REGULARITY OF PSEUDOGROUP ORBITS

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Let $\mathcal{G}$ be a Lie pseudogroup acting on a manifold M. In this paper we show that under a mild regularity condition the orbits of the induced action of $\mathcal{G}$ on the bundle $J^n(M,p)$ of nth order jets of p-dimensional submanifolds of M are immersed submanifolds of $J^n(M,p)$.

1. Introduction

Lie pseudogroups, roughly speaking, are infinite dimensional counterparts of local Lie groups of transformations. The first systematic study of pseudogroups was carried out at the end of the 19th century by Lie, whose great insight in the subject was to place the additional condition on the local transformations in a pseudogroup that they form the general solution of a system of partial differential equations, the determining equations for the pseudogroup. Nowadays these Lie or continuous pseudogroups play an important role in various problems arising in geometry and mathematical physics including symmetries of differential equations, gauge theories, Hamiltonian mechanics, symplectic and Poisson geometry, conformal geometry of surfaces, conformal field theory and the theory of foliations.

Since their introduction a considerable effort has been spent on develop-

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ing a rigorous foundation for the theory of Lie pseudogroups and the invariants of their action, and on their classification problem, see e.g. Refs. [5], [6], [7], [8], [9], [11] and the references therein. More recently, the authors of the paper at hand have employed a moving frames construction [3], [4] to establish a concrete theory for Lie pseudogroups amenable to practical computations. As applications, a direct method for uncovering the structure equations for Lie pseudogroups from the determining equations for the infinitesimal generators of the pseudogroup action is obtained (see, in particular, the work [1] on the structure equations for the KdV– and KP–equations) and systematic methods for constructing complete systems of differential invariants and invariant forms for pseudogroup actions are developed. Moreover, the new methods immediately yield syzygies and recurrence relations amongst the various invariant quantities which are instrumental in uncovering their structure, the knowledge of which is pivotal e.g. in the implementation of Vessiot's method of group splitting for obtaining explicit noninvariant solutions for systems of partial differential equations.

Let $\mathcal{G}$ be a Lie pseudogroup (a precise definition will be given in Sec. 2) acting of a manifold M. The action of $\mathcal{G}$ on M naturally induces an action of $\mathcal{G}$ on the extended jet bundle $J^n(M,p)$ of nth order jets of submanifolds of M by the usual prolongation process. Our goal in this paper is to prove that under a mild regularity condition on the action of the pseudogroup $\mathcal{G}$ on M the orbits of $\mathcal{G}$ in $J^n(M,p)$ are immersed submanifolds for n sufficiently large. We were originally lead to the problem in connection of the research reported in Ref. [4] and the result is of importance in the theoretical constructs therein. Interestingly, as we will see, the submanifold property of $\mathcal{G}$ orbits in $J^n(M,p)$ is closely related to local solvability of the determining equations for the infinitesimal generators of the pseudogroup action on M. The proof of our main result relies on classical work [12] on the structure of the orbits of a set of vector fields originally arising in the study of the accessibility question in the context of control theory.

In Sec. 2 we cover some background material on Lie pseudogroups and discuss the regularity condition for pseudogroup actions needed in our main result. Sec. 3 is dedicated to the proof of the submanifold property of orbits of the action of $\mathcal{G}$ on the extended jet bundles $J^n(M,p)$.

2. Tameness of Lie Pseudogroups

Let $\mathcal{D} = \mathcal{D}(M)$ denote the pseudogroup of all local diffeomorphisms of a manifold. We write $j_z^{(n)}\varphi$ for the nth order jet of $\varphi \in \mathcal{D}$ at z and $\sigma^{(n)} : \mathcal{D}^{(n)} \to M$ for the associated jet bundle, where $\sigma^{(n)}(j_z^n\varphi) = z$ stands for the source map. We furthermore write $\tau^{(n)} : \mathcal{D}^{(n)} \to M$, $\tau^{(n)}(j_z^n\varphi) = \varphi(z) = Z$ for the target map.

Definition 2.1. A subset $\mathcal{G} \subset \mathcal{D}$ is called a pseudogroup[3] acting on M if

(1) the restriction $\mathrm{id}_{|\mathcal{O}}$ of the identity mapping to any open $\mathcal{O} \subset M$ belongs to $\mathcal{G}$;
(2) if φ, $\psi \in \mathcal{G}$, then also the composition $\varphi \circ \psi \in \mathcal{G}$ where defined;
(3) if $\varphi \in \mathcal{G}$, then also the inverse mapping $\varphi^{-1} \in \mathcal{G}$.

A pseudogroup $\mathcal{G}$ is called a Lie pseudogroup if, in addition, there exists $N \geq 1$ so that the following conditions are satisfied for all $n \geq N$:

(4) $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is a smooth, embedded subbundle;
(5) $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle map;
(6) a local diffeomorphism φ of M belongs to $\mathcal{G}$ if and only if $z \to j_z^{(n)}\varphi$ is a local section of $\sigma^{(n)} : \mathcal{G}^{(n)} \to M$;
(7) $\mathcal{G}^{(n)} = \mathrm{pr}^{n-N}\mathcal{G}^{(N)}$ is obtained by prolongation.

We call the smallest N satisfying the above conditions the order of the pseudogroup, and unless otherwise specified, we will assume that $n \geq N$ in what follows. Note that by (1) and (2), the restriction $\varphi_{|\mathcal{O}}$ of a transformation $\varphi \in \mathcal{G}$ to any open subset $\mathcal{O}$ of the domain of φ is again a member of the pseudogroup.

Fix local coordinates $(z1, \ldots, z^m)$ about some $p \in M$, and let $(z, Z) = (z1, \ldots, z^m, Z1, \ldots, Z^m)$ denote the induced product coordinates about $(p, p) \in \mathcal{D}^{(0)} = M \times M$. Due to conditions (4) and (6) above, pseudogroup transformations are locally determined by a system

$$F_\alpha(z, Z^{(n)}) = 0, \qquad \alpha = 1, \ldots, k, \tag{1}$$

of partial differential equations, the determining equations for $\mathcal{G}$. Here $n \geq N$ is fixed and $(z, Z^{(n)})$ stands collectively for the coordinates of $\mathcal{D}^{(n)}$ induced by (z, Z). By Definition 2.1 the above equations are locally solvable, that is, given a jet $g_o^{(n)} = (z_o, Z_o^{(n)})$ satisfying (1), then there is a solution $\varphi \in \mathcal{G}$ of the equations so that $j_{z_o}^{(n)}\varphi = g_o^{(n)}$.

Let $\mathcal{X} = \mathcal{X}(M)$ denote the space of locally defined vector fields on M. Thus the domain $\mathcal{U}(\mathbf{v}) \subset M$ of $\mathbf{v} \in \mathcal{X}$ is an open subset of M. The vertical lift $\mathbf{V}^{(n)}$ of a vector field $\mathbf{v} \in \mathcal{X}$ to $\mathcal{D}^{(n)}$ is the infinitesimal generator of the local one-parameter group $\Phi_t^{(n)}$ of transformations acting on $\mathcal{D}^{(n)}$ defined by $\Phi_t^{(n)}(j_z^{(n)}\varphi) = j_z^{(n)}(\Phi_t \circ \varphi)$, where Φ_t stands for the flow map of $\mathbf{v}$. Note that the domain of $\mathbf{V}^{(n)}$ is $\tau^{(n)-1}(\mathcal{U}(\mathbf{v}))$.

Pick $\mathbf{v} = \sum_{a=1}^m \mathbf{v}^a(z)\partial_{z^a} \in \mathcal{X}$ and write $\mathbf{V} = \sum_{a=1}^m \mathbf{V}^a(Z)\partial_{Z^a}$ for the vertical counterpart of $\mathbf{v}$. Then $\mathbf{V}^{(n)}$ is simply given by the usual prolongation formula[2],

$$\mathbf{V}^{(n)}(z, Z^{(n)}) = \sum_{a=1}^m \sum_{|J| \le n} (D_J \mathbf{V}^a)(z, Z^{(n)})\partial_{Z_J^a}, \tag{2}$$

where $J = (j_1, \ldots, j_p)$ is stands for a multi-index of integers, $D_J = D_{j_1} \cdots D_{j_p}$ for the product of the total derivative operators $D_j = \partial_{z^j} + \sum_{a=1}^m \sum_{|J| \ge 0} Z_{Jj}^a \partial_{Z_J^a}$, and where the Z_J^a denote the components of the fiber coordinates on $\mathcal{D}^{(n)}$ induced by (z, Z). In particular, at the nth jet $\mathbb{I}_z^{(n)}$ of the identity mapping, Eq. (2) becomes

$$\mathbf{V}^{(n)}_{\mathbb{I}_z^{(n)}} = \sum_{a=1}^m \sum_{|J| \le n} \partial_{z^J}\mathbf{v}^a(z)\partial_{Z_J^a}, \tag{3}$$

where we have again used the obvious multi-index notation.

We denote the space of n jets of local diffeomorphisms with source at a fixed $z \in M$ by $\mathcal{D}^{(n)}{}_{|z}$. It is easy to see that $\mathcal{D}^{(n)}{}_{|z}$ is a regular submanifold of $\mathcal{D}^{(n)}$. Write $R_{h^{(n)}}$ for the right action of a jet $h^{(n)} \in \mathcal{D}^{(n)}$ on the source fiber $\mathcal{D}^{(n)}{}_{|\tau^{(n)}(h^{(n)})} = \sigma^{(n)-1}(\tau^{(n)}(h^{(n)}))$ by $R_{h^{(n)}} g^{(n)} = j^n_{\sigma^{(n)}(h^{(n)})}(\varphi \circ \psi)$, where $h^{(n)} = j^n_{\sigma^{(n)}(h^{(n)})}\psi$, $g^{(n)} = j^n_{\tau^{(n)}(h^{(n)})}\varphi$. Then, by differentiating the identity

$$R_{h^{(n)}}\Phi_t^{(n)}(g^{(n)}) = \Phi_t^{(n)}(R_{h^{(n)}}g^{(n)}),$$

it is easy to see that $\mathbf{V}^{(n)}$ is $R_{h^{(n)}}$-invariant, that is,

$$R_{h^{(n)}*}(\mathbf{V}^{(n)}(g^{(n)})) = \mathbf{V}^{(n)}(R_{h^{(n)}}g^{(n)}), \tag{4}$$

whenever $\sigma^{(n)}(g^{(n)}) = \tau^{(n)}(h^{(n)})$. Note that the action of $R_{h^{(n)}}$ on $\mathbf{V}^{(n)}(g^{(n)})$ is well defined since $\mathbf{V}^{(n)}$ is a vertical vector field.

Next let $\mathcal{G}$ be a Lie pseudogroup. A local vector field $\mathbf{v} \in \mathcal{X}$ on M is a $\mathcal{G}$ vector field if its flow map Φ_t is a member of $\mathcal{G}$ for all fixed t on some interval about 0. We denote the space of $\mathcal{G}$ vector fields by $\mathcal{X}_{\mathcal{G}}$.

The infinitesimal determining equations $L_\alpha(z, j_z^{(n)}\mathbf{v}) = 0$ for $\mathcal{G}$ vector fields $\mathbf{v}$ can be obtained by linearizing the determining equations (1) for $\mathcal{G}$

at $\mathbb{I}_z^{(n)}$, that is,

$$L_\alpha(z, j_z^{(n)}\mathbf{v}) = \frac{d}{dt} F_\alpha(z, j_z^{(n)}\Phi_t)_{|t=0} = 0 \quad \text{for all } z \in \mathcal{U}(\mathbf{v}). \tag{5}$$

By (2), this is equivalent to the condition

$$(\mathbf{V}^{(n)} F_\alpha)(z, \mathbb{I}_z^{(n)}) = 0 \quad \text{for all } z \in \mathcal{U}(\mathbf{v}).$$

As a consequence of our definition of a Lie pseudogroup, the infinitesimal determining equations completely characterize $\mathcal{G}$ vector fields.

Proposition 2.1. *A local vector field $\mathbf{v} \in \mathcal{X}$ is a $\mathcal{G}$ vector field if and only if*

$$L_\alpha(z, j_z^{(n)}\mathbf{v}) = 0 \qquad \textit{for all } z \in \mathcal{U}(\mathbf{v}) \quad \textit{with some } n \geq N. \tag{6}$$

Proof. We only need to show that a vector field $\mathbf{v}$ satisfying (6) is a $\mathcal{G}$ vector field. First note that equation (6) implies that $\mathbf{V}^{(n)}$ is tangent to $\mathcal{G}^{(n)}$ at $\mathbb{I}_z^{(n)}$ for all $z \in \mathcal{U}(\mathbf{v})$. Thus, by the right invariance (4), $\mathbf{V}^{(n)}$ is tangent to $\mathcal{G}^{(n)}$ at any $g^{(n)} \in \mathcal{G}^{(n)} \cap \tau^{-1}(\mathcal{U}(\mathbf{v}))$, and, consequently,

$$\Phi_t^{(n)}(\mathbb{I}_z^{(n)}) \in \mathcal{G}^{(n)} \quad \text{for all } t. \tag{7}$$

But (7) implies that

$$j_z^{(n)}\Phi_t \in \mathcal{G}^{(n)} \quad \text{for all } t \text{ and } z \in \mathcal{U}(\mathbf{v}),$$

and consequently, $\Phi_t \in \mathcal{G}$ for all t sufficiently small and thus $\mathbf{v}$ is a $\mathcal{G}$ vector field. □

Recall that the lift $\Phi_t^{(n)}$ of the flow of $\mathbf{v} \in \mathcal{X}$ to $\mathcal{D}^{(n)}$ is tangent to the fibers $\mathcal{D}^{(n)}{}_{|z}$. We thus obtain canonical mappings[10] $\lambda_{g^{(n)}}$ from the space of n-jets $\mathcal{X}_Z^{(n)}$, $Z = \tau^{(n)}(g^{(n)})$, of local vector fields into the tangent space $T_{g^{(n)}}\mathcal{D}^{(n)}{}_{|z}$ of $\mathcal{D}^{(n)}{}_{|z}$ at $g^{(n)}$ by

$$\lambda_{g^{(n)}}(j_Z^{(n)}\mathbf{v}) = \mathbf{V}^{(n)}(g^{(n)}). \tag{8}$$

Proposition 2.2. *The mappings*

$$\lambda_{g^{(n)}} : \mathcal{X}_Z^{(n)} \to T_{g^{(n)}}\mathcal{D}^{(n)}{}_{|z}, \quad \textit{where } z = \sigma^{(n)}(g^{(n)}),\ Z = \tau^{(n)}(g^{(n)}),$$

are well defined.

Proof. We only need to show that if $\mathbf{v}_1$, $\mathbf{v}_2$ are two local vector fields so that $j_Z^{(n)}\mathbf{v}_1 = j_Z^{(n)}\mathbf{v}_2$, then

$$\mathbf{V}_1^{(n)}(g^{(n)}) = \mathbf{V}_2^{(n)}(g^{(n)}) \quad \text{for all } g^{(n)} \text{ with } \tau^{(n)}(g^{(n)}) = Z. \tag{9}$$

By (3), Eq. (9) holds for $g^{(n)} = \mathbb{I}_Z^{(n)}$. But then, due to the invariance (4) of $\mathbf{V}_1^{(n)}$, $\mathbf{V}_2^{(n)}$ under the right translations $R_{g^{(n)}}$, Eq. (9) must hold for all $g^{(n)}$ with $\tau^{(n)}(g^{(n)}) = Z$. □

Let $\mathcal{G}$ be a Lie pseudogroup and write $\mathcal{G}^{(n)}{}_{|z} = \mathcal{G}^{(n)} \cap \mathcal{D}^{(n)}{}_{|z}$. Note that $\mathcal{G}^{(n)}{}_{|z}$ is a regular submanifold of $\mathcal{G}^{(n)}$. In fact, the source mapping $\sigma^{(n)}$ restricted to $\mathcal{G}^{(n)}{}_{|z}$ is of maximal rank as is seen by observing that for $\varphi \in \mathcal{G}$, the local section $j_z^{(n)}\varphi$ of $\mathcal{G}^{(n)} \to M$ yields a local right inverse for $\sigma^{(n)}$.

By the proof of Proposition 2.2, the mappings $\lambda_{g^{(n)}}$ for $g^{(n)} \in \mathcal{G}^{(n)}$ restrict to mappings from the n-jets of $\mathcal{G}$ vector fields $\mathcal{X}_{\mathcal{G},Z}^{(n)}$ at $Z = \tau^{(n)}(g^{(n)})$ into the tangent space $T_{g^{(n)}}\mathcal{G}^{(n)}{}_{|z}$ of the source fiber $\mathcal{G}^{(n)}{}_{|z}$ at $g^{(n)}$.

Definition 2.2. A Lie pseudogroup $\mathcal{G}$ is called *tame at order* n provided that the mappings

$$\lambda_{g^{(n)}} \colon \mathcal{X}_{\mathcal{G},Z}^{(n)} \to T_{g^{(n)}}\mathcal{G}^{(n)}{}_{|z}$$

are isomorphisms for all $g^{(n)} \in \mathcal{G}^{(n)}$.

Remark 2.1. By Eqs. (2) and (3), the mappings $\lambda_{g^{(n)}}$ are automatically monomorphisms. Thus, by the right invariance (4), $\mathcal{G}$ is a tame Lie pseudogroup provided that $\lambda_{\mathbb{I}_z^{(n)}}$ maps $\mathcal{X}_{\mathcal{G},z}^{(n)}$ onto $T_{\mathbb{I}_z^{(n)}}\mathcal{G}^{(n)}{}_{|z}$ for all $z \in M$. Thus in particular, when $\mathcal{G}$ is tame, the dimension of the space of n jets of $\mathcal{G}$ vector fields at $z \in M$ is constant in z.

Proposition 2.3. *A Lie pseudogroup $\mathcal{G}$ is tame at order n if and only if the nth order infinitesimal determining equations for $\mathcal{G}$ vector fields are locally solvable.*

Proof. We can use (5) to identify the solution manifold of the nth order infinitesimal determining equations with tangent vectors $\mathbf{w} \in T_{\mathbb{I}_z^{(n)}}\mathcal{G}^{(n)}{}_{|z}$. By Remark 2.1, a Lie pseudogroup is tame if and only if any $\mathbf{w} \in T_{\mathbb{I}_z^{(n)}}\mathcal{G}^{(n)}{}_{|z}$ can be represented by a solution of the equations, that is, provided that the infinitesimal determining equations are locally solvable. □

Remark 2.2. While the local solvability of the determining equations for $\mathcal{G}$ is built into Definition 2.1 of a Lie pseudogroup, as far as we know, the infinitesimal version of the equations does not necessarily possess this property. However, as of yet, we have been unable to construct an example of a Lie pseudogroup with infinitesimal determining equations that are not locally solvable.

3. Regularity of Orbits

Definition 3.1. Call $g^{(n)} \in \mathcal{G}^{(n)}$ *reachable* by $\mathcal{G}$ vector fields if there are $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathcal{X}_\mathcal{G}$ with flow maps $\Phi_t^{\mathbf{v}_1}, \ldots, \Phi_t^{\mathbf{v}_s}$ so that $g^{(n)} = j^{(n)}_{\sigma^{(n)}(g^{(n)})}(\Phi_{t_1}^{\mathbf{v}_1} \circ \cdots \circ \Phi_{t_s}^{\mathbf{v}_s})$ for some $t_1, \ldots, t_s$.

Write $\mathbb{I}^{(n)} = \cup_{z \in M} \mathbb{I}_z^{(n)} \in \mathcal{G}^{(n)}$ for the image of the section generated by the identity mapping and denote the connected component of $\mathcal{G}^{(n)}$ containing $\mathbb{I}^{(n)}$ by $\mathcal{G}_o^{(n)}$.

Theorem 3.1. *Let $\mathcal{G}$ be a tame Lie pseudogroup at order n. Then any $g^{(n)} \in \mathcal{G}_o^{(n)}$ is reachable by $\mathcal{G}$ vector fields.*

Proof. Denote by $\mathcal{R}_{|z_o} \subset \mathcal{G}_o^{(n)}{}_{|z_o}$ the set of jets with source at z_o that are reachable by $\mathcal{G}$ vector fields. Clearly $\mathcal{R}$ is non-empty. Our goal is to prove that $\mathcal{R}_{|z_o}$ is both open and closed in $\mathcal{G}_o^{(n)}{}_{|z_o}$.

First, let $g_o^{(n)} \in \mathcal{R}_{|z_o}$ and write $Z_o = \tau^{(n)}(g_o^{(n)})$. Choose a basis $\mathbf{w}_1, \ldots, \mathbf{w}_r$ for $T_{g_o^{(n)}}\mathcal{G}^{(n)}{}_{|z_o}$. By tameness, there are $\mathcal{G}$ vector fields $\mathbf{v}_j \in \mathcal{X}_\mathcal{G}$ defined in a neighborhood $\mathcal{N}_{Z_o}$ of Z_o in M so that

$$\mathbf{w}_j = \mathbf{V}_j^{(n)}(g_o^{(n)}), \quad j = 1, \ldots, r.$$

Due to linearity, any $\mathbf{w} = \sum_{i=1}^r a^i \mathbf{w}_i \in T_{g_o^{(n)}}\mathcal{G}^{(n)}{}_{|z_o}$ is a linear combination

$$\mathbf{w} = \sum_{i=1}^r a^i \mathbf{V}_i^{(n)}(g_o^{(n)})$$

of the vectors $\mathbf{V}_i^{(n)}(g_o^{(n)})$. Moreover, by shrinking $\mathcal{N}_{Z_o}$, if necessary, it is easy to verify that there is $\epsilon > 0$ so that for any $a = (a1, \ldots, a^r)$ with $\| a \| < 1$, the flow map $\Phi_t^{\mathbf{v}}$ of $\mathbf{v} = a1\mathbf{v}_1 + \cdots + a^r \mathbf{v}_r$ is defined on $(-\epsilon, \epsilon) \times \mathcal{N}_{Z_o}$.

By assumption,

$$g_o^{(n)} = j_{z_o}^{(n)}(\Phi_{t_1}^{\mathbf{y}_1} \circ \cdots \circ \Phi_{t_s}^{\mathbf{y}_s})$$

for some $\mathcal{G}$ vector fields $\mathbf{y}_1, \ldots, \mathbf{y}_s$ and for some $t_1, \ldots, t_s$. It is clear that there is a neighborhood $\mathcal{P}_{z_o}$ of z_o so that the composition

$$\Phi_{\epsilon/2}^{\mathbf{v}} \circ \Phi_{t_1}^{\mathbf{y}_1} \circ \cdots \circ \Phi_{t_s}^{\mathbf{y}_s}(z)$$

is defined for all $\mathbf{v} = a1\mathbf{v}_1 + \cdots + a^r\mathbf{v}_r$ with $\| a \| < 1$ and $z \in \mathcal{P}_{z_o}$.

Next consider the mapping

$$\Psi\colon \{(a1, \ldots, a^r)\} \to \mathcal{G}^{(n)}{}_{|z_o},$$
$$\Psi(a1, \ldots, a^r) = j^{(n)}_{z_o}(\Phi^{a1\mathbf{v}_1 + \cdots + a^r\mathbf{v}_r}_{\epsilon/2} \circ \Phi^{\mathbf{y}_1}_{t_1} \circ \cdots \circ \Phi^{\mathbf{y}_s}_{t_s}),$$

where $\| a \| < 1$. Then at $a1 = \cdots = a^r = 0$,

$$\Psi_*\,(\partial_{a^i}) = \frac{\epsilon}{2}\mathbf{w}_i. \tag{10}$$

Hence the Jacobian of Ψ at $a1 = \cdots = a^r = 0$ is non-degenerate, and so, in particular, the image of Ψ contains an open neighborhood $\mathcal{Q}_{g_o^{(n)}}$ of $g_o^{(n)}$ in $\mathcal{G}^{(n)}{}_{|z_o}$. Now by the definition of Ψ, any $g^{(n)} \in \mathcal{Q}_{g_o^{(n)}}$ is reachable by $\mathcal{G}$ vector fields and consequently, $\mathcal{R}_{|z_o} \subset \mathcal{G}_o^{(n)}{}_{|z_o}$ is open.

In order to show that $\mathcal{R}_{|z_o}$ is also closed in $\mathcal{G}_o^{(n)}{}_{|z_o}$, pick a sequence $\{g_i^{(n)}\} \subset \mathcal{R}_{|z_o}$ of jets in $\mathcal{R}_{|z_o}$ converging to $g_o^{(n)} \in \mathcal{G}_o^{(n)}{}_{|z_o}$. By the first part of the proof, there is a neighborhood $\mathcal{Q}_{g_o^{(n)}}$ of $g_o^{(n)}$ in $\mathcal{G}_o^{(n)}{}_{|z_o}$ so that any $g^{(n)} \in \mathcal{Q}^{(n)}_{g_o^{(n)}}$ can be expressed as

$$g^{(n)} = \Phi_t^{\mathbf{v}^{(n)}}(g_o^{(n)}) \tag{11}$$

for some $\mathcal{G}$ vector field $\mathbf{v}$. Choose i so large that $g_i^{(n)} \in \mathcal{Q}_{g_o^{(n)}}$. Then, by virtue of (11), we see that

$$g_o^{(n)} = \Phi_t^{\mathbf{v}^{(n)}}(g_i^{(n)})$$

for some $\mathcal{G}$ vector field $\mathbf{v}$. Consequently, $g_o^{(n)} \in \mathcal{R}_{|z_o}$ and $\mathcal{R}_{|z_o} \subset \mathcal{G}_o^{(n)}{}_{|z_o}$ is closed. This completes the proof of the Theorem. □

Next let $\mathcal{V}$ be a set of locally defined vector fields on a manifold M. An orbit $\mathcal{O}_p$ of $\mathcal{V}$ through $p \in M$ is the set of points that can be reached from p by a composition of flow maps of vector fields in $\mathcal{V}$,

$$\mathcal{O}_p = \{ q = \Phi^{\mathbf{v}_1}_{t_1} \circ \cdots \circ \Phi^{\mathbf{v}_r}_{t_r}(p) \,|\, t_1, \ldots, t_r \in \mathbb{R}, \mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathcal{V} \}.$$

We equip $\mathcal{O}_p$ with the strongest topology that makes all the maps

$$(t_1, \ldots, t_r) \in \mathbb{R}^r \to \Phi^{\mathbf{v}_1}_{t_1} \circ \cdots \circ \Phi^{\mathbf{v}_r}_{t_r}(p) \in \mathcal{O}_p \tag{12}$$

continuous. One can easily verify that this topology is independent of the point p on the orbit. We call $\mathcal{V}$ everywhere defined if every $p \in M$ is contained in the domain of at least one $\mathbf{v} \in \mathcal{V}$.

Theorem 3.2. *Let $\mathcal{V}$ be an everywhere defined set of local, smooth vector fields on M and let $\mathcal{O}_p$ be an orbit of $\mathcal{V}$. Then $\mathcal{O}_p$ admits a differentiable structure compatible with the topology defined by* (12) *in which it becomes an immersed submanifold of M.*

Proof. A proof can be found in Ref. [12]. □

We write $J^n = J^n(M, p)$ for the nth order extended jet bundle consisting of equivalence classes of p dimensional submanifolds of M under nth order contact and $\pi^n : J^n \to M$ for the canonical projection. We denote the action of a local diffeomorphism $\varphi \in \mathcal{D}$ on a jet $z^{(n)} \in J^n$ by $\varphi^{(n)} \cdot z^{(n)}$, where $z = \pi^n(z^{(n)})$ is contained in the domain of φ. This action factors into an action of the diffeomorphism jet bundle $\mathcal{D}^{(n)}$ on J^n in an obvious fashion.

The infinitesimal generators of the action of $\mathcal{D}$ on J^n are, by definition, the prolongations[2] to J^n of local vector fields on M obtained in local coordinates by the usual prolongation formula. Similarly, for a Lie pseudogroup $\mathcal{G}$ the infinitesimal generators of the action of $\mathcal{G}$ on J^n are, by definition, prolongations of $\mathcal{G}$ vector fields to J^n. We let $\mathfrak{g}^{(n)} \subset \mathcal{X}(J^n)$ stand for the Lie algebra of infinitesimal generators of $\mathcal{G}$.

We denote the $\mathfrak{g}^{(n)}$ orbit of a point $z^{(n)} \in J^n$ by $\mathcal{O}_{z^{(n)}}$ and the orbit of $z^{(n)}$ under the action of $\mathcal{G}$ on J^n by $\mathcal{O}'_{z^{(n)}}$. Then $\mathcal{O}_{z^{(n)}}$ consists of the points

$$\Phi_{t_1}^{\mathbf{v}_1(n)} \cdot \dots \cdot \Phi_{t_r}^{\mathbf{v}_r(n)} \cdot z^{(n)},$$

where each $\mathbf{v}_i$ is a $\mathcal{G}$ vector field. So obviously $\mathcal{O}_{z^{(n)}} \subset \mathcal{O}'_{z^{(n)}}$. Moreover, the orbit $\mathcal{O}'_{z^{(n)}}$ agrees with the orbit of $z^{(n)}$ under the induced action of the pseudogroup jet bundle $\mathcal{G}^{(n)}$, specifically,

$$\mathcal{O}'_{z^{(n)}} = \mathcal{G}^{(n)}{}_{|z} \cdot z^{(n)}.$$

We call a Lie pseudogroup $\mathcal{G}$ connected if the subbundle $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is connected.

Theorem 3.3. *Let $\mathcal{G}$ be a tame Lie pseudogroup at order n acting on M. Then the orbits of the action of $\mathcal{G}$ on J^n are immersed submanifolds of J^n.*

Proof. First assume that $\mathcal{G}$ is connected. By virtue of tameness of $\mathcal{G}$ and the prolongation formula, the Lie algebra $\mathfrak{g}^{(n)}$ of infinitesimal generators is everywhere defined on J^n. Thus by Theorem 3.2, the $\mathfrak{g}^{(n)}$ orbits $\mathcal{O}_{z^{(n)}}$ are immersed submanifolds of J^n. Hence, to conclude the proof of the Theorem for connected pseudogroups, we only need to show that $\mathcal{O}'_{z^{(n)}} \subset \mathcal{O}_{z^{(n)}}$.

For this, note that by Theorem 3.1, any jet $g^{(n)} \in \mathcal{G}^{(n)}$ can be expressed as

$$g^{(n)} = j_z^{(n)}(\Phi_{t_1}^{\mathbf{v}_1} \circ \cdots \circ \Phi_{t_r}^{\mathbf{v}_r}),$$

where the $\mathbf{v}_i$ are $\mathcal{G}$ vector fields and $z = \sigma^{(n)}(g^{(n)})$. Consequently,

$$g^{(n)} \cdot z^{(n)} = \Phi_{t_1}^{\mathbf{v}_1\,(n)} \cdot \cdots \cdot \Phi_{t_r}^{\mathbf{v}_r\,(n)} \cdot z^{(n)},$$

which shows that $\mathcal{O}'_{z^{(n)}} \subset \mathcal{O}_{z^{(n)}}$.

Next assume that $\mathcal{G}$ is not connected and let $\mathcal{G}_1^{(n)}$ be a connected component of $\mathcal{G}^{(n)}$ distinct from the connected component $\mathcal{G}_o^{(n)}$ containing $\mathbb{I}^{(n)}$. Write $\mathcal{G}_1^{(n)}{}_{|z} = \mathcal{G}_1^{(n)} \cap \mathcal{G}^{(n)}{}_{|z}$ and pick $g_o^{(n)} \in \mathcal{G}_1^{(n)}{}_{|z}$. Since $\mathcal{G}$ is tame, we can proceed as in the proof of Theorem 3.1 to show that any $g^{(n)} \in \mathcal{G}_1^{(n)}{}_{|z}$ can be expressed as

$$g^{(n)} = \Phi_{t_1}^{\mathbf{v}_1\,(n)} \circ \cdots \circ \Phi_{t_r}^{\mathbf{v}_s\,(n)}(g_o^{(n)}),$$

for some $\mathcal{G}$ vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_r$. This implies that the orbit of $z^{(n)} \in J^n$ under $\mathcal{G}_1^{(n)}$ coincides with the orbit of $g_o^{(n)} \cdot z^{(n)}$ under $\mathcal{G}_o^{(n)}$. This completes the proof of the Theorem. $\square$

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