2} - I_9^4 - I_9^3 I_{10} + I_9^2 I_{10}^2}{I_1^2}. \tag{5.29}$$

By this stage, the expressions have become so complicated that we will not even attempt to discuss Bianchi identities, etc.

(c) *Point Transformation Divergence Equivalence*

For the full divergence equivalence problem under point transformations, we use the inclusion $G_2 \subset G_4$. The lifted coframe is

$$\begin{aligned} \theta_1 &= d_1\eta_1, \\ \theta_2 &= d_2\eta_1 + d_3\eta_2, \\ \theta_3 &= d_4\eta_1 + d_5\eta_2 + d_6\eta_3, \\ \theta_4 &= d_7\eta_1 + (d_3 - 1)\eta_2 + \eta_4, \end{aligned} \tag{5.30}$$

where η_1, η_2, η_3 are given by (5.1), as normalized in part (a), and $\eta_4 = dw$.

Now

$$d\theta_1 = \delta_1 \wedge \theta_1 + \sigma_1,$$
$$d\theta_2 = \delta_2 \wedge \theta_1 + \delta_3 \wedge \theta_2 + \sigma_2,$$
$$d\theta_3 = \delta_4 \wedge \theta_1 + \delta_5 \wedge \theta_2 + \delta_6 \wedge \theta_3 + \sigma_3,$$
$$d\theta_4 = \delta_7 \wedge \theta_1 + \delta_3 \wedge \theta_2 + \sigma_4.$$

Using our standard notation

$$\sigma_i = \sum_{j,k} \tau_{ijk}\theta_j \wedge \theta_k, \qquad i = 1, ..., 4,$$

for the torsion components, we indicate how the absorption and reduction process runs:

Phase 1.

$$\tau_{123} = 1, \qquad d_1 = d_3\, d_6,$$
$$\tau_{223} = \tau_{423}, \qquad d_2 = d_7,$$

Phase 2.

$$\tau_{212} = \tau_{412}, \qquad d_5 = 0,$$
$$\tau_{213} = \tau_{413} - 1, \qquad d_3 = \frac{\varepsilon}{d_6^2},$$

Phase 3.

$$\tau_{112} = \tau_{323}, \qquad (*)$$
$$\tau_{223} = 2\tau_{113}, \qquad d_7 = -\frac{2\varepsilon\kappa J_1}{3d_6^2},$$

where (*) means that the equation is satisfied identically, and the signs ε, κ are as in part (a).

Phase 4. There are three distinct branches of the real-equivalence problem, depending on the sign of the derived invariant

$$\tilde{J}_4 \equiv \varepsilon - \frac{2}{3}\varepsilon\kappa J_{1,\eta_3} - \frac{2}{9}J_1^2 = \frac{L(4L_{ppp}^2 - 3L_{pp}L_{pppp})}{9L_{pp}^3}. \tag{5.31}$$

If $\tilde{J}_4 \equiv 0$, then we cannot obtain any further group reduction, and we will need to prolong the system; we discuss this case later. Otherwise, we restrict to a domain where $\tilde{J}_4 \neq 0$, and introduce $\hat{\varepsilon} = \operatorname{sign} \tilde{J}_4$. We then make the normalization

$$\tau_{413} = 1 + \hat{\varepsilon}, \qquad d_6 = \hat{\kappa} J_4,$$

where $J_4 = \sqrt{|\tilde{J}_4|}$, and $\hat{\kappa} = \pm 1$ denotes the resulting ambiguity in the sign of d_6.

Phase 5.

$$\tau_{323} = 0, \qquad d_4 = -\hat{\kappa} J_{4,\eta_2}.$$

Therefore, we obtain the structure equations

$$\begin{aligned}
d\theta_1 &= -\hat{\kappa} M_1 \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_3, \\
d\theta_2 &= \hat{\kappa} M_2 \theta_1 \wedge \theta_2 + \hat{\varepsilon}\, \theta_1 \wedge \theta_3 - 2\hat{\kappa} M_1 \theta_2 \wedge \theta_3, \\
d\theta_3 &= M_3 \theta_1 \wedge \theta_2 + \hat{\kappa} M_4 \theta_1 \wedge \theta_3, \\
d\theta_4 &= \hat{\kappa} M_2 \theta_1 \wedge \theta_2 + (1 + \hat{\varepsilon})\, \theta_1 \wedge \theta_3 - 2\hat{\kappa} M_1 \theta_2 \wedge \theta_3,
\end{aligned} \tag{5.32}$$

where the basic invariants (up to ambiguities in sign) are given by

$$\begin{aligned}
M_1 &= \frac{\frac{1}{3} J_1 J_4 - J_{4,\eta_3}}{J_4^2}, \\
M_2 &= -2 J_{4,\eta_1} + \frac{2}{3}\left(J_4 J_{1,\eta_2} - J_1 J_{4,\eta_2}\right) - 2\,\frac{J_{4,\eta_2} J_{4,\eta_3}}{J_4}, \\
M_3 &= J_4^4 J_2 + J_4^3 J_{4,\eta_2,\eta_2}, \\
M_4 &= J_{4,\eta_1} + J_1 J_{4,\eta_2} + J_{4,\eta_2,\eta_3} + J_3 J_4.
\end{aligned} \tag{5.33}$$

The simplest of these is

$$M_1 = \frac{9 L_{pp}^2 L_{ppppp} - 35 L_{pp} L_{ppp} L_{pppp} + 40 L_{ppp}^3}{2\,|4 L_{ppp}^2 - 3 L_{pp} L_{pppp}|^{3/2}}. \tag{5.34}$$

The derived invariants have the formulas

$$\begin{aligned}
M_{,\theta_1} &= \hat{\kappa}\{J_4 M_{,\eta_1} + \tfrac{2}{3} J_1 J_4 M_{,\eta_2} + J_{4,\eta_2} M_{,\eta_3} - \tfrac{2}{3} J_1 J_4 M_{,\eta_4}\}, \\
M_{,\theta_2} &= J_4^2 M_{,\eta_2} + (J_4^2 - 1)\, M_{,\eta_4}, \\
M_{,\theta_3} &= \hat{\kappa}\,\frac{M_{,\eta_3}}{J_4}, \\
M_{,\theta_4} &= M_{,\eta_4}.
\end{aligned}$$

(Of course, in our case, the derived invariants $M_{,\eta_4}$ always vanish.) The Bianchi identities are

$$M_{1,\theta_2} + M_2 + M_4 = 0,$$
$$M_{2,\theta_3} - 2M_{1,\theta_1} - M_1(M_2 + 2M_4) = 0, \tag{5.35}$$
$$M_{3,\theta_3} - M_{4,\theta_2} - (M_1 + 1)\, M_3 = 0.$$

The other case to consider is when the invariant (5.31) vanishes identically, i.e., $\tilde{J}_4 \equiv 0$. After absorbing all the nonconstant torsion, the resulting structure equations

$$\begin{aligned} d\theta_1 &= -\beta \wedge \theta_1 + \theta_2 \wedge \theta_3, \\ d\theta_2 &= -2\beta \wedge \theta_2, \\ d\theta_3 &= \alpha \wedge \theta_1 + \beta \wedge \theta_3, \\ d\theta_4 &= -2\beta \wedge \theta_2 + \theta_1 \wedge \theta_3, \end{aligned} \tag{5.36}$$

are isomorphic with the structure equations (5.16) for the prolonged fiber-preserving divergence equivalence problem, under the identification of ζ_i with θ_i, with α, β congruent to δ_4, δ_6 mod base. (This is somewhat surprising, since the cases (5.11), (5.32) which do not lead to prolongation *do not* have isomorphic structure equations. Even though (5.11) and (5.32) look very similar, they cannot be readily transformed into each other by different choices of normalization; indeed, in (5.32), it is impossible to normalize τ_{213} to 0, whereas, it must be 0 in (5.11).) Thus we can apply the identical intrinsic prolongation calculations as in part (b). The only task is to determine the new parametric expressions for the quantities appearing in the calculation. We find that the coefficient T of $\theta_2 \wedge \theta_3$ in $d\alpha$ is

$$T = -m + \tfrac{1}{3}\{2d_4^2 d_6 J_1 + d_6^3(2J_1 J_2 + J_{2,\eta_3})\},$$

which we translate to zero by prescribing the prolongation group parameter m in the obvious manner. In the resulting $\{e\}$-structure on $J^1 \times \tilde{G}_4$, the coefficient B of $\theta_1 \wedge \theta_2$ in $d\alpha$ is found to be

$$B = d_6^5(\tfrac{1}{3}J_{2,\eta_3,\eta_2} - J_{2,\eta_1}),$$

reconfirming the intrinsic result (5.22). As before, if $B \equiv 0$, i.e.,

$$J_{2,\eta_3,\eta_2} = 3J_{2,\eta_1}, \tag{5.37}$$

then we have the Maurer–Cartan equations for the symmetry algebra (5.24) of the Lagrangian $L = \frac{1}{2}p^2$. Otherwise, we can scale B to 1 by setting

$$d_6 = J_5 \equiv (\tfrac{1}{3}J_{2,\eta_3,\eta_2} - J_{2,\eta_1})^{-1/5}.$$

After this reduction, the coefficient A_2 of θ_2 in β is translated to 0 by setting

$$d_4 = -J_{5,\eta_2}.$$

The final structure equations are isomorphic to (5.28),

$$\begin{aligned} d\theta_1 &= \theta_2 \wedge \theta_3, \\ d\theta_2 &= 2A_1\theta_1 \wedge \theta_2, \\ d\theta_3 &= -B_2\theta_1 \wedge \theta_2 - 2A_1\theta_1 \wedge \theta_3, \\ d\theta_4 &= 2A_1\theta_1 \wedge \theta_2 + \theta_1 \wedge \theta_3, \end{aligned} \tag{5.38}$$

where the invariants are given explicitly by

$$\begin{aligned} A_1 &= J_{5,\eta_1} + \tfrac{2}{3}J_1 J_{5,\eta_2} + \tfrac{1}{3}J_3 J_5, \\ B_2 &= J_5^3 J_{5,\eta_2,\eta_2} - J_5^4 J_2 - 2J_5^2 J_{5,\eta_2}. \end{aligned}$$

Alternatively, we can rederive the invariants for the full divergence equivalence problem from those of the fiber-preserving divergence equivalence problem, based on the inclusion $G_3 \subset G_4$. For simplicity, we just do the case when the invariant I_4 does not vanish, so we did not need to prolong for the G_3-equivalence problem. Now the lifted coframe is

$$\begin{aligned} \theta_1 &= d_1\zeta_1, \\ \theta_2 &= d_2\zeta_1 + d_3\zeta_2, \\ \theta_3 &= d_4\zeta_1 + d_5\zeta_2 + d_6\zeta_3, \\ \theta_4 &= d_7\zeta_1 + (d_3 - 1)\,\zeta_2 + \zeta_4. \end{aligned}$$

The absorption and reduction process works just as before. Phases 1 and 2 lead to the same formulae for the group parameters d_1, d_2, d_3, d_5 as above; phase 3 is also the same, except now the formula for d_7 becomes

$$d_7 = -\frac{1}{3d_6^2}.$$

Proceeding to phase 4, we still have the normalization $\tau_{413} = \hat{\varepsilon} + 1$, where $\hat{\varepsilon} = \operatorname{sign}(3K_1 - 2)$. This gives the reduction $d_6 = \hat{\kappa} K_4$, where $\hat{\kappa} = \pm 1$ is another undetermined sign, and K_4 denotes the invariant

$$K_4 \equiv \sqrt{\left|\frac{1}{3}K_1 - \frac{2}{9}\right|} = \sqrt{\frac{|4L_{pp}^2 - 3L_{pp}L_{pppp}|}{9L_{pp}^2}}.$$

Assuming that K_4 does not vanish (otherwise we need to prolong), phase 5 leads to the final reduction

$$d_4 = -\hat{\kappa} K_{4,\zeta_2}.$$

We obtain exactly the same structure equations (5.32) as before, where the basic invariants are now given by the alternative expressions

$$\begin{aligned} M_1 &= \frac{K_1 K_4 - \frac{1}{3} K_4 - K_{4,\zeta_3}}{K_4^2}, \\ M_2 &= -2K_{4,\zeta_1} + (2K_1 - \tfrac{2}{3} - K_{4,\zeta_3})\, K_{4,\zeta_2} + K_2 K_4, \\ M_3 &= K_4^4 K_3 + K_4^3 K_{4,\zeta_2,\zeta_2}, \\ M_4 &= K_{4,\zeta_1} + K_1 K_{4,\zeta_2} + K_{4,\zeta_2,\zeta_3} + K_4^2, \end{aligned} \tag{5.39}$$

in terms of the fiber-preserving divergence invariants. The derived invariants also have alternative formulas

$$M_{,\theta_1} = K_4 M_{,\zeta_1} + \tfrac{2}{3} K_4 M_{,\zeta_2} + K_{4,\zeta_2} M_{,\zeta_3} - \tfrac{1}{3} K_4 M_{,\zeta_4},$$

$$M_{,\theta_2} = K_4^2 M_{,\zeta_2} + (K_4^2 - 1)\, M_{,\zeta_4}, \qquad M_{,\theta_3} = \frac{M_{,\zeta_3}}{K_4}, \qquad M_{,\theta_4} = M_{,\zeta_4}.$$

This concludes our discussion of the various equivalence problems. It is worthwhile to review some of the subtle features in the equivalence method which have arisen in the course of our presentation. Four points stand out: First is the appearance of ambiguous signs, as occurs in Section 5(a), which is essential for a correct solution to the equivalence problem, but has not been adequately dealt with in other treatments of the problem. All other treatments have just taken one branch of the square root, ignoring problems associated with its multiple-valuedness. Second is the fact that the fundamental invariants of an equivalence problem can be functionally dependent as functions of the base variables, but independent differential functions of the Lagrangian and its derivatives. The divergence equivalence problems both illustrate this phenomenon. Third, as mentioned after (3.14), is the fact that the determining functions for the reduced $\{e\}$-structure can themselves be multiply-valued functions, which serves to complicate the practical implementation of the solution. Fourth is the fact that the expressions for the invariants can have a much simpler expression in terms of covariant derivatives than has appeared before; the new formula (3.8) are surprisingly compact. The inductive method also gives manageable expressions for invariants that tax even the most sophisticated symbolic manipulation computer program. All these (and more) must be taken into account when one is practically implementing the solution to any equivalence problem.

6. Examples and Applications

We now illustrate how the results in the preceding sections are applied, by considering some representative examples of physically interesting Lagrangians. These are by no means intended to provide a complete collection of possible applications, but have been selected so as to give a flavor of what is possible. The main difficulty here is that the calculations tend to get very complicated, although this is unfortunately an unavoidable feature of the rather involved nature of the Lagrangian equivalence problem itself.

Example 6.1. Consider the Lagrangian

$$L = a(x,u)\sqrt{1+p^2}, \tag{6.1}$$

which arises in optics, in which $a(x, u)$ represents the refractive index of the optical medium, and, what is essentially the same problem, is a special case of the equation for geodesics on a surface; it also includes the classical brachistochrone problem as a special case. The Euler–Lagrange equation is

$$u'' = \frac{a_u - u'a_x}{a}(1+u'^2).$$

We compute the fiber-preserving invariants (3.8) for L,

$$I_1 = \frac{1}{p^2},$$

$$I_2 = \frac{a_{xu}(p^3-p) - 2a_{uu}p - a_x a_u(3p^3+p) + a_u^2(4p^2+2)}{a^4 p^2},$$

$$I_3 = 2\frac{\sqrt{1+p^2}}{p}\frac{a_u - pa_x}{a^2}.$$

One determining function is easy to determine explicitly; we find

$$I_{1,\xi_3} = -2p^{-4} - 2p^{-2} = -2I_1^2 - 2I_1. \tag{6.2}$$

The others are considerably more complicated.

In order to analyze this problem further, we note that the invariants I_1, I_2, I_3 will be functionally independent, and the structure equations will have maximal rank 3 (and order 0) if the collection of functions

$$\frac{a_{xu}}{a^3}, \quad \frac{a_{uu}}{a^3}, \quad \frac{a_x}{a^2}, \quad \frac{a_u}{a^2} \tag{6.3}$$

has rank 2. For illustration, let us look at the degenerate cases, when the rank is less than 2.

Rank 0. This occurs when all the functions (6.3) are constants. A simple calculation shows that this only occurs when

$$a(x, u) = \frac{1}{kx + lu + m},$$

where k, l, m are constants.

Rank 1. If only one of the three functions (6.3) is independent, then there are three possible cases,

$$a(x, u) = \frac{1}{kx + \psi(u)},$$

$$a(x, u) = \frac{1}{ku + \psi(x)},$$

$$a(x, u) = \psi(kx + mu),$$

where k, m are constants, and $\psi(t)$ an arbitrary non-affine analytic function. In all three cases, the order is always 0. These are the only Lagrangians of type (6.1) which lead to nonmaximal rank equivalence problems in the standard fiber-preserving case.

EXAMPLE 6.2. Even more basic is the Lagrangian

$$L = \tfrac{1}{2} p^2 + a(x, u), \tag{6.4}$$

with Euler–Lagrange equation

$$u'' = \frac{\partial a}{\partial u}(x, u).$$

This problem describes the one-dimensional motion of a particle subjected to a conservative force field, with x presenting the time variable, and $a(x, u)$ the potential function. We compute the fiber-preserving invariants for L:

$$I_1 = \frac{1}{2} + \frac{a}{p^2} = \frac{L}{p^2},$$

$$I_2 = \frac{2a_u^2 - p(a_{xu} + 2pa_{uu})}{\cdot p^2 L^2},$$

$$I_3 = \frac{2aa_u - pa_x - p^2 a_u}{pL^2}.$$

Note that

$$I_{1,\xi_3} = -2\frac{a^2}{p^4} - \frac{a}{p^2} = -2I_1^2 + I_1. \tag{6.5}$$

Comparing (6.2) and (6.5), we immediately deduce that a quadratic Lagrangian (6.4) can never be equivalent to an optical Lagrangian (6.1) under a fiber-preserving transformation. In order to analyze this problem further, we introduce the equivalent invariant

$$\hat{I}_1 \equiv I_1 - \frac{1}{2} = \frac{a}{p^2},$$

in terms of which the other two invariants are given by

$$I_2 = 2\frac{a_u^2}{a^3}\hat{I}_1^3 - \frac{a_{xu}}{a^{5/2}}\hat{I}_1^{5/2} - 2\frac{a_{uu}}{a^2}\hat{I}_1^2,$$

$$I_3 = 2\frac{a_u}{a^{3/2}}\hat{I}_1^{5/2} - \frac{a_x}{a^2}\hat{I}_1^2 - \frac{a_u}{a^{3/2}}\hat{I}_1^{3/2}.$$

Clearly, $\hat{I}_1$, I_2, I_3 will be functionally independent, and the structure equations will have maximal rank 3 (and order 0) if the collection of functions

$$\frac{a_{xu}}{a^{5/2}}, \quad \frac{a_{uu}}{a^2}, \quad \frac{a_x}{a^2}, \quad \frac{a_u}{a^{3/2}} \tag{6.6}$$

has rank 2. The degenerate cases here are now:

Rank 0.

$$a(x, u) = \frac{1}{kx + l} \quad \text{or} \quad a(x, u) = \frac{1}{(ku + l)^2},$$

where k and l are constants.

Rank 1.

$$a(x, u) = \frac{1}{kx + \psi(u)},$$

$$a(x, u) = \frac{1}{(ku + \psi(x))^2},$$

$$a(x, u) = \psi(kx + mu),$$

where k, m are constants, and $\psi(t)$ an arbitrary nonconstant analytic function.

Actually, we can say quite a bit more about these particular Lagrangians. We note that they lead to prolonged problems under both divergence equivalence problems. To shorten the discussion, we jump ahead to the most general case of divergence equivalence under point transformations ($m=4$), and determine the entire class of Lagrangians for which we are required to prolong our equivalence problem.

THEOREM 6.3. *Let $L(x, u, p)$ be a nonaffine Lagrangian. Then the corresponding coframe* (5.30) *for 4-equivalence leads to a prolongation structure if and only if the Lagrangian is equivalent to a quadratic Lagrangian of the elementary form*

$$L_0 = \pm\tfrac{1}{2}p^2 + a(x, u) \tag{6.7}$$

considered in Example 6.2.

Proof. According to Section 5c, we are lead to a prolonged equivalence problem for 4-equivalence if and only if the invariant $\tilde{J}_4$ (cf. (5.31)) vanishes, i.e., if and only if L satisfies the ordinary differential equation

$$3L_{pp}L_{pppp} = 4L_{ppp}^2.$$

It is not hard to see that L is a solution of this ordinary differential equation if and only if it is given either by the general solution

$$L = \frac{1}{\alpha p + \beta} + \gamma p + \delta, \tag{6.8}$$

or by the singular solution

$$L = \tfrac{1}{2}\rho p^2 + \sigma p + \tau. \tag{6.9}$$

Here α, β, γ, δ, ρ, σ, τ are arbitrary functions of x and u. We have shown that any Lagrangian leading to a prolonged structure must be of the form (6.8) or (6.9). Therefore, we need only show how these particular Lagrangians can be transformed into the elementary form (6.7).

As a first step, note that the affine terms in these Lagrangians can be integrated by parts to remove the coefficient of p; specifically, the affine Lagrangian $\gamma(x, u)p + \delta(x, u)$ is divergence equivalent to the p-independent Lagrangian $a(x, u)$, where $a = \delta - \varphi_x$, and $\varphi(x,u)$ is any potential for γ, i.e., $\varphi_u = \gamma$. Therefore, we can take γ and σ in (6.8), (6.9) to be zero without loss of generality.

The next step is to show how to transform a Lagrangian of the form

$$L = \frac{1}{\alpha p + \beta} + \delta \tag{6.10}$$

into one of the form

$$L = \tfrac{1}{2}\rho p^2 + \tau. \tag{6.11}$$

Consider the hodograph-like transformation

$$x \to \varphi(x, u), \qquad u \to x.$$

Under this transformation, the Lagrangian (6.11) gets transformed into

$$\tilde{L} = \frac{\tilde{\rho}}{2(\varphi_u p + \varphi_x)} + \tilde{\tau}(\varphi_u p + \varphi_x),$$

where

$$\tilde{\rho}(x, u) = \rho(\varphi(x, u), x), \qquad \tilde{\tau}(x, u) = \tau(\varphi(x, u), x). \tag{6.12}$$

Integrating the second summand by parts, we have transformed (6.11) into

$$\tilde{L} = \frac{\tilde{\rho}}{2(\varphi_u p + \varphi_x)} + \gamma^*, \qquad \text{where} \quad \gamma^* = \tilde{\tau}\varphi_x - \left\{\int^u \tilde{\tau}\varphi_u \, du\right\}_x.$$

Comparing this expression with (6.10), we see that we can complete the transformation provided we

(i) let $\tilde{\rho}(x, u)$ be an integrating factor for the one-form $\alpha(x, u)\, du + \beta(x, u)\, dx$,

(ii) let $\varphi(x, u)$ be the resulting first integral, $d\varphi = \frac{1}{2}\tilde{\rho}(\alpha\, du + \beta\, dx)$, and

(iii) let $\tilde{\tau}(x, u)$ be a solution to the first order partial differential equation

$$\varphi_x \tilde{\tau}_u - \varphi_u \tilde{\tau}_x = \gamma_u.$$

We recover ρ and τ by inverting (6.12).

The final step in the proof is to show how a Lagrangian of type (6.11) can be transformed into one of type (6.7). Consider the transformation

$$x \to x, \qquad u \to \psi(x, u).$$

Applying this transformation to the Lagrangian (6.7), and performing an integration by parts, we derive a Lagrangian of the form (6.11), with

$$\rho = \pm\frac{1}{2}\psi_u^2, \qquad \tau = \tilde{a} \pm \frac{1}{2}\psi_x^2 - \left\{\int^u \psi_x \psi_u \, du\right\}_x,$$

where

$$\tilde{a}(x, u) = a(\varphi(x, u), x).$$

Clearly, outside the singular points where ρ vanishes, we can solve for a suitable ψ and $\tilde{a}$, and thereby complete the transformation. The proof is complete.

In particular, Theorem 6.3 shows that an optical Lagrangian (6.1) is not even divergence equivalent to a quadratic Lagrangian (6.7), reconfirming the calculations in Examples 6.1 and 6.2.

Once we have a prolonged Lagrangian equivalence problem, there are two further possible branches, depending on whether or not condition (5.37) is satisfied. Using our "canonical form" (6.7) for a prolongation Lagrangian, we can distinguish the cases of maximal and nonmaximal symmetry.

THEOREM 6.4. *A Lagrangian of the form* (6.7) *is equivalent to the simple Lagrangian* $\frac{1}{2}p^2$ *if and only if* $a(x, u) = \lambda(x)u^2 + \mu(x)u + \nu(x)$.

The proof follows either by computing the invariants in (5.37) for the Lagrangian (6.7), or, more simply, by a direct analysis of the possible transformations. Incidentally, the latter method shows that we can always choose the canonical form (6.7) such that $c(x, 0) = c_u(x, 0) = c_{uu}(x, 0) = 0$, i.e., c is at least cubic in u.

7. TRESSE'S EQUIVALENCE PROBLEM FOR EULER–LAGRANGE EQUATIONS

In his thesis [24], Tresse considered the equivalence problem for second order ordinary differential equations

$$u'' = Q(x, u, u'),$$

under the pseudo-group of point transformations

$$\bar{x} = \varphi(x, u), \qquad \bar{u} = \psi(x, u). \tag{7.1}$$

Although his work predated Cartan's powerful techniques, he nevertheless was able to determine a complete set of relative invariants for such an equation and thereby completely solve this problem. It is easy to recast Tresse's problem in a form amenable to Cartan's method (cf. [6]). The appropriate coframe on the base is given by

$$\varpi_1 = du - p\,dx, \qquad \varpi_2 = dx, \qquad \varpi_3 = dp - Q\,dx,$$

and the pseudo-group of point transformations which maps solutions to solutions is encoded by the structure group

$$H_2 = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ h_2 & h_3 & 0 \\ h_4 & 0 & h_6 \end{pmatrix} : h_i \in \mathbb{R},\ h_1 \cdot h_3 \cdot h_6 \neq 0 \right\},$$

leading to the lifted coframe

$$v = h \cdot \varpi, \qquad h \in H_2. \tag{7.2}$$

This equivalence problem has been solved by a number of researchers, but, curiously, the complete solution based on Cartan's methods still has not appeared in the published literature. A partial solution, detailing the necessary prolongation, and providing the reduction to an $\{e\}$-structure in the prolonged problem, but stopping short of reduction to an $\{e\}$-structure on the base, has recently become available [14]. The corresponding problem for fiber-preserving transformations, which corresponds to the subgroup

$$H_1 = \left\{ \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_3 & 0 \\ h_4 & 0 & h_6 \end{pmatrix} : h_i \in \mathbb{R},\ h_1 \cdot h_3 \cdot h_6 \neq 0 \right\},$$

has been treated by Kamran *et al.* [17], and leads to interesting differential-geometric characterizations of the Painlevé equations.

Note that the Lagrangian equivalence problem under point transformations (with or without divergence) is a special case of Tresse's equivalence problem, since the equivalence of two Lagrangians implies the equivalence of their Euler–Lagrange equations. The converse, however, is not true, and leads to the interesting possibility of inequivalent Lagrangians that give rise to the same Euler–Lagrange equation. It is well known that a scalar second order ordinary differential equation can always be identified with the Euler–Lagrange equation for an infinite collection of inequivalent variational problems [9; 15; 18, Exercise 5.38]. Thus it is of interest to relate the Lagrangian equivalence problem with Tresse's equivalence problem for a second order ordinary differential equation. For instance, in light of Theorem 6.3, it would be of great interest to see how the special structure of the Euler–Lagrange equation $u'' = Q(x, u)$, corresponding to a Lagrangian of the form (6.7), manifests itself in the Tresse problem.

In this section, we will content ourselves with setting up the connection between the various Lagrangian equivalence problems and Tresse's problem. We will not, for reasons of space, complete the solution to the

Tresse problem so as to determine the explicit relations between our Lagrangian invariants and Tresse's invariants; however, the reader will have all the necessary tools to complete this calculation by this point.

In accordance with our basic philosophy, we seek to induce the solution to the Tresse problem from our solutions to the Lagrangian problem. Here, however, the construction is not quite as straightforward as it was in Section 5 since (i) the base coframes are not the same, and (ii) the Lagrangian groups G_m are not subgroups of H_1 or H_2, so our naïve inductive approach is no longer applicable. However, it is not too hard to see what we must do is take into account some of the reductions resulting from Cartan's procedure so as to relate the coframes and the groups. The easiest approach is to explicitly indicate the connection in the case of fiber-preserving standard equivalence, from which the more complicated cases will follow. Looking at the invariant coframe (3.4), we see that it is of the form

$$\xi = h_0 \cdot \varpi,$$

where

$$h_0 = \begin{pmatrix} L_p & 0 & 0 \\ 0 & L & 0 \\ \dfrac{L_u}{L} & 0 & \dfrac{L_p}{L} \end{pmatrix} \in H_1 \subset H_2.$$

In a sense, the invariance of the Euler–Lagrange expression dictates what some of the group reductions in the Cartan algorithm for the Lagrangian equivalence problem must be. The resulting normalizations must necessarily be adaptable to the Tresse equivalence problem, in order that the Lagrangian problem be considered as a "subproblem" of the Tresse problem for its Euler–Lagrange equation. Moreover, Cartan's method automatically produces the proper adapted coframe for use in the induced version of the latter problem. Clearly, we can now work with the H_2-adapted coframe

$$\nu = h \cdot \xi = \tilde{h} \cdot \varpi, \qquad h = \tilde{h} \cdot h_0^{-1} \in H_2,$$

and use the structure equations (3.5), as in Section 5, so as to determine the connection between the Tresse invariants and the Lagrangian invariants.

The connections for the other problems work similarly, since in each case the group reductions prescribed by the equivalence method are precisely those needed to ensure that some suitable reduced group is a subgroup of H_2 (or H_1 in the fiber-preserving cases). For example, in the point transformation standard equivalence problem, we find that the coframes are related by

$$\eta = g_2 \cdot \xi = g_2 \cdot h_0 \cdot \varpi,$$

where $g_2 \in G_2 \cap H_2$, and h_0 is as above, since the second loop through the procedure has required that $b_5 = 0$, $b_6 = b_1/b_3$. Similarly, in the full divergence equivalence problem,

$$\theta = g_4 \cdot \eta = g_4 \cdot g_2 \cdot h_0 \cdot \varpi = \hat{h} \cdot \varpi,$$

where $g_4 \in G_4 \cap H_2$, since we must have $d_5 = 0$, $d_6 = d_1/d_3$, and hence $\hat{h} \in H_2$. Therefore, we can safely use our inductive proceudre on the adapted coframe

$$\sigma = h \cdot \theta, \qquad h \in H_2,$$

to solve the Lagrangian version of the Tresse equivalence problem. The details, however, are sufficiently complicated that we will defer them to a future publication.

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