Prolonged Analytic Connected Group Actions are Generically Free

Scot Adams
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
adams@math.umn.edu
http://www.math.umn.edu/∼adams

Peter J. Olver†
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@umn.edu
http://www.math.umn.edu/∼olver

Abstract. We prove that an effective, analytic action of a connected Lie group G on an analytic manifold M becomes free on a comeager subset of an open subset of M when prolonged to a frame bundle of sufficiently high order. We further prove that the action of becomes free on a comeager subset of an open subset of a submanifold jet bundle over M of sufficiently high order, thereby establishing a general result that underlies Lie's theory of symmetry groups of differential equations and the equivariant method of moving frames.

1. Introduction.

The analysis of prolonged transformation groups on jet bundles is of fundamental importance in Lie's theory of symmetry groups of differential equations and differential invariants, cf. [**16**, **19**], and in Cartan's method of moving frames, [**7**, **9**] and its equivariant generalization, [**8**, **14**, **18**]. However, despite its long pedigree, a number of basic issues in this subject remain unresolved.

A key question that arises in the moving frame construction is whether the prolonged transformation group action is free on submanifold jet bundles of sufficiently high order. (See Definition 2.2 for basic terminology.) It has been known for some time, [**16**, **17**], that a smooth Lie group action that is locally effective on subsets becomes locally free on a dense open subset of a sufficiently high order jet bundle. Until recently, in all known

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examples, such transformation group actions became, in fact, free on such a dense open subset of a possibly higher order jet bundle. This observation was thus conjectured to hold in general by the second author. However, recent results of the first author demonstrate that this conjecture is false in general for both smooth, [**2**], and analytic, [**3**], actions, but true for algebraic actions, [**1**]. The frame bundle conjecture is true for smooth actions of connected Lie groups with compact center, [**4**]. The goal of this paper is to prove a version of the freeness conjecture for analytic actions of *connected* Lie groups. For simplicity, we restrict our attention to global group actions here, although extensions to local actions are straightforward. Furthermore, in view of [**5**], these results can be further extended to C^∞ actions. We will also prove an analogous result for prolonged group actions on higher order frame bundles.

We begin in Section 2 by introducing basic terminology and constructions for Lie group actions and their prolongations to submanifold jet bundles, using [**16**] as a basic reference. Section 3 contains several required lemmas concerning abelian Lie groups. Section 4 is devoted to the study of meager subsets of topological spaces and their behavior under maps. Section 5 states and proves our first main result, Theorem 5.3, that a connected analytic effective Lie transformation group acts "generically freely" when prolonged to a frame bundle of sufficiently high order, meaning that the action is free on those fibers sitting over a comeager subset of an open subset of the base manifold. In Section 6 we present a key technical lemma that if a local diffeomorphism fixes the n jets of all submanifolds passing through a point $z \in M$, then it has the same k jet as the identity map for some smaller k upon which n depends. In the final section, we prove, in Theorem 7.1, a similar freeness result for prolonged connected, analytic, effective group actions on submanifold jet bundles, except here the freeness is established on a comeager subset of an open subset of a sufficiently high order jet bundle.

Basic notation: Throughout, $\mathbb{R}$ will denote the real numbers. We will work over the real number field, although our results apply equally well to complex group actions on complex manifolds. We use $\mathbb{Z}$ to denote the integers, $\mathbb{N} = \{\, n \in \mathbb{Z} \mid n \geq 0 \,\}$ for the natural numbers, and $\mathbb{N}^+ = \{\, n \in \mathbb{Z} \mid n \geq 1 \,\} = \mathbb{N} \setminus \{\, 0 \,\}$ for the positive integers.

2. Prolonged Group Actions on Jet Bundles.

Let G be an r-dimensional Lie group acting smoothly on a smooth, meaning C^∞, m-dimensional manifold M. The identity element of G is denoted by e. We use G° to denote the connected component of the group G containing e, so that G is connected if and only if $G = G^\circ$. The action of G on M is denoted by $z \mapsto g \cdot z$ for $g \in G$ and $z \in M$, and we let $\overline{g}\colon M \to M$ be the diffeomorphism defined by the group element $g \in G$, so that $\overline{g}(z) = g \cdot z$.

If $S \subset M$, we let $g \cdot S = \{\, g \cdot z \mid z \in S \,\}$ be its image under the group element $g \in G$. Thus, S is *G-invariant* if and only if $G \cdot S := \bigcup_{g \in G} g \cdot S = S$. The *orbits* are the minimal non-empty G-invariant subsets of M, and the action is *transitive* if M is the only orbit. Most Lie group actions of interest in applications of moving frames are transitive, although our results apply equally well to intransitive actions.

Definition 2.1. The *stabilizer* or *isotropy subgroup* of a subset $S \subset M$ is

$$G_S = \mathrm{Stab}_G(S) = \{\, g \in G \mid \ g \cdot S = S \,\}\,. \tag{2.1}$$

In particular, if $S = \{\, z \,\}$ is a singleton, we write G_z for $G_{\{z\}}$. Further, the *global stabilizer* of S is the subgroup

$$G_S^* = \bigcap_{z \in S} G_z = \{\, g \in G \mid \ g \cdot z = z \text{ for all } z \in S \,\} \tag{2.2}$$

consisting of group elements which fix *all* points in S.

Definition 2.2. The group G acts

- *freely* if $G_z = \{e\}$ for all $z \in M$;
- *locally freely* if G_z is a discrete subgroup of G for all $z \in M$, or, equivalently, all the orbits of G are r-dimensional submanifolds of M;
- *effectively* or *faithfully* if $G_M^* = \{e\}$;
- *effectively on subsets* if $G_U^* = \{e\}$ for every nonempty open subset $U \subset M$;
- *locally effectively* if G_M^* is a discrete subgroup of G;
- *locally effectively on subsets* if G_U^* is a discrete subgroup of G for every nonempty open subset $U \subset M$.

Note also that, for analytic actions on connected manifolds, (locally) effective implies (locally) effective on subsets, which is not the case for C^∞ actions.

Remark: While the subset $S = \{\, z \in M \mid \dim G_z = 0 \,\} \subset M$ where the Lie group acts locally freely is necessarily open, the same cannot, in general, be said of the subset $S^* = \{\, z \in M \mid \ G_z = \{e\} \,\} \subset M$ where the action is free. An elementary example is the action $(x, y, z) \longmapsto (x + t, y + t\,z, z) \bmod 1$ of the Lie group $G = \mathbb{R}$ on $M = S^1 \times S^1 \times S^1 \simeq (\mathbb{R}/\mathbb{Z})^3$. This action is locally free everywhere, but is free only on the tori with irrational $z \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$. Thus, the subsets where the action is free and not free are both dense in M. This example can be easily adapted to an action of $\mathbb{R}$ on $\mathbb{R}^3$ with similar freeness properties by use of the Hopf fibration by tori.

Let us fix an integer $1 \leq p < m = \dim M$. Let $\mathrm{J}^n = \mathrm{J}^n(M, p)$ denote the n^{th} order (extended) jet bundle on M defined by p-dimensional submanifolds $N \subset M$ under the equivalence relation of n^{th} order contact at a common point, cf. [**16**]. We are not assuming that M itself has any sort of bundle structure, but if $M = E \to X$ is a fiber bundle over a p-dimensional base manifold X, then the n^{th} order jet bundle $\mathrm{J}^n E$ of sections of E forms a dense open subbundle of the submanifold jet bundle $\mathrm{J}^n(E, p)$ traced out by those p-dimensional submanifolds that intersect the fibers of E transversally. Our results apply equally well in both contexts.

Since any (local) diffeomorphism Φ of M preserves contact between submanifolds, it induces a (local) diffeomorphism of the n^{th} order submanifold jet bundle J^n, called its n^{th} *order prolongation*, and denoted by $\Phi^{(n)} \colon \mathrm{J}^n \to \mathrm{J}^n$. We will be concerned with the prolonged actions on J^n of Lie groups G acting on M. The basic stabilization theorem of Ovsiannikov, [**19**], rigorously stated and proved in [**16**], proves local freeness of the prolonged action on an open subset of a sufficiently high order jet bundle; the following version appears in [**17**; Theorem 4.2].

Theorem 2.3. *Let G be an r-dimensional Lie group that acts smoothly and locally effectively on subsets of M. Then, for $r \leq n \in \mathbb{N}$, the prolonged action is locally free on a dense open subset $\varnothing \neq V^n \subset \mathrm{J}^n$ whose intersection with each fiber $V^n|_z = V^n \cap \mathrm{J}^n|_z$, $z \in M$, is both dense and open therein.*

The existing proofs of Theorem 2.3 all rely on infinitesimal techniques, and thus cannot be easily extended to remove the "local" caveat. A primary goal of this paper is to state and prove a global counterpart to Theorem 2.3 for analytic actions of connected Lie groups, which appears as Theorem 7.1 below.

Example 2.4. To see that freeness does not necessarily hold globally, consider the one-parameter group Lie group action

$$(x, y) \longmapsto \big(x, u + t\, f(x)\big), \qquad t \in G = \mathbb{R}, \qquad (x, u) \in M = \mathbb{R}^2,$$

where $f \colon \mathbb{R} \to \mathbb{R}$ is an analytic function which has a zero of order k at the point $x_k \in \mathbb{R}$ for $k = 1, 2, 3, \ldots$, and so that $x_k \to \infty$ as $k \to \infty$. Such a function can be constructed using a Weierstrass product expansion, cf. [**6**; p. 194]. Let $\mathrm{J}^n = \mathrm{J}^n(M, 1)$ be the curve (one-dimensional submanifold) jet space. Then G acts trivially on the fiber $\mathrm{J}^n|_{(x_k,u)}$ for all $k > n$. On the other hand, G does act freely on the dense open subset of J^n that omits the fibers over these points. ■

A related, non-analytic example, suggested by J. Pohjanpelto, [**21**], shows that a smooth action can fail to be free on entire fibers of the submanifold jet spaces of arbitrarily high order:

Example 2.5. Let $M = \mathbb{R} \times S^1$ be a cylinder, with coordinates (x, u), where we use the additive structure on $S^1 = \mathbb{R}/\mathbb{Z}$, so that u is taken modulo 1. Let $f \colon \mathbb{R} \to S^1$ be a smooth function with $0 < f(x) < 1$ for all x, and such that $f(x)$ is not constant on any open subinterval $I \subset \mathbb{R}$. Consider the one-parameter transformation group

$$(x, u) \longmapsto (x, u + t\, f(x) \bmod 1), \qquad t \in G = \mathbb{R}, \qquad (x, u) \in M = \mathbb{R} \times S^1.$$

The orbits are the one-dimensional vertical circles $S_{x_0} = \{x = x_0\} \simeq S^1$. The isotropy subgroup of a point $(x, u) \in M$ is

$$G_{(x,u)} = \{\, t = n/f(x) \mid n \in \mathbb{Z} \,\} \subset G = \mathbb{R}.$$

Thus, the action is locally free everywhere on M, and also locally effective on subsets since we assume that $f(x)$ is not constant on open subintervals.

Local coordinates on the n^{th} order curve jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, 1)$ are given by $(x, u, u_1, \ldots, u_n)$, where u_k correspond to the k^{th} order derivative of u with respect to x, viewing the curve as the graph of a function† $u = h(x)$, where $h \colon \mathbb{R} \to S^1$. Keep in mind

† Thus, we are ignoring curves $C \subset M$ with vertical tangents. Or, equivalently, we are viewing $M = \mathbb{R} \times S^1 \to \mathbb{R}$ as a circle bundle, and restricting our attention to the jet bundle $\mathrm{J}^n M$ of sections.

that, while u is a coordinate on the circle S^1, the derivative coordinates u_k for $k \geq 1$ take values in $\mathbb{R}$. The prolonged action on J^n is thus given by

$$(x, u, u_1, \ldots, u_n) \longmapsto \big(x, u + t\, f(x) \bmod 1, u_1 + t\, f'(x), \ldots, u_n + t\, f^{(n)}(x) \big).$$

Consequently, if $f'(x_0) \neq 0$ for some $x_0 \in R$, the action is free at the 1 jets $(x_0, u, u_1) \in \mathrm{J}^1 \mid S_{x_0}$, and hence also at all $(x_0, u, u_1, \ldots, u_n) \in \mathrm{J}^n \mid S_{x_0}$. More generally, if $0 = f'(x_0) = f''(x_0) = \cdots = f^{(k-1)}(x_0) = 0$ while $f^{(k)}(x_0) \neq 0$, the action remains locally free on $\mathrm{J}^n \mid S_{x_0}$ for $0 \leq n \leq k-1$, but is free on $\mathrm{J}^\ell \mid S_{x_0}$ for any $\ell \geq k$. Thus, if $f'(x)$ has a zero of infinite order at x_0, so $f^{(k)}(x_0) = 0$ for all $k \geq 1$, then the action remains only locally free on $\mathrm{J}^n \mid S_{x_0}$ for *any* order $n \geq 0$, and hence never becomes free on such fibers. ■

Let us call a jet $z^{(n)} \in \mathrm{J}^n$ *locally regular* if G acts locally freely at $z^{(n)}$. Let $V^n \subset \mathrm{J}^n$ denote the open subset containing all locally regular jets. According to Theorem 2.3, $V^n \neq \varnothing$ provided $n \geq r = \dim G$. The *singular subset* $\Sigma^n := \mathrm{J}^n \setminus V^n$ is an algebraic subvariety of J^n defined by the vanishing of a (generalized) Lie determinant, [**16**, **17**].

Definition 2.6. A p-dimensional submanifold $S \subset M$ is *totally singular at a point* $z \in S$ if, for all $n \in \mathbb{N}$, its n^{th} order jet at Z is singular: $\mathrm{j}_n S|_z \subset \Sigma^n$. A *totally singular* submanifold is one all of whose points are totally singular.

Theorem 2.7. *Suppose G acts analytically. An analytic submanifold S is totally singular at a point $z_0 \in S$ if and only if its stabilizer subgroup G_S does not act locally freely on S at z_0.*

This result, along with a smooth counterpart, is proved in [**17**], which also contains a Lie algebraic characterization of the totally singular submanifolds as (unions of) orbits of suitable subgroups of G.

Example 2.8. Consider the (special) Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ consisting of all rigid motions — rotations and translations — acting on space curves $C \subset M = \mathbb{R}^3$. The totally singular curves are the straight lines, which is a consequence of Theorem 2.7. Indeed, the Euclidean stabilizer $G_L \subset \mathrm{SE}(3)$ of a straight line $L \subset \mathbb{R}^3$ is two-dimensional, consisting of all translations in the direction of the line and all rotations having the line as axis. Since $2 = \dim G_L > \dim L = 1$, the group G_L does not act locally freely on L, and so Theorem 2.7 implies that the prolonged action of SE(3) can never be free on the jets of straight lines of any order. ■

One important application of the freeness of prolonged group actions is to the Cartan theory of moving frames, as extended by the second author and M. Fels, [**8**, **18**]. If G acts on M, an n^{th} order (left) *moving frame* is defined as a smooth G-equivariant mapping $\rho : U \to G$ for some open $U \subset \mathrm{J}^n$, meaning that

$$\rho(g \cdot z^{(n)}) = g \cdot \rho(z^{(n)}) \tag{2.3}$$

for all $z^{(n)} \in U$ and $g \in G$ such that $g \cdot z^{(n)} \in U$. All classical moving frames, e.g., those in [**7**, **9**], fit naturally into this equivariant framework, which enables one to systematically and algorithmically extend the classical theory to general smooth Lie group actions. One easily proves that such a moving frame map exists if and only if G acts freely and regularly[†] on an open subset of the jet bundle. Therefore, once we are assured that the prolonged group actions are free on $V^n \subset \mathrm{J}^n$, we can construct a locally equivariant n^{th} order moving frame in a neighborhood of any regular jet $z^{(n)} \in \mathrm{J}^n$. Only the totally singular submanifolds fail to admit an equivariant moving frame of any order.

3. Abelian Groups.

Our results rely on some basic lemmas about abelian Lie groups. Given a Lie group G, let

$$Z(G) = \{\, h \in G \mid \ h\, g = g\, h \text{ for all } g \in G \,\}$$

denote its *center*. Recall that G° denotes the connected component of G containing the identity element e.

Lemma 3.1. *If $Z \subseteq Z(G)$ is a closed subgroup of the center of a connected Lie group G, then Z/Z° is finitely generated.*

Proof: Replacing G by G/Z° and Z by Z/Z°, we may assume that Z is a discrete subgroup. Replacing G by its universal cover and replacing Z by its preimage in the universal cover, we may assume that G is simply connected. Then Z is isomorphic to the fundamental group of the Lie group G/Z. Since every connected Lie group is homotopy equivalent to a maximal compact subgroup, [**10**], it follows that the fundamental group of any Lie group is finitely generated. *Q.E.D.*

Lemma 3.2. *Let Y be an abelian Lie group such that Y/Y° is finitely generated. If X is a closed subgroup of Y, then X/X° is finitely generated.*

Proof: Replacing X by X/X° and Y by Y/X°, we may assume that X is discrete. Let $\pi: Y \to Y/Y^\circ$ be the canonical projection. Then $\widehat{X} = \pi(X)$ is a subgroup of a finitely generated abelian group, and is therefore finitely generated. Moreover, the kernel K of $\widehat{\pi} = \pi | X : X \to \widehat{X}$ is a discrete subgroup of $Z(Y^\circ)$, and hence, by Lemma 3.1 is also finitely generated. Thus, X itself is finitely generated. *Q.E.D.*

Recall that g is called a *torsion element* if $g^n = e$ for some $0 \neq n \in \mathbb{Z}$. A group is called *torsion-free* if it contains no torsion elements. The *rank* of an abelian group A is defined as the largest integer $r \geq 0$ such that the additive group $\mathbb{Z}^r$ is isomorphic to a subgroup of A.

Lemma 3.3. *Let $A \subseteq B \subseteq C$ be subgroups of a finitely generated abelian group C. If C/A and C/B are both torsion-free and have the same rank, then $A = B$.*

[†] The action is called *regular* if its orbits form a regular foliation. We will not address this global regularity condition here.

Proof: Let $\chi: C/A \to C/B$ be the canonical map. Our assumption on C/A and C/B being torsion-free abelian groups having equal rank combined with the surjectivity of χ immediately implies that χ is an isomorphism. *Q.E.D.*

The next result is the descending chain condition (DCC) for "co-torsion-free" subgroups of abelian Lie groups.

Proposition 3.4. *Suppose A_1 is an abelian Lie group such that A_1/A_1° is finitely generated. Let $A_2, A_3, \ldots$ be closed subgroups of A_1 such that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$. Suppose that A_1/A_k is torsion-free for all $k \in \mathbb{N}^+$. Then there exists $n \in \mathbb{N}^+$ such that $A_n = A_{n+1} = A_{n+2} = \cdots$.*

Proof: Lemma 3.2 implies that A_k/A_k° is finitely generated for all k. Furthermore, $\dim A_1 \geq \dim A_2 \geq \cdots \geq 0$, and hence there exists $d \in \mathbb{N}$ and $m \in \mathbb{N}^+$ such that, when $m \leq k \in \mathbb{N}$, we have $\dim A_k^\circ = d$, and hence $A_k^\circ = A_m^\circ$. Replacing each such A_k by $A_k/A_k^\circ = A_k/A_m^\circ$, we may assume that A_k is a countable discrete group for $k \geq n$. Let r_m be the rank of A_m, and let s_k be the rank of A_m/A_k. Since $s_m \leq s_{m+1} \leq \cdots \leq r_m$, we may choose an integer $n \geq m$ such that $s_n = s_{n+1} = \cdots$. The result then follows from Lemma 3.3. *Q.E.D.*

4. Meager Sets.

In this section, we establish some general facts about meager subsets of topological spaces and their behavior under maps. The results are stated in very general form, but our motivational examples are the higher order frame bundles and submanifold jet bundles associated with a smooth manifold. Basic topological definitions and facts can be found in the standard reference [**12**].

Let X be a topological space. We say that a subset $A \subseteq X$ is *interior-free in X* if $X \setminus A$ is dense in X, *i.e.*, if the interior in X of A is empty. Note that, for any base $\mathcal{B}$ of the topology on X,

$$[\ A \text{ is interior-free in } X\ ] \quad \text{if and only if} \quad [\ \text{for all } U \in \mathcal{B},\ (U \subseteq A) \Rightarrow (U = \varnothing)\ ].$$

Let $\overline{A}$ be the closure in X of A. For any open subset U of X, we say A is *X-dense in U* if $U \subseteq \overline{A}$. We say that A is *somewhere dense in X* if there exists a nonempty open subset U of X such that A is X-dense in U. We say that A is *nowhere dense in X* if A is not somewhere dense in X. Note that:

$$[\ A \text{ is nowhere dense in } X\ ] \quad \text{if and only if} \quad [\ \overline{A} \text{ is interior-free in } X\ ].$$

Definition 4.1. Let X be a topological space and let $S \subseteq X$. The subset S is said to be *meager in X* if there exists a sequence $A_1, A_2, \ldots$ of nowhere dense subsets of X such that $S = A_1 \cup A_2 \cup \cdots$.

Note that S is meager in X if and only if there exists a sequence $C_1, C_2, \ldots$ of interior-free closed subsets of X such that $A \subseteq C_1 \cup C_2 \cup \cdots$. That is, a set is meager in X if and only if it is in the σ-ideal generated by the interior-free closed subsets of X.

We further say that

- S is *nonmeager in* X if S is not meager in X;
- S is *comeager in* X if $X \setminus S$ is meager in X;
- X is *meager-in-itself* if X is a meager subset of X;
- X is *nonmeager-in-itself* if X is not meager-in-itself.

For any $A, B \subseteq X$, the statement "$A \equiv B$ in X", means that $(A \setminus B) \cup (B \setminus A)$ is meager in X. By the phrase "for essentially all $x \in X$", we will mean "there exists a comeager subset $S \subseteq X$ such that, for all $x \in S$".

A topological space X is said to have the *Baire property* if every nonempty open subset of X is nonmeager in X, and hence, in particular, if nonempty, X is nonmeager-in-itself. A *Polish space* is a separable completely metrizable topological space; any Polish space has the Baire property. In particular, any manifold is Polish.

For any $S \subseteq X$, we say that S is *almost open in* X if there exists an open subset U of X such that $S \equiv U$ in X, while S is *almost closed in* X if there exists a closed subset C of X such that $S \equiv C$ in X. Thus, "almost open" is another way of saying that a set has the Baire property. The boundary of any closed set is interior-free and closed, so any closed set is $\equiv$ its interior, and, therefore, is almost open. Thus, any almost closed set is also almost open, and it follows that the complement of any almost open set is again almost open. The collection $\mathcal{B}$ of almost open subsets of X is therefore closed under complement. Since $\mathcal{B}$ is also closed under countable union, we see that $\mathcal{B}$ is a σ-algebra, namely that generated by the open sets and meager sets. In measure theory, the σ-algebra generated by open sets and null (measure zero) sets is the σ-algebra of measurable sets, and $\mathcal{B}$ is the Baire categorical analogue. That is, "almost open" is the Baire categorical analogue of "measurable". For this reason, "Baire measurable" is sometimes used instead of "almost open". Construction of sets that are *not* almost open requires some version of the Axiom of Choice, [**23**, **24**].

Absoluteness of nowhere dense and meager

Let $V \subset X$ be an open subset with the subspace topology inherited from X, and let $Z \subseteq V$.

Lemma 4.2. *[Z is somewhere dense in X] $\Longleftrightarrow$ [Z is somewhere dense in V].*

Proof: Let $\overline{Z}$ be the closure in X of Z. Then $\overline{Z} \cap V$ is the closure in V of Z.

To prove $\Longrightarrow$: Choose a nonempty open subset U of X such that Z is X-dense in U. Then $\varnothing \neq U \subseteq \overline{Z}$, and so, as U is open, $\varnothing \neq U \cap Z \subseteq U \cap V \subseteq \overline{Z} \cap V$. Therefore, Z is V-dense in $U \cap V$, and hence is somewhere dense in V.

To prove $\Longleftarrow$: Choose a nonempty open subset U of V such that Z is V-dense in U. Then $\varnothing \neq U \subseteq \overline{Z} \cap V$. Since U is open in V and V is open in X, we see that U is open in X. Therefore, Z is X-dense in U, and hence somewhere dense in X. *Q.E.D.*

Lemma 4.2 says that "nowhere dense in X" is the same as "nowhere dense in V". Since a meager set is, by definition, a countable union of nowhere dense sets, we deduce:

Corollary 4.3. *[Z is meager in X] $\Longleftrightarrow$ [Z is meager in V.]*

Open map preimages of small sets are small

Let X and Y be topological spaces. Let $f: X \to Y$ be *continuous*, meaning that the preimage of any open subset $V \subset Y$ is an open subset $U = f^{-1}(V) \subset X$, and *open*, meaning that it maps open subsets $U \subset X$ to open subsets $V = f(U) \subset Y$.

Lemma 4.4. *Let C be an interior-free closed subset of Y. Then its preimage $f^{-1}(C)$ is an interior-free closed subset of X.*

Proof: Because f is continuous, and C is closed in Y, then $f^{-1}(C)$ is closed in X. Because f is open, and C is interior-free in Y, then $f^{-1}(C)$ is interior-free in X. *Q.E.D.*

Lemma 4.5. *If Z is a meager subset of Y, then $f^{-1}(Z)$ is a meager subset of X.*

Proof: Choose a sequence $C_1, C_2, \ldots$ of interior-free closed subsets of Y such that $Z \subseteq C_1 \cup C_2 \cup \cdots$. Then, by Lemma 4.4, $f^{-1}(C_1), f^{-1}(C_2), \ldots$ is a sequence of interior-free closed subsets of X such that $f^{-1}(Z) \subseteq f^{-1}(C_1 \cup C_2 \cup \cdots) = f^{-1}(C_1) \cup f^{-1}(C_2) \cup \cdots$, proving that $f^{-1}(Z)$ is meager in X. *Q.E.D.*

Small sets contain few fibers

Lemma 4.6. *Let U and V be topological spaces. Let $g: U \to V$ be continuous, open, and onto. Let P be an interior-free closed subset of U. Then there exists a dense open subset $W \subset V$ such that $g^{-1}\{w\} \not\subseteq P$ for all $w \in W$.*

Proof: Since P is interior-free in U, its complement $U \setminus P$ is dense in U and hence $W := g(U \setminus P)$ is dense in V. Since P is closed in U, it follows that $U \setminus P$ is open in U, and thus W is open in V. Finally, given $w \in W$, choose $u \in U \setminus P$ such that $g(u) = w$. Then $u \in g^{-1}\{w\} \setminus P$, and hence $g^{-1}\{w\} \not\subseteq P$, as desired. *Q.E.D.*

When fibers of small sets are small, generically

From here on in this section, X and Y will be topological spaces, with X second countable. Let $f: X \to Y$ be continuous and open. For all $y \in Y$, let $X_y := f^{-1}\{y\}$, and give X_y the subspace topology inherited from X. For all $S \subseteq X$ and $y \in Y$, let $S_y := S \cap X_y$.

Lemma 4.7. *Let Q be an interior-free closed subset of X. Then $Q_y = Q \cap X_y$ is interior-free and closed in $X_y = f^{-1}\{y\}$ for essentially all $y \in Y$.*

Proof: First, since Q is closed in X, it follows that Q_y is closed in X_y for all $y \in Y$. If $\mathcal{B}$ is a countable base for the topology on X, then $\{\, U_y = U \cap f^{-1}\{y\} \mid U \in \mathcal{B} \,\}$ is a base for the topology on X_y. It therefore suffices to prove that $(U_y \subseteq Q_y) \Rightarrow (U_y = \varnothing)$ for essentially all $y \in Y$ and all $U \in \mathcal{B}$. As $\mathcal{B}$ is countable, we may interchange "for essentially all $y \in Y$" and "for all $U \in \mathcal{B}$". Let $U \in \mathcal{B}$ be given, so $V = f(U)$ is open in Y. Give U and V the subspace topologies inherited from X and Y, respectively. Because f is continuous and open, it follows that $g = f \mid U: U \to V$ is continuous, open, and surjective. Since Q is an interior-free, closed subset of X, and since U is open in X, it follows that $P := Q \cap U$ is an interior-free, closed subset of U. So, by Lemma 4.6, choose a dense open subset $W \subset V$ such that $g^{-1}\{w\} \nsubseteq P$ for all $w \in W$. Then $Z = V \setminus W$ is an interior-free closed subset of V, and hence meager in V. Corollary 4.3 then implies that Z is meager in Y.

It therefore suffices to prove, for all $y \in Y \setminus Z$, that $(U_y \subseteq Q_y) \Rightarrow (U_y = \varnothing)$. We assume, for a contradiction, that both $U_y \subseteq Q_y$ and $U_y \neq \varnothing$, and so $y \in V = f(U)$. So, since $g^{-1}\{y\} = U \cap f^{-1}\{y\} = U \cap X_y = U_y \subseteq Q_y \cap U \subseteq Q \cap U = P$, we deduce that $y \notin W$. Thus $y \in V \setminus W = Z$, which contradicts the fact that $y \in Y \setminus Z$. *Q.E.D.*

Lemma 4.8. *Let Z be a meager subset of X. Then $Z_y = Z \cap X_y$ is meager in X_y for essentially all $y \in Y$.*

Proof: Choose countable sequence $Q^1, Q^2, \ldots$, of interior-free closed subsets of X such that $Z \subseteq Q^1 \cup Q^2 \cup \cdots$, whence $Z_y \subseteq Q^1_y \cup Q^2_y \cup \cdots$ for all $y \in Y$. Lemma 4.7 implies that each Q^j_y is interior-free and closed in X_y for essentially all $y \in Y$ and all $j \in \mathbb{N}$. *Q.E.D.*

Lemma 4.9. *Let $f: X \to Y$ be continuous and open, where X is second countable and Y has the Baire property. Assume, for all $y \in Y$, that $X_y = f^{-1}\{y\}$ is almost open in X. Let R be an almost open subset of X. If $R_y = R \cap X_y$ is meager in X_y for essentially all $y \in Y$, then R is meager in X.*

Proof: Choose a meager subset $Z \subset Y$ such that R_y is meager in X_y for all $y \in Y \setminus Z$. As R is almost open in X, choose an open subset U of X such that $R \equiv U$ in X. Then $S := (R \setminus U) \cup (U \setminus R)$ is meager in X, and it thus suffices to show that $U = \varnothing$, and hence $R = S$. By Lemma 4.8, we can choose a meager subset $T \subset Y$ such that $S_y = S \cap X_y$ is meager in X_y for all $y \in Y \setminus T$. Let $V := f(U)$, which is open in Y since f is open. We claim that $V \subseteq Z \cup T$, which implies that V is also meager in Y. But V is also open in Y, hence $V = \varnothing$, which implies $U = \varnothing$ as desired.

To prove the claim, let $y \in V$ be given. We assume that $y \notin Z \cup T$, and seek a contradiction. Since $y \in Y \setminus Z$ and $y \in Y \setminus T$, both R_y and S_y are meager in X_y, so $R_y \cup S_y$ is meager in X_y. It follows that $S_y = (R_y \setminus U_y) \cup (U_y \setminus R_y)$ where $U_y = U \cap X_y \subseteq R_y \cup S_y$ is thus also meager in X_y. On the other hand, since U is open in X, it follows that U_y is open in X_y, which is itself almost open. Together these imply $U_y = \varnothing$. However, $y \in V = f(U)$, so $U_y \neq \varnothing$, which is a contradiction. *Q.E.D.*

Remark: Thanks to our colleague Karel Prikry for explaining that there exists a function $h : \mathbb{R} \to \mathbb{R}$ whose graph is nonmeager in $\mathbb{R}^2$, [**22**]; see also [**20**; Theorem 15.5, p. 57]. This demonstrates the necessity of the assumption in Lemma 4.9 that R be almost open. Without that assumption, letting R be the graph of such an h and f be the projection map $(x, y) \mapsto x$ from $\mathbb{R}^2 \to \mathbb{R}$ would provide a counterexample.

The next result is motivated by the Kuratowski-Ulam Theorem. An equivalent result appears as Theorem A.1 in Appendix A of [**15**].

Theorem 4.10. *Assume that X and Y are both Polish. Let $R \subseteq X$. Then the following are equivalent*: (i) *R is meager in X.* (ii) *R is almost open in X and R_y is meager in X_y for essentially all $y \in Y$.*

Proof: The empty set is open, so any meager set is almost open. Thus Lemma 4.8 implies that $(i) \Rightarrow (ii)$. Vice versa, since f is continuous, it follows that X_y is closed in X for all $y \in Y$. Any closed subset of a Polish topological space is Polish, and hence X_y is almost open for all $y \in Y$. Therefore, Lemma 4.9 implies that $(ii) \Rightarrow (i)$. *Q.E.D.*

When preimages are meager

Lemma 4.11. *Let $f: X \to Y$ be continuous, open and surjective and let $S \subseteq Y$. Assume that $X_y = f^{-1}\{y\}$ is a nonmeager-in-itself for all $y \in Y$. Then the following are equivalent: (i) S is meager in Y. (ii) $f^{-1}(S)$ is meager in X.*

Proof: Lemma 4.5 implies that $(i) \Rightarrow (ii)$. To prove that $(ii) \Rightarrow (i)$, we assume that $Z := f^{-1}(S)$ is meager in X. Lemma 4.8 implies $Z_y := Z \cap X_y$ is meager in X_y for essentially all $y \in Y$. Then $W := \{ y \in Y \mid Z_y \text{ is nonmeager in } X_y \}$ is meager in Y, so it suffices to prove that $S \subseteq W$. Let $y \in S$ be given. Then $X_y = f^{-1}\{y\} \subseteq f^{-1}(S) = Z$, hence $Z_y = Z \cap X_y = X_y$. But X_y is nonmeager-in-itself, and hence $y \in W$. *Q.E.D.*

Corollary 4.12. *Assume that X is Polish and that Y is T_1. Let $S \subseteq Y$. Then the following are equivalent: (i) S is meager in Y. (ii) $f^{-1}(S)$ is meager in X.*

Proof: By Lemma 4.11, it suffices to show, for all $y \in Y$, that $X_y = f^{-1}\{y\}$ is nonmeager-in-itself. As f is continuous and surjective, and Y is T_1, it follows that X_y is nonempty and closed. Since any closed subset of a Polish topological space is Polish, and hence Baire, it follows that X_y is nonmeager-in-itself, as desired. *Q.E.D.*

5. Generic Freeness on Frame Bundles.

Let M be a m-dimensional connected analytic manifold. For $n \in \mathbb{N}$, let $\pi_n: \mathcal{F}^{(n)} \to M$ denote the n^{th} order frame bundle over M, cf. [**13**]. Any local diffeomorphism $\Phi: M \to M$ has an induced action on $\mathcal{F}^{(n)}$, called its n^{th} order *frame bundle prolongation*. We are interested in the freeness of the frame bundle prolongations of Lie group actions on M.

It turns out that there are examples of smooth, [**2**], and analytic, [**3**], effective transformation groups that, in each frame bundle, are not free on any nonempty open set. The main result in this section, Theorem 5.3, asserts that freeness does occur "generically" meaning on the entire fibers of $\mathcal{F}^{(n)}$ sitting over a comeager invariant subset of a nonempty open invariant subset of M. This indicates that construction of a counterexample is complicated. It involves the development of a transformation that has a dense but meager set of periodic orbits whose return maps have arbitrarily high order agreement with the identity. This kind of "threading the needle" is found in [**2**] and [**3**].

Let $\mathcal{D} = \mathcal{D}(M)$ be the pseudo-group of all local analytic diffeomorphisms of M. Given $z \in M$ and $n \in \mathbb{N}$, let $\mathcal{D}_z^{(n)} \subset \mathcal{D}$ denote the subgroup containing those diffeomorphisms that fix z to order n.

Lemma 5.1. *For any integer $1 \le k \le n$, the quotient group $\mathcal{D}_z^{(k)}/\mathcal{D}_z^{(n)}$ is torsion-free.*

Proof: By inspection of the Taylor series, $\mathcal{D}_z^{(n)}/\mathcal{D}_z^{(n+1)}$ is seen to be isomorphic to the additive Lie group $\mathbb{R}^N$ where $N = m\binom{n+m-1}{n}$. Therefore, $\mathcal{D}_z^{(k)}/\mathcal{D}_z^{(n)}$ is a (nilpotent) extension of torsion-free groups, and hence torsion-free. *Q.E.D.*

Theorem 5.2. *Let G be a connected Lie group acting analytically and effectively on a connected manifold M. Let $z \in M$ and let G_z be its stabilizer subgroup. Then there exists an integer $n \in \mathbb{N}^+$ such that the homomorphism $G_z \to \mathcal{D}_z^{(0)}/\mathcal{D}_z^{(n)}$ is injective.*

Proof: For all $k \geq 0$, let $G_z^{(k)}$ be the kernel of the homomorphism $G_z \to \mathcal{D}_z^{(0)}/\mathcal{D}_z^{(k)}$, whereby

$$G_z^{(0)} \supseteq G_z^{(1)} \supseteq G_z^{(2)} \supseteq \ \cdots \ . \tag{5.1}$$

By analyticity and effectiveness, we conclude that $\bigcap_{k=0}^{\infty} G_z^{(k)}$ is trivial. To show that $G_z^{(n)}$ is trivial for some $n \geq 0$, it suffices to show that the sequence (5.1) stabilizes. Fix $\ell \in \mathbb{N}^+$ such that $G_z^{(k)}$ is discrete for all $k \geq \ell$. Let $\mu_0 \in \mathcal{F}^{(\ell)}|_z$ be a fixed frame of order ℓ at the point z. Let $\varepsilon = \pi(e) \in G/G_z^{(\ell)}$ denote the image of the identity e under the canonical map $\pi\colon G \to G/G_z^{(\ell)}$. There is a G_z-equivariant injection of $G/G_z^{(\ell)}$ into $\mathcal{F}^{(\ell)}|_z$ which sends ε to μ_0. It follows that $G_z^{(\ell+1)}$ fixes ε to order 1, and so its action on the tangent space $T(G/G_z^{(\ell)})|_\varepsilon$ is trivial. This implies that the adjoint representation $\operatorname{Ad} G_z^{(\ell+1)}$ on the Lie algebra $\mathfrak{g}$ of G is trivial, and therefore $G_z^{(\ell+1)}$ is contained in the center of G.

Lemma 3.1 implies that $G_z^{(\ell+1)}/(G_z^{(\ell+1)})^\circ$ is a finitely generated abelian group. For any integer $k \geq \ell + 1$, the quotient $G_z^{(\ell+1)}/G_z^{(k)}$ injects in $\mathcal{D}_z^{(\ell)}/\mathcal{D}_z^{(k)}$, which, according to Lemma 5.1, is torsion-free; therefore, $G_z^{(\ell+1)}/G_z^{(k)}$ is itself torsion-free. Proposition 3.4 implies that the sequence (5.1) stabilizes. *Q.E.D.*

We can now state and prove our main result concerning freeness of group actions on higher order frame bundles.

Theorem 5.3. *Let G be a connected real Lie group that acts analytically and effectively on a connected analytic manifold M. Then there exist*

- *an integer $n \geq 0$,*
- *a nonempty G-invariant open subset $U \subset M$, and*
- *a closed G-invariant meager subset $Z \subset M$,*

such that the prolonged action of G is free on the G-invariant subset $\pi_n^{-1}(U \setminus Z) \subset \mathcal{F}^{(n)}$ contained in the frame bundle of order n sitting over the comeager subset $U \setminus Z \subset M$.

Remark: Replacing M by an open submanifold thereof, we see that the conclusion of the theorem holds in a neighborhood of any $z \in M$, although the order of freeness n may vary from point to point. Note that $U \setminus Z$ is not necessarily open; indeed, in the examples constructed in [**2**, **3**] the meager set Z where G does not act freely is dense in U.

Proof: For all $k \in \mathbb{N}$, let M_k denote the G-invariant subset consisting of all $z \in M$ such that $G_\mu = \{e\}$ for all $\mu \in \pi_k^{-1}\{z\} \subset \mathcal{F}^{(k)}$. Theorem 5.2 implies $\bigcup\limits_{k=0}^{\infty} M_k = M$. By the Baire Category Theorem, fix an integer $n \geq 0$ such that M_n is nonmeager in M. Because G and $\mathcal{F}^{(n)}$ are σ-compact, the Arsenin-Kunugui Uniformization Theorem, cf. [**11**; Theorem 35.46(ii), p. 297], implies that M_n is a Borel subset of M, and is therefore almost open in M. That is, there is an open subset $V \subset M$ as well as meager subsets $Z', Z'' \subset M$, such that $M_n = (V \setminus Z') \cup Z''$. Since M_n is nonmeager in M, we see that $M_n \not\subseteq Z''$, so $V \neq \varnothing$. Thus $Z_0 = V \setminus M_n \subseteq Z'$ is meager in M, and so $g \cdot Z_0$ is meager in M for all $g \in G$.

Let $\Delta \subset G$ be a countable dense subset. Then $Z = \Delta \cdot Z_0 = \bigcup_{g \in \Delta} g \cdot Z_0$ is meager in M. We claim that $Z = G \cdot Z_0$. Since $Z = \Delta \cdot Z_0 \subseteq G \cdot Z_0$, we need only prove that $G \cdot Z_0 \subseteq Z$. In other words, given $g \in G$ and $z \in Z_0$, we need to show that $g \cdot z \in Z$.

The orbit map $\psi_z : G \to M$ given by $\psi_z(g) = g \cdot z$ is continuous, and so, since V is open in M, it follows that $H_z := \psi_z^{-1}(V) \subset G$ is a nonempty open subset containing e. By density of Δ in G, choose $\delta \in \Delta \cap g \cdot H_z^{-1}$, so that $h = \delta^{-1} g \in H_z$. Since $z \in Z_0 = V \setminus M_n \subseteq M \setminus M_n$ and $M \setminus M_n$ is G-invariant, it follows that $h \cdot z \in M \setminus M_n$. In addition, $h \cdot z = \psi_z(h) \in \psi_z(H_z) \subseteq V$, hence $h \cdot z \in V \setminus M_n = Z_0$. Then $g \cdot z = \delta \cdot h \cdot z \in \Delta \cdot Z_0 = Z$, thus establishing our claim that Z is G-invariant.

Consider the nonempty G-invariant open subset $U = G \cdot V \subset M$. It remains to show that the G-action on $\pi_n^{-1}(U \setminus Z) \subset \mathcal{F}^{(n)}$ is free, i.e., that $U \setminus Z \subseteq M_n$. Let $u = g \cdot v \in U \setminus Z$ with $g \in G$ and $v \in V$. Since $u \notin Z = G \cdot Z_0$, we must have $v \in V \setminus Z_0 = V \setminus (V \setminus M_n) \subseteq M_n$. Finally, G-invariance of M_n implies $u = g \cdot v \in M_n$. *Q.E.D.*

6. Jets of Submanifolds and Diffeomorphisms.

We now turn our attention to submanifold jet bundles, referring back to Section 2 for basic terminology. The goal of this section is to prove a key lemma that states that if a local diffeomorphism fixes the n jets of all (or, in fact, a particular class of) submanifolds passing through a point $z \in M$, then the diffeomorphism must have the same k jet as the identity map at z, for some (smaller) k depending on n. The proof relies on a detailed analysis of Taylor expansions. To explain the basic idea, we first treat the simplest case: curves in the plane. The general result will then follow by a fairly straightforward adaptation of the univariate calculations.

The Planar Case

Let

$$\Phi(x,u) = (\varphi(x,u), \psi(x,u)), \qquad (x,u) \in \mathbb{R}^2, \tag{6.1}$$

be a local diffeomorphism of $M = \mathbb{R}^2$ such that $\Phi(0,0) = (0,0)$. In particular, we denote the identity diffeomorphism by $\mathbb{1}(x,u) \equiv (x,u)$. Given $n \in \mathbb{N}^+$, suppose Φ fixes the n jets at the origin of a family of curves defined by elementary monomials:

$$C_a^d = \left\{ \, (x,u) \;\middle|\; u = a\, x^d \, \right\}, \qquad \text{where} \qquad d \in \mathbb{N}, \qquad a \in \mathbb{R}, \tag{6.2}$$

meaning that

$$\Phi^{(n)}(j_n C_a^d|_0) = j_n C_a^d|_0. \tag{6.3}$$

Note in particular that $\mathrm{j}_{d-1} C_a^d|_0 = 0$. Our goal is to prove that if (6.3) holds for certain values of d and for all $a \in I \subset \mathbb{R}$, an open subinterval, then, for some (smaller) $k \in \mathbb{N}^+$, depending on n and d, the k jet of the diffeomorphism Φ at the origin is the identity jet:

$$J^k \Phi|_0 = J^k \mathbb{1}|_0, \qquad \text{or, equivalently,} \qquad \Phi \in \mathcal{D}_0^{(k)}. \tag{6.4}$$

We use standard "big O" notation, so that given $n \in \mathbb{N}^+$ and smooth real-valued functions f, g defined in a neighborhood of $x = 0$,

$$f(x) = g(x) + \mathrm{O}(n) \qquad \text{means} \qquad f(x) = g(x) + x^n h(x)$$

for some smooth $h(x)$ and all x in some neighborhood of $x = 0$. Similarly,

$$f(x,u) = g(x,u) + \mathrm{O}(n) \qquad \text{means} \qquad f(x,u) = g(x,u) + \sum_{i=0}^{n} x^i u^{n-i} h_i(x,u)$$

for smooth $h_i(x,u)$ and all x, u in a neighborhood of the origin. Thus, condition (6.4) is equivalent to

$$\varphi(x,u) = x + \mathrm{O}(k+1), \qquad \psi(x,u) = u + \mathrm{O}(k+1). \tag{6.5}$$

Observe that the diffeomorphism (6.1) preserves the n–jet at the origin of a curve given by a graph of a smooth function $h\colon \mathbb{R} \to \mathbb{R}$, namely,

$$C = \{\, u = h(x) \,\} \qquad \text{with} \qquad h(0) = 0,$$

provided

$$\psi(x, h(x)) = h\big[\, \varphi(x, h(x)) \,\big] + \mathrm{O}(n+1).$$

Thus, condition (6.3) is equivalent to

$$\psi(x, a\,x^d) = a\,\varphi(x, a\,x^d)^d + x^{n+1} h(x,a), \tag{6.6}$$

where h depends smoothly on a and on x near the origin.

Lemma 6.1. *Let $2 \le \ell, n, d \in \mathbb{N}$ and $0 \in I \subset \mathbb{R}$ an open subinterval. If (6.6) holds for $n \ge (\ell - 1)\,d$ and all $a \in I$, then*

$$x^d \psi(x,u) = u\,\varphi(x,u)^d + \mathrm{O}(\ell + 1). \tag{6.7}$$

Proof: Let c_{ij} denote the coefficient of the monomial $x^i u^j$ in the Taylor expansion of the bivariate function

$$F(x,u) := x^d \psi(x,u) - u\,\varphi(x,u)^d$$

at the origin. Replacing $u \longmapsto a\,x^d$ and dividing by x^d produces

$$g_a(x) := x^{-d} F(x, a\,x^d) = \psi(x, a\,x^d) - a\,\varphi(x, a\,x^d)^d. \tag{6.8}$$

The monomial $c_{ij}\, x^i u^j$ in the Taylor expansion of $F(x,u)$ at $x = u = 0$ is mapped to $c_{ij}\, a^j x^{i + (j-1)d}$ in that of $g_a(x)$ at $x = 0$, and thus, condition (6.6) implies that

$$c_{ij} = 0 \qquad \text{whenever} \qquad i + (j-1)\,d \le n. \tag{6.9}$$

Moreover, by our hypothesis, if $i + j \le \ell$, then

$$i + (j-1)\,d \le (\ell - 1)\,d \le n,$$

by hypothesis. Thus, (6.9) implies that all the Taylor coefficients of order $\le \ell$ in $F(x,u)$ vanish, thereby proving (6.7). *Q.E.D.*

Next, let us assume that condition (6.3) holds for the two curve families C_a^d and C_a^{d+1} for some $r \ge 2$ and all a in some interval containing 0. Assuming

$$n \ge (\ell - 1)\,(d+1), \tag{6.10}$$

Lemma 6.1 implies that (6.7) holds and, in addition,

$$x^{d+1}\psi(x,u) = u\,\varphi(x,u)^{d+1} + \mathrm{O}(\ell+1). \tag{6.11}$$

Multiplying (6.7) by x and subtracting from (6.11) yields

$$u\,\varphi(x,u)^d \big[\,\varphi(x,u) - x\,\big] = \mathrm{O}(\ell+1). \tag{6.12}$$

Now write

$$\varphi(x,u) = \alpha\,x + \beta\,u + \mathrm{O}(2), \qquad \psi(x,u) = \gamma\,x + \delta\,u + \mathrm{O}(2),$$

where the local diffeomorphism condition requires

$$\alpha\,\delta - \beta\,\gamma \ne 0. \tag{6.13}$$

Then, provided $\ell \ge d+2$, the lowest order terms in (6.12) are

$$u\,(\alpha\,x + \beta\,u)^d \,\big[\,(\alpha - 1)\,x + \beta\,u\,\big] = 0.$$

This implies that $\beta = 0$, and hence, in view of (6.13), $\alpha = 1$, which establishes (6.5) when $k = 1$. Next, suppose, by induction on k, that we have proved

$$\varphi(x,u) = x + \mathrm{O}(k), \qquad \psi(x,u) = u + \mathrm{O}(k).$$

for some $k \ge 2$. In particular,

$$\varphi(x,u) = x + \sum_{i=0}^{k} c_i\,x^i u^{k-i} + \mathrm{O}(k+1)$$

for some $c_0, \ldots, c_k \in \mathbb{R}$. Assuming

$$\ell \ge k + d + 1, \tag{6.14}$$

the terms of order $k+d+1$ in (6.12) are

$$\sum_{i=0}^{k} c_i\,x^{i+d} u^{k+1-i} = 0,$$

which implies all $c_i = 0$ and hence $\varphi(x,u) = x + \mathrm{O}(k+1)$. Substituting this result into (6.7), we deduce that $\psi(x,u) = u + \mathrm{O}(k+1)$, and therefore (6.5) holds. Recalling the inequalities (6.10, 14), we find that the preceding induction step remains valid provided

$$n \geq (\ell - 1)\,(d+1) \geq (k+d)\,(d+1). \tag{6.15}$$

We have thus proved the desired result:

Lemma 6.2. *Let $2 \leq k, d \in \mathbb{N}$. If the diffeomorphism* (6.1) *satisfies*

$$\Phi^{(n)}(j_n C^d_a|_0) = j_n C^d_a|_0, \qquad \Phi^{(n)}(j_n C^{d+1}_a|_0) = j_n C^{d+1}_a|_0, \tag{6.16}$$

for $(k+d)\,(d+1) \leq n \in \mathbb{N}$ and all a in some interval containing 0*, then $\Phi \in \mathcal{D}^{(k)}_0$.*

The General Case

The proof of the multidimensional version of Lemma 6.2 proceeds in a similar fashion. We use the notation $(x,u) = (x_1, \ldots, x_p, u_1, \ldots, u_q)$ for coordinates on $M = \mathbb{R}^m$ with $m = p + q$. The graphs of functions $u = h(x)$ are thus p-dimensional submanifolds $S \subset M$. Let

$$\Phi(x,u) = (\varphi(x,u), \psi(x,u)) = \bigl(\, \varphi_1(x,u), \ldots, \varphi_p(x,u), \psi_1(x,u) \ldots, \psi_q(x,u) \,\bigr) \tag{6.17}$$

be a local diffeomorphism of $\mathbb{R}^m$ such that $\Phi(0,0) = (0,0)$. Again, $\mathbb{1}$ denotes the identity diffeomorphism, so $\mathbb{1}(x,u) \equiv (x,u)$.

Let $D = (D^1, \ldots, D^q) \in (\mathbb{N}^p)^q$ be a collection of multi-indices, with entries $D^\sigma = (d^\sigma_1, \ldots, d^\sigma_p) \in \mathbb{N}^p$ for $\sigma = 1, \ldots, q$. Let $|\, D^\sigma \,| = d^\sigma_1 + \cdots + d^\sigma_p$. Define the *minimum* and *maximum orders* of D as

$$\operatorname{min\,ord} D = \min\bigl\{\, |\, D^1 \,|, \ldots, |\, D^q \,| \,\bigr\}, \qquad \operatorname{max\,ord} D = \max\bigl\{\, |\, D^1 \,|, \ldots, |\, D^q \,| \,\bigr\}. \tag{6.18}$$

Given $1 \leq i \leq p$, $1 \leq \sigma \leq q$, let $\widetilde{D}^\sigma_i \in (\mathbb{N}^p)^q$ be the collection obtained by adding 1 to the i^{th} entry of D^σ only, and so its $(\tau, j)^{\text{th}}$ entry is $d^\tau_j + \delta^\tau_\sigma \delta^i_j$, where δ is the usual Kronecker symbol. Note that $\operatorname{min\,ord} \widetilde{D}^\sigma_i \geq \operatorname{min\,ord} D$, while $\operatorname{max\,ord} \widetilde{D}^\sigma_i \leq \operatorname{max\,ord} D + 1$, with equality in the latter case if and only if $\operatorname{max\,ord} D = |\, D^\sigma \,|$.

Given $a = (a_1, \ldots, a_q) \in \mathbb{R}^q$, we abbreviate a q-tuple of elementary monomials by

$$a\, x^D = \bigl(\, a_1 x^{D^1},\ \ldots\ , a_q\, x^{D^q} \,\bigr) = \bigl(\, a_1 x_1^{d^1_1} \cdots x_p^{d^1_p},\ \ldots\ , a_q\, x_1^{d^q_1} \cdots x_p^{d^q_p} \,\bigr). \tag{6.19}$$

Consider the corresponding p-dimensional "monomial submanifolds"†

$$S^D_a = \bigl\{\, (x,u) \bigm|\ u = a\, x^D \,\bigr\} = \bigl\{\, (x,u) \bigm|\ u_\sigma = a_\sigma\, x^{D^\sigma},\ \ \sigma = 1, \ldots, q \,\bigr\} \subset M. \tag{6.20}$$

Note that S^D_a passes through the origin, where

$$\mathrm{j}_\ell S^D_a|_0 = 0 \qquad \text{for} \qquad 0 \leq \ell < \operatorname{min\,ord} D. \tag{6.21}$$

The multidimensional version of Lemma 6.2 is:

† No summations are implied by the repeated indices.

Lemma 6.3. *Let $2 \leq k \in \mathbb{N}$, and let $D \in (\mathbb{N}^p)^q$ with $d = \max \operatorname{ord} D \geq 2$. Suppose*[‡]

$$(k+d+1)\,(d+1) \leq n \in \mathbb{N}. \tag{6.22}$$

Let $0 \in U \subset \mathbb{R}^q$ be an open set. If the diffeomorphism (6.17) satisfies

$$\begin{aligned} \Phi^{(n)}(j_n S_a^D|_0) &= j_n S_a^D|_0, \\ \Phi^{(n)}(j_n S_a^{\widetilde{D}_i^\sigma}|_0) &= j_n S_a^{\widetilde{D}_i^\sigma}|_0, \end{aligned} \qquad \text{for all} \qquad \begin{aligned} i &= 1, \ldots, p, \\ \sigma &= 1, \ldots, q, \end{aligned} \qquad a \in U, \tag{6.23}$$

then $\Phi \in \mathcal{D}_0^{(k)}$.

Proof: We begin with the analog of Lemma 6.1.

Lemma 6.4. *Let $2 \leq \ell \in \mathbb{N}$, and let $D \in (\mathbb{N}^p)^q$ with $d = \max \operatorname{ord} D \geq 2$. Suppose $\ell\, d \leq n \in \mathbb{N}$. If, as a function of x,*

$$x^{D^\sigma} \psi_\sigma(x, a\,x^D) = a_\sigma\, \varphi(x, a\,x^D)^{D^\sigma} + \mathrm{O}(n+1), \tag{6.24}$$

for some $1 \leq \sigma \leq q$ and all $a \in U$, then, as a function of x, u,

$$x^{D^\sigma} \psi_\sigma(x, u) = u_\sigma\, \varphi(x, u)^{D^\sigma} + \mathrm{O}(\ell+1). \tag{6.25}$$

Proof: Let c^σ_{IJ} denote the coefficient of the monomial $x^I u^J$ in the Taylor expansion of

$$F^\sigma(x,u) := x^{D^\sigma} \psi_\sigma(x,u) - u_\sigma\, \varphi(x,u)^{D^\sigma}$$

at the origin. Replacing $u \longmapsto a\,x^D$ and dividing by x^{D^σ} produces

$$g^\sigma_a(x) := F^\sigma(x, a\,x^D)/x^{D^\sigma} = \psi_\sigma(x, a\,x^D) - a_\sigma\, \varphi(x, a\,x^D)^{D^\sigma}.$$

Under this transformation, the monomial $c^\sigma_{IJ}\, x^I u^J$ in the Taylor expansion of $F^\sigma(x,u)$ at the origin is mapped to the following monomial in the Taylor expansion of $g^\sigma_a(x)$ at the origin:

$$c^\sigma_{IJ}\, a^J x^{I+JD-D^\sigma} \qquad \text{where} \qquad JD := j_1 D^1 + \cdots + j_q D^q. \tag{6.26}$$

Thus, condition (6.24) implies that

$$c^\sigma_{IJ} = 0 \qquad \text{whenever} \qquad 0 \leq |\, I + JD - D^\sigma \,| \leq n. \tag{6.27}$$

(Note that every term in $F^\sigma(x, a\,x^D)$ has a factor of x^{D^σ} and hence $c^\sigma_{IJ} = 0$ ab initio if any entry of $I + J\,D - D^\sigma$ is negative.) Moreover, the monomial (6.26) has order

$$|\, I + JD - D^\sigma \,| = |\, I \,| + \sum_{\tau=1}^{q} (j_\tau - \delta^\sigma_\tau)\, |\, D^\tau \,|.$$

‡ This inequality is slightly different than the univariate version, owing to the extra complications involving several multi-indices.

Under the constraint $|\,I\,| + |\,J\,| \leq \ell$, the maximal such order is $\leq \ell\, d$ and is obtained[†] by setting all $i_\nu = j_\tau = 0$ except for $j_\rho = \ell$ where ρ is such that $d = \max\operatorname{ord} D = |\,D^\rho\,|$. Thus, assuming $n \geq \ell\, d$, (6.27) implies that all the Taylor coefficients of order $\leq \ell$ in (6.25) vanish, proving the result. *Q.E.D.*

Turning to the proof of Lemma 6.3, the first condition in (6.23) implies (6.25) holds for all $\sigma = 1, \ldots, q$ provided $n \geq \ell\, d$. The corresponding condition for $\widetilde{D}_i^\sigma$ yields, in particular,

$$x_i\, x^{D^\sigma} \psi_\sigma(x,u) = u_\sigma\, \varphi(x,u)^{D^\sigma} \varphi_i(x,u) + \mathrm{O}(\ell+1), \tag{6.28}$$

provided $n \geq \ell\,(d+1)$. On the other hand, multiplying (6.25) by x_i and subtracting yields

$$u_\sigma\, \varphi(x,u)^{D^\sigma} \left[\, \varphi_i(x,u) - x_i \,\right] + \mathrm{O}(\ell+1). \tag{6.29}$$

By the same inductive reasoning as in the planar case, using the assumed inequality (6.22), we deduce that

$$\varphi_i(x,u) = x_i + \mathrm{O}(k+1), \qquad i = 1, \ldots, p.$$

Substituting this result back into (6.25) also implies

$$\psi_\sigma(x,u) = u_\sigma + \mathrm{O}(k+1), \qquad \sigma = 1, \ldots, q,$$

which establishes the lemma. *Q.E.D.*

Theorem 6.5. *Let $k, \ell, n \in \mathbb{N}^+$ satisfy*

$$(k+\ell+2)\,(\ell+2) \leq n. \tag{6.30}$$

Given an ℓ^{th} order jet $z_0^{(\ell)} \in \mathrm{J}^\ell|_{z_0}$ based at a point $z_0 \in M$, let $Q_0^n := (\pi_\ell^n)^{-1}\{\, z_0^{(\ell)}\,\} \subset \mathrm{J}^n$. Suppose that $\Phi\colon M \to M$ is a diffeomorphism such that

$$\Phi^{(\ell)}(z_0^{(\ell)}) = z_0^{(\ell)} \quad \textit{and} \quad \Phi^{(n)}(z^{(n)}) = z^{(n)} \quad \textit{for all} \quad z^{(n)} \in Q_0^n. \tag{6.31}$$

Then $\Phi \in \mathcal{D}_{z_0}^{(k)}$, i.e., $J^k\Phi|_{z_0} = J^k \mathbb{1}|_{z_0}$.

Proof: Choose local coordinates on M such that $z_0 = 0$ and $z_0^{(\ell)} = 0$. Then, under the hypothesis of the theorem, $\Phi^{(n)}$ fixes the n jets of all p-dimensional submanifolds $S \subset M$ passing through $z_0 = 0$ such that $j_\ell S|_0 = 0$, and hence, in particular, those given by $S = S_a^D$, as defined in (6.20), provided $\ell < \min\operatorname{ord} D$, as in (6.21). Fix D so that

$$\ell + 1 = d = |\,D^1\,| = \ \cdots\ = |\,D^q\,| = \min\operatorname{ord} D = \max\operatorname{ord} D,$$

and let $U \subset \mathbb{R}^q$ be open. Then

$$j_n S_a^D|_{z_0},\ j_n S_a^{\widetilde{D}_i^\sigma}|_{z_0} \in Q_0^n \quad \text{whenever} \quad i = 1, \ldots, p, \quad \sigma = 1, \ldots, q, \quad a \in U,$$

and the conclusion follows immediately from Lemma 6.3. *Q.E.D.*

[†] If $\rho = \sigma$, this may not be the index that gives the maximal order, but the maximal order bound remains valid in all cases.

7. Generic Freeness of Prolonged Group Actions on Jet Bundles.

We are now ready to prove our main result concerning the generic freeness of prolonged group actions on higher order submanifold jet bundles.

Theorem 7.1. *Let G be a connected Lie group that acts on a m-dimensional manifold M analytically and effectively on subsets. Then, for sufficiently large $n \in \mathbb{N}^+$, there exists a nonempty G-invariant open subset of the n^{th} order submanifold jet bundle, $W^n \subset \mathrm{J}^n$, and a G-invariant meager subset $Z^n \subset \mathrm{J}^n$ such that the n^{th} order prolonged action of G on $W^n \setminus Z^n$ is free.*

Proof: Let $r = \dim G$. By Theorem 2.3, we can find a nonempty G-invariant dense open subset $V^r \subset \mathrm{J}^r$ such that the G-action on V^r is locally free.

Next, given $g \in G$ with induced diffeomorphism $\overline{g}: M \to M$, set

$$A^i_g := \{\, z \in M \mid \overline{g} \in \mathcal{D}^{(i)}_z \,\} \qquad \text{for} \qquad i \in \mathbb{N}^+. \tag{7.1}$$

Further, let $G^\times = G \setminus \{e\}$, and define

$$A^i = \bigcup_{g \in G^\times} A^i_g, \qquad\qquad B^i = M \setminus A^i. \tag{7.2}$$

Observe that $A^i_g \supseteq A^{i+1}_g$, and hence $A^i \supseteq A^{i+1}$, $B^i \subseteq B^{i+1}$. Further, since $h \cdot A^i_g = A^i_{hgh^{-1}}$ for any $h \in G$, it follows that A^i and B^i are all G-invariant subsets of M. By Theorem 5.2, $A^0 \cap A^1 \cap A^2 \cap \cdots = \varnothing$, hence $B^0 \cup B^1 \cup B^2 \cup \cdots = M$. As M is Polish, we can find $2 \leq k \in \mathbb{N}$ such that, for all $j \geq k$, the set B^j is nonmeager in M.

Recalling Theorem 6.5, we now claim that

$$n = (k + r + 2)\,(r + 2) \tag{7.3}$$

satisfies the conditions described in the Theorem.

Claim 1: The set of points in J^n where G acts freely, namely,

$$S^n = \{\, z^{(n)} \in \mathrm{J}^n \mid G_{z^{(n)}} = \{e\} \,\},$$

is nonmeager and G-invariant in J^n.

Proof of Claim 1: The G-invariance of S^n is clear. We suppose that S^n is meager in J^n, and seek a contradiction.

The canonical projection $\pi^n_r: \mathrm{J}^n \to \mathrm{J}^r$ is continuous and open. Given $z^{(r)} \in \mathrm{J}^r$, let $Q^n|_{z^{(r)}} := (\pi^n_r)^{-1}\{z^{(r)}\} \subset \mathrm{J}^n$, which has the subspace topology inherited from J^n. By Lemma 4.8, $S^n|_{z^{(r)}} := S^n \cap Q^n|_{z^{(r)}}$ is meager in $Q^n|_{z^{(r)}}$ for essentially all $z^{(r)} \in \mathrm{J}^r$. Therefore,

$$T^r := \{\, z^{(r)} \in \mathrm{J}^r \mid S^n|_{z^{(r)}} \text{ is nonmeager in } Q^n|_{z^{(r)}} \,\}$$

is meager in J^r.

Let $\widehat{\pi}^r_0 = \pi^r_0 \mid V^r: V^r \to M$, which is continuous, open, and surjective. Since B^k is nonmeager in M, Corollary 4.12 implies that $U^r := (\widehat{\pi}^r_0)^{-1}(B^k)$ is nonmeager in V^r, and, by Corollary 4.3, also nonmeager in J^r. Thus, $U^r \setminus T^r \neq \varnothing$ and there exists $z_0^{(r)} \in U^r \setminus T^r$, meaning that $S_0^n := S^n|_{z_0^{(r)}}$ is meager in $Q_0^n := Q^n|_{z_0^{(r)}}$. We set $z_0 = \widehat{\pi}_0^r(z_0^{(r)})$, so that $z_0 \in B^k$ by construction. Since $z_0^{(r)} \in V^r$, its isotropy subgroup $\Gamma := G_{z_0^{(r)}}$ is a discrete subgroup of G and hence countable. Clearly $\Gamma \cdot Q_0^n \subseteq Q_0^n$. Moreover, $G_{z^{(n)}} \subseteq \Gamma$ for any $z^{(n)} \in Q_0^n$ since $\pi_r^n(z^{(n)}) = z_0^{(r)}$. Given $g \in \Gamma$, let

$$K_g^n = \left\{ z^{(n)} \in Q_0^n \,\middle|\, g \cdot z^{(n)} = z^{(n)} \right\}.$$

Set $\Gamma^\times = \Gamma \setminus \{e\}$. Then

$$\begin{aligned} \bigcup_{g \in \Gamma^\times} K_g^n &= \left\{ z^{(n)} \in Q_0^n \,\middle|\, \text{there exists } g \in \Gamma \text{ such that } g \neq e \text{ and } g \cdot z^{(n)} = z^{(n)} \right\} \\ &= \left\{ z^{(n)} \in Q_0^n \,\middle|\, G_{z^{(n)}} \neq \{e\} \right\} = Q_0^n \setminus S_0^n. \end{aligned}$$

Since Q_0^n is closed in J^n, which is Polish, it follows that Q_0^n is Polish, and therefore almost open, and hence nonmeager-in-itself. Thus $Q_0^n \setminus S_0^n$ is nonmeager in Q_0^n. We can hence choose $g_0 \in \Gamma^\times$ such that $K_{g_0}^n$ is nonmeager in Q_0^n. Since $K_{g_0}^n$ is also closed in Q_0^n, it is not interior free in Q_0^n, so let $\widehat{K}_{g_0}^n \neq \varnothing$ be its interior in Q_0^n. For any $K \subseteq Q_0^n$, let $\sigma(K)$ denote the set of its limit points in Q_0^n. Because Q_0^n admits a C^ω-manifold structure under which the prolonged action of g_0 is analytic, we conclude that $\sigma(\widehat{K}_{g_0}^n) \subseteq \sigma(K_{g_0}^n) \subseteq \widehat{K}_{g_0}^n \subseteq K_{g_0}^n$. Thus, $\widehat{K}_{g_0}^n$ is both closed and open in Q_0^n and hence, since Q_0^n is connected, $\widehat{K}_{g_0}^n = Q_0^n$.

Consequently, according to (7.3), the diffeomorphism $\Phi = \overline{g}_0$ of M determined by g_0 satisfies all the hypotheses of Theorem 6.5, and hence $\overline{g}_0 \in \mathcal{D}_{z_0}^{(k)}$. Hence, by (7.1), $z_0 \in A_{g_0}^k \subset A^k$. On the other hand, we already established that $z_0 \in B^k = M \setminus A^k$, which produces our desired contradiction. *End of proof of Claim 1.*

Next, consider the closed subset

$$\left\{ (g, z^{(n)}) \in G^\times \times \mathrm{J}^n \,\middle|\, g \cdot z^{(n)} = z^{(n)} \right\} \subset G^\times \times \mathrm{J}^n = (G \setminus \{e\}) \times \mathrm{J}^n.$$

Under the projection mapping $G^\times \times \mathrm{J}^n \to \mathrm{J}^n$, its image is $\mathrm{J}^n \setminus S^n$, which is thus an analytic subset of J^n, and hence, by, *e.g.*, [**11**; Theorem 21.6, p. 153], almost open in J^n. Thus, its complement S^n is also almost open in J^n. Choose an open subset $W_0^n \subset \mathrm{J}^n$ such that $W_0^n \equiv S^n$ in J^n. As a consequence of Claim 1, W_0^n is also nonmeager in J^n, and so $W_0^n \neq \varnothing$. Let Δ be a countable dense subset of G, and set $W^n := \Delta \cdot W_0^n \neq \varnothing$.

Claim 2: $G \cdot W_0^n \subseteq W^n$.

Proof of Claim 2: Let $g \in G$ and $z_0^{(n)} \in W_0^n$ be given. We wish to show that $g \cdot z_0^{(n)} \in W^n$. Since the orbit map $\psi_{z_0^{(n)}}\colon g \mapsto g \cdot z_0^{(n)}$ from G to J^n is continuous, and W_0^n is open in J^n, we can choose an open neighborhood H of e in G such that $H \cdot z_0^{(n)} \subseteq W_0^n$. By density of Δ in G, choose $\delta \in \Delta \cap gH^{-1}$. Set $h := \delta^{-1} g \in H$, so that $g \cdot z_0^{(n)} = \delta \cdot h \cdot z_0^{(n)} \in \Delta \cdot H \cdot z_0^{(n)} \subseteq \Delta \cdot W_0^n = W^n$, as desired. *End of proof of Claim 2.*

Claim 2 implies $G \cdot W^n = G \cdot \Delta \cdot W_0^n = G \cdot W_0^n \subseteq W^n$, and so W^n is G-invariant. Since Δ is countable and $W_0^n \equiv S^n$ in J^n, we have $W^n = \Delta \cdot W_0^n \equiv \Delta \cdot S^n = S^n$ in J^n. Thus, $Z^n = W^n \setminus S^n$ is both G-invariant and meager in J^n, as desired. *Q.E.D.*

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