Symmetry Groups of Partial Differential Equations

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by

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Introduction

The application of the theory of local transformation groups to the study of differential equations has its origins in the original work of Sophus Lie on the foundations of the theory of continuous groups. One of Lie's primary motivations for introducing the concept of a continuous group of transformations was the incisive observation that many seemingly dispatate special methods that had been devised for integrating ordinary differential equations were in fact all special cases of his general method of integrating ordinary differential equations invariant under one-parameter groups of transformations acting on the space of independent and dependent variables in the equations. Thus, the standard method used to integrate by quadratures a homogeneous first order ordinary differential equation is a direct consequence of its invariance under the scale group, which acts via multiplication of the independent and dependent variables by the same arbitrary nonzero scalar. The above observation alone, however, would be of negligible importance for practical purposes, were it not for the fact that the invariance of a specific equation under a transformation group can be readily checked by the consideration of the infinitesimal generators of the group, rather than the group per se. In fact, for a given equation, the Lie algebra of all vector fields (i.e. infinitesimal generators of local one-parameter groups) which leave it invariant can be straightforwardly found via the solution of a number of auxillary differential equations of an elementary nature, whereas the "symmetry group" of the equation cannot itself be found by any easy computational methods, short of exponentiating the algebra.†

Lie further generalized the concept of group invariance from ordinary to partial differential equations, and could by similar calculations construct the symmetry group of a given system of partial differential equations. Although these symmetry groups still possessed the property of transforming solutions to other solutions, they did not provide much help towards the problem of constructing the general solution, in contrast to the case of ordinary differential equations. (A notable exception is the long neglected work of Vessiot [VE] on the group fibering of an equation. This has been rarely applied to any practical situations.) After the turn of the century, the emphasis in Lie group theory shifted away from the local transformation group to the more abstract global theory, leaving behind the results on differential equations, which are of an essentially local

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†Throughout this thesis, systems of partial differential equations will be viewed as subvarieties of extended jet bundles, which are fiber bundles over a smooth manifold, and whose local coordinates are the various partial derivatives of the dependent variables. The advantage of this abstract point of view is that the arguments take on a geometric flavor that allows us to apply the standard differential-geometric results of transformation group theory.

The term "symmetry group" will be taken to mean the widest local group of transformations acting on the manifold whose prolongation to the extended jet bundle, i.e. whose action on the partial derivatives, leaves the subvariety representing the equations invariant. It should be noted that this group may be properly contained in the symmetry group of the solution set -- the widest transformation group taking solutions of the equations to other solutions, especially if the equations have very few solutions. However, these two groups are necessarily equal if the equations have solutions passing through every point of their corresponding subvariety.

character. The early investigators also failed to discover the concept of a group invariant solution, and for these reasons the applications of Lie group theory to partial differential equations was neglected for almost half a century. Thus it was not until after 1940, beginning with the work of L.I. Sedov [SE], and G. Birkhoff [B], on a general theory of dimensional analysis with applications to the equations of continuum mechanics, and particularly hydrodynamics, that essentially new research into this area was started. The concept of a group invariant or self-similar solution† to a system of partial differential equations, in the special case of the scale groups of dimensional analysis, was apparently first consciously considered at this time. It was soon realized that group invariant solutions could be found for arbitrary local transformation groups, and their construction involved the solution of partial differential equations in fewer independent variables. (See, for instance, [MO] for an early version of this theorem.) Finally, in the early 1960's the fundamental work of L.V. Ovsjannikov on group invariant solutions demonstrated the power and generality of these methods for the construction of explicit solutions to complicated systems of partial differential equations. While only local in nature and not fully rigorous in proof, Ovsjannikov's methods provided the theoretical framework to commence a systematic study of the groups of well-known

†The terms symmetric and automodel solutions have also been used in the literature. I shall exclucively use the term group invariant (of G invariant, if G is the particular group) to describe all of these equivalent concepts.

partial differential equations of mathematical physics. This work is being pursued by Ovsjannikov, Bluman, Cole, Ames and others, and has provided many new explicit solutions to important equations.

One of the primary purposes of this thesis is to provide a rigorous foundation for the theory of symmetry groups of differential equations, and to demonstrate the global counterparts (and counter-examples) to the local results of Ovsjannikov. This will be accomplished primarily in the language of differential geometry, utilizing a new theory of partial differential equations on arbitrary smooth manifolds which generalizes the theory of differential equations for vector bundles. To give some intuitive ideas of the relevant concepts, this introduction contains a brief nonrigorous sketch of the theory, which shall be illustrated by one of the most important examples -- the heat equation. For the heat equation, it should be noted that while the general ideas were first developed in [B], the general similarity solution was rigorously discussed only as recently as 1969 in [BC1].

The main computational tool introduced by Lie group theory was the reduction of questions of invariance to the infinitesimal or Lie algebra level, making them amenable to algebraic techniques.† Thus,

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†Here the major problem with the previous definition of the symmetry group of a system of equations becomes apparent; namely, that there may be little correspondence between the group and its infinitesimal generators. First, it should be remarked that we will not consider discrete symmetries not contained in continuous subgroups, i.e. we will only deal with the local group. The second problem is that in the case that the group is infinite dimentional, the introduction of the more complicated concept of a Lie pseudogroup of transformations becomes necessary to maintain any reasonable correspondence between the infinitesimal algebra and the group -- see [SS] and [KU] for detailed discussions of this.

a necessary and sufficient condition for the invariance of a subvariety given by the vanishing of a smooth function is that the differential of the function annihilate all the infinitesimal generators of the group action. The concrete realization of a system of partial differential equations as a subvariety of some appropriate space of partial derivatives together with the formulas for the prolongation of the transformation group and hence its infinitesimal generators to this space reduces the question of invariance of the system of differential equations to the invariance of the corresponding subvariety under the prolonged group action. Here the infinitesimal criterion of invariance becomes crucial in that the prolonged vector fields are readily computable, whereas the prolonged group action is much more difficult to get a grasp on.

More specifically, let Z be a smooth manifold representing the independent and dependent variables in the equations under consideration. In the classical case Z will be an open subset of the Euclidean space $\mathbb{R}^p \times \mathbb{R}^q$ with coordinates $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$, where the x's are the independent variables and the u's the dependent variables. Fibered over Z will be a fiber bundle $J_k^*(Z,p)$ (p denoting the number of independent variables) corresponding to the various partial derivatives of the u's with respect to the x's of order $\leq k$. This bundle will be called the extended jet bundle, and will, in the special case that Z is a vector bundle, be the "completion" of the usual k-jet bundle $J_k Z$ in the same sense that projective space is the "completion" of affine space. In this context, a system of partial differential equations of k-th order will be described by a closed subvariety Δ_0 of $J_k^*(Z,p)$. A solution of Δ_0 will be a p dimensional submanifold of Z, called a p-section, whose extended k-jet, a p dimensional submanifold of $J_k^*(Z,p)$ lying over the original submanifold of Z, is entirely contained in Δ_0. As a matter of fact, the fiber $J_k^*(Z,p)|_z$ over a point $z \in Z$ is given by the equivalence classes of p dimensional submanifolds of Z passing through z having k-th order contact. Now suppose that G is a local Lie group of transformations acting smoothly on Z. The transformations in G prolong to transformations on $J_k^*(Z,p)$ under their action on p-sections. The corresponding prolongation of the infinitesimal generators of G has a relatively simple expression in local coordinates. We derive this expression using the techniiques of symmetric algebra; it does not seem to have appeared previously in the literature. Its use allows a much more unified and straightforward derivation of the symmetry groups of higher order partial differential equations than has been done in other work in this subject.

Now suppose that G acts regularly on Z in the sense of Palais [P1] (see the appendix for the correct definitions); this implies that there is a natural manifold structure on the quotient space Z/G. The resulting system of partial differential equations for the G invariant solutions to Δ_0 will be a subvariety $\Delta_0/G \subset J_k^*(Z/G,p-\ell)$, where ℓ is the dimension of the orbits of G. The solutions of Δ_0/G are $(p-\ell)$-sections of Z/G, which when lifted back to Z provide all the G invariant solutions to the original system Δ_0. The important point is that the number of independent variables is reduced by ℓ, making the reduced system Δ_0/G in some sense easier to solve. (Although this does not always hold true in practice; there are examples where Δ_0 is, say, linear, whereas Δ_0/G is a messy nonlinear equation.) It is this fact that makes the symmetry group method so useful for finding exact solutions to complicated differential equations.

To illustrate the application of these abstract concepts in a concrete situation, consider the heat equation

$$u_t = u_{xx}$$

where (x,t) are the independent variables and u the dependent variable. Therefore we take $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u). The bundle $J_2^*(Z,2)$ will be a bundle of twice prolonged Grassmann manifolds, whose structure we will not attempt to describe here. Suffice it to say that the usual 2-jet bundle $J_2Z \approx \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^3$, with coordinates $(x,t;u;u_x,u_t;u_{xx},u_{xt},u_{tt})$, is an open dense subbundle of $J_2^*(Z,2)$, so we are justified in restricting our attention to it. The heat equation in our language is given by the closure of the subvariety

$$\Delta_0 = \{u_t = u_{xx}\} \subset J_2Z$$

in $J_2^*(Z,2)$. Using the prolongation formula for the smooth vector fields on Z and solving the auxilliary equations resulting from the infinitesimal criterion of invariance of Δ_0 (a procedure that will be described in great detail in chapter III) it can be seen that the infinitesimal symmetry algebra of the heat equation is spanned by the vectorfields

$$\vec{v}_1 = \partial_x$$
$$\vec{v}_2 = \partial_t$$
$$\vec{v}_3 = u\partial_u$$
$$\vec{v}_4 = x\partial_x + 2t\partial_t$$
$$\vec{v}_5 = 2t\partial_x - xu\partial_u$$
$$\vec{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$$
$$\vec{v}_\alpha = \alpha(x,t)\partial_u$$

where $\alpha(x,t)$ is any solution of the heat equation. The vector fields $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_\alpha$ merely reflect the fact that the heat equation is a linear, constant coefficient partial differential equation. The other three vector fields reflect more nontrivial symmetries.

Now consider some of the one-parameter subgroups generated by single vector fields in the symmetry algebra. For example, the group G_1 generated by $\vec{v}_1$ is just translation in the x coordinate

$$G_1: \quad (x,t,u) \mapsto (x+\lambda,t,u) \qquad \lambda \in \mathbb{R}$$

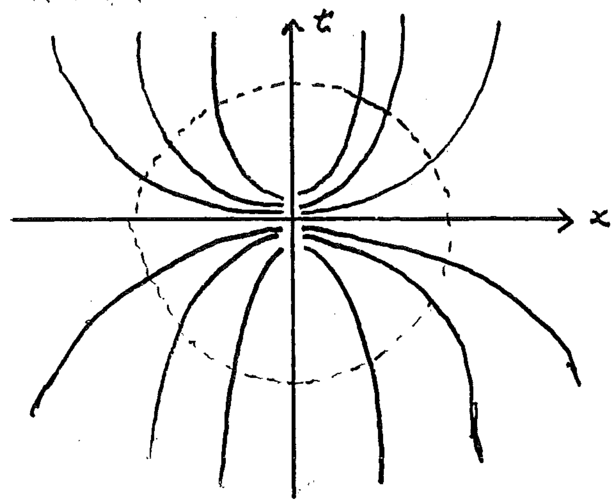
and is therefore a global, regular one-parameter group of transformations on Z. The quotient manifold $Z/G_1 \simeq \mathbb{R}^2$ with coordinates (t,u), so the reduced equation Δ_0/G_1 is just

$$u' = 0$$

so the only G_1 invariant solutions to the heat equation are the constants. Secondly consider the one-parameter group generated by the vector field $\vec{v}_4$

$$G_4: \quad (x,t,u) \mapsto (e^{\lambda}x,\ e^{2\lambda}t,\ u) \qquad \lambda \in \mathbb{R}.$$

In this case G_4 acts regularly on $Z' = Z - \{(x,t,u): x=t=0\}$ and $Z'/G_4 \simeq S^1 \times \mathbb{R}$ since on any horizontal plane $\{u=\text{constant}\}$ the orbits of G_4 are the halves of parabolas passing through the origin. In terms of the local coordinates $(\xi = x^2/t, u)$ on Z'/G_4 the reduced equation Δ_0/G_4 is



$$4\xi u'' + (2+\xi)u' = 0$$

and therefore the G_4 invariant solutions to the heat equation are

$$u(x,t) = \begin{cases} c_1 \operatorname{erf}(\tfrac{1}{2}xt^{-1/2}) + k_1 & t > 0 \\ c_1 + k_1 = c_2 + k_2 & t = 0,\ x \neq 0 \\ c_2 \operatorname{erf}(\tfrac{1}{2}x(-t)^{-1/2}) + k_2 & t < 0 \end{cases}$$

where erf is the error function and c_1, k_1, c_2, k_2 are arbitrary constants subject to the constraint of the middle equation. Finally, the vector field $\vec{v}_6$ can be seen to generate only a local group

$$G_6: \quad (x,t,u) \mapsto \left(\frac{x}{4\lambda t+1}, \frac{t}{4\lambda t+1}, u\sqrt{4\lambda t+1}\, \exp \frac{\lambda x^2}{4\lambda t+1} \right) \qquad \lambda \in \mathbb{R}$$

where λ is restricted so that $4\lambda t+1 > 0$. The group G_6 can be globalized, cf. [P1], but this does not seem to have any practical consequences. The G_6 invariant solutions to the heat equation are

$$u(x,t) = \begin{cases} t^{-3/2}(ax+bt)\exp(-\tfrac{1}{4}x^2t^{-1}) & t > 0 \\ (-t)^{-3/2}(ax+bt)\exp(\tfrac{1}{4}x^2t^{-1}) & t < 0 \end{cases}$$

where a and b are arbitrary constants.

One might reasonably ask whether there is anything to be gained (or lost) by considering the equivalent first order system of equations

$$u_x = v \qquad\qquad u_t = v_x$$

rather than the second order heat equation. Here the new equations form a subvariety of $J_1^*(Z_1,2)$, where $Z_1 = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates $(x,t;u,v)$. In chapter III, we show that the group of the heat equation is just the projection (along the v-axis) of the group of the above system, and conversely, the group of the system is in some sense the prolongation of the group of the heat equation. In general, the symmetry groups of a higher order equation and the equivalent first order system are isomorphic, barring the presence of symmetries that depend on the derivatives ("higher order symmetries.") The main difference is that the group is easier to compute for the single higher order equation than for the first order system.

The various problems involved in putting Ovsjannikov's theory in a rigorous, global setting now become more readily apparent. The most important is the presence of singular orbits in the group action. The first aspect of this problem concerns the algebra of infinitesimal generators not having constant dimensionality over the whole manifold Z. Here we can invoke the theorem on the group invariance of the sets of constant dimensionality to break up the original manifold, so that the algebra can be considered to have constant dimension. Thus, for instance, in the case of the group G_4 of the heat equation, the line $\{x=t=0\}$ where $\vec{v}_4$ vanishes must be discarded to construct the G_4

invariant solutions. This simplified approach, however, ignores the more complicated question of how one patches together the group invariant solutions on each set of constant dimensionality; in general it seems that the solutions must have singularities, although this is not necessarily true. The other aspect of irregular group action is the presence of irregular orbits, the quintessential example of this phenomenon being the irrational flow on the torus. Again, using the fact that the regular orbits form an open submanifold, one can make the simplistic choice of discarding the irregular part of the group action, but this again ignores the more subtle question of describing group invariant solutions near irregular orbits. Both of these problems do not have satisfactory answers at this time.

Less crucial in terms of theoretical difficulties, but still important in the development of a global theory is the determination of explicit group invariant solutions. If the manifold Z is given an a priori distinction between the independent and the dependent variables, (for instance, Z might be an open subset of $\mathbb{R}^p \times \mathbb{R}^q$, or more generally an open submanifold of a vector bundle, where the base manifold represents the independent variables) then one might reasonably ask when the group invariant sections actually give the dependent variables as global single-valued smooth functions of the independent variables. There are two aspects to this question: local and global. If the group action is transversal to the dependent variable fibers, then there is an induced fibration on the quotient manifold Z/G. By the implicit function theorem, any explicit section of Z/G will lift back to a locally explicit section of Z. The global counterpart is more subtle and involves a detailed analysis of the orbits. This is perhaps best illustrated by the following "spiral group" action on $Z = (\mathbb{R}^2-\{0\})\times\mathbb{R}$:

$$G: \quad (x,y,u) \mapsto (x\cos\lambda - y\sin\lambda,\ x\sin\lambda + y\cos\lambda,\ u+\lambda) \quad \lambda\in\mathbb{R},$$

with infinitesimal generator $-y\partial_x + x\partial_y + \partial_u$. The group action is obviously transversal to the fibers $\{(x,y)=\text{constant}\}$ and the orbits are spirals sitting over the circles centered at the origin of the (x,y)-plane. Therefore the G invariant sections of Z are in general locally explicit sections, but the variable u will always be a multiple valued function of x and y. Thus, for example, the G invariant solutions of Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

are the multiple valued logarithmic potentials

$$u(x,y) = a\log(x^2+y^2) + b\tan^{-1}(y/x) + 2n\pi$$

where a and b are constants and n is allowed to assume any integral value.

Now consider the important case in which Z is a fiber bundle over a p-dimensional base manifold with q dimensional fiber. A diffeomorphism $f: Z \to Z$ will be called projectable if it preserves the fibers of Z. In local coordinates, f must take the form

$$f(x,u) = (f_1(x), f_2(x,u)).$$

A local group of transformations acting on Z is projectable if all of its transformations are. The projectable groups form the most important and by far the commonest groups arising in the subject. Note that the symmetry group of the heat equation is entirely projectable. As an application of our theory, we shall prove that any linear partial differential equation of order ≥ 3 has only projectable symmetries in its local symmetry group. This result is a generalization of a result of Ovsjannikov for second order linear equations. The advantages of considering only projectable symmetries are great. The prolongation fromula for projectable vector fields has a much simpler expression, which makes the calculation of the projectable symmetry group of an equation much more tractable. Moreover, the projectable symmetries transform explicit sections to explicit sections. Using the fact that the quotient manifold of a fiber bundle under a projectable, regular group action which projects to a regular group action on the base manifold X is again a fiber bundle (with possibly different fiber) over the quotient manifold of the base space, the preceding formalism of extended jet bundles over arbitrary manifolds can be dispensed with in favor of the more well-known theory of jet bundles of fiber bundles. (Note that we still cannot just consider vector bundles due to the possibility of change in the fibers.) This again is overly restrictive, since it is easy to construct groups that act regularly on the total space Z, but whose projections to X contain irregular orbits. For instance, consider $Z = (\mathbb{R}^2-\{0\})\times\mathbb{R}$ and the group with infinitesimal generator (in polar coordinates) $(1-r)\partial_r + \partial_\theta + \partial_u$. Here the circle $r = \sqrt{x^2+y^2} = 1$ is an irregular orbit of the projected group action. Note that there are no group invariant sections that extend continuously across the irregular orbit.

The preceding discussion should indicate some of the results that this global theory of symmetry groups of partial differential equations will contain. Much of the literature up to the present time has been written in the imprecise and local style of Lie's original work, failing to utilize any modern terminology or results. Ovsjannikov's works, [O1], [O2], are written in the language of the classical differential geometers, cf. Eisenhart, [E1], [E2], with (unstated) concentration on purely local results. Bluman and Cole's latest book, [BC2], is essentially a restatement of results known at the turn of the century, cf. Cohen's monograph [CO], together with their more recent work on group invariant solutions and applications to many interesting equations. Ames, [AW], has also done some work in this field using similar concepts, but with a slightly more cumbersome fromalism. In all of the above works, the results are strictly local in nature; moreover, the proofs are often not entirely rigorous -- all of which is indicative of the need for a rigorous and global theory, which this thesis hopefully provides.

My main inspirations for developing such a theory came from Palais' monograph [P1] on the global theory of Lie transformation groups and the exposition in Federer's book [F] on symmetric algebra and its applications to studying the differentials of smooth maps between vector spaces. The relevant results from Palais have been summarized in the appendix for convenient reference. The main theorem from symmetric algebra is the Faa-di-Bruno formula for the higher order differentials of the composition of smooth maps, which is quoted in section II.2. This theorem forms the basis of many of the new theoretical tools that appear in the ensuing sections. In particular it is used to derive a new natural explicit matrix representation of the prolongations of the general linear group. The extended jet bundles and prolonged Grassmann manifolds are new ideas, although they have been developed in the spirit of Ehresmann's original theory of jet bundles and frames of higher order contact, cf. [EH] and [KO1]. The material on total derivatives is fairly standard, but it is put here into the framework of symmetric algebra. The definitions of systems of partial differential equations over arbitrary manifolds form a straightforward generalization of the corresponding concepts in the category of vector bundles.

The first major application of these new theoretical concepts is the derivation of a new local coordinate expression for the prolongation of an infinitesimal generator of a one-parameter group of transformations, given in theorem III.1.3. This is applied to derive results on the symmetry groups of linear eauations and the relationship between symmetry groups of equivalent systems of partial differential equations. The fundamental theorem of chapter IV is a rigorized and globalized version of the basic theorem of Ovsjannikov on the construction of group invariant solutions via the reduction of the number of independent variables. As far as examples go, the results on the heat equation are for the most part contained in the paper [BC1], although their derivation here is considerably simplified. Most of the results on the Korteweg-deVries equation and Burgers' equation and the telegraph equation appear to be new, especially as regards to their group-theoretical interpretation.

A few brief comments on the organization of the material in this thesis are in order; more complete discussions appear in the introductory sections at the beginning of each chapter. Chapter I deals with the special case of first order systems of equations, and develops the theory from a somewhat different standpoint than is done in the general higher order case. Here the main ideas are able to be presented unencrusted by the computational details that complicate the general case. Chapters II-IV treat the general case of systems of arbitrary order, and form a unit independent of chapter I. The mathematical machinery of symmetric algebra, extended jet bundles, prolongations and total derivatives is developed in chapter II. Chapter III derives the prolongation formula for vector fields and applies this to discuss some specific examples of symmetry groups. Also included in this chapter are discussions of the symmetry groups of linear equations and the relationship between the symmetry group of a higher order equation and its equivalent first order system. In chapter IV, the fundamental theorem relating the extended jet bundle of the quotient manifold to the subbundle of extended jets of group invariant sections is proven and used to find group invariant solutions to some specific partial differential equations. The last section of this chapter takes up the topic of explicit sections. The applications of the general theory are mainly to be found in sections I.5, III.2 and IV.2.

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I. Symmetries of First Order Partial Differential Equations

The first chapter of this thesis provides a complete development of the theory of symmetry groups and the method of construction of group invariant solutions for the case of first order systems of partial differential equations. This material is essentially independent of the remaining three chapters of the thesis, and the reader could just as well begin on the general case starting with chapter 2 immediately. However, there are two good reasons for the inclusion of this first chapter. The first is that most of the crucial ingredients needed to develop a rigorous theory of symmetry groups of partial differential equations already make their appearance in the first order case, but there is a minimum of the complicated and technical mathematical machinery, which only serves to obscure the basic issues, that is needed to discuss the general, higher order case. It was therefore thought that an exposition of the first order case would set the ideas for the general construction in their proper perspective. The second purpose behind this chapter is that for the first order case much of the theory can be based on the concept of a graphic differential equation, an object of independent interest. Basically, these are partial differential equations whose solutions form implicit solutions to other partial differential equations. At the present time there does not seem to be any satisfactory generalization of this concept to higher order partial differential equations. Much of the discussion in this chapter is not of a completely precise or rigorous nature. The rigor will for the most part be deferred until the general case is taken up in the subsequent chapters.

Section 1 begins with a heuristic discussion of sections of vector bundles and the construction of the first jet bundles from a geometrical point of view. This serves to motivate the the definition of the extended first jet bundle over an arbitrary smooth manifold and the generalization of the concept of a system of first order partial differential equations for a manifold. Section 2 discusses graphic differential equations in preparation for their application to the construction of group invariant solutions of first order partial differential equations. The prolongation of smooth local transformation group actions to the extended first jet bundle is done in the next section. Here, the well-known formula for the first prolongation of a vector field is derived and applied to find the symmetry group of a first order system which is equivalent to the heat equation. In section 4 the relationship between the symmetry groups of first order systems and their graphic equivalents is described, and this is applied to construct the system of equations for the group invariant solutions. Section 5 discusses the examples of the heat equation, the telegraph equation and a general system of quasi-linear equations in one dependent variable. These serve to illustrate the theory developed in the preceding four sections.

I.1 Extended First Jet Bundles

To provide some motivation for the concept of a system of first order partial differential equations defined on an arbitrary smooth manifold, this section will begin with a brief recapitulation of the modern approach to differential operators and equations. Suppose π: $Z \to X$ is a vector bundle over the p-dimensional smooth manifold X with q-dimensional fiber, which shall be denoted by U. In the classical treatment of systems of partial differential equations in p independent and q dependent variables, $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$ and Z is the trivial bundle $\mathbb{R}^p \times \mathbb{R}^q$. In general X will represent the dependent variables; smooth solutions to differential equations on Z will then be smooth sections s: $X \to Z$. For our purposes, it will be more natural to regard a section of Z not as a map s: $X \to Z$, but rather as a smooth p-dimensional submanifold $s \subset Z$ that satisfies two additional conditions: firstly the submanifold s must satisfy a local condition of transversality to the fibers of Z, and secondly s must satisfy the global condition of intersecting each fiber of Z in exactly one point. A local section s of Z is a section that is defined only over an open subset of X. It again satisfies the local condition and the less restrictive global condition of intersecting each fiber in at most one point. It is readily seen that any p-dimension submanifold of Z satisfying the local transversality condition and one of the two global conditions is a local or global section of Z, as the case may be.

The first jet bundle of a vector bundle Z - J_1Z - is that vector bundle over X whose fiber over a point $x \in X$ is given by the equivalence classes of local sections having first order contact over x. In terms of our geometrical interpretation of sections, the condition of first order contact translates into the condition of two submanifolds having equal tangent spaces at a common point. The first jet bundle has fiber dimension $q + pq$, the fiber coordinates representing the dependent variables and their first order partial derivatives with respect to the independent variables. Given a section s of Z, there is a corresponding section $j_1 s$ of $J_1 Z$, called the first jet of s, whose value at a point $x \in X$ is prescribed by $s(x)$ and the tangent plane to s at x. In local coordinates, if $s(x)=(x,f(x))$, then $j_1 s(x) = (x,f(x),df(x))$, where df is the Jacobian matrix of f. A first order differential operator on Z is a smooth map $\Delta: J_1 Z \to W$ where W is some other vector bundle over X; Δ can be required to have the property that it project to the identity map on X. (Note that since we are interested in linear equations, Δ is only required to take fibers to fibers, not be a vector bundle morphism.) The differential equation Δ_0 associated to Δ is the inverse image of the zero section of W, i.e. $\Delta_0=\Delta^{-1}\{0\}$.† A solution to Δ_0 is a smooth section s of Z such that $j_1 s \subset \Delta_0$ or equivalently $\Delta \circ j_1 s=0$. Note that an arbitrary closed subset of $J_1 Z$ can be described as the differential equation associated with some first order differential operator. The advantage of the preceding interpretation of differential equations is that they are realized geometrically as subvariaties of vector bundles, rather than some conditions on the derivatives of smooth

†In general Δ_0 will be given in local coordinates as a system of first order partial differential equations.

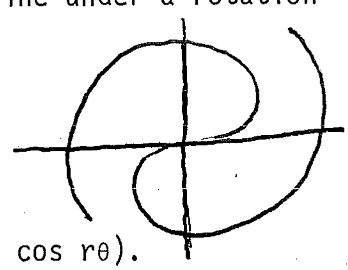
functions. This will be of importance when it comes time to apply the conditions of invariance of subsets under the actions of local groups of transformations.

Now consider what happens when G is a local Lie group of transformations acting smoothly on Z. The important point to keep in mind is that in general G will not consist solely of transformations that preserve the fibers of Z - the so-called projectable transformations. Thus while the transformations of G will take p-dimensional submanifolds of Z to p-dimensional submanifolds, neither the local nor the global conditions required for the submanifold to be a section will necessarily be preserved.

Example 1.1 Consider the trivial line bundle $Z = \mathbb{R}\times\mathbb{R}$ with coordinates (x,u). If $G = S^1$ is the rotation group

$$G: (x,u) \mapsto (x \cos\theta - u \sin\theta,\ x \sin\theta + u \cos\theta)\ .\ \theta \in S^1$$

then any constant section becomes a vertical line under a rotation of $\frac{\pi}{2}$. Even worse is the case in which the rotations speed up as the radius $r = \sqrt{x^2 + y^2}$ approaches ∞; for example



$$G': (x,u) \mapsto (x \cos r\theta - u \sin r\theta,\ x \sin r\theta + u \cos r\theta).$$

In this case any global section of Z is transformed into a "spiral" that has both vertical tangents and intersects each fiber infinitely many times.

In spite of all this, the transformations of G will still locally carry local sections of Z to local sections of Z, i.e. the condition of transversality to the fiber is maintained under transformations sufficiently close to the identity. Moreover there is an induced local action of G on J_1Z, denoted by $G^{(1)}$ and called the first prolongation of G, since G preserves the condition of first order contact. Given $z \in Z$ let G_z denote the nucleus of elements of G defined at z. If π_0^1: $J_1Z \to Z$ is the natural projection and $j \in (\pi_0^1)^{-1}\{z\}$, then the above discussion shows that $G_j^{(1)} \subset G_z$ but in general there is not equality. If a differential equation $\Delta_0 \subset J_1Z$ is an invariant subset under the action of the first prolongation of G, then the group G itself will (locally) transform solutions of Δ_0 to solutions of Δ_0. It will be shown that there is a concise expression for the first prolongation of the infinitesimal generators of G, so that the invariance of a differential equation under a group can be readily checked by the standard infinitesimal criteria of a subvariety of a manifold being invariant under a local group of transformations. Indeed, the full symmetry group of a system of partial differential equations can be determined via the solution of a number of easy auxillary partial differential equations involving the infinitesimal generators.

The solutions of Δ_0 that will be of greatest interest here are those (local) sections of Z that are (locally) invariant under the action of G. These will be called G-invariant or self-similar solutions. The theorem we are driving towards is that the G-invariant solutions of a system of partial differential equations Δ_0 which is invariant under the prolonged action of G can be all obtained by the solution of a new system of partial differential equations Δ_0/G in a fewer number of independent variables. If G satisfies certain mild regularity conditions so that the quotient space Z/G is a smooth manifold, and ℓ denotes the dimension of the leaves (orbits) of G, then the equation Δ_0/G for the G-invariant solutions to Δ_0 will naturally live on Z/G and will involve ℓ fewer independent variables. The main problem is that the quotient space Z/G is not necessarily endowed with any bundle structure, which necessitates a definition of a differential equation on an arbitrary smooth manifold that will reduce to the above definition when the manifold happens to be a vector bundle. (It should be remarked here that if G is a projectable group of transformations, then Z/G has the structure of a fiber bundle, so we must at least generalize the theory to include fiber bundles.)

It should be clear from the preceding discussion what is needed to be done. The G-invariant p dimensional submanifolds of Z are in one-to-one correspondence with the $p-\ell$ dimensional submanifolds of Z/G via the projection $\pi_G\colon Z \to Z/G$. Furthermore, the condition of first order contact is preserved under this correspondence. The only conditions that have no analogues in Z/G are the local and global conditions on a submanifold for it to be a section. Keeping this in mind, the generalization of the notion of a section to an arbitrary smooth manifold is straightforward.

Definition 1.2 Let Z be a smooth manifold and let p be a positive integer less than the dimension of Z. A smooth p-section of Z is an arbitrary smooth p dimensional submanifold of Z.

In the case of a bundle, we have enlarged the collection of sections to include those with "vertical tangents." Given a point $z \in Z$, let $C^\infty(Z,p)|_z$ denote the space of germs of smooth p-sections of Z at z. In other words $C^\infty(Z,p)|_z$ is just the set of all p-dimensional submanifolds of Z passing through z modulo the equivalence relation that two submanifolds define the same germ at z iff they are identical in some neighborhood of z. A <u>parametrization</u> of a p-section $s \in C^\infty(Z,p)|_z$ is an embedding $f: \Upsilon \to Z$, where $\Upsilon \subset \mathbb{R}^p$ is an open neighborhood of the origin, $f(0) = z$ and $\operatorname{im} f$ agrees with s in some neighborhood of z. In general, since most of our considerations are of a strictly local character, we will be a bit sloppy notationally and write $f: \mathbb{R}^p \to Z$ even when f might only be defined on an open subset of $\mathbb{R}^p$. We will also sometimes use the shorthand notation $f \in C^\infty(Z,p)|_z$ to mean that f is a parametrization of some p-section through z. Note that by the inverse function theorem, f and f' parametrize the same section of $C^\infty(Z,p)|_z$ iff there exists a local diffeomorphism $\psi: \mathbb{R}^p \to \mathbb{R}^p$ such that $f \circ \psi = f'$. (Again ψ may be defined and the equation may hold only in a suitably small neighborhood of the origin in $\mathbb{R}^p$.)

<u>Definition</u> 1.3 Given $z \in Z$, the <u>extended first jet bundle</u> if p-sections at z, $J_1^*(Z,p)|_z$ is the quotient space of $C^\infty(Z,p)|_z$ modulo the equivalence relation of first order contact.

In other words, two p-dimensional submanifolds s, s' passing through z define the same first jet at z iff $Ts|_z = Ts'|_z$. For $s \in C^\infty(Z,p)|_z$ let $j_1^* s|_z$ denote the image of s in $J_1^*(Z,p)|_z$. Let

$$J_1^*(Z,p) = \bigcup_{z \in Z} J_1^*(Z,p)|_z$$

be the extended first jet bundle of p-sections of Z. It has the structure of a fiber bundle over Z, such that for any smooth p-section s of Z, $j_1^* s$ is a smooth p-section of $J_1^*(Z,p)$.

Given a real vector space V, let $\text{Grass}(V,p)$ denote the Grassmann manifold of p-dimensional subspaces of V. Given a vector bundle $E \to Z$, let $\text{Grass}(E,p)$ denote the associated bundle of Grassmannians whose fiber at $z \in Z$ is just $\text{Grass}(E|_z,p)$. By the definition of first order contact, there is an isomorphism

$$J_1^*(Z,p) \simeq \text{Grass}(T(Z),p)$$

which assigns to each section its tangent space. This can serve to define the bundle structure of the first jet bundle.

<u>Definition</u> 1.4 Let $\mathcal{U}$ be an involutive q dimensional differential system† on Z, a smooth p+q-dimensional manifold. A p-section $s \subset Z$ is <u>transversal</u> to $\mathcal{U}$ if $Ts|_z \cap \mathcal{U}|_z = \{0\}$ for all $z \in s$. Let $C^\infty(Z,p;\mathcal{U})|_z$ denote the space of germs of p-sections passing through z transversal to $\mathcal{U}$ and let $J_1^*(Z,p,\mathcal{U})$ be the subbundle of the extended first jets of transversal p-sections.

Note that if V is an n dimensional vector space and $U \subset V$ a $q = n-p$ dimensional subspace, then

$$\text{Grass}(V,p;U) = \{\Lambda \in \text{Grass}(V,p) : \Lambda \cap U = \{0\}\}$$

is a Euclidean space of dimension pq. Therefore

$$J_1^*(Z,p;\mathcal{U}) = \text{Grass}(T(z),p;\mathcal{U})$$

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†This is sometimes called an involutive distribution on Z. It consists of a smooth family of q dimensional subspaces $\mathcal{U}|_z \subset T(Z)|_z$ for each $z \in Z$. Moreover for any smooth vector fields v,v' contained in $\mathcal{U}$, their Lie bracket $[v,v']$ is also contained in $\mathcal{U}$. Frobenius' theorem implies that such a differential system is integrable, so that there is a q dimensional foliation of Z with $\mathcal{U}$ forming the tangent spaces to the leaves. See [W].

is a Euclidean bundle over Z with fiber dimension pq. For this reason, this bundle will be called the <u>trivialized Grassmann</u> or <u>trivialized extended first jet bundle</u> with respect to $\mathcal{U}$.

<u>Lemma 1.5</u> Let $Z \to X$ be a vector bundle over a smooth p dimensional manifold X and let $\mathcal{U}$ denote the differential system given by the tangent spaces to the fibers of Z. Then

$$J_1 Z \simeq J_1^*(Z,p;\mathcal{U})$$

This lemma shows that we have indeed generalized the notion of the first jet bundle correctly. The proof is a direct consequence of the fact that any p-dimensional submanifold of Z transversal to $\mathcal{U}$ is locally a section of Z, and only local sections are needed to construct $J_1 Z$. Therefore the extended first jet bundle can be considered as the "completion" of the usual first jet bundle in the same sense that projective space is the "completion" of affine space, the completion being obtained by allowing sections with "vertical tangents."

Given local coordinates $\chi: Z_0 \xrightarrow{\sim} \mathbb{R}^p \times \mathbb{R}^q$ for some open $Z_0 \subset Z$, so that $\chi(z) = (x,u) = (x^1,\ldots,x^p, u^1,\ldots,u^q)$, there is a naturally defined q-dimensional involutive differential system $\mathcal{U}$ on Z_0 spanned by the vector fields $d\chi^{-1}(\partial/\partial u^i)$. In this context, the x's should be viewed as the local independent variables and the u's as the local dependent variables.

<u>Definition 1.6</u> Given local coordinates on Z and the corresponding differential system $\mathcal{U}$, then for any p-section $s \subset Z_0$

transversal to $\mathcal{U}$, a <u>normal parametrization</u> of s is a smooth map $\hat{f}\colon \mathbb{R}^p \to Z_0$ with $\operatorname{im}\hat{f} = s$ and $\chi \circ \hat{f}(x) = (x, f(x))$ for some function $f\colon \mathbb{R}^p \to \mathbb{R}^q$.

Note that the inverse function theorem assures the local existence of normal parametrizations for transversal sections. Indeed, if $\hat{g}\colon \mathbb{R}^p \to Z$ is any transversal section, so that in local coordinates $\chi \circ \hat{g}(x) = (g_1(x), g_2(x))$ then $\hat{f}$, the <u>normalization</u> of $\hat{g}$, is given in these coordinates by $\chi \circ \hat{f}(x) = (x, g_2 \circ g_1{}^{-1}(x))$, the inverse of g_1 always existing locally. When using local coordinates, we shall often suppress the map χ and identify z with $\chi(z)$. Now if $\hat{f}\colon \mathbb{R}^p \to Z$ is the normal parametrization of a transversal p-section s, then Ts is spanned by the vector fields

$$d\hat{f}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \frac{\partial f^j}{\partial x^i} \frac{\partial}{\partial u^j} \qquad i = 1,\ldots,p$$

so we can regard the components $u_i^j = \partial f^j/\partial x^i$ as local coordinates in $J_1^*(Z,p;\mathcal{U})$ corresponding to the local coordinates (x,u) on Z. Often the Jacobian matrix (u_i^j) will be abbreviated by the symbol $u^{(1)}$. Thus if a p-section s has $j_1^* s|_z = (x,u,u^{(1)})$ in these local coordinates, then $Ts|_z$ is spanned by the vector fields $\partial/\partial x^i + \sum_j u_i^j \partial/\partial u^j$ for $i = 1,\ldots, p$. In summary:

<u>Lemma</u> 1.7 Let $\chi\colon Z_0 \xrightarrow{\sim} \mathbb{R}^p \times \mathbb{R}^q$ be a system of local coordinates on Z and let $\mathcal{U}$ denote the differential system $d\chi^{-1}[T\mathbb{R}^q]$, then there is an induced local coordinate system

$$\chi^{(1)}\colon J_1^*(Z_0,p;\mathcal{U}) \xrightarrow{\sim} \mathbb{R}^p \times \mathbb{R}^q \times \operatorname{Hom}(\mathbb{R}^p,\mathbb{R}^q) \text{ such that}$$

if $\hat{g}\colon \mathbb{R}^p \to Z_0$ is any parametrization of a p-section s with

$\chi \circ \hat{g} = g_1 \times g_2$ then

$$\chi^{(1)} \circ j_1^* \hat{g} = 1\!\!1_p \times (g_2 \circ g_1^{-1}) \times d(g_2 \circ g_1^{-1}) .$$

Note that in the case of a vector bundle, these coordinates are the usual coordinates in $J_1 Z$. (See, for instance, [P2; chapter 4] for a fairly complete discussion of this.)

Definition 1.8 A first order differential equation for p-sections of a smooth manifold Z is a closed subset $\Delta_0 \subset J_1^*(Z,p)$. A solution to Δ_0 is a p-section $s \subset Z$ with $j_1^* s \subset \Delta_0$.

The definition of differential operators and their relation to differential equations will be deferred until section II.5. If

$$\Delta^i(x,u,u^{(1)}) = 0 \quad i = 1,\ldots,\alpha \qquad (*)$$

is any system of first order partial differential equations on $Z = \mathbb{R}^p \times \mathbb{R}^q$, then (*) defines a subvariety of $J_1 Z$ and the first order differential equation corresponding to the above system will just be the closure of this subvariety in $J_1^*(Z,p)$. In other words, we are allowing solutions to (*) with vertical tangents as long as these tangents are limits of tangents that satisfy (*). These concepts will be explored in more detail in section II.5.

I.2 Graphic Differential Equations

Given a system of first order partial differential equations, a graphic equivalent to this system is another system of first order partial differential equations whose solutions are the implicit solutions to the original system. If the first system is on the smooth manifold Z, the graphic equivalent is on the trivial bundle $\mathbb{I}^q = Z \times \mathbb{R}^q$. The advantage of considering the graphic equivalent rather than the original system is that a transformation group acting on Z will naturally correspond to a group acting trivially on the fibers. In addition any differential equation on $\mathbb{I}^q$ satisfying certain symmetry conditions is the graphic equivalent of some equations on Z (It is this last fact that does not have an analogue for higher order differential equations, and restricts the use of this technique to first order equations.) These two properties will be used to provide an easy proof of the theorem on the existence of symmetric implicit solutions to a system of first order partial differential equations.

Let Z be a smooth manifold of dimension $p+q$ and $s \subset Z$ a p-dimensional submanifold. Locally s can be described either parametrically as the image of a smooth embedding $f: \mathbb{R}^p \to Z$ or implicitly as a level set of a smooth submersion $F: Z \to \mathbb{R}^q$. It will be more convenient to regard the implicit function F as a section of the trivial bundle $\mathbb{I}^q = Z \times \mathbb{R}^q$. Let $\mathrm{Hom}_0(TZ, \mathbb{I}^q)$ denote the subbundle of $\mathrm{Hom}(TZ, \mathbb{I}^q)$ of linear maps of maximal rank. Consider the jet bundle exact sequence

$$0 \to \mathrm{Hom}(TZ, \mathbb{1}^q) \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} J_1\mathbb{1}^q \to \mathbb{1}^q \to 0 \qquad (2.1)$$

which splits since $\mathbb{1}^q$ is trivial. Let $J_1(\mathbb{1}^q)_0 = \pi_1^{-1}[\mathrm{Hom}_0(TZ, \mathbb{1}^q)]$. Note that F is a submersion iff $j_1F \subset J_1(\mathbb{1}^q)_0$ since $\pi_1(j_1F|_z) = dF|_z$. There is an action of $GL(q)$ on $\mathrm{Hom}(T(Z),\mathbb{1}^q)$ given by left matrix multiplication; the action is nonsingular on $\mathrm{Hom}_0(T(Z),\mathbb{1}^q)$. This corresponds to the action of diffeomorphisms of $\mathbb{R}^q$ on sections of $\mathbb{1}^q$. Note that if $G: \mathbb{R}^q \to \mathbb{R}^q$ is a diffeomorphism and $F:Z \to \mathbb{R}^q$ is a submersion, then F and $G \circ F$ define the same implicit sections of Z. This is reflected in the next lemma. Let

$$\Pi: \mathrm{Hom}_0(TZ, \mathbb{1}^q) \to \mathrm{Hom}_0(TZ, \mathbb{1}^q)/GL(q)$$

be the projection onto the quotient bundle.

<u>Lemma</u> 2.1 There is a natural isomorphism

$$\mathrm{Grass}(TZ,p) \simeq \mathrm{Hom}_0(TZ,\mathbb{1}^q)/GL(q)$$

such that if $F:Z \to \mathbb{R}^q$ is a submersion and $F(z) = c \in \mathbb{R}^q$, then

$$j_1^* F^{-1}\{c\}|_z = \Pi[\pi_1(j_1F(z))]. \qquad (2.2)$$

<u>Proof</u>

Let $\pi: T(\mathbb{R}^q) = \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q$ be projection onto the second factor. Then

$$\pi \circ dF = \pi_1[j_1F]$$

in $\mathrm{Hom}_0(TZ, \mathbb{1}^q)$. Therefore it suffices to show that if F,F' are submersions and $z \in F^{-1}\{c\} \cap F'^{-1}\{c'\}$ for $c,c' \in \mathbb{R}^q$ then $T F^{-1}\{c\}|_z = T F'^{-1}\{c'\}|_z$ iff there exists a matrix $A \in GL(q)$ with

$$A\cdot\pi \circ dF = \pi \circ dF' .$$

This in turn follows straightforwardly from the fact that $TF^{-1}\{c\}|_z = \ker[\pi \circ dF|_z]$ and the maximality of the rank of $\pi \circ dF$.

In other words the isomorphism of the lemma identifies $B \in \mathrm{Hom}_0(TZ, \mathbb{1}^q)|_z$ with $\ker B$ which is a p-plane in $TZ|_z$.

Q.E.D.

Definition 2.2 Let $\Delta_0 \subset J_1^*(Z,p)$ be a differential equation. A (local) implicit solution to Δ_0 is a (local) submersion $F: Z \to \mathbb{R}^q$ such that the level sets $F^{-1}\{c\}$, $c \in \mathbb{R}^q$ are all solutions to Δ_0.

For the remainder of this chapter we will be just concerned with implicit solutions to differential equations. There is a small loss in generality since such anomalous solutions as envelope solutions do not lie in this class, but these will be covered when we turn to the general case of higher order equations.

Definition 2.3 Let $\Delta_0 \subset J_1^*(Z,p)$ be a first order system of partial differential equations. A graphic equivalent to Δ_0 is a first order differential equation $\hat{\Delta}_0 \subset J_1(\mathbb{1}^q)_0$ such that any $F: Z \to \mathbb{R}^q$ is a local implicit solution to Δ_0 iff it is a solution to $\hat{\Delta}_0$.

For a given differential equation Δ_0 on Z there is an essentially unique graphic equivalent to Δ_0, namely

$$\hat{\Delta}_0 = \pi_1^{-1}[\Pi^{-1}(\Delta_0)] \tag{2.3}$$

where π_1 is the projection in the exact sequence (2.1). $\hat{\Delta}_0$ is not unique since any differential equation in $\mathbb{1}^q$ with an empty solution set could be added to $\hat{\Delta}_0$. A graphic differential equation will be an equation in $\mathbb{1}^q$ that is the graphic equivalent of some equation Z.

There is a natural action of $\mathbb{R}^q$ on $\mathbb{1}^q$ given by translation in the fibers. By the splitting of (2.1), we obtain an action of the group $GL(q) \times \mathbb{R}^q$ on $J_1\mathbb{1}^q$.

<u>Definition</u> 2.4 A differential equation $\hat{\Delta}_0 \subset J_1\mathbb{1}^q$ is of <u>purely first order</u> if $\hat{\Delta}_0$ is invariant under $\mathbb{R}^q$. In other words, $\hat{\Delta}_0$ does not depend on the dependent variables, only on their derivatives.

<u>Lemma</u> 2.5 Let $\hat{\Delta}_0 \subset J_1(\mathbb{1}^q)_0$ be a first order system of partial differential equations invariant under $GL(q) \times \mathbb{R}^q$, then $\hat{\Delta}_0$ is a graphic equation. Conversely, if $\hat{\Delta}_0$ is a graphic equation which possesses solutions whose first jets pass through every point of $\hat{\Delta}_0$ (or, alternately, $\hat{\Delta}_0 = \pi_1^{-1}\Pi^{-1}\Delta_0$ for some equation $\Delta_0 \subset J_1^*(Z,p)$) then $\hat{\Delta}_0$ is invariant under $GL(q) \times \mathbb{R}^q$.

<u>Proof</u>

The first statement follows from lemma 2.1. Namely, let

$\Delta_0 = \Pi \circ \pi_1[\hat{\Delta}_0]$ and note that $\hat{\Delta}_0 = \pi_1^{-1}\Pi^{-1}[\Delta_0]$

by the conditions on $\hat{\Delta}_0$. The fact that $\hat{\Delta}_0$ is a graphic equivalent to Δ_0 follows from equation (2.2). To show the converse, note that if F is any solution to $\hat{\Delta}_0$ then so is $G \circ F$ for any diffeomorphism $G : \mathbb{R}^q \to \mathbb{R}^q$. Letting G be translation by a fixed vector gives the fact that $\hat{\Delta}_0$ is of purely first order.

Using the fact that

$$\pi_1 j_1(G \circ F) = dG \cdot \pi_1 j_1 F$$

gives the invariance of $\pi_1\hat{\Delta}_0$ under $GL(q)$.

Q.E.D.

To see what is involved in the preceding construction, it is helpful to look at the local coordinate description. Let $(x,u,w) = (x^1,\ldots,x^p,u^1,\ldots,u^q,w^1,\ldots,w^q)$ be local coordinates on $\mathbb{R}^q$ with corresponding coordinates $u^{(1)} = (u^i_j)$ on the fibers of $J^*_1(Z,p;\mathcal{U})$ and $w^{(1)} = (w^{(1)}_1 | w^{(1)}_2) = (w^i_j | w^i_\sigma)$ on Hom $(T(Z), \mathbb{R}^q)$, where the w^i_j 's correspond to partial derivatives in the x^j direction and w^i_σ's to derivatives in the u^σ direction. Then the map Π is given by the formula

$$u^{(1)} = \Pi(w^{(1)}_1 | w^{(1)}_2) = -(w^{(1)}_2)^{-1} w^{(1)}_1 \tag{2.4}$$

whenever the square matrix $w^{(1)}_2$ is invertible, and in fact $\Pi^{-1}[J^*_1(Z,p;\mathcal{U})]$ is just the set of those $(w^{(1)}_1 | w^{(1)}_2)$ with $w^{(1)}_2 \in GL(q)$. This follows directly from the identification of $\Pi(w^{(1)})$ with $\ker w^{(1)}$, as in the proof of lemma 2.1. Thus if an equation $\Delta_0 \subset J^*_1(Z,p)$ is given by

$$\Delta^i(x,u,u^{(1)}) = 0 \qquad i = 1,\ldots,\alpha$$

then the graphic equivalent is just

$$\Delta^i(x,u,-(w^{(1)}_1)^{-1} w^{(1)}_2) = 0 \qquad i = 1,\ldots,\alpha\ .$$

This procedure could be generalized to higher order equations, but there would be no immediate analogue of proposition 2.5, which will be crucial in what follows.

<u>Example</u> 2.6 Here the graphic equivalent to the first order system corresponding to the heat equation will be derived. Consider the equations

$$u_x = v \qquad\qquad v_x = u_t \tag{*}$$

on $Z = \mathbb{R}^2 \times \mathbb{R}^2$. Then with w, w' the fiber coordinates on $\mathbb{I}^2 = Z \times \mathbb{R}^2$

$$\begin{pmatrix} u_x & u_t \\ v_x & v_t \end{pmatrix} = \begin{pmatrix} w_u & w_v \\ w'_u & w'_v \end{pmatrix}^{-1} \begin{pmatrix} w_x & w_t \\ w'_x & w'_t \end{pmatrix}$$

$$= \frac{1}{d} \cdot \begin{pmatrix} w'_v w_x - w_v w'_x & w'_v w_t - w_v w'_t \\ -w'_u w_x + w_u w'_x & -w'_u w_t + w_u w'_t \end{pmatrix}$$

where $d = w_u w'_v - w'_u w_v$, hence a graphic equivalent to (*) is

$$\begin{aligned} w'_v w_x - w_v w'_x &= v(w_u w'_v - w'_u w_v) \\ w_u w'_x - w'_u w_x &= w'_v w_t - w'_u w'_t \end{aligned} \quad .$$

This is not overly useful for practical purposes, of course, but the theoretical implications are the main purpose behind the introduction of these ideas.

I.3 Prolongation of Group Actions

In this section the prolonged action of a local group of transformations on the extended first jet bundle will be discussed. In particular a useful formula for the prolonged infinitesimal operators will be derived, which will be applied to find the symmetry groups of some interesting systems of first order partial differential equations in section I.5. Throughout this section the standard results on local groups of transformations acting on smooth manifolds will be assumed; consult the appendix or [P1] for the relevant definitions and theorems.

Let G be a local Lie group of transformations acting smoothly on the manifold Z. For $z \in Z$, G_z will denote the nucleus of transformations defined at z. Correspondingly, for $g \in G$, Z_g will denote the open submanifold of Z where the transformation g is defined. For simplicity it will always be assumed that $Z_{g^{-1}} = g \cdot Z_g$. Given $g \in G$ and a p dimensional submanifold $s \subset Z_g$ then gs is again a p-dimensional submanifold of Z. In particular, for $s \in C^\infty(Z,p)|_z$ a p-section, there is a well-defined p-section $gs \in C^\infty(Z,p)|_{gz}$ for all $g \in G_z$ given by $g(s \cap Z_g)$. This local action of G on the space of p-sections induces an action of G on $J_1^*(Z,p)$ since the transformations in G preserve the condition of first order contact. This action of G is called the <u>first prolongation of</u> G and will be denoted by $pr^{(1)}G$. Note that for $g \in G$, $J_1^*(Z,p)_g = (\pi_0^1)^{-1}[Z_g]$ where $\pi_0^1: J_1^*(Z,p) \to Z$ is the standard projection.

To see that $pr^{(1)}G$ actually is a smooth local group action on $J_1^*(Z,p)$, the local coordinate picture will be discussed. Let

(x,u) and $(\hat{x},\hat{u})$ be local coordinates near $z \in Z_g$ and gz respectively, so that the transformation g is given by

$$(\hat{x},\hat{u}) = g(x,u) = (\Phi(x,u),\Psi(x,u)).$$

Let $u^{(1)} = (u^i_j)$ and $\hat{u}^{(1)} = (\hat{u}^i_j)$ be the induced fiber coordinates in $J^*_1(Z,p)$ and suppose $j = (x,u,u^{(1)}) \in J^*_1(Z,p;\mathcal{U})$ is represented by the normal parametrized section $\hat{f}: \mathbb{R}^p \to Z$, i.e. for $x \in \mathbb{R}^p$, $\hat{f}(x) = (x,f(x)) = (x,u)$ and $u^{(1)} = df(x)$. The transformed section $g\hat{f}$ is given by

$$g\hat{f}: x \mapsto (\Phi(x,f(x)), \Psi(x,f(x)) = (\hat{\Phi}(x),\hat{\Psi}(x)).$$ Assuming that $\hat{\Phi}$ is a diffeomorphism (which can always be done by appropriate choice of local coordinates $(\hat{x},\hat{u})$) the normalization of the parametrized section $g\hat{f}$ is given by $\hat{x} \mapsto (\hat{x},\hat{\Psi}\circ\hat{\Phi}^{-1}(\hat{x}))$. For any function $\psi: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^k$ let $D\psi: \mathbb{R}^p \times \mathbb{R}^q \times \mathrm{Hom}(\mathbb{R}^p,\mathbb{R}^q) \to \mathrm{Hom}(\mathbb{R}^p,\mathbb{R}^k)$ be the total derivative given by

$$D\psi(x,u,u^{(1)}) = d_x\psi(x,u) + u^{(1)}d_u\psi(x,u) \tag{3.1}$$

so that if $f: \mathbb{R}^p \to \mathbb{R}^q$ is any smooth function, then

$$d_x\psi(x,f(x)) = D\psi(x,f(x),df(x)).$$ (This concept will be discussed in greater detail in section II.4). Then differentiating the normal parametrization of $g\hat{f}$ gives

$$pr^{(1)}g(x,u,u^{(1)}) = (\Phi(x,u),\Psi(x,u),D\Psi(x,u,u^{(1)})\cdot[D\Phi(x,u,u^{(1)})]^{-1}) \tag{3.2}$$

which shows the smoothness of the prolonged group action.

There is a corresponding prolongation of infinitesimal generators of local groups to $J_1^*(Z,p)$ given by

$$pr^{(1)}\vec{v} = \frac{d}{dt}\Big|_{t=0} \; pr^{(1)} \; (\exp t\vec{v})$$

for $\vec{v}$ a smooth vector field on Z. To see the formula for the prolongation of a vector field in local coordinates $(x,u,u^{(1)})$, suppose

$$\vec{v} = \sum_{j=1}^{p} \xi^j(x,u)\,\frac{\partial}{\partial x^j} + \sum_{i=1}^{q} \phi_i(x,u)\frac{\partial}{\partial u^i} \tag{3.3}$$

then

$$pr^{(1)}\vec{v} = \vec{v} + \sum_{i,j} \phi_i^j\,(x,u,u^{(1)})\frac{\partial}{\partial u_j^i} \tag{3.4}$$

Given $z = (x,u)$ then for t sufficiently small $\exp t\vec{v}(z)$ is still in the same coordinate patch and is given by

$$\exp t\vec{v}\,(x,u) = (\Phi_t(x,u),\, \Psi_t(x,u)).$$

If $j = (x,u,u^{(1)}) \in J_1^*(Z,p;\mathcal{U})|_z$ is represented by a p-section s, then for t sufficiently small $pr^{(1)}(\exp t\vec{v})\cdot s$ is again a section transverse to $\mathcal{U}$. Let $\vec{\phi}$ and $\vec{\xi}$ denote the column vectors ${}^t(\phi_1,\ldots,\phi_q)$ and ${}^t(\xi^1,\ldots,\xi^p)$ respectively. Then the coefficient functions ϕ_i^j are the matrix entries in

$$\frac{d}{dt}\Big|_{t=0} D\Psi_t(x,u,u^{(1)})\cdot[D\Phi_t(x,u,u^{(1)})]^{-1} = D\vec{\phi}\,(x,u,u^{(1)}) - u^{(1)}D\vec{\xi}\,(x,u,u^{(1)})$$

which follows from (3.1), (3.2) and because $D\Psi_0 = u^{(1)}$, $D\Phi_0 = \mathbb{1}_p$ -- the identity map of $\mathbb{R}^p$. Therefore we get as

$$\phi_i^j(x,u,u^{(1)}) = D_j\phi_i(x,u,u^{(1)}) - \sum_{\sigma=1}^{p} u_\sigma^i\, D_j\xi^\sigma(x,u,u^{(1)}) \tag{3.5}$$

first order prolongation formula for vector fields on Z.

Now consider a system of first order partial differential equations on Z given by a subvariety $\Delta_0 \subset J_1^*(Z,p)$. The equation Δ_0 is said to be invariant under G if the first prolongation $pr^{(1)}G$ leaves Δ_0 invariant. Explicitly, if Δ_0 is given by the equations

$$\Delta^1(x,u,u^{(1)}) = \ldots = \Delta^\alpha(x,u,u^{(1)}) = 0 \tag{3.6}$$

then using the infinitesimal criterion, Δ_0 is invariant under G iff

$$pr^{(1)}\vec{v}[\Delta^j(x,u,u^{(1)})] = 0 \quad j = 1,\ldots,\alpha \tag{3.7}$$

whenever (3.6) holds for all infinitesimal generators $\vec{v}$ of G.

<u>Example</u> 3.1 Consider the heat equation

$$u_x = v \qquad v_x = u_t \tag{3.8}$$

on $Z = \mathbb{R}^2 \times \mathbb{R}^2$ will coordinates (x,t,u,v). To calculate the symmetry group of (3.8) we look for all vector fields

$$\vec{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u + \Psi\partial_v$$

whose first prolongation leave (3.8) infinitesimally invariant. Now

$$pr^{(1)}\vec{v} = \vec{v} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} \qquad \text{where}$$

$$\begin{aligned} \phi^x &= D_x\phi - u_xD_x\xi - u_tD_x\tau \\ \phi^t &= D_t\phi - u_xD_t\xi - u_tD_t\tau \\ \Psi^x &= D_x\Psi - v_xD_x\xi - v_tD_x\tau \\ \Psi^t &= D_t\Psi - v_xD_t\xi - v_tD_t\tau \end{aligned} \tag{3.9}$$

by formula (3.5). The equations to be satisfied are

$$\phi^x = \Psi \qquad \Psi^x = \phi^t \tag{3.10}$$

whenever (3.8) holds. The first equation is equivalent to

$$-u_t^2\tau_v + u_t(\phi_v - v\xi_v - \tau_x - v\tau_u) + \phi_x + v\phi_u - v\xi_x - v^2\xi_u = \Psi$$

where subscripts are used to denote partial derivatives. Hence, on equating the coefficients of the various powers of u_t

$$\tau_v = 0 \tag{3.11}$$

$$\phi_v - v\xi_v - \tau_x - v\tau_u = 0 \tag{3.12}$$

$$\phi_x + v\phi_u - v\xi_x - v^2\xi_u = \Psi \tag{3.13}$$

Similarly treating the second equation and equating the coefficients of the various powers of u_t and v_t gives

$$\left.\begin{aligned} \tau_v &= 0 \\ \xi_v &= \tau_u \end{aligned}\right. \tag{3.14}$$

$$-\tau_x - v\tau_u = \phi_v - v\xi_v \tag{3.15}$$

$$\Psi_v - \xi_x = \phi_u - \tau_t \tag{3.16}$$

$$\Psi_x + v\Psi_u = \phi_t - v\xi_t \tag{3.17}$$

Now equations (3.11), (3.12), (3.14) and (3.15) all imply

$\tau_x = \tau_u = \tau_v = \xi_v = \phi_v = 0$. Then (3.16) gives $\Psi_{vv} = 0$, so (3.13) shows $\xi_u = 0$ and $\Psi = \phi_x + v(\phi_u - \xi_x)$.

Together with (3.16) this shows $\tau_t = 2\xi_x$ and hence $\xi_{xx} = 0$.

Finally (3.17) shows that $\phi_{uu} = 0$, $2\phi_{xu} = -\xi_t$ and $\phi_{xx} = \phi_t$.

Let $\phi = \alpha(x,t) + u\beta(x,t)$

then $\alpha_{xx} = \alpha_t$ and $\beta_{xx} = \beta_t$, $2\beta_x = -\xi_t$, hence

$\beta_{xxx} = \beta_{xt} = \xi_{tt} = \tau_{ttt} = 0$.

What this all shows is that the algebra of infinitesimal symmetries of the heat equation is spanned by the six operators

$$
\begin{aligned}
\vec{v}_1 &= \partial_x \\
\vec{v}_2 &= \partial_t \\
\vec{v}_3 &= u\partial_u + v\partial_v \\
\vec{v}_4 &= x\partial_x + 2t\partial_t - v\partial_v \\
\vec{v}_5 &= 2t\partial_x - xu\partial_u - (u + xv)\partial_v \\
\vec{v}_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u - (2xu + (x^2 + 6t)v)\partial_v
\end{aligned}
$$

and the infinite dimensional subalgebra spanned by the operators

$$\vec{v}_\alpha = \alpha(x,t)\partial_u + \alpha_x(x,t)\partial_v \qquad \alpha_{xx} = \alpha_t \tag{3.19}$$

This symmetry group will be discussed in more detail in section I.5.

Lemma 3.2 If a system of first order partial differential equations Δ_0 is invariant under a local group of transformations G, then G takes solutions of Δ_0 to solutions of Δ_0.

The proof follows directly from the definition of the prolonged group action. Next consider the action of a group of transformations on implicit sections. If $F: Z \to \mathbb{R}^q$ is a submersion define $gF: Z_{g^{-1}} \to \mathbb{R}^q$ by $gF(z) = F(g^{-1}z)$. Under this definition

$$g \cdot F^{-1}\{c\} = (gF)^{-1}\{c\} \qquad \text{for } c \in \mathbb{R}^q$$

so the action does indeed correspond to the action of G on p-sections. Readopting the notation of section I.2; the following lemma is easy to prove.

<u>Lemma</u> 3.3 Consider the bundles

$$J_1(\mathbb{R}^q)_0 \xrightarrow{\pi_1} \mathrm{Hom}_0(TZ, \mathbb{R}^q) \xrightarrow{\Pi} J_1^*(Z,p) \ .$$

There are natural actions of G on these bundles given respectively by

i) $g \cdot j_1 F = j_1 gF$ $\qquad F: Z \to \mathbb{R}^q$

ii) $gA(v) = A(dg^{-1}(v))$ $\qquad A \in \mathrm{Hom}_0(TZ, \mathbb{R}^q)|_z$

$\qquad v \in TZ|_{gz}$

iii) $pr^{(1)}G$

The actions of G commute with the two projections:

$$g\pi_1(j) = \pi_1(gj) \qquad\qquad g\Pi(A) = \Pi(gA) \quad .$$

<u>Corollary</u> 3.4 A system of first-order partial differential equations Δ_0 is invariant under a group G iff its graphic equavalent $\pi_1^{-1}(\Pi^{-1}\Delta_0)$ is invariant under G.

I.4 Group Invariant Solutions

We now turn to a discussion of symmetric or group invariant solutions of systems of first order partial differential equations. Using the theorem of the invariance of sets of constant dimensionality of quasi-differential systems (theorem A.8 of the appendix) it is possible to assume without local loss of generality that G is a local group of transformations acting on Z all of whose orbits are of the same dimension ℓ. We further assume that G acts regularly on Z so that the quotient space Z/G can be given the structure of a smooth manifold. (Here it should be remarked that Z/G will not necessarily be Hausdorff: consider the example of $Z = \mathbb{R}^2 \sim \{0\}$ with G acting as translations in the first coordinate. This does not add significant complications; the interested reader should consult the appendix for a fuller explanation.)

Suppose $\Delta_0 \subset J_1^*(Z,p)$ is a differential equation which is invariant under the prolonged action of G. A G-invariant solution to Δ_0 is a p-section of Z which is invariant under the action of G and which is a solution to Δ_0. Define the invariant subbundle of $J_1^*(Z,p)$

$$\mathrm{Inv}(G,p)|_z = \{\Lambda \in \mathrm{Grass}(TZ,p)|_z : \Lambda \supset \mathfrak{g}|_z\}$$ where $\mathfrak{g}|_z$ denotes the ℓ-dimensional subspace of $TZ|_z$ spanned by the algebra of infinitesimal generators of G at the point z. It is easy to see that $\mathrm{Inv}(G,p)$ forms a smooth subbundle of $J_k^*(Z,p)$ and that $s \in C^\infty(Z,p)|_z$ is a G invariant section iff $j_1^* s \subset \mathrm{Inv}(G,p)$, which is a direct consequence of the infinitesimal criterion of invariance. Note that in particular $p \geq \ell$ for there to exist G-invariant p-sections.

There is a corresponding subvariety $I \subset J_1\mathbb{1}^q$ given by the first jets of G invariant functions $F\colon Z \to \mathbb{R}^q$. It is obviously invariant under the action of $GL(q) \times \mathbb{R}^q$ on $J_1\mathbb{1}^q$ and its projection $\pi_1 I \subset \mathrm{Hom}(TZ, \mathbb{1}^q)$ is just the set of all matrices $A \in \mathrm{Hom}(TZ, \mathbb{1}^q)|_Z$ which vanish on $\mathfrak{g}|_Z$. Using lemma 2.1, we have

$$\Pi \circ \pi_1^{-1}(I) = \mathrm{Inv}(G, p).$$

The following theorem in its local formulation is due to Ovsjannikov [O1], and shows that the G invariant solutions to a system of first order partial differential equations Δ_0 can all be found by solving a system of partial differential equations Δ_0/G in ℓ fewer independent variables. In this section, this will only be proven for implicit solutions; the proof for the general theorem will be deffered until section IV.1, when it will be proven for differential equations of arbitrary order. Let $\pi_G\colon Z \to Z/G$ be the projection.

<u>Theorem 4.1</u> Let Δ_0 be a system of partial differential equations of first order in p independent variables on the manifold Z. Let G be a regular group of transformations acting on Z with ℓ dimensional orbits. If Δ_0 is invariant under the prolonged group action $pr^{(1)}G$, then there exists a system of first order partial differential equations Δ_0/G in $p-\ell$ independent variables on the quotient manifold Z/G such that $F\colon Z/G \to \mathbb{R}^q$ is an implicit solution to Δ_0/G iff $F \circ \pi_G\colon Z \to \mathbb{R}^q$ is a G invariant implicit solution to Δ_0.

Proof

Let $\hat{\Delta}_0 = \pi_1^{-1}\Pi^{-1}\Delta_0 \subset J_1(\mathbb{1}_Z^q)_0$ be the graphic equivalent of Δ_0. By lemma 3.3, $\hat{\Delta}_0$ is invariant under the action of G on $J_1(\mathbb{1}_Z^q)$. Let $I \subset J_1(\mathbb{1}_Z^q)$ be the subbundle of G-invariant sections. There is a one-to-one correspondence between G-invariant maps $\hat{F}: Z \to \mathbb{R}^q$ and maps $F: Z/G \to \mathbb{R}^q$ given by $\hat{F} = F \circ \pi_G$. This induces a map

$$\chi_1: \ I \to J_1(\mathbb{1}^q_{Z/G})$$

such that for any $z \in Z$ with $z' = \pi_G(z)$

$$\chi_1: \ I|_z \xrightarrow{\sim} J_1(\mathbb{1}^q_{Z/G})|_{z'}$$

is an isomorphism of fibers. This follows from the fact that $j_1\hat{F}|_z$ is uniquely determined by $\hat{F}(z)$ and $d\hat{F}(z)$. Now $\hat{F}(z)$ uniquely determines $F(z')$ and $d\hat{F}(z)$ vanishes on $\mathfrak{g}|_z$, hence $d\hat{F}(z)$ uniquely determines $dF(z')$. Moreover, $\hat{F}$ is a submersion iff F is a submersion. Note that χ_1 commutes with the action of $GL(q) \times \mathbb{R}^q$ on the fibers of the two first jet bundles.

Now let $\hat{\Delta}_0/G = \chi_1(\hat{\Delta}_0 \cap I)$. Since $\hat{\Delta}_0 \cap I$ is G-invariant

$$\hat{\Delta}_0 \cap I|_z \simeq \hat{\Delta}_0/G|_{z'}$$

hence given F and $\hat{F}$ as above, $j_1\hat{F} \subset \hat{\Delta}_0 \cap I$ iff $j_1F \subset \hat{\Delta}_0/G$. Now propositon 2.5 imples $\hat{\Delta}_0/G$ is a graphic equation, hence there is an equation $\Delta_0/G \subset J_1^*(Z/G, p-\ell)$ with graphic equivalent $\hat{\Delta}_0/G$. The fact that Δ_0/G satisfies the properties of the theorem is a straightforward consequence of the properties of graphic equations and of the map χ_1. Q.E.D.

Note that the theorem does not guarantee the existence of group invariant solutions. Indeed the "dimensionality" of the space of these solutions depends on the dimension of $I \cap \hat{\Delta}_0$, or, equivalently, the dimension of $\mathrm{Inv}(G,p) \cap \Delta_0$. The following example demonstrates that symmetric solutions do not necessarily exits, and gives a good illustration of the process of finding the reduced equation Δ_0/G. More interesting examples will appear in section I.5.

Example 4.2 Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,y,u) and consider the equation

$$\Delta_0: \quad xu_x + yu_y = 1 \tag{*}$$

which is invariant under the group $G = \mathbb{R}$ whose action on Z is given by

$$\begin{aligned} gx &= e^\lambda x \\ gy &= e^\lambda y \qquad\qquad \lambda \in \mathbb{R} \\ gu &= u \end{aligned}$$

G acts regularly on $Z' = Z \sim \{x = y = 0\}$. Then $Z'/G \simeq S^1 \times \mathbb{R}$ and local coordinates on Z'/G are given by the functionally independent invariants of G

$$t = \frac{x}{y}$$
$$\zeta = u$$

Considering ζ as a function of t, we have

$$u_x = \frac{1}{y}\ \zeta'(t)$$
$$u_y = -\frac{x}{y^2}\ \zeta'(t)$$

so the reduced system Δ_0/G is given by

$$\frac{x}{y} \zeta' - \frac{x}{y} \zeta' = 1$$

$$\text{or} \qquad 0 = 1$$

which is vacuous. Indeed $\mathrm{Inv}(G,2)$ is given by the equation

$$xu_x + yu_y = 0$$

so $\Delta_0 \cap \mathrm{Inv}\,(G,2) = \emptyset$. Note that G still takes solutions of (*) to solutions - it just leaves no solution invariant.

I.5 Examples of Group Invariant Solutions

The theory developed in the previous sections will now be applied to finding some interesting group invariant solutions for the telegraph equation and the heat equation. The main step in the construction of the system Δ_0/G for the G-invariant solutions to a system of partial differential equations Δ_0 is finding local coordinates for the quotient manifold Z/G. These will be provided by what are classically known as a complete set of functionally independent invariants for the group G, which are functions of the coordinates (x,u) on Z which are left invariant by the group action. Once these have been determined, it is necessary to decide which of the invariants will be the new dependent variables; this will often be determined from a priori considerations. The system Δ_0/G is then constructed by substituting for the derivatives of u in terms of the invariants. This process will be clearer in the examples that follow; see also section IV.2. Finally, at the end of this section, we find the symmetry group of a system of quasi-linear first order partial differential equations in one dependent variable, which gives an application of the theory of graphic differential operators.

Example 5.1 The Telegraph Equation

Let $Z = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x,t,u,v) and consider the system of partial differential equations

$$\Delta_0 : \begin{array}{l} u_t + u_x = v \\ v_t - v_x = u \end{array} \tag{5.1}$$

which are equivalent to the telegraph equation, the function u being the corresponding solution to the telegraph equation $u_{tt} = u_{xx} + u$, cf. [BL], [CH; p. 192]. If the variables x and t are interchanged the resulting equation is known as the Klein-Gordon equation and is important in particle physics, cf. [MF; p. 139]. This example will compute the symmetry group of this system and demonstrate some interesting group invariant solutions.

To find the symmetry group, let

$$\vec{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u + \psi\partial_v$$

be an arbitrary vector field on Z, so that ξ, τ, ϕ, ψ are unknown functions of (x,t,u,v). The prolongation of $\vec{v}$ is

$$pr^{(1)}\vec{v} = \vec{v} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \psi^x\partial_{v_x} + \psi^t\partial_{v_t}$$

where the coefficients are given by the prolongation formula. Using the infinitesimal criterion of group invariance of the subvariety of $J_1^*(Z,2)$ given by Δ_0, we have

$$\begin{aligned} \phi^t + \phi^x &= \psi \\ \psi^t - \psi^x &= \phi \end{aligned} \tag{5.2}$$

whenever (5.1) holds. Substituting for u_t and v_t according to (5.1) in the prolongation formula, we see that

$$\begin{aligned} \phi^x = (\phi_x - v\tau_x) &+ u_x(\phi_u + \tau_x - \xi_x - v\tau_u) + u_x^2(\tau_u - \xi_u) \\ &+ v_x(\phi_v - v\tau_v) + v_x u_x(\tau_v - \xi_v) \end{aligned}$$

$$\phi^t = (\phi_t+v\phi_u+u\phi_v-v\tau_t-v^2\tau_u-uv\tau_v) +$$

$$+ u_x(\tau_t+2v\tau_u+u\tau_v-\phi_u-\xi_t-v\xi_u-u\xi_v)$$

$$+ u_x^2(\xi_u-\tau_u) + v_x(\phi_v-v\tau_v) + u_xv_x(\tau_v-\xi_v)$$

$$\psi^x = (\psi_x-u\tau_x) + u_x(\psi_u-u\tau_u) + v_x(\psi_v-\xi_x-\tau_x-u\tau_v)$$

$$+ v_x^2(-\tau_v-\xi_v) + v_xu_x(-\xi_u-\tau_u)$$

$$\psi^t = (\psi_t+v\psi_u+u\psi_v-u\tau_t-uv\tau_u-u^2\tau_v)$$

$$+ u_x(-\psi_u+u\tau_u) + v_x(\psi_v-\xi_t-v\xi_u-u\xi_v-\tau_t -$$

$$- v\tau_u-2u\tau_v) + v_x^2(-\xi_v-\tau_v) + u_xv_x(\xi_u+\tau_u) \quad .$$

Substituting these expressions into (5.2) and equating the coefficients of the various terms involving u_x and v_x to 0, after some elementary manipulations we get

$$\tau_v = \xi_v \tag{5.3}$$

$$\tau_u = -\xi_u \tag{5.4}$$

$$\tau_x + \tau_t + 2v\tau_u - \xi_x - \xi_t = 0 \tag{5.5}$$

$$\psi_u = u\tau_u \tag{5.6}$$

$$\phi_v = v\tau_v \tag{5.7}$$

$$\tau_x - \tau_t - 2u\tau_v + \xi_x - \xi_t = 0 \tag{5.8}$$

$$\phi_x - v\tau_x + \phi_t + v\phi_u - v\tau_t - v^2\tau_u = \psi \tag{5.9}$$

$$\psi_t + u\psi_v - u\tau_t - u^2\tau_v - \psi_x + u\tau_x = \phi \quad . \tag{5.10}$$

By (5.3) and (5.4) we have $\tau_{uv} = 0 = \xi_{uv}$ so differentiation of (5.5) and (5.8) with respect to v and u respectively shows that $\tau_u = \xi_u = \tau_v = \xi_v = 0$, hence

$$\tau_x = \xi_t \, , \tau_t = \xi_x$$
$$\psi_u = 0 = \phi_v \quad .$$

Now differentiating (5.9) and (5.10) with respect to v and u respectively and adding the resulting equations shows that $\tau_t = 0 = \xi_x$, which implies that

$$\begin{aligned} \xi &= c_1 + c_3 t \\ \tau &= c_2 + c_3 x \end{aligned} \tag{5.11}$$

where c_1 , c_2 and c_3 are arbitrary constants. Furthermore, since $\phi_{uu} = 0 = \psi_{vv}$, again by (5.9) and (5.10) it is seen that

$$\begin{aligned} \phi &= (c_4 + \tfrac{1}{2}c_3)u + \alpha \\ \psi &= (c_4 - \tfrac{1}{2}c_3)v + \beta \end{aligned} \tag{5.12}$$

where c_4 is a constant and (α,β) form an arbitrary solution to the telegraph equation (5.1). The infinitesimal symmetry algebra of the telegraph equation is therefore spanned by the four vector fields

$$
\begin{aligned}
v_1 &= \partial_x \\
v_2 &= \partial_t \\
v_3 &= t\partial_x + x\partial_t + \tfrac{1}{2}u\partial_u - \tfrac{1}{2}v\partial_v \\
v_4 &= u\partial_u + v\partial_v
\end{aligned}
\tag{5.13}
$$

and the infinite dimensional ideal given by

$$v_{\alpha,\beta} = \alpha\partial_u + \beta\partial_v$$

where (α,β) is an arbitrary solution to the telegraph equation. The commutation table for the symmetry algebra is

	v_1	v_2	v_3	v_4	$v_{\alpha,\beta}$
v_1	0	0	v_2	0	v_{α_x,β_x}
v_2	0	0	v_1	0	v_{α_t,β_t}
v_3	$-v_2$	$-v_1$	0	0	$v_{\alpha',\beta'}$
v_4	0	0	0	0	$-v_{\alpha,\beta}$
$v_{\alpha,\beta}$	$-v_{\alpha_x,\beta_x}$	$-v_{\alpha_t,\beta_t}$	$-v_{\alpha',\beta'}$	$v_{\alpha,\beta}$	0

(5.14)

where

$$
\begin{aligned}
\alpha' &= t\alpha_x + x\alpha_t - \tfrac{1}{2}\alpha \\
\beta' &= t\beta_x + x\beta_t + \tfrac{1}{2}\beta \quad .
\end{aligned}
$$

In this table the entry in the row opposite v_i and the column under v_j is $[v_i, v_j]$. Note that as an immediate consequence of the fact that the infinitesimal symmetries of any system of equations form a Lie algebra, if (α,β) is a solution to the telegraph equation, so are (α_x,β_x) , (α_t,β_t) and (α',β') given by the expression in (5.14). Note further that while the exponentiation of the infinitesimal symmetry group gives a connected neighborhood of the identity in the entire symmetry group, it does not give all the symmetries of (5.1); for instance the transformation

$$(x,t,u,v) \mapsto (-x,t,v,u)$$

leaves (5.1) invariant, but is not in the connected component of the group which contains the identity.

We now construct some group invariant solutions corresponding to one-parameter subgroups of the symmetry group. As a first example, consider the one-parameter group given by the vector field $c\partial_x + \partial_t$ for some constant c , i.e.

$$G_c : (x,t,u,v) \mapsto (x+c\lambda,t+\lambda,u,v) \qquad \lambda \in \mathbb{R} .$$

Here $Z/G_c \simeq \mathbb{R}^3$ with coordinates given by $(\xi = x-ct,u,v)$, which are the invariants of G_c . Then $u_x = u'$, $v_x = v'$, $u_t = -cu'$, $v_t = -cv'$ where the primes denote differentiation with respect to ξ , so the system Δ_0/G_c for the G_c-invariant solutions is

$$(1-c)u' = v$$
$$-(1+c)v' = u$$

If $c = \pm 1$, the only G_c-invariant solution is $u = v = 0$. If

$c^2 - 1 > 0$ then let $\hat{c}^2 = (c^2-1)^{-1}$ so the G_c-invariant solutions are

$$u = c_1 \sinh \hat{c}(x-ct) + c_2 \cosh \hat{c}(x-ct)$$

$$v = c_1 \hat{c}(1-c)\sinh \hat{c}(x-ct) + c_2 \hat{c}(1-c)\cosh \hat{c}(x-ct)$$

where c_1 and c_2 are arbitrary constants. For $c^2 - 1 < 0$ let $\hat{c}^2 = -(c^2-1)^{-1}$ and the G_c-invariant solutions are

$$u = c_1 \sin \hat{c}(x-ct) + c_2 \cos \hat{c}(x-ct)$$

$$v = c_1 \hat{c}(1-c)\cos \hat{c}(x-ct) - c_2 \hat{c}(1-c)\sin \hat{c}(x-ct) \quad .$$

More generally consider the one-parameter subgroup corresponding to the vector field $a\partial_x + b\partial_t + u\partial_u + v\partial_v$ given by

$$G_{a,b} : (x,t,u,v) \mapsto (x+a\lambda, t+b\lambda, e^{\lambda}u, e^{\lambda}v) \qquad \lambda \in \mathbb{R} \quad .$$

Assume $a \neq 0$, hence $G_{a,b}$ acts regularly on Z and $Z/G_{a,b} \simeq \mathbb{R}^3$ with coordinates $(\xi,\rho,\sigma) = (bx-at, e^{-\frac{1}{a}x}u, e^{-\frac{1}{a}x}v)$. Then

$$u_x = (b\rho' + \tfrac{1}{a}\rho)e^{\frac{1}{a}x} \qquad u_t = -a\rho' e^{\frac{1}{a}x}$$

$$v_x = (b\sigma' + \tfrac{1}{a}\sigma)e^{\frac{1}{a}x} \qquad v_t = -a\sigma' e^{\frac{1}{a}x}$$

and the equations $\Delta_0/G_{a,b}$ are

$$(b-a)\rho' + \tfrac{1}{a}\rho = \sigma$$

$$-(b+a)\sigma' - \tfrac{1}{a}\sigma = \rho \quad .$$

Solving for ρ gives the second order ordinary differential equation

$$(b^2-a^2)\rho'' + 2\frac{b}{a}\rho' + (\frac{1}{a^2} + 1)\rho = 0$$

which has either hyperbolic or trigonometric solutions depending on the discriminant. The remaining details are left to the reader.

Finally we consider the one parameter group corresponding to the vector field v_3 which is given by

$$G_3 : (x,t,u,v) \mapsto (x \cosh\lambda + t \sinh\lambda, x \sinh\lambda + t \cosh\lambda, e^{\frac{\lambda}{2}}u, e^{-\frac{\lambda}{2}}v) \quad .$$

Here we must consider the submanifold $Z' = Z \sim \{(0,0,0,0)\}$ to have G_3 acting regularly. Local coordinates on Z'/G_3 are given by, for instance, when $x + t > 0$

$$(\xi,\rho,\sigma) = (x^2-t^2, uv, u(x+t)^{-\frac{1}{2}}) \quad .$$

For these coordinates,

$$u_x = \frac{1}{2}(x+t)^{-\frac{1}{2}}\sigma + 2(x+t)^{\frac{1}{2}}x\sigma'$$

$$u_t = \frac{1}{2}(x+t)^{-\frac{1}{2}}\sigma - 2(x+t)^{\frac{1}{2}}t\sigma'$$

$$v_x = 2x\rho'u^{-1} - \rho u_x u^{-2}$$

$$v_t = -2t\rho'u^{-1} - \rho u_t u^{-2}$$

hence the symmetry equations Δ_o/G_3 are, in these coordinates,

$$\sigma^2 + 2\xi\sigma\sigma' = \rho$$

$$2\sigma'\rho - 2\sigma\rho' = \sigma^3 \quad .$$

Differentiating the first of these and substituting in the second shows that σ satisfies the linear ordinary differential equation

$$4\xi\sigma'' + 6\sigma' + \sigma = 0 \quad .$$

Now let $\sigma = \xi^{-3/4}\hat{\sigma}$, so $\hat{\sigma}$ satisfies

$$4\hat{\sigma}'' + (\tfrac{3}{4}\xi^{-2} + \xi^{-1})\hat{\sigma} = 0$$

the solutions of which (see (9.1.50) in [NBS]) are half-integer order Bessel functions, which are expressible in terms of elementary functions. Thus for $\xi > 0$

$$\sigma = \xi^{-\frac{1}{2}}(c_1\sin\sqrt{\xi} + c_2\cos\sqrt{\xi})$$

for arbitrary constants c_1, c_2. This gives the solutions

$$u(x,t) = \frac{c_1\sin\sqrt{x^2-t^2} + c_2\cos\sqrt{x^2-t^2}}{\sqrt{x-t}} \qquad \begin{array}{l} x - t > 0 \\ x + t > 0 \end{array}$$

$$v(x,t) = \frac{c_1\cos\sqrt{x^2-t^2} - c_2\sin\sqrt{x^2-t^2}}{\sqrt{x+t}}$$

Similar solutions are obtained for the other four quadrants of the (x,t) plane.

Example 5.2 The Heat Equation

Let $Z = \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x,t,u,v) and consider the first order system of equations

$$\Delta_o : \quad \begin{aligned} u_x &= v \\ v_x &= u_t \end{aligned} \quad . \tag{5.15}$$

It can be seen that this system is equivalent to the heat equation $u_{xx} = u_t$ in the sense that if $(u(x,t)\ ,\ v(x,t))$ is any smooth solution to (5.15), then $u(x,t)$ is a solution to the heat equation and conversely, if $u(x,t)$ is any smooth solution to the heat equation, then $(u(x,t)\ ,\ u_x(x,t))$ solves (5.15). It was shown in example 3.1 that the symmetry algebra of the heat equation is spanned by the vector fields

$$\begin{aligned} v_1 &= \partial_x \\ v_2 &= \partial_t \\ v_3 &= u\partial_u + v\partial_v \\ v_4 &= x\partial_x + 2t\partial_t - v\partial_v \\ v_5 &= 2t\partial_x - xu\partial_u - (u+xv)\partial_v \\ v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u - (2xu + (x^2+6t)v)\partial_v \end{aligned} \tag{5.16}$$

and the infinite dimensional ideal given by

$$v_\alpha = \alpha(x,t)\partial_u + \alpha_x(x,t)\partial_v \qquad \alpha_{xx} = \alpha_t$$

corresponding to solutions α of the heat equation. The commutator table for these vector fields is

$$\begin{array}{c|ccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_\alpha \\
\hline
v_1 & 0 & 0 & 0 & v_1 & -v_3 & 2v_5 & v_{\alpha_x} \\
v_2 & 0 & 0 & 0 & 2v_2 & 2v_1 & 4v_4-2v_3 & v_{\alpha_t} \\
v_3 & 0 & 0 & 0 & 0 & 0 & 0 & -v_\alpha \\
v_4 & -v_1 & -2v_2 & 0 & 0 & -v_5 & -2v_6 & v_{\alpha'} \\
v_5 & v_3 & -2v_1 & 0 & v_5 & 0 & 0 & v_{\alpha'} \\
v_6 & -2v_5 & 2v_3-4v_4 & 0 & 2v_6 & 0 & 0 & v_{\alpha''} \\
v_\alpha & -v_{\alpha_x} & -v_{\alpha_t} & v_\alpha & -v_{\alpha'} & -v_{\alpha'} & -v_{\alpha''} & 0
\end{array} \tag{5.17}$$

where

$$\alpha' = x\alpha_x + 2t\alpha_t$$

$$\alpha'' = 4tx\alpha_x + 4t^2\alpha_t + (x^2+2t)\alpha \quad .$$

Note that for all the vector fields the coefficient ψ is the same as ϕ^x when v is substituted for u_x .

Next, some interesting group-invariant solutions corresponding to one-parameter subgroups of the symmetry group will be calculated. More complete results in this direction can be found in [BC1], [BC2]. As a first example, the "travelling wave" solutions correspond to the vector field $c\partial_x + \partial_t$ where c denotes the velocity of the wave having one-parameter group

$$G_c : (x,t,u,v) \longmapsto (x+\lambda c, t+\lambda, u, v) \qquad \lambda \in \mathbb{R} \quad .$$

Thus $Z/G_c \simeq \mathbb{R}^3$ with natural coordinates $(\xi = x-ct, u, v)$. Letting primes denote derivatives with respect to ξ , then $u_x = u'$, $u_t = -cu'$, $v_x = v'$, so the equations Δ_0/G_c for G_c-invariant solutions are

$$u' = v$$
$$v' = -cu'$$

hence $u'' = -cu'$ and we have

$$u(x,t) = \begin{cases} c_1 \sin\sqrt{c}(x-ct) + c_2 \cos\sqrt{c}(x-ct) & c > 0 \\ c_1 x + c_2 & c = 0 \\ c_1 \sinh\sqrt{-c}(x-ct) + c_2 \cosh\sqrt{-c}(x-ct) & c < 0 \end{cases}$$

as the G_c invariant solutions, where c_1 and c_2 are arbitrary constants.

Next consider the one parameter group generated by the vector field v_4 :

$$G_4 : (x,t,u,v) \mapsto (e^{\lambda}x, e^{2\lambda}t, u, e^{-\lambda}v) \qquad \lambda \in \mathbb{R} \quad .$$

Here, for G_4 to act regularly, we must consider the manifold $Z' = Z \sim \{(0,0,0,0)\}$. Local coordinates on Z' are given by $\xi = x^2/t$, u , $\zeta = xv$, so that

$$u_x = 2xt^{-1}u' \qquad u_t = -x^2t^{-2}u'$$

$$v_x = -x^{-2}\zeta + 2t^{-1}\zeta'$$

and the equations Δ_0/G_4 are, in this coordinate system,

$$2\xi u' = \zeta$$

$$2\xi\zeta' - \zeta = -\xi^2 u'$$

so $u(\xi)$ must satisfy the ordinary differential equation

$$4\xi u'' + (2+\xi)u' = 0 \quad .$$

The solutions of this equation are

$$u(\xi) = c_1 \operatorname{erf}(\tfrac{1}{2}\sqrt{\xi}) + c_2 \qquad \xi > 0$$

where erf denotes the error function. Thus the G_4-invariant solutions for $t > 0$ are

$$u(x,t) = c_1 \operatorname{erf} \frac{x}{2\sqrt{t}} + c_2 \qquad t > 0 \quad .$$

More generally, consider the vector field $x\partial_x + 2t\partial_t + au\partial_u + (a-1)v\partial_v$ with group

$$G_a : (x,t,u,v) \mapsto (e^{\lambda}x, e^{2\lambda}t, e^{a\lambda}u, e^{(a-1)\lambda}v) \qquad \lambda \in \mathbb{R}$$

and invariants $\xi = x^2/t$, $\eta = x^{-a}u$, $\zeta = x^{1-a}v$. Solving for the derivatives of u and v , it can be seen that the equations Δ_o/G_a are

$$a\eta + 2\xi\eta' = \zeta$$

$$(a-1)\zeta + 2\xi\zeta' = -\xi^2\eta'$$

so that η must solve the linear equation

$$4\xi^2\eta'' + [(2a+4)\xi + \xi^2]\eta' + a(a-1)\eta = 0 \ .$$

Transforming $\hat{\xi} = \frac{1}{4}\xi$, $\hat{\eta} = \xi^{\frac{a}{2}+\frac{1}{4}} e^{\frac{1}{8}\xi}\eta$, we see that $\hat{\eta}$ satisfies

$$\hat{\eta}'' + [-\tfrac{1}{4} - (\tfrac{a}{2}+\tfrac{1}{4})\hat{\xi}^{-1} + \tfrac{3}{16}\hat{\xi}^{-2}]\hat{\eta} = 0$$

which is Whittaker's equation, cf. (13.1.31) in [NBS]. The solutions are parabolic cylinder functions, [NBS; Chapter 19] or [MI], so for $\hat{\xi} > 0$

$$\hat{\eta}(\hat{\xi}) = \begin{cases} c_1\hat{\xi}^{\frac{1}{4}}U(a+\frac{1}{2},(2\hat{\xi})^{\frac{1}{2}}) + c_2\hat{\xi}^{\frac{1}{4}}U(a+\frac{1}{2},-(2\hat{\xi})^{\frac{1}{2}}) & a \geq \frac{1}{2} \\ c_1\hat{\xi}^{\frac{1}{4}}U(a+\frac{1}{2},(2\hat{\xi})^{\frac{1}{2}}) + c_2\hat{\xi}^{\frac{1}{4}}\overline{U}(a+\frac{1}{2},(2\hat{\xi})^{\frac{1}{2}}) & a < \frac{1}{2} \end{cases}$$

The G_a-invariant solutions to the heat equation are therefore given by

$$u(x,t) = t^{a/2}e^{-x^2/8t}[c_1U(a+\tfrac{1}{2},x/\sqrt{2t})+c_2U(a+\tfrac{1}{2},-x/\sqrt{2t})] \quad t > 0 \ , \ a \geq \tfrac{1}{2}$$

and similarly for $a < \frac{1}{2}$. Note that the parabolic cylinder functions generalize the error function solution found for G_4 .

As an application of the theory of graphic differential operators, we consider the group classification problem for quasi-linear systems of first order partial differential equations. Roughly speaking, group classification means that one is given a system of partial differential equations which involve some arbitrary elements - e.g. functions and/or constants - and the problem is to determine which specific assignments of these arbitrary elements lead to equivalent symmetry groups - cf. [01], [02]. The main question is to determine the values of the arbitrary elements that give the largest possible symmetry groups, these in some sense being the most physically interesting systems of equations.

In this example, consider the system of quasi-linear equations

$$\Delta_0 : \sum_{j=1}^{p} K_\sigma^j(x,u)u_j + K_\sigma(x,u) = 0 \qquad \sigma = 1,\cdots,m \tag{5.18}$$

on $Z = \mathbb{R}^p \times \mathbb{R}$. Recall first that if $\mathcal{U}$ is a differential system on a smooth manifold, then its involutive completion is the smallest involutory differential system $\hat{\mathcal{U}} \supset \mathcal{U}$.

<u>Proposition</u> 5.3 The symmetry group of the system of equations (5.18) is contained in the "group" of transformations on Z preserving the fibration given by the involutive completion of the differential system

$$\mathcal{U} = \{v_\sigma = \sum K_\sigma^j(x,u) \frac{\partial}{\partial x^j} - K_\sigma(x,u) \frac{\partial}{\partial u} : \sigma = 1,\cdots,m\}$$

and is equal to that group if $\mathcal{U}$ is itself involutive.

<u>Proof</u>

Let $\hat{\Delta}_0$ denote the graphic equivalent to Δ_0. Then $\hat{\Delta}_0$ is invariant under a group of transformations on Z iff Δ_0 is invariant under the group. If w denotes the fiber coordinate of $\mathbb{1}_Z$, the trivial line bundle over Z, then $\hat{\Delta}_0$ is given by the equations

$$\sum K_\sigma^j(x,u)w_j - K_\sigma(x,u)w_u = 0$$

and the solutions are w=const along the leaves of $\hat{\mathcal{U}}$, hence the leaves of $\hat{\mathcal{U}}$ must be preserved by G. In the case that $\mathcal{U}$ is itself involutive, choose local coordinates $(z^1,\cdots,z^{p+1})$ so that $\mathcal{U}$ is spanned by $\{\partial/\partial z^1,\cdots,\partial/\partial z^m\}$. In these coordinates, $\hat{\Delta}_0$ is given by

$$\frac{\partial w}{\partial z^\sigma} = 0 \qquad \sigma = 1,\cdots,m$$

and it is easy to see that these equations admit every transformation leaving the leaves invariant. Q.E.D.

Example 5.3 To see that the full group is not always obtained, consider the case $p = 3$ with coordinates (x,y,z,u) and the equations

$$\Delta_0 : \quad \begin{aligned} u_y &= 0 \\ yu_x + u_z &= 0 \end{aligned}$$

so that $\mathcal{U}$ is spanned by $\{\partial/\partial y, y\partial/\partial x + \partial/\partial z\}$. This is not involutive, the involutive completion being spanned by $\{\partial/\partial x,\partial/\partial y,\partial/\partial z\}$, so the infinitesimal symmetry group is given by all vector fields of the form

$$\xi(x,y,z,u)\frac{\partial}{\partial x} + \eta(x,y,z,u)\frac{\partial}{\partial z} + \zeta(x,y,z,u)\frac{\partial}{\partial z} + \phi(u)\frac{\partial}{\partial u} \ .$$

However, the symmetry equations for Δ_1 give the restriction that

$$\xi_y - y\zeta_y = 0$$

$$\xi_x + \xi_z - y(\zeta_x + \zeta_z) = \eta$$

so the full symmetry group is not obtained. This is also an illustration of the fact that the group that leaves the solution set invariant may be different from the group of the system of equations as we have defined it.

II. Extended Jet Bundles

Now that an adequate theory of symmetry groups and group invariant solutions for first order partial differential equations has been developed, we direct our attention to higher order systems of partial differential equations. This chapter is fairly technical in nature; its purpose is to develop the appropriate mathematical machinery, which includes symmetric algebra, higher order extended jet bundles and total derivatives. This machinery will be applied to the problems of symmetry groups and group invariant solutions in chapters 3 and 4, respectively. Although some of these ideas have been touched upon in the first chapter, we shall recapitulate any of the necessary material, so that chapters 2-4 can be read independently of chapter 1.

Section one is a brief discussion of the concept of a p-section of a manifold, and recalls the definition of the k-th order tangent bundle of a smooth manifold. Before being able to do any interesting computations, we need to have the tool of the symmetric algebra of a vector space firmly in hand. The second section recalls the relevant parts of this theory, which is then applied to discuss the higher order derivatives of smooth maps between vector spaces. The most important result is the general Faa-di-Bruno formula for the k-th order differential of the composition of two functions. This gives in particular an explicit matrix representation of the k-th order prolongation of the general linear group of a vector space. Section 3 tackles the problem of local coordinate and fiber bundle descriptions of the extended jet bundles. Their fibers are prolonged Grassmann manifolds, which are discussed in some detail. In section 4 the computational tool of the total derivative is defined, and a expression for it derived. This becomes of importance in the general prolongation formula for vector fields, to be derived in chapter III. The last section of this chapter shows how systems of partial differential equations fit into the jet bundle scheme. We define and discuss differential operators and equations, and their prolongations. Some of this material is not essential for the remainder of the thesis, but was included for the sake of completeness. The most important concept to be introduced here is the prolongation of a diffeomorphism of a manifold to its extended jet bundles.

II.1 Sections and Jet Bundles

The first step in the development of a comprehensive theory of systems of partial differential equations over arbitrary smooth manifolds representing both the independent and dependent variables is the construction of an appropriate fiber bundle over the manifold whose points represent the partial derivatives of the dependent variables of order $\le k$, called the extended k jet bundle of the manifold. To give some motivation for the definitions to follow, the construction of the jet bundles of a vector bundle will be briefly recalled following [GG]. Then some preliminary definitions for the more general case will be made; the machinery needed to fully explore these ideas will be developed in subsequent sections. Section I.1 should be read for additional detail and more motivational material.

Suppose $\pi: Z \to X$ is a vector bundle over a p dimensional base manifold X, representing the independent variables, with q dimensional fiber U, representing the dependent variables. Sections of Z are usually defined to be smooth maps $f: X \to Z$ such that $\pi \circ f = \mathbb{1}_X$, the identity map of X. It will be more convenient, however, to view sections geometrically as p dimensional submanifolds of Z that satisfy a condition of transversality to the fibers of Z. In addition, for the submanifold to truly be a section, it must satisfy a further global condition of intersecting each fiber exactly once; in the construction of the jet bundles, this condition can be safely ignored, since only local sections are needed. The k-jet bundle J_kZ is given by the equivalence classes of sections of Z agreeing up to k-th order. It is a vector bundle with fiber

$$\mathbb{R}^q \times \bigoplus_{i=1}^{k} \odot^i(\mathbb{R}^p, \mathbb{R}^q)$$

where $\odot^i(\mathbb{R}^p, \mathbb{R}^q)$ denotes the space of all i-linear symmetric maps from $\mathbb{R}^p$ to $\mathbb{R}^q$.† Note that

$$\dim \odot^i(\mathbb{R}^p, \mathbb{R}^q) = q\binom{p+i-1}{i},$$

which is the correct dimension for it to represent the space of all partial derivatives of order i of sections of Z. The following geometric characterization of the condition of k-th order contact between sections will be useful. First, recall the definition of the k-th order tangent bundle of a smooth manifold.

<u>Definition 1.1</u> Let M be a smooth manifold and let $m \in M$. Let $C^\infty(M, \mathbb{R})|_m$ denote the algebra of germs of smooth real valued functions on M at the point m. Let $I_m \subset C^\infty(M, \mathbb{R})|_m$ be the ideal of germs of functions which vanish at m, and let I_m^k denote its k-th power, which consists of all finite linear combinations of k-fold products of elements of I_m. The <u>k-th order cotangent bundle</u> of M at m is

$$\mathfrak{T}_k^* M|_m = I_m / I_m^{k+1}.$$

The <u>k-th order tangent bundle</u> to M at m is the dual space

$$\mathfrak{T}_k M|_m = [\mathfrak{T}_k^* M|_m]^*.$$

Alternatively, $\mathfrak{T}_k M|_m$ can be defined directly as the vector space of all k-th order linear derivatives of the algebra $C^\infty(M, \mathbb{R})|_m$.

There are natural differentiable structures on the tangent and

†See section II.2.

cotangent bundles

$$\mathfrak{J}_k M = \bigcup_{m \in M} \mathfrak{J}_k M|_m \qquad \mathfrak{J}_k^* M = \bigcup_{m \in M} \mathfrak{J}_k^* M|_m.$$

The interested reader is advised to consult [W; §1.26] for the details of this construction. In terms of local coordinates,

$$\mathfrak{J}_k M|_m \simeq \odot_k^* TM|_m = \bigoplus_{i=1}^{k} \odot_i TM|_m$$

where $\odot_i V$ denotes the i-th symmetric power of a vector space V. If $\{\partial_1, \ldots, \partial_n\}$ forms a basis of $TM|_m$, with ∂_i denoting the partial derivative in the i-th coordinate direction, then

$$\{\partial_I : I = (i_1, \ldots, i_n) \quad \Sigma I = i_1 + \ldots + i_n \leq k\}$$

forms a basis for $\mathfrak{J}_k M|_m$, with ∂_I denoting the partial derivative $\partial_1^{i_1} \partial_2^{i_2} \ldots \partial_n^{i_n}$. Now if S and S' are submanifolds of M, then S and S' have k-th order contact at $m \in S \cap S'$ iff $\mathfrak{J}_k S|_m = \mathfrak{J}_k S'|_m$. It can be readily checked that for the case of sections of a vector bundle this definition of k-th order contact agrees with all the other definitions, and has the advantage of an immediate geometrical interpretation.

Suppose M and M' are smooth manifolds and $F: M \to M'$ is a smooth map. There is an induced bundle morphism $d^k F: \mathfrak{J}_k M \to \mathfrak{J}_k M'$ given by the formula $d^k F(v)(f) = v(f \circ F)$ for $v \in \mathfrak{J}_k M$ and $f \in C^\infty(M', \mathbb{R})$. It is readily verified that if $G: M' \to M''$ is another smooth map, then

$$d^k G \circ d^k F = d^k (G \circ F) \tag{1.1}$$

This formula will be subsequently seen to contain the general Faa-di-Bruno formula for the partial derivatives of the composition of smooth maps, restated in the general context of smooth manifolds. It will be of primary importance for what is to follow; unfortunately, to properly understand its implications, it is necessary to develop the complex machinery of symmetric algebra. It should be remarked here that the k-th order tangent and cotangent bundles are vector bundles of a special type, since their group is the k-th prolongation of the general linear group, which will be defined and discussed in the next section.

<u>Definition 1.2</u> Given a smooth manifold Z and a point $z \in Z$, a <u>p-section</u> of Z passing through z is an arbitrary smooth p dimensional submanifold of Z containing z. The space of germs of p-sections of Z passing through z, $C^\infty(Z,p)|_z$ is the set of all smooth p-dimensional submanifolds of Z passing through z modulo the equivalence relation that s and s' define the same germ at z iff there is a neighborhood V of z with $s \cap V = s' \cap V$.

<u>Definition 1.3</u> The <u>space of extended k-jets of p-sections</u> of Z at a point $z \in Z$, $J_k^*(Z,p)|_z$ is given by the space of germs of p-sections of Z passing through z modulo the equivalence relation of k-th order contact. In other words, two p dimensional submanifolds s and s' define the same extended k-jet at z iff $\mathfrak{J}_k s|_z = \mathfrak{J}_k s'|_z$. Note that $J_0^*(Z,p) \simeq Z$. Given a p-section s passing through z, let $j_k^* s|_z$ denote the equivalence class in $J_k^*(Z,p)|_z$ defined by s, which will be called the <u>extended k-jet</u> of s at z.

Alternately, a p-section of Z can be described locally parametrically by a smooth embedding $f\colon U \to Z$ where U is an open subset of $\mathbb{R}^p$.

Such an embedding will be called a parametrization of the p-section defined by its image. In general, since all considerations here are of a local nature, we shall be a bit sloppy notationally and write $f: \mathbb{R}^p \to Z$ even though f might only be defined on an open subset of $\mathbb{R}^p$. Note that two embeddings f and g are parametrizations of the same germ $s \in C^\infty(Z,p)|_z$ iff there exists a (local) diffeomorphism $\psi: \mathbb{R}^p \to \mathbb{R}^p$ such that $f \circ \psi = g$ near the point z. The notation $j_k^* f|_z$ will be taken to mean the extended k-jet of the p-section given by im f at z.

It will be shown in detail in section II.3 that the extended k-jet bundle

$$J_k^*(Z,p) = \bigcup_{z \in Z} J_k^*(Z,p)|_z$$

has the structure of a smooth fiber bundle over Z such that if s is any smooth p dimensional submanifold of Z, then its extended k-jet

$$j_k^* s = \bigcup_{z \in Z} j_k^* s|_z$$

is a smooth p dimensional submanifold of $J_k^*(Z,p)$ which projects back onto s. The fibers of the extended k-jet bundle will be "prolonged Grassmann manifolds" which shall be defined and described in section II.3.

II.2 Symmetric Algebra and Derivatives

The first part of this section is essentially a recapitulation of some notation and results to be found in [F; chapters 1,3] on symmetric algebra, and its applications to discussing the higher order derivatives of smooth functions between vector spaces. The most important result is the general Faa-di-Bruno formula given in theorem 2.2 for the higher order differentials of the composition of functions. This serves to motivate the definition of the Faa-di-Bruno injection, which is applied to giving an explicit matrix representation of the k-th prolongation of the general linear group. The reader is advised to consult Federer's book for a much more complete exposition of these ideas.

Let V be a real vector space. Let

$$\bigodot_* V = \bigoplus_{i=0}^{\infty} \bigodot_i V$$

denote the graded symmetric algebra based on V. It is realised as the tensor algebra of V modulo the two-sided ideal generated by all elements of the form $v \otimes v' - v' \otimes v$. The product in $\bigodot_* V$ will be denoted by the symbol $\odot$; thus if $v \in \bigodot_i V$ and $v' \in \bigodot_j V$, then their symmetric product $v \odot v' = v' \odot v \in \bigodot_{i+j} V$. If W is another real vector space, then let

$$\bigodot^*(V,W) = \bigoplus_{i=0}^{\infty} \bigodot^i (V,W)$$

denote the graded vector space of W-valued symmetric linear forms on V. In other words, $\bigodot^i(V,W)$ is the vector space of all i-linear symmetric functions $A: \overbrace{V\times\ldots\times V}^{i} \to W$, hence there is an identification

$$\odot^i(V,W) = \mathrm{Hom}(\odot_i V,W).$$

This identification will be used extensively in what is to follow, usually without further comment. (By convention we define $\odot^0(V,W)=W$.) Before proceeding further, we have need of some multi-index notation, which is collected together in the following definition for convenience.

<u>Definition 2.1</u> Let

$$S_k^n = \{I = (i_1,\ldots,i_n);\ 0\le i_\sigma \in \mathbb{Z}, \sigma=1,\ldots,n;\ \Sigma I = i_1+\ldots+i_n = k\}$$

be the set of n multi-indices of rank $k\ge 0$, and let

$$S^n = \bigcup_{k=0}^{\infty} S_k^n$$

be the set of all n multi-indices. Given $I \in S_k^n$, $J \in S_\ell^n$ let $I+J \in S_{k+\ell}^n$ be the multi-index with components $i_\sigma + j_\sigma$. Introduce a partial ordering on S^n by defining $I\le J$ iff $i_\sigma \le j_\sigma$ for all $\sigma=1,\ldots,n$. In case $I\le J$ let $J-I$ be the multiindex with components $j_\sigma - i_\sigma$. Define

$$I! = i_1!i_2!\ldots i_n! \qquad I \in S^n$$

$$\binom{J}{I} = \frac{J!}{I!(J-I)!} \qquad I\le J \in S^n.$$

Let $\delta^j \in S_1^n$ be the multi-index with components δ_σ^j, the second δ being the Kronecker symbol.

Now suppose V is a finite dimensional real vector space with basis $\{e_1,\ldots,e_n\}$. In this case $\odot_k V$ has a corresponding basis given by

$$\{e_I : I \in S_k^n\} \qquad \text{where} \qquad e_I = e_1^{i_1} \odot \ldots \odot e_n^{i_n}.$$

(The powers are of course taken in the symmetric algebra of V.) If W is a real algebra, then there is a naturally defined product on $\odot^*(V,W)$ making it into a graded commutative algebra. This product can be reconstructed from the following fundamental formula:

$$\phi \odot \psi(e_J) = \sum_{J \geq I \in S_p^n} \binom{J}{I} \phi(e_I) \cdot \psi(e_{J-I}) \tag{2.1}$$

for $\phi \in \odot^p(V,W)$, $\psi \in \odot^q(V,W)$, $J \in S_{p+q}^n$. In the particular case that $W = \mathbb{R}$, suppose $\{e^1,\ldots,e^n\}$ forms a dual basis of $V^* = \mathrm{Hom}(V, \mathbb{R})$, then $\{e^I : I \in S_k^n\}$ forms a basis of $\odot^k V = \odot^k(V, \mathbb{R}) = \odot_k V^*$. These two bases are dual to within a factor in the sense that

$$\langle e_I, e^{I'} \rangle = \begin{cases} 0 & I \neq I' \\ I! & I = I'. \end{cases}$$

Now suppose that V and W are real normed vector spaces and $f: V \to W$ is a smooth function. The k-th order differential of f at a point $x \in V$, which we shall denote by the symbol $\partial^k f(x)$,† is that symmetric k-linear W-valued form on V whose matrix entries are just the k-th order partial derivatives of f. In other words, we have $\partial^k f(x) \in \odot^k(V,W)$ and if $\{e_1,\ldots,e_n\}$ is a basis of V, then

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†The reason for the symbol $\partial^k f$ rather than the more standard $D^k f$ or $d^k f$ will become clearer in what follows. Suffice it to say that $d^k f$ will be reserved for the action of a smooth map on k-th order tangent vectors to a manifold, and $D^k f$ for the total derivative. The relationship between these concepts will be made clear.

$$\langle e_I, \partial^k f(x)\rangle = \partial_I f(x) \qquad I \in S_k^n$$

where $\partial_I = \partial_1^{i_1}\partial_2^{i_2}\cdots\partial_n^{i_n}$, ∂_i denoting the partial derivative in the e_i direction. Let

$$\odot^{k+\ell}(V,W) \subset \odot^{\ell}(V,\odot^k(V,W))$$

be the natural inclusion given by

$$\langle v, \langle v', w\rangle\rangle = \langle v \odot v', w\rangle \qquad v \in \odot_\ell V,\ v' \in \odot_k V,\ w \in \odot^{k+\ell}(V,W)$$

It can be seen that

$$\partial^\ell(\partial^k f)(x) = \partial^{k+\ell} f(x) \qquad x \in V$$

under this inclusion.

Theorem 2.2 (The Faa-di-Bruno formula) Let V,W, and X be normed real vector spaces, and let $f: V \to W$ and $g: W \to X$ be smooth maps. Given any $x \in V$, let $y = f(x) \in W$. For any positive integer k

$$\partial^k(g\circ f)(x) = \sum_{I \in \mathcal{A}_k} \partial^{\Sigma I} g(y) \circ \frac{\partial f(x)^{i_1} \odot \partial^2 f(x)^{i_2} \odot \cdots \odot \partial^k f(x)^{i_k}}{I!} \qquad (2.2)$$

where

$$\mathcal{A}_k = \{I \in S^k : \sum_{\sigma=1}^{k} \sigma i_\sigma = k\}.$$

Note that in formula (2.2) the powers and symmetric products of the differentials of f are taken in the algebra $\odot^*(V,\odot_* W)$. The proof of this theorem can be found in [F; page 222].

Let us define the vector spaces

$$\odot^*_k V = \bigoplus_{i=1}^{k} \odot_i V$$

$$\odot^k_*(V,W) = \bigoplus_{i=1}^{k} \odot^i(V,W)$$

for notational convience. There is a natural projection

$$\pi_k : \odot^k_*(V, \odot_* W) \to \odot^k_*(V,W)$$

given by composition with the projection $\odot_* W \to \odot_1 W = W$.

<u>Definition 2.3</u> The <u>Faà-di-Bruno</u> <u>injection</u> is that map

$$\varepsilon_k : \odot^k_*(V,W) \to \odot^k_*(V, \odot^*_k W) \subset \quad {}^k_*(V, \quad {}_*W)$$

such that given a matrix $A \in \odot^k_*(V,W)$ with $A|\odot_i V = A_i$, then

$$\varepsilon_k(A) = \sum_{j=1}^{k} \sum_{I \in \mathcal{A}_j} \frac{A_1^{i_1} \odot A_2^{i_2} \odot \ldots \odot A_j^{i_j}}{I!} \quad . \tag{2.3}$$

Note that $\pi_k \circ \varepsilon_k = \mathbb{1}$, the identity map of $\odot^k_*(V,W)$.

<u>Example 2.4</u> To get some idea of what the matrix $\varepsilon_k(A)$ looks like in block format, consider the case k=4. Given $A \in \odot^4_*(V,W)$, then A has the block matrix form

$$A = (\ A_1 \quad A_2 \quad A_3 \quad A_4).$$

Note that

$$\varepsilon_k(A) : \odot^*_4 V \to \odot^*_4 W$$

by the definition of the set of multi-indices $\mathcal{A}_4$. Using formula (2.3), we see that $\varepsilon_k(A)$ has block matrix form

$$\varepsilon_4(A) = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & \frac{1}{2}A_1 \odot A_1 & A_1 \odot A_2 & A_1 \odot A_3 + \frac{1}{2}A_2 \odot A_2 \\ 0 & 0 & \frac{1}{3!}A_1 \odot A_1 \odot A_1 & \frac{1}{2}A_1 \odot A_1 \odot A_2 \\ 0 & 0 & 0 & \frac{1}{4!}A_1 \odot A_1 \odot A_1 \odot A_1 \end{pmatrix}$$

where the (i,j)th entry is the part taking $\odot_i V$ to $\odot_j W$.

Returning to our discussion of the differentials of smooth functions between vector spaces, define

$$\partial_*^k f(x) = \partial f(x) + \partial^2 f(x) + \ldots + \partial^k f(x).$$

Then the Faa-di-Bruno formula (2.2) can be restated concisely as

$$\partial_*^k (g\circ f)(x) = \partial_*^k g(y) \circ \varepsilon_k[\partial_*^k f(x)]. \tag{2.4}$$

Define

$$d^k f(x) = \varepsilon_k[\partial_*^k f(x)].$$

(The reason for this notation will become clear in the context of higher order tangent vectors on manifolds, cf. lemma 3.1.) Applying ε_k to (2.4) yields

$$d^k(g\circ f)(x) = d^k g(y) \circ d^k f(x). \tag{2.5}$$

Definition 2.5 Let V be a real normed vector space of dimension n and let k be a positive integer. Let

$$\odot_*^k(V,V)_0 = \{A \in \odot_*^k(V,V): A|V \in GL(V)\}$$

be the set of all polynomial functions from V to itself of order $\leq k$ which are invertible when restricted to $\odot_1 V = V$. Define the

<u>k-th prolongation of the general linear group of V</u> to be

$$GL^{(k)}(V) = \epsilon_k[\odot_*^k(V,V)_0]$$

which is a matrix subgroup of $GL(\odot_k^* V)$.

Note that this definition coincides with that given in [KO2; page 139], since $GL^{(k)}(V)$ can be realized as the set of all matrices $d^k f(0)$ corresponding to all local diffeomorphisms f of V with f(0)=0, the group multiplication being induced by composition of diffeomorphisms according to (2.5). In addition we have given an explicit matrix representation of $GL^{(k)}(V)$. In fact, given $A \in GL^{(k)}(V)$, let (A_j^i) be the block matrix form of A so that $A_j^i\colon \odot_j V \to \odot_i V$. By the definition, A has the following properties:

i) A is block upper triangular; i.e. $A_j^i = 0$ for i>j.

ii) Let

$$\mathcal{A}_j^i = \{I \in \mathcal{A}_j : \Sigma I = i\} = \{I \in S^j : \Sigma I = i, \textstyle\sum \sigma i_\sigma = j\}$$

then

$$A_j^i = \sum_{I \in \mathcal{A}_j^i} \frac{(A_1^1)^{i_1} \odot (A_2^1)^{i_2} \odot \cdots \odot (A_k^1)^{i_k}}{I!}$$

iii) If $A \in GL^{(k)}(V)$, then for $\ell<k$, $A|\odot_\ell^* V \in GL^{(\ell)}(V)$.

In particular, if $A = d^k f(x)$, then we will abbreviate $A_j^i = \partial_j^i f(x)$. This gives $\partial_j^1 f(x) = \partial^j f(x)$, which is a little confusing, but the symbol ∂_j has been reserved for the partial derivative in the x^j direction.

II.3 Grassmann Manifolds and Bundles

The purpose of this section is to provide a detailed description of the extended jet bundles of a smooth manifold, both in terms of local coordinates and in terms of their structure as fiber bundles. This will rely heavily on the symmetric algebra that was developed in section II.2. The fibers of the extended jet bundles will be called, in analogy with the first order case that was discussed in chapter I, prolonged Grassmann manifolds. In addition to the definition of these manifolds as the set of all k-th order tangent spaces to p dimensional submanifolds passing through the origin of a vector space, two additional useful characterizations will be discussed -- one being a multiple of the standard quotient bundle over a regular Grassmann manifold, the other being the quotient space of a set of matrices under the action of a prolonged general linear group. These will then be applied towards the local coordinate description of the extended jet bundles.

Before proceeding to the prolonged Grassmannians, we need to show that the two previously introduced notations d^kF have the same meaning when F happens to be a smooth function between vector spaces.

<u>Lemma 3.1</u> Let $F:V \to W$ be a smooth map between finite dimensional real vector spaces. Let $x \in V$ and $w=F(x)$. Then the map

$$d^kF|_x : \mathfrak{J}_kV|_x \to \mathfrak{J}_kW|_w$$

under the identification $\mathfrak{J}_kV|_x \simeq \mathcal{O}_k^*V$ is the same as the map

$$d^kF(x) = \varepsilon_k[\partial^k_* F(x)] : \odot^*_k V \to \odot^*_k W.$$

Proof

Let $\{e_1,\ldots,e_n\}$ be a basis of V and let $\{e'_1,\ldots,e'_m\}$ be a basis for W. The identification $\mathfrak{J}_k V|_x \cong \odot^k_* V$ takes $\partial_I|_x$ to e_I for any multi-index I. Now suppose $f \in C^\infty(W,\mathbb{R})$ then

$$d^kF(\partial_I|_x)f = \partial_I(f\circ F)(x)$$
$$= \sum_{\Sigma I \le \Sigma J} \partial'_J f(w)\cdot[\varepsilon_k \partial^k_* F(x)]^J_I$$

which follows from the Faa-di-Bruno identity.† Therefore

$$d^kF(\partial_I|_x) = \sum\, [\varepsilon_k \partial^k_* F(x)]^J_I\, \partial'_J|_w$$

hence the lemma follows. Q.E.D.

Notice that $\partial^k_* F(x)$ has matrix entries $\partial_I F^j(x)$ for $j=1,\ldots,m$ and $1 \le \Sigma I \le k$ showing that d^kF is uniquely determined by the partial derivatives of F of order $\le k$. If $F:Z \to Z'$ is a smooth map between smooth manifolds then the local coordinate descriptions of Z and Z' identify them with open subsets of real vector spaces, so the local coordinate description of d^kF is given by the Faa-di-Bruno formula. This will be important in computations to come. Note that the matrix $\partial^k_* F$ provides local coordinates for $J_k(Z,Z')$, the space of k-jets of maps between Z and Z', hence we may identify $j_kF = \partial^k_* F$.

Let V be a real n dimensional vector space and let 0 denote the origin in V. Throughout this section, the identifications $TV|_0 \cong V$,

---

†For a matrix $A:\odot^*_k V \to \odot^*_k W$, A^J_I is the matrix entry given by $A(e_I) = \sum A^J_I\, e'_J$

$\mathfrak{J}_k V|_0 \simeq \odot_k^* V$ will be used without further comment.

Definition 3.2 The k-th order prolonged Grassmann manifold of prolonged p-planes in V is given by

$\mathrm{Grass}^{(k)}(V,p) = \{ \mathfrak{J}_k s|_0 \subset \odot_k^* V : s \in C^\infty(V,p)|_0 \}$

where $C^\infty(V,p)_0$ denotes the set of smooth p dimensional submanifolds of V passing through the origin.

It can therefore be seen that, assuming $J_k^*(Z,p)$ really is a fiber bundle over Z, the fiber must be $\mathrm{Grass}^{(k)}(n,p)$ where n is the dimension of Z. (Here we have abbreviated $\mathrm{Grass}^{(k)}(\mathbb{R}^n,p)$ by just $\mathrm{Grass}^{(k)}(n,p)$.)

Lemma 3.3 Let $\odot_*^k(\mathbb{R}^p,V)_0$ denote the open subset of $\odot_*^k(\mathbb{R}^p,V)$ consisting of those maps $A: \odot_k^*\mathbb{R}^p \to V$ whose restriction to $\odot_1\mathbb{R}^p$ has maximal rank equal to p. There is a natural action of $GL^{(k)}(p)$ on $\odot_*^k(\mathbb{R}^p,V)_0$ given by right multiplication of matrices and

$$\mathrm{Grass}^{(k)}(V,p) \simeq \odot_*^k(\mathbb{R}^p,V)_0 \,/\, GL^{(k)}(p)$$

is the quotient space.

Proof

Given a p dimensional submanifold $s \subset V$ passing through 0, there always exists a local embedding $f: \mathbb{R}^p \to V$ with im f = s and f(0)=0. Moreover, since f is an embedding

$$d^k f[\mathfrak{J}_k \mathbb{R}^p|_0] = \mathfrak{J}_k s|_0$$

$$\partial_*^k f(0) \in \odot_*^k(\mathbb{R}^p,V)_0.$$

Therefore, it suffices to show that if f and f' are smooth embeddings of $\mathbb{R}^p$ into V with f(0)=f'(0)=0 then

$$\operatorname{im} d^k f|_0 = \operatorname{im} d^k f'|_0 \qquad (*)$$

iff there exists a matrix $A \in GL^{(k)}(p)$ satisfying

$$\partial_*^k f(0)\cdot A = \partial_*^k f'(0).$$

First, given such a matrix A, (*) follows immediately since A is invertible on $\bigodot_k^* \mathbb{R}^p = \mathfrak{J}_k \mathbb{R}^p|_0$. Conversely, given (*), let $V' \subset V$ be an n-p dimensional subspace such that in a neighborhood of the origin $T(\operatorname{im} f)\cap V' = T(\operatorname{im} f')\cap V' = \{0\}$. Let $\pi : V \to V/V' \simeq \mathbb{R}^p$ be the projection, so by the inverse function theorem both $\pi\circ f$ and $\pi\circ f'$ are invertible in a neighborhood of the origin in $\mathbb{R}^p$. Now since (*) holds, there exists a linear transformation $A: \bigodot_k^* \mathbb{R}^p \to \bigodot_k^* \mathbb{R}^p$ satisfying

$$d^k f|_0 \cdot A = d^k f'|_0$$

therefore

$$d^k\pi\cdot d^k f\cdot A = d^k\pi\cdot d^k f'$$
$$A = d^k(\pi\circ f')|_0 \cdot d^k(\pi\circ f)^{-1}|_0$$

proving that $A \in GL^{(k)}(p)$. Q.E.D.

For positive integers $k>\ell$ there is a canonical projection

$$\pi_\ell^k : \operatorname{Grass}^{(k)}(V,p) \to \operatorname{Grass}^{(\ell)}(V,p)$$

given by

$$\pi_\ell^k[\mathfrak{J}_k s|_0] = \mathfrak{J}_\ell s|_0.$$

It will soon be shown that π_ℓ^k makes $\operatorname{Grass}^{(k)}(V,p)$ into a Euclidean fiber bundle over $\mathrm{Grass}^{(\ell)}(V,p)$. Suppose $\Lambda \in \mathrm{Grass}^{(k)}(V,p)$ and $\{e_1,\ldots,e_p\}$ is a basis for the p-plane $\Lambda_1 = \pi_1^k(\Lambda) \in \mathrm{Grass}(V,p)$. (Given Λ as an abstract subspace of $\bigodot_k^* V$, Λ_1 is the unique p-plane in V such that $\bigodot_k \Lambda_1 = \Lambda \cap \bigodot_k V$.) It is claimed that there is a unique set of elements $e_J^! \in V/\Lambda_1$ for each multi-index $J \in S^p$ with $1<\Sigma J \le k$, such that for any representatives $e_J \in V$ of the $e_J^!$, the vectors

$$\hat{e}_J \equiv \sum \frac{J!}{\bar{J}!} e_{\bar{J}} \quad \text{and} \quad e_1,\ldots,e_p \tag{3.1}$$

form a basis of Λ. In (3.1) the sum is taken over all unordered sets of multi-indices $\bar{J} = \{J_1,\ldots,J_\ell\}$ such that $J_1+J_2+\ldots+J_\ell = J$, and

$$e_{\bar{J}} = e_{J_1} \odot e_{J_2} \odot \ldots \odot e_{J_\ell},$$

$$\bar{J}! = J_1!\, J_2! \,\ldots\, J_\ell!.$$

The proof of (3.1) follows from the Faa-di-Bruno formula. Let $f: \mathbb{R}^p \to V$ be an embedding such that $\mathfrak{J}_k(\mathrm{im}\, f)|_0 = \Lambda$. Note that from equation (2.1) if $\phi_\sigma: \bigodot_{i_\sigma} \mathbb{R}^p \to \bigodot_* V$ for $\sigma=1,\ldots,m$, and $i=i_1+\ldots+i_m$, then for any $I \in S_i^p$

$$\phi_1 \odot \phi_2 \ldots \odot \phi_m(e_I) = \sum \frac{I!}{J_1!J_2!\ldots J_m!} \phi_1(e_{J_1}) \odot \ldots \odot \phi_m(e_{J_m}) \tag{3.2}$$

where the sum is taken over all ordered sets of multi-indices $(J_1,\ldots,J_m)$ with $J_\sigma \in S_{i_\sigma}^p$ and $J_1+J_2+\ldots+J_m = I$. From this formula, we conclude that

$$\hat{e}_J = d^k f(\partial_J)$$

from which (3.1) immediately follows.

Conversely, given a basis $\{e_1,\ldots,e_p\}$ of a p-plane $\Lambda_1 \subset V$ and elements

$e'_J \in V/\Lambda_1$ for $J \in S^p$ with $1<\Sigma J\leq k$, it is not hard to see that the subspace spanned by $\{\hat{e}_J\colon 1\leq\Sigma J\leq k\}$ as given by (3.1) is an element of $\mathrm{Grass}^{(k)}(V,p)$. In fact, if $\{\varepsilon_1,\ldots,\varepsilon_n\}$ is a basis for V and $e_J = \sum c_J^i \varepsilon_J$, then the polynomial $f\colon\mathbb{R}^p \to V$

$$f^i(x^1,\ldots,x^p) = \sum c_J^i \frac{x^J}{J!}$$

satisfies $\mathfrak{J}_k(\mathrm{im}\ f)|_0 = \Lambda$. Moreover Λ is uniquely determined by the e'_J once $e_1,\ldots,e_p$ are prescribed. Let

$$N_{k,p} = \dim \odot_k^* \mathbb{R}^p = p + \binom{p+1}{2} + \ldots + \binom{p+k-1}{k}. \tag{3.3}$$

We have thus proven the following:

Proposition 3.4 Let Q denote the standard quotient bundle over $\mathrm{Grass}(V,p)^{\dagger}$, then

$$\mathrm{Grass}^{(k)}(V,p) \simeq (N_{k,p}-p)\ Q \tag{3.4}$$

are diffeomorphic as smooth manifolds.

In particular, this proposition shows that the bundle

$$\pi_\ell^k\colon \mathrm{Grass}^{(k)}(V,p) \to \mathrm{Grass}^{(\ell)}(V,p)$$

has Euclidean fiber of dimension $N_{k,p} - N_{\ell,p}$. These bundles are not

---

† If U denotes the universal bundle over $\mathrm{Grass}(V,p)$ whose fiber over a p-plane Λ is just Λ itself, and I denotes the trivial bundle $V \times \mathrm{Grass}(V,p)$, then Q is given as the quotient bundle

$$0 \to U \to I \to Q \to 0$$

whose fiber over a p-plane Λ is the quotient space V/Λ. The notation jQ for j an integer just means $Q\oplus\ldots\oplus Q$ j times.

necessarily trivial. For instance

$$\mathrm{Grass}^{(2)}(2,1) \to \mathrm{Grass}(2,1) \simeq S^1$$

is the Möbius line bundle over S^1.

For computational purposes, some natural coordinate charts on $\mathrm{Grass}^{(k)}(V,p)$ similar to the standard coordinate charts on Grass(V,p) will be introduced. Introducing a basis, we may identify $V=\mathbb{R}^n$. Given a matrix $A \in \bigodot_*^k(\mathbb{R}^p,\mathbb{R}^n)_0$ let $A^j = A|\bigodot_j \mathbb{R}^p$ so that A has the block matrix form

$$A = (A^1|A^2|\ldots|A^k)$$

where each A^j is an $n \times \binom{p+j-1}{j}$ matrix. Let $\{A_\alpha^1\}$ denote the set of all minors of the matrix A^1, where for $\alpha=(\alpha_1,\ldots,\alpha_p)$ the minor A_α^1 is the $p \times p$ matrix consisting of rows $\alpha_1,\ldots,\alpha_p$ of A^1. Similarly, let A_α and A_α^j denote the matrices consisting of rows $\alpha_1,\ldots,\alpha_p$ of A and A^j. Let

$$\Pi: \bigodot_*^k(\mathbb{R}^p, \mathbb{R}^n)_0 \to \mathrm{Grass}^{(k)}(n,p)$$

be the projection as given in lemma 3.3, and let

$$U_\alpha = \Pi\{A: \det A_\alpha^1 \neq 0\}$$

which is an open subset of $\mathrm{Grass}^{(k)}(n,p)$. The U_α's cover $\mathrm{Grass}^{(k)}(n,p)$.

Given $A \in \bigodot_*^k(\mathbb{R}^p, \mathbb{R}^n)_0$ with A_α^1 nonsingular, there is a unique matrix $K \in GL^{(k)}(p)$ such that the matrix B=AK is of the form

$$B_\alpha = (\mathbb{1}_p|0|0|\ldots|0).$$

In fact, K is found by recursion as follows. Suppose

$$K = \varepsilon_k(K^1|K^2|\ldots|K^k),$$

then the Faa-di-Bruno formula shows that

$$
\begin{aligned}
K^1 &= (A^1_\alpha)^{-1} \\
K^2 &= -K^1 \cdot A^2_\alpha \cdot \tfrac{1}{2} K^1 \odot K^1 \\
K^3 &= -K^1 \cdot [A^2_\alpha \cdot K^1 \odot K^2 + A^3_\alpha \cdot \tfrac{1}{3!} K^1 \odot K^1 \odot K^1] \\
&\vdots \\
K^k &= -K^1 \cdot \sum_{j=2}^{k} A^j_\alpha \sum_{I \in \mathcal{A}^j_k} \frac{(K^1)^{i_1} \odot \dots \odot (K^k)^{i_k}}{I!}
\end{aligned}
\tag{3.5}
$$

(Note that no term on the right hand side of the last equation actually contains K^k.) This procedure gives a well-defined map

$$h_\alpha : U_\alpha \xrightarrow{\cong} \mathbb{R}^{N_{k,p}}$$

where $h_\alpha[\Pi A]$ is the matrix $\hat{B}_\alpha$ consisting of the rows of B not in B_α. Moreover, it is easy to see using (3.5) that the transition functions $h_\beta \circ h_\alpha^{-1}$ are smooth maps, so the U_α's do indeed form a coordinate atlas on $\mathrm{Grass}^{(k)}(n,p)$.

The preceding construction of local coordinates is just a special case of the trivialization of the prolonged Grassmann manifolds.

Lemma 3.5 Let $W \subset V$ be an n-p dimensional subspace. Then the trivialized prolonged Grassmannian with respect to W is given by

$$\mathrm{Grass}^{(k)}(V,p;W) = \{ \mathfrak{J}_k s|_0 : Ts|_0 \cap W = \{0\} \}$$

and is diffeomorphic to $\mathbb{R}^{N_{k,p}}$.

The trivialized Grassmannian $\mathrm{Grass}^{(k)}(V,p;W)$ is just the space of k-th order tangent spaces of sections transversal to W. For the above coordinate charts $U_\alpha = \mathrm{Grass}^{(k)}(V,p;W_\alpha)$ where W_α is the orthogonal complement to the subspace spanned by $\{e_{\alpha_1},\dots,e_{\alpha_p}\}$.

There is a natural action of the Lie group $GL^{(k)}(n)$ induced by the action of diffeomorphisms of $\mathbb{R}^n$ on sections. Namely, if $A \in GL^{(k)}(n)$ and $\Lambda \in \mathrm{Grass}^{(k)}(n,p)$, then given any p-section $s \in C^\infty(\mathbb{R}^n,p)|_0$ with $\mathfrak{J}_k s|_0 = \Lambda$ and a local diffeomorphism $G\colon \mathbb{R}^n \to \mathbb{R}^n$ with $G(0)=0$, $d^kG(0)=A$ then

$$A\cdot\Lambda = \mathfrak{J}_k[G(s)]|_0.$$

This just corresponds to left matrix multiplication

$$A\cdot\Lambda = \Pi(A\cdot B) = \Pi(\pi_k(A\cdot\varepsilon_k B)) \qquad B \in \bigodot_*^k(\mathbb{R}^p,\mathbb{R}^n)_0$$

where π_k is the projection inverse to the Faa-di-Bruno injection ε_k and B is any matrix such that $\Pi B = \Lambda$. (Note that the action of $GL^{(k)}(n)$ commutes with the action of $GL^{(k)}(p)$ on $\bigodot_*^k(\mathbb{R}^p,\mathbb{R}^n)$.) $GL^{(k)}(n)$ acts transitively on $\mathrm{Grass}^{(k)}(n,p)$.

Now suppose that Z is a smooth manifold and $E \to Z$ is a bundle with fiber $\bigodot_k^*\mathbb{R}^n$ and group $GL^{(k)}(n)$. (See [ST] for the details on fiber bundles.) By the general theory of fiber bundles, for $0<p<n$ there exists a unique bundle $\mathrm{Grass}^{(k)}(E,p) \to Z$ having fiber $\mathrm{Grass}^{(k)}(E|_z,p)$ over $z \in Z$ such that if $E_0 \subset E$ is any smooth (local) subbundle of prolonged p-planes, the E_0 defines a smooth (local) section of $\mathrm{Grass}^{(k)}(E,p)$. If the transition functions of E are given by $A_{\alpha\beta} \in GL^{(k)}(n)$, then the transition functions for $\mathrm{Grass}^{(k)}(E,p)$ are given by the images of the $A_{\alpha\beta}$ in $PGL^{(k)}(n)$, the k-th prolongation of the projective linear group, which is obtained as the quotient group of $GL^{(k)}(n)$ by its center - the group consisting of all matrices

$\lambda \mathbb{1}$ for $\lambda \in \mathbb{R}$, where $\mathbb{1}$ is the identity map of $\bigodot_k^* \mathbb{R}^n$. Actually, to do the preceding construction, we need the following lemma.

Lemma 3.6 The action of $PGL^{(k)}(n)$ on $Grass^{(k)}(n,p)$ for $0<p<n$ is effective; i.e. if $A \in GL^{(k)}(n)$ is such that $A \cdot \Lambda = \Lambda$ for all prolonged p-planes $\Lambda \in Grass^{(k)}(n,p)$, then $A = \lambda \mathbb{1}$ for some $\lambda \in \mathbb{R}$.

The proof is a direct consequence of the corresponding statement for ordinary Grassmannians and the following elementary lemma from symmetric algebra.

Lemma 3.7 If V is a real finite dimensional vector space, then

$$\{v \odot v \odot \ldots \odot v \in \bigodot_k V : \; v \in V\}$$

spans $\bigodot_k V$.

As a corollary of these more abstract considerations, we obtain an alternate characterization of the extended jet bundles of a smooth manifold as appropriate prolonged Grassmann bundles. This is perhaps the most convenient characterization of the bundle structure of the extended jet bundles.

Proposition 3.8 There is an identification

$$J_k^*(Z,p) \simeq Grass^{(k)}(\mathcal{J}_k Z,p)$$

giving $J_k^*(Z,p)$ the structure of a fiber bundle over Z with fiber $Grass^{(k)}(n,p)$ where $n = \dim Z$ and group $PGL^{(k)}(n)$, such that if $s \subset Z$ is any smooth p dimensional submanifold, then $j_k^* s \subset J_k^*(Z,p)$ is also a smooth p dimensional submanifold.

Proposition 3.9 Let $k>\ell$ be positive integers. Then

$$\pi^k_\ell \colon J^*_k(Z,p) \to J^*_\ell(Z,p)$$

is a fiber bundle with Euclidean fiber of dimension $N_{k,p} - N_{\ell,p}$ and group $PGL^{(k)}(n)/PGL^{(\ell)}(n)$.

Suppose that $\mathfrak{U}$ is an involutive n-p dimensional differential system on the n dimensional manifold Z. The trivialized extended k-jet bundle of Z with respect to is the open subbundle

$$J^*_k(Z,p;\mathfrak{U}) = \mathrm{Grass}^{(k)}(\mathfrak{J}_k Z,p;\mathfrak{U})$$

consisting of the k-th order tangent spaces of sections transverse to $\mathfrak{U}$. By lemma 3.5, $J^*_k(Z,p;\mathfrak{U})$ is a bundle with Euclidean fiber of dimension $N_{k,p}$. We now propose to introduce "canonical" coordinates on $J^*_k(Z,p;\mathfrak{U})$ associated with a coordinate system on Z that is flat with respect to $\mathfrak{U}$.

Let $\chi\colon Z \to \mathbb{R}^p \times \mathbb{R}^q$ be a local coordinate system on Z with the coordinates on $\mathbb{R}^p \times \mathbb{R}^q$ denoted by $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ so that the differential system is spanned by $\{\partial/\partial u^1,\ldots,\partial/\partial u^q\}$. The case to keep in mind is when $Z \to X$ is a vector bundle and $\mathfrak{U}$ is the differential system given by the tangent spaces to the fibers, so that $(x^1,\ldots,x^p)$ are the coordinates on the base manifold X (independent variables) and $(u^1,\ldots,u^q)$ are the fiber coordinates (dependent variables). In this case we can identify the trivialized jet bundle with the ordinary jet bundle corresponding to the vector bundle Z, since both are constructed by consideration of p-sections transverse to the fibers.

Proposition 3.10 Let $Z \to X$ be a fiber bundle over a p dimensional manifold X and let $\mathcal{U}$ denote the involutive differential system of tangent spaces to the fibers of Z, then

$$J_k^*(Z,p;\mathcal{U}) \simeq J_kZ.$$

This shows that the extended jet bundle can be regarded as the "completion" of the ordinary jet bundle in the same manner that projective space is the completion of affine space. In this case the completion is obtained by allowing sections with vertical tangents.

Let s be a p-section of Z transverse to the differential system $\mathcal{U}$. Recall that the normal parametrization of s relative to the coordinate system χ is that map $\hat{f}\colon \mathbb{R}^p \to Z$ with im f = s and $\chi \circ \hat{f} = \mathbb{1}_p x f$ for some (uniquely determined) smooth $f\colon \mathbb{R}^p \to \mathbb{R}^q$. Then

$$\partial_*^k(\chi \circ f)(x) = \begin{pmatrix} \mathbb{1}_p & 0 & \dots & 0 \\ \partial f(x) & \partial^2 f(x) & \dots & \partial^k f(x) \end{pmatrix}$$

so that $\partial_*^k f(x)$ can be regarded as the local coordinates of the extended k-jet of s at the point f(x).

Proposition 3.12 Let $\chi\colon Z \to \mathbb{R}^p x \mathbb{R}^q$ be a local coordinate system on the smooth manifold Z and let $\mathcal{U}$ denote the local differential system $(d\chi)^{-1}T\mathbb{R}^q$, then there is an induced local coordinate system

$$\chi^{(k)}\colon J_k^*(Z,p;\mathcal{U}) \to \mathbb{R}^p \times \mathbb{R}^q \times \bigodot_*^k(\mathbb{R}^p, \mathbb{R}^q)$$

such that if $\hat{f}\colon \mathbb{R}^p \to Z$ is the normal parametrization of a p-section of Z with $\chi \circ f = \mathbb{1}_p x f$, then

$$\chi^{(k)} \circ j_k^* f = \mathbb{1}_p x f x \partial_*^k f.$$

If (x,u) are the local coordinates on Z, then the local coordinates on $J_k^*(Z,p)$ will usually be denoted by $(x,u,u^{(k)})$. Here $u^{(k)}$ is a matrix with entries u_K^i for $1\le i\le q$ and $K \in S^p$, $1\le \Sigma K\le k$, so that if $u^i=f^i(x)$, then $u_K^i=\partial_K f^i(x)$. Note that if $\hat{g}: \mathbb{R}^p \to Z$ is any parametrization of a p-section of Z transversal to $\mathfrak{U}$ so that $\chi\circ\hat{g} = g_1 \times g_2$, then g_1 is locally invertible, so the normalized representative of im $\hat{g}$ is given by $\chi^{-1}\circ(\mathbb{1}_p \times g_2\circ g_1^{-1})$ and

$$\chi^{(k)}\circ j_k^* g = \mathbb{1}_p \times g_2\circ g_1^{-1} \times \partial_*^k(g_2\circ g_1^{-1}).$$

II.4 Total Derivatives

Consider a function $F(x,u)$ where the x's are viewed as independent and the u's as dependent variables. Then for given $u = f(x)$, the various partial derivatives of $F(x,f(x))$ can be evaluated and re-expressed in terms of the derivatives of u with respect to x. This heuristic idea is called the total derivative, and will be discussed with precision in this section.

Definition 4.1 Let X be a fixed real normed vector space. Let U and W be normed vector spaces and let $Z = X \times U$. Suppose $F:Z \to W$ is a smooth function, then the k-th order total differential of F is the unique map

$$D^kF : Z \times \odot_*^k(X,U) \to \odot^k(X,W)$$

such that for any smooth $F:X\to U$,

$$D^kF(x,f(x),\ \partial_*^k f(x)) = \partial^k[F\circ(\mathbb{I}_X \times f)](x)$$

Existence and uniqueness of the total differential follows from the Faa-di-Bruno formula. In fact

$$D^kF(x,f(x),\partial_*^k f(x)) = \sum_{\ell=1}^{k} \partial^k F(x,f(x))\partial_\ell^k(\mathbb{I}\times f)(x)$$

where $\partial_\ell^k(\mathbb{I}\times f)$ are the matrix blocks in $d^k(\mathbb{I}\times f)$ and are thus expressed in terms of $\partial_*^k f(x)$.

As in the case of the usual differentials, let

$$D_*^kF = D^1F + D^2F + \ldots + D^kF: Z\times \odot_*^k(X,U) \to \odot_*^k(X,W)$$

Lemma 4.2 Let k and ℓ be positive integers, then

$$D^\ell(D^kF)|Z\times\odot_*^{k+\ell}(X,U) = D^{k+\ell}F$$

where $Z\times\odot_*^{k+\ell}(X,U)$ is naturally a subspace of $Z\times\odot_*^k(X,U)\times\odot_*^\ell(X,U\times\odot_*^k(X,U))$.

Proof

Using the fact that

$$\bigodot_*^{\ell}(X,U\times V) \simeq \bigodot_*^{\ell}(X,U)\times \bigodot_*^{\ell}(X,V)$$

and that $\bigodot_*^{k+\ell}(X,U)$ is already a subspace of $\bigodot_*^{\ell}(X, \bigodot_*^{k}(X,U))$,[†] the embedding described in the theorem is given by the map

$$(z,\partial_*^{k+\ell} f) \to (z,\partial_*^{k}f,\partial_*^{\ell}f, \partial_*^{k+\ell} f)$$

for any smooth function $f: X \to U$. Given such an f and using the fact that

$$\partial^{\ell}(\partial^{k}f)(x) = \partial^{k+\ell}f(x)$$

we have

$$\begin{aligned} D^{\ell}(D^{k}F)(x,f(x), \partial_*^{k+\ell} f(x)) &= \partial^{\ell}[D^{k}F\circ(\mathbb{1}\times f\times\partial_*^{\ell}f)](x) \\ &= \partial^{\ell}[\partial^{k}[F\circ(\mathbb{1}\times f)](x) \\ &= \partial^{k+\ell}[F\circ(\mathbb{1}\times f)](x) \\ &= D^{k+\ell}F(x,f(x),\partial_*^{k+\ell}f(x)) \end{aligned}$$

Q.E.D.

In the case that X and U are finite dimensional, with respective bases $\{e_1,\ldots,e_p\}$ and $\{\hat{e}_1,\ldots,\hat{e}_q\}$, the matrix elements of $D^kF(z,u^{(k)})$ where $z \in Z$, $u^{(k)} \in \bigodot_*^{k}(X,U)$, are given by

$$D_IF(z,u^{(k)}) = \langle e_I,D^kF(z,u^{(k)})\rangle \ , \quad I \in S_k^n \ .$$

Given an element $u^{(k)} \in \bigodot_*^{k}(X,U)$ we will let $u^{(\ell)} \in \bigodot_*^{\ell}(X,U)$ denote its restriction to $\bigodot_{\ell}^{*}X$. The coordinates of $u^{(k)}$ are given by

$$\sum_{\ell=1}^{q} u_J^{\ell}\, \hat{e}_{\ell} = \langle e_J, u^{(k)}\rangle \quad \text{for} \quad k \geq \textstyle\sum J \ .$$

[†] see section II.2.

Lemma 4.3 Let $F\colon X \times U \to W$ be smooth. Then the matrix entries of $D^{k+1}F$ are given recursively by

$$D_{I+\delta^j}F(x,u,u^{(k+1)}) = D_j[D_I F(x,u,u^{(k)})]$$

for $I \in S_k$, $1 \le j \le p$, $x \in X$, $u \in U$, $u^{(k+1)} \in \bigodot_*^{k+1}(X,U)$, where D_j is the total derivative operator in the e_j direction which is given by

$$D_j = \frac{\partial}{\partial x^j} + \sum_{\ell=1}^{q} \{u_j \frac{\partial}{\partial u^\ell} + \sum_{0<I \in S^n} u^\ell_{I+\delta^j} \frac{\partial}{\partial u^\ell_I}\} .$$

Note that for any fixed k only finitely many terms in the expression for D_j are necessary.

Proof

This will be demonstrated by induction on k. Let $f\colon X \to U$ be any smooth function. In case k=0 we have

$$\begin{aligned} DF(x,f(x),\partial f(x)) &= \partial(F\circ(\mathbb{1}\times f))(x) \\ &= \partial F(x,f(x)) \circ \partial(\mathbb{1}\times f)(x) \end{aligned}$$

hence

$$\begin{aligned} D_jF(x,f(x),\partial f(x)) &= \langle e_j, \partial F(x,f(x)) \circ \partial(\mathbb{1}\times f)(x)\rangle \\ &= \sum_{\ell=1}^{p+q} \partial_\ell F(x,f(x))\cdot\partial_j(\mathbb{1}\times f)^\ell(x) \\ &= \partial_j F(x,f(x)) + \sum_{\ell=1}^{q} \partial_j f^\ell(x)\cdot\partial_{\ell+p}F(x,f(x)) \end{aligned}$$

proving the lemma for k=0. Next, for k>0, by the previous lemma

$$DD^kF|Z \times \bigodot_*^{k+1}(X,U) = D^{k+1}F.$$

Therefore, given $u^{(k+1)}$, and using the k=0 case, it suffices to verify that the coordinates

$$\sum_\ell u^\ell_{I+\delta^j}\,\hat{e}_\ell = \langle e_{I+\delta^j}, u^{(k+1)}\rangle$$

for $u^{(k+1)}$ are the same as the coordinates

$$\sum_{\ell} (u_I^{\ell})_j \ \hat{e}_{\ell} = \langle e_I, \langle e_j, u^{(k+1)} \rangle\rangle .$$

But this is a direct consequence of the way the embedding

$$\odot^{k+1}(X,U) \subset \odot^k(X, \odot^1(X,U))$$

was constructed in section II.2. Q.E.D.

Example 4.4 Consider the case dim X = 1 = dim U. Then

$$\begin{aligned}
DF(x,u,u_x) &= F_x(x,u) + u_x F_u(x,u) \\
D^2F(x,u,u_x,u_{xx}) &= D(DF)(x,u,u_x,u_{xx}) \\
&= F_{xx} + 2u_x F_{xu} + u_x^2 F_{uu} + u_{xx} F_u \\
D^3F(x,u,u_x,u_{xx},u_{xxx}) &= D(D^2F)(x,u,u_x,u_{xx},u_{xxx}) \\
&= F_{xxx} + 3u_x F_{xxu} + 3u_x^2 F_{xuu} + u_x^3 F_{uuu} + \\
&\quad + 3u_{xx} F_{xu} + 3u_x u_{xx} F_{uu} + u_{xxx} F_u .
\end{aligned}$$

It is easily checked that these are the correct expressions; i.e. if $f\colon X \to U$ is smooth, then

$$\frac{d^i}{dx^i} F(x,f(x)) = D^i F(x,f(x),\ f'(x), f''(x), \ldots).$$

II.5 Differential Operators and Equations

The next step in our theory of extended jet bundles over smooth manifolds is to describe how the notions of a differential operator and equation become reformulated in this context. These concepts will be direct generalizations of the corresponding objects for vector bundles and will include as special cases what are classically meant by systems of partial differential equations. First of all, a differential operator on an extended jet bundle will be defined and some important properties described. Next, we proceed to a discussion of differential equations, which will be connected with the previously mentioned differential operators. Recall that a differential operator in the category of vector bundles is given by a smooth fiber preserving map from a jet bundle to another vector bundle, cf. [P2; chapter 4]. In strict analogy we make the definition of a differential operator on an arbitrary smooth manifold.

Definition 5.1 Let Z be a smooth manifold of dimension p+q. A k-th order differential operator (for p-sections) is a fiber bundle morphism (i.e. a fiber preserving map)

$$\Delta\colon J_k^*(Z,p) \to F,$$

where $\rho\colon F \to Z$ is a fiber bundle over Z, such that $\rho\circ\Delta$ is the projection $\pi_0^k\colon J_k^*(Z,p) \to Z$.

Proposition 5.2 Let $\Delta\colon J_k^*(Z,p) \to F$ be a k-th order differential operator and let ℓ be a nonnegative integer. Then there exists (k+ℓ)-th order differential operator

$$pr^{(\ell)}\Delta\colon J^*_{k+\ell}(Z,p) \to J^*_\ell(F,p)$$

called the <u>ℓ-th</u> <u>prolongation</u> of Δ, such that for any p-section parametrized by $f\colon \mathbb{R}^p \to Z$ the following diagram commutes:

$$\begin{tikzcd}
 & J^*_{k+\ell}(Z,p) \arrow[r, "pr^{(\ell)}\Delta"] \arrow[d, "\pi^{k+\ell}_k"] & J^*_\ell(F,p) \arrow[d, "\rho^\ell"] \\
\mathbb{R}^p \arrow[ur, "j^*_{k+\ell}f"] \arrow[r, "j^*_k f"] \arrow[dr, "f"'] & J^*_k(Z,p) \arrow[r, "\Delta"] \arrow[d, "\pi^k_0"] & F \arrow[dl, "\rho"] \\
 & Z &
\end{tikzcd}$$

In terms of local coordinates (x,u) on Z and (x,u,w) on F,

$$pr^{(\ell)}\Delta(x,u,u^{(k+\ell)}) = (x,u,\Delta(x,u,u^{(k)}),D^\ell_*\Delta(x,u,u^{(k+\ell)}))$$

for $(x,u) \in Z$, $(x,u,u^{(k+\ell)}) \in J^*_{k+\ell}(Z,p)$ with $(x,u,u^{(k)}) = \pi^{k+\ell}_k(x,u,u^{(k+\ell)})$.

<u>Proof</u>

Given $z \in Z$ and $j \in J^*_{k+\ell}(Z,p)|_z$, let $f\colon \mathbb{R}^p \to Z$ represent j so that $f(0) = z$ and $j^*_{k+\ell}f|_z = j$. Note that $\Delta\circ j^*_k f\colon \mathbb{R}^p \to F$ is a p-section of F since $\rho\circ\Delta\circ j^*_k f = \pi^k_0\circ j^*_k f = f$, hence it makes sense to define

$$pr^{(\ell)}\Delta(j) = j^*_\ell[\Delta\circ j^*_k f]|_{\Delta(j')} \qquad j' = \pi^{k+\ell}_k(j).$$

Now given local coordinates (x,u) on Z that are compatible with a local q-dimensional differential system $\mathfrak{U}$, and corresponding local coordinates (x,u,w) on F compatible with the q+r dimensional differential system $\mathfrak{U}\times\mathfrak{F}$, where $\mathfrak{F}$ denotes the tangents to the fibers of F and r is the dimension of these fibers, there are induced identifications

$$J^*_k(Z,p;\mathcal{U}) \approx \mathbb{R}^p \times \mathbb{R}^q \times \mathcal{O}^k_*(\mathbb{R}^p,\mathbb{R}^q)$$

$$J^*_\ell(F,p;\mathcal{U}\times\mathcal{F}) \approx \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r \times \mathcal{O}^\ell_*(\mathbb{R}^p,\mathbb{R}^q\times\mathbb{R}^r)$$

In terms of these induced coordinates

$$\begin{aligned} j^*_\ell(\Delta\circ j^*_k f)|_Z &= \partial^\ell_*[\Delta\circ(\mathbb{1}_p \times f\times\partial^k_* f)](x) \\ &= D^\ell_*\,[x,f(x),\partial^{k+\ell}_* f(x)] \end{aligned}$$

for $f\colon \mathbb{R}^p \to Z$ any normalized representative of $j = (x,u,u^{(k+\ell)})$. (Note that $\Delta\circ j^*_k f$ is therefore a normal parametrization of the corresponding p-section of f with respect to $\mathcal{U}\times\mathcal{F}$.) The remainder of the proposition follows directly from this formula. Q.E.D.

<u>Corollary 5.3</u> Letting

$$\Delta = \mathbb{1} : J^*_k(Z,p) \to J^*_k(Z,p)$$

be the identity map, then for nonnegative integers ℓ there is a natural embedding

$$i^{k+\ell}_k = pr^{(\ell)}\mathbb{1} : J^*_{k+\ell}(Z,p) \hookrightarrow J^*_\ell(J^*_k(Z,p),p).$$

In terms of local coordinates (x,u) on Z and the induced coordinates $(x,u,u^{(k+\ell)})$ on $J^*_{k+\ell}(Z,p)$ and $(x,u,u^{(k)},u^{(\ell)},(u^{(k)})^{(\ell)})$ on $J^*_\ell(J^*_k(Z,p),p)$, $J^*_{k+\ell}(Z,p)$ is the subbundle of $J^*_\ell(J^*_k(Z,p),p)$ given by

$$\{\ (u^i_I)_J = (u^i_{I'})_{J'} : I+J = I'+J',\ I,J,I',J' \in S^p,\ 0\le\Sigma I,\Sigma I'\le k,$$
$$0\le\Sigma J,\Sigma J'\le\ell,\ i=1,\ldots,q\ \}$$

where u^i_0 denotes the coordinate u^i and $(u^i_I)_0$ denotes the coordinate u^i_I.

Corollary 5.4 If Δ, k, ℓ are as in the proposition and ℓ' is another nonnegative integer, then

$$\mathrm{pr}^{(\ell')}\,\mathrm{pr}^{(\ell)}\,\Delta = i_{\ell}^{\ell+\ell'} \circ \mathrm{pr}^{(\ell+\ell')}\,\Delta.$$

Corollary 5.5 If $\Phi\colon Z \to Z'$ is a smooth diffeomorphism, then there is a unique smooth diffeomorphism

$$\mathrm{pr}^{(k)}\Phi\colon J_k^*(Z,p) \to J_k^*(Z',p)$$

called the k-th prolongation of Φ such that for any parametrization $f\colon \mathbb{R}^p \to Z$ of a p-section of Z, the following diagram commutes:

$$\begin{tikzcd}
 & J_k^*(Z,p) \arrow[r, "\mathrm{pr}^{(k)}\Phi"] \arrow[d] & J_k^*(Z',p) \arrow[d] \\
\mathbb{R}^p \arrow[ur, "j_k^* f"] \arrow[r, "f"'] & Z \arrow[r, "\Phi"] & Z'
\end{tikzcd}$$

Moreover

$$\mathrm{pr}^{(k')}\,\mathrm{pr}^{(k)}\,\Phi = i_k^{k+k'} \circ \mathrm{pr}^{(k+k')}\,\Phi.$$

The last corollary will be especially important when the prolongation of transformation group actions to extended jet bundles is discussed in chapter III.

We now proceed to describe the concept of a system of partial differential equations for an extended jet bundle. Recall that in the category of vector bundles a differential equation is usually given as $\Delta^{-1}\{0\}$ where Δ is a differential operator and 0 denotes the zero cross-section of the image bundle. In the case of fiber bundles, there is no such intrinsically defined cross-section. (Indeed, the bundle may have no smooth global sections.) Even if we restricted our attention to bundles with a distinguished cross-section, there would be problems in relating the prolonged differential equation to the prolonged differential operator. In view of these observations, the following definition can be seen to give an appropriate generalization of the notion of a system of partial differential equations.

<u>Definition 5.6</u> Let $\Delta\colon J_k^*(Z,p) \to F$ be a smooth differential operator and let $F_0 \subset F$ be a subbundle over Z. Then the differential equation corresponding to F_0 is the subvariety $\Delta^{-1}\{F_0\} \subset J_k^*(Z,p)$.

Note that if $\Delta_0 = \Delta^{-1}\{0\} \subset J_k^*(Z,p)$ is a subvariety of $J_k^*(Z,p)$ given by the zero set of some smooth map $\Delta\colon J_k^*(Z,p) \to \mathbb{R}^\alpha$, then Δ_0 is the differential equation corresponding to the zero cross-section of the trivial bundle $Z\times\mathbb{R}^\alpha$ under the differential operator $\pi_0^k \times \Delta$. Therefore any closed subvariety of $J_k^*(Z,p)$ can be viewed as the differential equation corresponding to some smooth differential operator. Suppose

$$\Delta^i(x,u,u^{(k)}) = 0 \qquad\qquad i = 1,\ldots,\alpha$$

is a system of partial differential equations in some coordinate system on Z, then the closure of the subset of $J_k^*(Z,p)$ given by these equations (which are only defined on an appropriate trivialized jet bundle) will be the differential equation corresponding to this system. Thus all classical systems of partial differential equations are included in our definition, with the added feature that solutions with "vertical tangents" are allowed, provided these tangents are in some sense the limits of "tangents" that satisfy the system of equations. In general, the interesting object will be the subvariety in the extended jet bundle and not the particular differential operator used to define it. Therefore a k-th order system of partial differential equations over a smooth manifold Z will be taken to mean an arbitrary closed subset $\Delta_0 \subset J_k^*(Z,p)$. A solution to a system of equations is a p-section $s \subset Z$ satisfying $j_k^* s \subset \Delta_0$.

Given a subset $S \subset Z$, let $J_k^*(S,p) \subset J_k^*(Z,p)$ denote the subset of all extended k-jets of p-sections of Z wholly contained in S. Note that this set will be empty if S contains no p-dimensional submanifolds.

<u>Definition 5.7</u> Let $\Delta_0 \subset J_k^*(Z,p)$ be a k-th order system of partial differential equations. Then the <u>ℓ-th prolongation</u> of Δ_0 is the $(k+\ell)$-th order differential equation

$$\mathrm{pr}^{(\ell)}\Delta_0 = J_\ell^*(\Delta_0,p) \cap J_{k+\ell}^*(Z,p)$$

where $J_{k+\ell}^*(Z,p) \subset J_\ell^*(J_k^*(Z,p),p)$ via the injection given in corollary 5.2.

The next proposition shows that $\mathrm{pr}^{(\ell)}\Delta_0$ is indeed a differential equation and corresponds to the prolongation of the differential operator defining Δ_0, providing that this operator is in some sense "irreducible".

<u>Proposition 5.8</u> Let $\Delta\colon J_k^*(Z,p) \to F$ be a smooth differential operator and let $F_0 \subset F$ a subbundle such that Δ is transversal to F_0. Let $\Delta_0 = \Delta^{-1}\{F_0\}$ be the differential equation corresponding to F_0. Then

$$\mathrm{pr}^{(\ell)}\,\Delta_0 = (\mathrm{pr}^{(\ell)}\,\Delta)^{-1}[J_\ell^*(F_0,p)].$$

Recall that a smooth map $f: M \to N$ is transversal to a submanifold $N_0 \subset N$ if for any $y \in f(M) \cap N_0$ with $f(x)=y$, $TN|_y = TN_0|_y + df[TM|_x]$. The transversality of Δ to F_0 is necessary for the proposition to hold. For instance, if the differential equation $u_x = 0$ is defined by the operator $\Delta = u_x^2$, then $pr^{(1)}\Delta = (u_x^2, 2u_x u_{xx})$, so the equation in the second jet bundle given by $pr^{(1)}\Delta$ is just $u_x=0$, which is not the prolonged equation -- $u_x = 0 = u_{xx}$. It can be seen that for polynomial operators, transversality corresponds to irreducibility.

Given a fiber bundle F over Z, a p-section of F will be called vertical at a point if its tangent space at that point has non-zero intersection with the tangent space to the fiber of F. Define $V_\ell^k \subset J_\ell^*(J_k^*(Z,p),p)$ to be the subset given by all ℓ-jets of vertical sections of $J_k^*(Z,p)$.

<u>Lemma 5.9</u> For each k, ℓ,

$$J_{k+\ell}^*(Z,p) \cap V_\ell^k = \emptyset$$

<u>Lemma 5.10</u> If s is a smooth p-section of $J_k^*(Z,p)$ such that $j_\ell^* s|_j \in J_{k+\ell}^*(Z,p)$ for some $j \in J_k^*(Z,p)$, then for any smooth differential operator $\Delta: J_k^*(Z,p) \to F$

$$pr^{(\ell)}\,\Delta(j_\ell^* s|_j) = j_\ell^*(\Delta \circ s)|_{\Delta(j)}.$$

<u>Lemma 5.11</u> Suppose $F: Z \to Z'$ is a smooth map between manifolds and $Z_0' \subset Z'$ a submanifold transversal to F. Given $z \in Z$ with $F(z) = z' \in Z_0'$ and a p-section $s \in C^\infty(Z,p)|_z$ such that

$$d^{\ell}F[\mathfrak{J}_{\ell}s|_z] \subset \mathfrak{J}_{\ell}Z_0'|_{z'}$$

then there exists another p-section $\hat{s} \in C^{\infty}(Z,p)|_z$ with $\mathfrak{J}_{\ell}\hat{s}|_z = \mathfrak{J}_{\ell}s|_z$ and $F(\hat{s}) \subset Z_0'$.

Proof

Choose local coordinates (x,y,t) around z such that $s = \{y=0,\ t=0\}$ and $dF(\partial/\partial y^i)$ form a basis for $TZ'|_{z'} \big/ TZ_0'|_{z'}$. Choose local coordinates (ξ,η) around z' such that $Z_0' = \{\eta=0\}$. Let

$$F(x,y,t) = (F_1(x,y,t), F_2(x,y,t))$$

in these coordinates and using the implicit function theorem let $y = y(x)$ be the smooth solution to the equation $F_2(x,y(x),0) = 0$ near $z=(0,0,0)$. Then $\hat{s} = \{(x,y(x),0)\}$ satisfies the criteria of the lemma. Indeed, differentiating the equation that implicitly defines $y(x)$ gives

$$\frac{\partial^{\Sigma K}}{\partial x^K} F_2(0,0,0) + \frac{\partial^{\Sigma K}}{\partial x^K} y^j(x)\cdot\frac{\partial}{\partial y^j} F_2(0,0,0) + A_K = 0$$

where A_K is a sum of terms involving derivatives of the $y^j(x)$ of orders $<\Sigma K$, hence by induction all derivatives of $y(x)$ up to and including k-th order vanish. Q.E.D.

Proof of 5.8

Let $\Delta_0^{(\ell)} = [pr^{(\ell)}\Delta]^{-1}\{J_{\ell}^*(F_0,p)\}$. To show that $\Delta_0^{(\ell)} \supset pr^{(\ell)}\Delta_0$ let s be a p-section contained in Δ_0 with $j_{\ell}^*s \cap V_{\ell}^k = \emptyset$. This means that (locally) $\pi_0^k[s]$ is a smooth p-section of Z, hence $\Delta[s]$ is a smooth p-section of F_0 since $\rho\circ\Delta[s] = \pi_0^k[s]$. Therefore $j_{\ell}^*\ \Delta[s] \subset J_{\ell}^*(F_0,p)$. In particular, if $j_{\ell}^*s|_{j_0} \in J_{k+\ell}^*(Z,p)$, then by lemma 5.10

$$pr^{(\ell)}\ \Delta(j_\ell^* s|_{j_0}) = j_\ell^*(\Delta[s])|_{\Delta(j_0)} \in J_\ell^*(F_0, p).$$

Now lemma 5.9 implies $\Delta_0^{(\ell)} \supset pr^{(\ell)}\ \Delta_0$.

Conversely, suppose $j \in \Delta_0^{(\ell)}$ and let $\pi_k^{k+\ell}(j) = j_0$ and $\pi_0^{k+\ell}(j) = z$. Let $s \in C^\infty(Z,p)|_z$ represent j so $pr^{(\ell)}\Delta(j) = j_\ell^*(\Delta[j_k^* s])|_{\Delta(j_0)}$, hence $d^\ell\Delta[\mathfrak{J}_\ell(j_k^* s)|_{j_0}] \subset \mathfrak{J}_\ell F_0$. By lemma 5.11 there exists $\hat{s} \in C^\infty(J_k^*(Z,p),p)|_{j_0}$ with $\mathfrak{J}_\ell \hat{s}|_{j_0} = \mathfrak{J}_\ell s|_{j_0}$ and $\hat{s} \subset \Delta_0$. In addition $j_\ell^* \hat{s}|_{j_0} = j$, hence $j \in pr^{(\ell)}\Delta_0$. Q.E.D.

III. Symmetry Groups of Differential Equations

Now that the exposition of the appropriate mathematical machinery has been completed in chapter II, it is time to begin the discussion of symmetry groups of higher order systems of partial differential equations. The first item to be dealt with is the derivation of the local coordinate formula for the k-th order prolongation of a vector field. This is done in the first section of this chapter, where we also prove that the prolongation operation preserves the Lie algebra structure on the space of smooth vector fields on a manifold. The prolongation formula appears to be new, although Eisenhart [E2] gives a recursive relation for calculating the prolongations, that is derived here as a corollary of the general formula.

The remaining three sections of this chapter constitute different applications of the prolongation formula. Section 2 gives concrete derivations of the symmetry groups of some interesting partial differential equations: the heat equation, Laplace's equation, Burgers' equation and the Korteweg-deVries equation. Section 3 is a discussion of the symmetry groups of linear partial differential equations. The main result here is that for an equation of order ≥ 3 there are no nonprojectable symmetries. The projectable symmetries of a linear partial differential equation form a subgroup of the conformal group of the top order symbol of the equation. Section 4 considers the problem of when the symmetry groups of a higher order equation and its equivalent first order system are the same. It is shown that the group of the first order system is a prolongation of the group of the higher order equation, except possibly for the presence of "higher order symmetries" which depend on the derivatives as well as the function values. An example of an equation which possesses higher order symmetries, the wave equation, will be discussed in detail.

III.1 The Prolongation Formula

We are finally in a position to derive the fundamental prolongation formula for the infinitesimal generators for a local group of transformations on a manifold Z. Since the action of each generator will be uniquely determined by its corresponding one-parameter subgroup, it suffices to consider the case where G is a local one parameter group of transformations with generating vector field $\xi: Z \to T(Z)$, so that the local transformations of G are given by $\exp(t\xi)$.

Given any local p-section of Z, $f: \mathbb{R}^p \to Z$, for t sufficiently small $\exp(t\xi)\circ f$ will also be a local p-section hence there is a local prolongation of $\exp(t\xi)$ to $J_k^*(Z,p)$, given by corollary II.5.4. The derivative of $pr^{(k)}[\exp t \xi]$ with respect to t evaluated at $t=0$ will then give the prolongation of ξ on $J_k^*(Z,p)$. More precisely:

Definition 1.1 Given a vector field ξ on Z, the k-th prolongation of ξ is the vector field on $J_k^*(Z,p)$ given by

$$pr^{(k)}\xi|_j = \frac{d}{dt}\Big|_{t=0} pr^{(k)}[\exp t\xi](j) \quad \text{for } j \in J_k^*(Z,p).$$

It is readily checked that $pr^{(k)}\xi$ is a smooth vector field on $J_k^*(Z,p)$ since $pr^{(k)}[\exp t \xi]$ is a family of smooth maps depending smoothly on the parameter t.

Lemma 1.2 Let ξ and η be the vector fields on Z. Given $0 \le k \in \mathbb{Z}$ and constants a,b, then

$$pr^{(k)}(a\xi + b\eta) = a\, pr^{(k)}\xi + b\, pr^{(k)}\eta$$

$$pr^{(k)}[\xi,\eta] = [pr^{(k)}\xi, pr^{(k)}\eta] .$$

Proof

By use of the projection

$$\Pi\colon J_k(\mathbb{R}^p, Z) \to J_k^*(Z,p)$$

given by lemma II.3.3, it suffices to show that

$$\frac{d}{dt}\Big|_{t=0} j_k(\exp[t(a\xi + b\eta)]\circ f) = a\frac{d}{dt}\Big|_{t=0} j_k(\exp(t\xi)\circ f) + b\frac{d}{dt}\Big|_{t=0} j_k(\exp(t\eta)\circ f)$$

for any smooth $f\colon \mathbb{R}^p \to Z$ to prove the first formula. This, however is a straightforward consequence of the composition rule $d^k(g\circ f) = d^k g\circ d^k f$ and the local coordinate expression for $j_k f$. To prove the second equation, recall that the Lie bracket is given by

$$[\xi,\eta]|_z = \lim_{t\to 0} \frac{d[\exp(t\xi)]\eta|_{z'} - \eta|_z}{t}$$

for $z \in Z$ with $z' = \exp(t\xi)z$. Therefore given $j \in J_k^*(Z,p)$

$$[\mathrm{pr}^{(k)}\xi, \mathrm{pr}^{(k)}\eta]|_j = \lim_{t\to 0} \frac{d[\exp(t\,\mathrm{pr}^{(k)}\xi)]\mathrm{pr}^{(k)}\eta|_{j'} - \mathrm{pr}^{(k)}\eta|_j}{t}$$

and it suffices to show that

$$d[\exp(t\,\mathrm{pr}^{(k)}\xi)]\mathrm{pr}^{(k)}\eta = \mathrm{pr}^{(k)}[d(\exp t\xi)\eta] \qquad (*)$$

since the formula follows from the smooth dependence of the prolongations on parameters and the first formula in the lemma:

$$[\mathrm{pr}^{(k)}\xi, \mathrm{pr}^{(k)}\eta]|_j = \lim_{t\to 0} \frac{\mathrm{pr}^{(k)}[d(\exp t\xi)\eta]|_{j'} - \mathrm{pr}^{(k)}\eta|_j}{t}$$

$$= \mathrm{pr}^{(k)} \lim_{t\to 0} \frac{d(\exp t\xi)\eta - \eta}{t}\Big|_j$$

$$= \mathrm{pr}^{(k)}[\xi,\eta]|_j .$$

To verify (*), choose local coordinates $(z^1,\dots,z^{p+q})$ on Z such that $\xi = \partial/\partial z^1$ in these coordinates. Thus $\exp(t\xi)z = z + te_1$, where

$e_1 = (1,0,\ldots,0)$, and therefore $pr^{(k)}(\exp t\xi)$ acts trivially on the fiber coordinates of $J_k^*(Z,p)$. Using the local trivialization

$$J_k^*(Z,p) \simeq Z \times \mathrm{Grass}^{(k)}(p+q,p)$$

suppose $z \in Z$, $\Lambda \in \mathrm{Grass}^{(k)}(p+q,p)$ and $F\colon J_k^*(Z,p) \to \mathbb{R}$ is a smooth function. Then

$$\begin{aligned} d[\exp(t\, pr^{(k)}\xi)]pr^{(k)}\eta\ F(z,\Lambda) &= d[pr^{(k)}\exp t\xi]pr^{(k)}\eta\ F(z,\Lambda) \\ &= pr^{(k)}\eta\ F(z+te_1,\Lambda). \end{aligned}$$

Therefore

$$d[\exp(t\, pr^{(k)}\xi)]pr^{(k)}\eta|_{(z,\Lambda)} = pr^{(k)}\eta|_{(z+te_1,\Lambda)}.$$

On the other hand

$$d[\exp t\xi]\eta|_z = \eta|_{z} + te_1$$

hence

$$pr^{(k)}[d(\exp t\xi)\eta]|_{(z,\Lambda)} = pr^{(k)}\eta|_{(z+te_1,\Lambda)}$$

completing the proof of the lemma. Q.E.D.

The next step is to derive the explicit formula for the prolonged vector field in local coordinates. This could be used to provide an alternate, computational proof of lemma 1.2.

<u>Theorem 1.3</u> Let $\chi\colon Z \to \mathbb{R}^p \times \mathbb{R}^q$ be a local coordinate chart on Z with induced coordinates $\chi^{(k)}\colon J_k^*(Z,p;\mathcal{U}) \to \mathbb{R}^p \times \mathbb{R}^q \times \mathbf{O}_*^k(\mathbb{R}^p,\mathbb{R}^q)$ on the extended jet bundle. Let ξ be a smooth vector field on Z given in local coordinates by $\xi = \xi^1(x,u)\partial_{x^1} + \ldots + \xi^p(x,u)\partial_{x^p} + \phi_1(x,u)\partial_{u^1} + \ldots + \phi_q(x,u)\partial_{u^q}$, where $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ are the coordinates on $\mathbb{R}^p \times \mathbb{R}^q$. Then in the induced coordinates $(x,u,\ldots,u_J^i,\ldots)$

$$\mathrm{pr}^{(k)}\xi = \mathrm{pr}^{(k-1)}\xi + \sum_{J \in S_k^p} \phi_i^J(x,u,\ldots,u_K^i,\ldots)\partial_{u_J^i}$$

where

$$\phi_i^J = D_J\phi_i - \sum_{0<K\leq J} \sum_{\sigma=1}^{p} \binom{J}{K} u^i_{J-K+\delta^\sigma} D_K\xi^\sigma \tag{1.1}$$

where the D_K's denote total derivatives.

Proof

Let $\hat{\Phi}_t\colon Z \to Z$ denote the local transformation $\exp(t\xi)$ so that in local coordinates for t sufficiently small

$$\hat{\Phi}_t(x,u) = (\Phi_t(x,u),\Psi_t(x,u))$$

with

$$\frac{d}{dt}\Big|_{t=0} \Phi_t^j(x,u) = \xi^j(x,u) \qquad j=1,\ldots,p$$

$$\frac{d}{dt}\Big|_{t=0} \Psi_t^i(x,u) = \phi_i(x,u) \qquad i=1,\ldots,q.$$

Let $j=(z,u^{(k)})=(x,u,\ldots,u_J^i,\ldots) \in J_k^*(Z,p;\mathcal{U})$ and let $\hat{f}\colon \mathbb{R}^p \to Z$ be a normalized parametrization of a section representing j. In local coordinates $\hat{f}(x) = (x,f(x))$ and $u^{(k)} = \partial_*^k f(x)$. Let $\hat{g}_t = \hat{\Phi}_t \circ \hat{f}$ for t sufficiently small so that the sections given by $\hat{g}_t$ are transversal to $\mathcal{U}$ and let $\hat{f}_t\colon \mathbb{R}^p \to Z$, $\hat{f}_t(x) = (x,f_t(x))$ be the renormalized parametrization of im $\hat{g}_t$. It follows that

$$f_t = \Psi_t \circ \hat{f} \circ (\Phi_t \circ \hat{f})^{-1}$$

for t sufficiently small so the inverse exists. Therefore

$$\mathrm{pr}^{(k)}\,\hat{\Phi}_t(z,u^{(k)}) = (\hat{\Phi}_t(z),\ \partial_*^k[\Psi_t \circ \hat{f} \circ (\Phi_t \circ \hat{f})^{-1}](x)).$$

Taking the derivative of this expression with respect to t and setting t=0 and making use of the fact that

$$\frac{d}{dt} A^{-1}(t) = -A^{-1}(t)\cdot\frac{d}{dt}A(t)\cdot A^{-1}(t)$$

for any differentiable matrix valued function of t, we have

$$\begin{aligned} pr^{(k)}\xi|_j &= \frac{d}{dt}\Big|_{t=0} pr^{(k)}\,\hat{\Phi}_t(z,u^{(k)}) \\ &= \xi + \frac{d}{dt}\Big|_{t=0}\partial_*^k(\Psi_t\circ f) - \partial_*^k f\circ\frac{d}{dt}\Big|_{t=0} d^k(\Phi_t\circ f) \qquad (*) \end{aligned}$$

since $\Psi_0\circ\hat{f} = f$ and $\Phi_0\circ\hat{f} = 1\!\!1_p$. The second term in (*) is just the total derivative matrix of $\vec{\phi} = (\phi_1,\ldots,\phi_q)$ -- $D_*^k\vec{\phi}(x,u,u^{(k)})$ -- since the entries of $\partial_*^k(\Psi_t\circ\hat{f})$ are just the various partial derivatives $\partial_K(\Psi_t^i\circ\hat{f})$, hence we are allowed to interchange the differentiations

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\partial_*^k(\Psi_t\circ\hat{f}) &= \partial_*^k\Big(\frac{d}{dt}\Big|_{t=0}\Psi_t\circ\hat{f}\Big) \\ &= \partial_*^k(\vec{\phi}\circ(1\!\!1_p x f)) \\ &= D_*^k\vec{\phi}(x,u,u^{(k)}) \end{aligned}$$

by the definition of total derivative. The third term requires more careful analysis since the matrix entries do not depend linearly on functions of t. Representing the matrix $d^k g$ in the usual block form $(\partial_j^i g)$ then

$$\Big[\partial_*^k f\circ\frac{d}{dt}\Big|_{t=0} d^k(\Phi_t\circ\hat{f})\Big]_m^1 = \sum_{i=1}^{m} \partial_i^1 f\cdot\frac{d}{dt}\Big|_{t=0}\partial_m^i(\Phi_t\circ\hat{f})$$

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\partial_m^i(\Phi_t\circ\hat{f}) &= \frac{d}{dt}\Big|_{t=0}\sum_{J\in\mathcal{A}_m^i}\frac{\partial(\Phi_t\circ\hat{f})^{j_1}\odot\ldots\odot\partial^k(\Phi_t\circ\hat{f})^{j_k}}{J!} \\ &= \frac{1\!\!1_p^{i-1}}{(i-1)!}\odot D^{m-i+1}\vec{\xi}(x,u,u^{(k)}) \end{aligned}$$

where $\vec{\xi} = (\xi^1,\ldots,\xi^p)$. The proof of the last equality is a consequence

$\Phi_0 \circ f = 1\!\!1_p$ so $\partial(\Phi_0 \circ f) = 1\!\!1_p$ and $\partial^2(\Phi_0 \circ f) = \ldots = \partial^k(\Phi_0 \circ f) = 0$ and therefore the only multi-indices $J \in \mathcal{A}_m^i$ contributing anything to the sum upon evaluating the derivative at t=0 are $(i-1)\delta^i + \delta^{m-i+1}$ and $i\delta^1$. Now the matrix entries of (*) are

$$\phi_i^J = D_J\phi_i - \sum_{0<L\leq J} u_L^i \left[\frac{1\!\!1_p^{\ell-1}}{(\ell-1)!} \odot D^{j-\ell+1}\vec{\xi} \right]_J^L \qquad (**)$$

where we have abbreviated $\ell = \Sigma L$, $j = \Sigma J$. Now recall that

$$\frac{1}{(\ell-1)!} 1\!\!1_p^{\ell-1} = 1\!\!1_{\odot_{\ell-1}\mathbb{R}^p}$$

hence by (II.2.1)

$$\left\langle e_J, \frac{1\!\!1_p^{\ell-1}}{(\ell-1)!} \odot D^{j-\ell+1}\vec{\xi} \right\rangle = \sum_{J\geq K \in S^p} \binom{J}{K} e_{J-K} \left\langle e_K, D^{j-\ell+1}\vec{\xi} \right\rangle$$

$$= \sum_{J\geq K} \sum_{\sigma=1}^{p} \binom{J}{K} D_K \xi^\sigma e_{J-K+\delta^\sigma}.$$

Substituting back into (**) we conclude that (1.1) is true. Q.E.D.

<u>Example 1.4</u> Consider the special case p=2, q=1 with coordinates (x,y,u) and vector field $\xi\partial_x + \eta\partial_y + \phi\partial_u$. The second prolongation of this vector field is given by

$$\xi\partial_x + \eta\partial_y + \phi\partial_u + \phi^x\partial_{u_x} + \phi^y\partial_{u_y} + \phi^{xx}\partial_{u_{xx}} + \phi^{xy}\partial_{u_{xy}} + \phi^{yy}\partial_{u_{yy}}$$

where

$$\phi^x = D_x\phi - u_x D_x\xi - u_y D_x\eta$$

$$= \phi_x + u_x(\phi_u - \xi_x) - u_x^2\xi_u - u_y\eta_x - u_x u_y \eta_u$$

$$\begin{aligned}
\phi^{y} &= D_y\phi - u_x D_y \xi - u_y D_y \eta \\
&= \phi_y + u_y(\phi_u - \eta_y) - u_y^2\eta_u - u_x \xi_y - u_x u_y \xi_u \\
\phi^{xx} &= D_{xx}\phi - 2u_{xx} D_x \xi - 2u_{xy} D_x \eta - u_x D_{xx} \xi - u_y D_{xx}\eta \qquad (1.2) \\
\phi^{xy} &= D_{xy}\phi - u_{xx} D_y \xi - u_{xy} D_y \eta - u_{xy} D_x \xi - u_{yy} D_x \eta - u_x D_{xy} \xi - u_y D_{xy}\eta \\
\phi^{yy} &= D_{yy}\phi - 2u_{xy} D_y \xi - 2u_{yy} D_y \eta - u_x D_{yy} \xi - u_y D_{yy}\eta.
\end{aligned}$$

The next corollary can be found in [E2;p.106] and provides a useful recursion formula to compute the functions ϕ_i^J.

Corollary 1.5 There is a recursion formula for the coefficient functions ϕ_i^J given by

$$\phi_i^{J+\delta^k} = D_k \phi_i^J - \sum_{\sigma=1}^{p} u^i_{J+\delta^\sigma} D_k \xi^\sigma. \qquad (1.3)$$

III.2 Calculating Groups of Differential Equations

Consider a k-th order system of partial differential equations $\Delta_0 \subset J_k^*(Z,p)$. The "symmetry group" of Δ_0 will be taken to mean the local group of all smooth local transformations of Z whose k-th prolongation to $J_k^*(Z,p)$ leaves Δ_0 invariant. The algebra of infinitesimal symmetries of Δ_0 will be the space of smooth vector fields on Z whose k-th prolongations leave Δ_0 infinitesimally invariant. Note that by lemma 1.2, the infinitesimal symmetries form a Lie algebra. In general, it is to be expected that the symmetry algebra exponentiates to form the connected component of the identity of the symmetry group; a technical problem arises in the case the symmetries form an infinite dimensional algebra: some condition such as that of a Lie pseudogroup must be imposed to provide the correspondence between the group and the algebra. It shall be seen that the infinitesimal symmetries must satisfy a large number of partial differential equations so that under some appropriate weak conditions on Δ_0 the Lie pseudogroup criteria will be satisfied. However, as these are rather technical in nature and shed little additional light on the subject, they shall not be investigated here. Besides, in practice we shall only be concerned with finite dimensional subgroups of the symmetry group and problems of this nature will not arise.[†]

[†]Another technical problem arises if the manifold is not Hausdorff, so that a vector field might not generate any local one-parameter group. See the appendix for a discussion of this phenomenon.

In practice, Δ_0 will not be given as a subvariety of $J_k^*(Z,p)$, but will be given, in local coordinates (x,u) on Z, as a system of equations

$$\Delta^i(x,u,u^{(1)},\cdots,u^{(k)}) = 0 \quad i = 1,\cdots,\alpha \tag{2.1}$$

where the Δ^i's are smooth real valued functions on $J_k^*(Z,p;\mathcal{U})$. Here $\mathcal{U}$ denotes the differential system spanned by $\{\partial/\partial u^1,\ldots,\partial/\partial u^q\}$ and the $u^{(i)}$ are the induced coordinates on the trivialized jet bundle. (See section II.3 for the details.) In the classical case $Z = \mathbb{R}^p \times \mathbb{R}^q$ and equations (2.1) are given on $J_k(\mathbb{R}^p,\mathbb{R}^q) = J_k^*(Z,p;\mathcal{U})$. Then Δ_0 will denote the closure of the subvariety of $J_k^*(Z,p;\mathcal{U})$ given by (2.1) in $J_k^*(Z,p)$. Note that to check the invariance of Δ_0 it suffices to check the local invariance of the subvariety defined by equations (2.1), so we can effectively restrict our attention to the trivialized jet bundle.

Let $\Delta : J_k^*(Z,p;\mathcal{U}) \to \mathbb{R}^\alpha$ denote the map with components Δ^i. If Δ is a submersion, then the infinitesimal criterion of invariance gives that Δ_0 is invariant under the group G iff

$$pr^{(k)}\vec{v}[\Delta] = 0 \text{ whenever } \Delta = 0 \tag{2.2}$$

for all infinitesimal generators $\vec{v}$ of G. In the case that Δ_0 is "irreducible" in the sense that any real valued function f vanishing on Δ_0 must be of the form $f = \sum \lambda_i \Delta^i$ where the λ_i's are smooth real valued functions on $J_k^*(Z,p)$, then condition (2.2) becomes

$$pr^{(k)}\vec{v}[\Delta] = \sum \lambda_i \Delta^i \ . \tag{2.3}$$

In practice (2.3) is the condition most frequently used - note that a priori the λ_i's can depend on all the derivatives as well as the dependent and independent variables. To calculate the symmetry group of such a system of equations, $\vec{v}$ is allowed to be an arbitrary vector field so that

$$\vec{v} = \sum_1^p \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_1^q \phi^j(x,u) \frac{\partial}{\partial u^j}$$

where the ξ^i's and ϕ^j's are unknown functions of the variables (x,u) . Using the prolongation formula of section III.1, the vector field $pr^{(k)}\vec{v}$ is computed in terms of the ξ^i , ϕ^j and their derivatives. Then condition 2.3 provides a large system of partial differential equations that these functions must satisfy, the general solution of which is the desired (infinitesimal) symmetry group.

<u>Example</u> 2.1 Burgers' Equation

Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u) and consider the second order quasi-linear equation

$$u_t + uu_x + u_{xx} = 0 \tag{2.4}$$

known as Burgers' equation. It is important in nonlinear wave theory, being the simplest equation that contains both nonlinear propagation and diffusion. See, for instance, [WH; chapter 4] for a fairly complete discussion of its properties. Let $\vec{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u$ be a smooth vector field on Z, with second prolongation

$$\mathrm{pr}^{(2)}\vec{v} = \vec{v} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}} + \phi^{xt}\partial_{u_{xt}} + \phi^{tt}\partial_{u_{tt}}$$

where the coefficient functions are given at the end of section III.1. Using criterion 2.3, we have that

$$\phi^t + u\phi^x + u_x\phi + \phi^{xx} = \lambda(u_t+uu_x+u_{xx}) \qquad (2.5)$$

must be satisfied for some function λ which might depend on $(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})$. However, as the second order derivatives u_{xx},u_{xt},u_{tt} occur only linearly on both sides of (2.5), λ can only depend on (x,t,u,u_x,u_t). The coefficient of u_{xt} in (2.5) is

$$-2D_x\tau = 0$$

implying that τ is a function t alone. The coefficient of u_{xx} is

$$\phi_u - 2\xi_x - 3u_x\xi_u = \lambda \qquad (2.6)$$

giving the form of λ. The coefficient of u_t is, since λ is independent of u_t,

$$\phi_u - \tau_t - u_x\xi_u = \lambda$$

which implies that $\xi_u = 0$, $\tau_t = 2\xi_x$, so $\xi_{xx} = 0$ and λ depends only on (t,u). The coefficient of u_x^2 is now $\phi_{uu} = 0$, hence

$$\phi(x,t,u) = \alpha(x,t) + u\beta(x,t) \ .$$

The coefficient of u_x is now

$$-\xi_t + u(\beta-\xi_x) + \alpha + \beta u + 2\beta_x = \lambda$$

hence using (2.6)

$$\beta = -\xi_x \quad \beta_x = 0 \quad \alpha = \xi_t$$

Finally, the terms in (2.5) not involving any derivatives are

$$\phi_t + u\phi_x + \phi_{xx} = 0$$

hence $\alpha_t = 0$, $\beta_t + \alpha_x = 0$, which implies $\xi_{tt} = 0$. Therefore the general solution to (2.5) is given by

$$\xi = c_1 + c_3 x + c_4 t + c_5 xt$$

$$\tau = c_2 + 2c_3 t + c_5 t^2$$

$$\phi = c_4 + c_5 x - (c_3+c_5 t)u$$

$$\lambda = -3c_3 - 3c_5 t$$

where c_1 , c_2 , c_3 , c_4 , c_5 are arbitrary real constants. The infinitesimal symmetry algebra of Burgers' equation is five dimensional with basis consisting of the vector fields

$$\vec{v}_1 = \partial_x$$
$$\vec{v}_2 = \partial_t$$
$$\vec{v}_3 = x\partial_x + 2t\partial_t - u\partial_u \qquad (2.7)$$
$$\vec{v}_4 = t\partial_x + \partial_u$$
$$\vec{v}_5 = xt\partial_x + t^2\partial_t + (x-tu)\partial_u \quad .$$

The commutation relations between these vector fields is given by the following table, the entry at column i and row j representing $[\vec{v}_i, \vec{v}_j]$.

	$\vec{v}_1$	$\vec{v}_2$	$\vec{v}_3$	$\vec{v}_4$	$\vec{v}_5$
$\vec{v}_1$	0	0	$\vec{v}_1$	0	$\vec{v}_4$
$\vec{v}_2$	0	0	$2\vec{v}_2$	$\vec{v}_1$	$\vec{v}_3$
$\vec{v}_3$	$-\vec{v}_1$	$-2\vec{v}_2$	0	$\vec{v}_4$	$2\vec{v}_5$
$\vec{v}_4$	0	$-\vec{v}_1$	$-\vec{v}_4$	0	0
$\vec{v}_5$	$-\vec{v}_4$	$-\vec{v}_3$	$-2\vec{v}_5$	0	0

(2.8)

Let G_i denote the one-parameter local group associated with $\vec{v}_i$. Then G_1 and G_2 are just translations in the x and t directions respectively

$$G_1 : (x,t,u) \mapsto (x+\lambda,t,u)$$
$$\lambda \in \mathbb{R} \qquad (2.9a)$$
$$G_2 : (x,t,u) \mapsto (x,t+\lambda,u)$$

and represent the fact that Burgers' equation has no dependence on x or t . The group G_3 is a scale transformation

$$G_3 : (x,t,u) \mapsto (e^{\lambda}x, e^{2\lambda}t, e^{-\lambda}u) \qquad \lambda \in \mathbb{R} \tag{2.9b}$$

The groups G_4 and G_5 represent less trivial symmetries:

$$\left.\begin{aligned} G_4 &: (x,t,u) \mapsto (x+\lambda t, t, u+\lambda) \\ G_5 &: (x,t,u) \mapsto \left(\frac{t}{1-\lambda t}, \frac{x}{1-\lambda t}, u + \lambda(x-tu)\right) \end{aligned}\right\} \quad \lambda \in \mathbb{R} \tag{2.9c}$$

These will be discussed in more detail in example IV.2.1.

In the next examples, some of the results to be proven in section III.3 will imply that no nonprojectable transformations are in the symmetry group, meaning that the functions ξ^i in the expression for a vector field v do not depend on the dependent variables u. This will help simplify the calculations.

Example 2.2 The Heat Equation.

As in the previous example $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u) ; we consider the second order linear equation

$$u_t = u_{xx} \tag{2.10}$$

Letting $\vec{v}$ and $pr^{(2)}\vec{v}$ be as in example 1, then the infinitesimal invariance criterion is

$$\phi^t - \phi^{xx} = \lambda(u_t - u_{xx}) \ . \tag{2.11}$$

Using the results of section III.3, we have $\xi_u = \tau_u = 0$ and λ

depends only on (x,t,u) . The coefficient of u_{xt} in (2.11) implies $\tau_x = 0$. The remaining coefficients of the various derivatives give the equations

$$\phi_u - 2\xi_x = \lambda$$

$$\phi_u - \tau_t = \lambda$$

$$\phi_{uu} = 0$$

$$2\phi_{xu} - \xi_{xx} + \xi_t = 0$$

$$\phi_{xx} - \phi_t = 0 \ .$$

A little work shows that the general solution to these equations is

$$\xi = c_1 + 2c_5 t + c_4 x + 4c_6 xt$$

$$\tau = c_2 + 2c_4 t + 4c_6 t^2$$

$$\phi = \alpha(x,t) + (c_3 - 2c_6 t - c_5 x - c_6 x^2)u$$

$$\lambda = c_3 - 2c_4 - 10c_6 t - c_5 x - c_6 x^2$$

where c_1 , c_2 , c_3 , c_4 , c_5 , c_6 are arbitrary real constants and $\alpha(x,t)$ is an arbitrary solution of the heat equation. Therefore the symmetry algebra of the heat equation is spanned by the six vector fields

$$\vec{v}_1 = \partial_x$$
$$\vec{v}_2 = \partial_t$$
$$\vec{v}_3 = u\partial_u$$
$$\vec{v}_4 = x\partial_x + 2t\partial_t \qquad (2.12)$$
$$\vec{v}_5 = 2t\partial_x - xu\partial_u$$
$$\vec{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$$

and the infinite dimensional abelian subalgebra

$$\vec{v}_\alpha = \alpha(x,t)\partial_u \quad \alpha_t = \alpha_{xx} .$$

The commutator table is

	$\vec{v}_1$	$\vec{v}_2$	$\vec{v}_3$	$\vec{v}_4$	$\vec{v}_5$	$\vec{v}_6$	v_α
$\vec{v}_1$	0	0	0	$\vec{v}_1$	$-\vec{v}_3$	$-2\vec{v}_5$	$\vec{v}_{\alpha_x}$
$\vec{v}_2$	0	0	0	$2\vec{v}_2$	$2\vec{v}_1$	$4\vec{v}_4-2\vec{v}_3$	$\vec{v}_{\alpha_t}$
$\vec{v}_3$	0	0	0	0	0	0	$-\vec{v}_\alpha$
$\vec{v}_4$	$-\vec{v}_1$	$-2\vec{v}_2$	0	0	$-\vec{v}_5$	$-2\vec{v}_6$	$\vec{v}_{\alpha'}$
$\vec{v}_5$	$\vec{v}_3$	$-2\vec{v}_1$	0	$\vec{v}_5$	0	0	$\vec{v}_{\alpha'}$
$\vec{v}_6$	$2\vec{v}_5$	$2\vec{v}_3-4\vec{v}_4$	0	$2\vec{v}_6$	0	0	$\vec{v}_{\alpha''}$
$\vec{v}_\alpha$	$-\vec{v}_{\alpha_x}$	$-\vec{v}_{\alpha_t}$	$\vec{v}_\alpha$	$-\vec{v}_{\alpha'}$	$-\vec{v}_{\alpha'}$	$-\vec{v}_{\alpha''}$	0

(2.13)

where
$$\alpha' = x\alpha_x + 2t\alpha_t$$
$$\alpha'' = 4tx\alpha_x + 4t^2\alpha_t + (x^2+2t)\alpha \quad .$$

Note that since the infinitesimal operators form a Lie algebra, if α is any solution to the heat equation, so are α' and α''. Note that the symmetries given by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_\alpha$ merely reflect the fact that the heat equation is a constant coefficient linear partial differential equation. The well-known invariance under a scale transformation in x and t is given by $\vec{v}_4$. The one parameter groups are

$$\begin{aligned}
G_1 &: (x,t,u) \mapsto (x+\lambda, t, u) \\
G_2 &: (x,t,u) \mapsto (x, t+\lambda, u) \\
G_3 &: (x,t,u) \mapsto (x, t, e^{\lambda}u) \qquad \lambda \in \mathbb{R} \;. \qquad (2.14) \\
G_4 &: (x,t,u) \mapsto (e^{\lambda}x, e^{2\lambda}t, u) \\
G_5 &: (x,t,u) \mapsto (x-2\lambda t, t, u\exp(x\lambda-\lambda^2 t)) \\
G_6 &: (x,t,u) \mapsto \left(\frac{x}{4\lambda t+1}, \frac{t}{4\lambda t+1}, u\sqrt{4\lambda t+1}\exp\left[\frac{-\lambda x^2}{4\lambda t+1}\right]\right)
\end{aligned}$$

<u>Example</u> 2.3 Laplace's Equation in the plane

Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,y,u) and consider the equation

$$\Delta u = u_{xx} + u_{yy} = 0 \;. \qquad (2.15)$$

If $\vec{v} = \xi\partial_x + \eta\partial_y + \phi\partial_u$ is an arbitrary infinitesimal symmetry of (2.15), then

$$\phi^{xx} + \phi^{yy} = \lambda(u_{xx}+u_{yy})$$

As in example 2, we use the results of section III-3 to imply that $\xi_u = \eta_u = 0$ and $\lambda = \lambda(x,y,u)$. The relevant equations for ξ , η , ϕ and λ are

$$\xi_y + \eta_x = 0$$
$$\phi_u - 2\xi_x = \lambda$$
$$\phi_u - 2\eta_y = \lambda$$
$$\phi_{uu} = 0$$
$$2\phi_{xu} - \xi_{xx} + \xi_{yy} = 0$$
$$2\phi_{yu} - \eta_{yy} + \eta_{xx} = 0$$
$$\phi_{xx} + \phi_{yy} = 0 \ .$$

The general solution to these equations gives the familiar conformal group in the (x,y) - plane. Namely ξ , η form an arbitrary solution to the Cauchy-Riemann equations and

$$\phi = \beta(x,t)u + \alpha(x,t)$$

where $\beta = \xi_x + \eta_y$ and α is an arbitrary solution to Laplace's equation.

<u>Example</u> 2.4 The Korteweg-deVries Equation

Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u) and consider the quasi-linear evolution equation

$$u_t + uu_x + u_{xxx} = 0 \ . \qquad (2.16)$$

The equation is used to describe long waves in shallow water, and exhibits many surprising phenomena; see [WH; chapter 17] for some details. Criterion 2.3 gives

$$\phi^t + u\phi^x + \phi u_x + \phi^{xxx} = \lambda(u_t + uu_x + u_{xxx}) ,$$

and the prolongation formula gives

$$\begin{aligned}\phi^{xxx} = \phi_{xxx} &+ u_x(3\phi_{xxu} - \xi_{xxx}) + 3u_x^2\phi_{xuu} + u_x^3\phi_{uuu} \\ &+ 3u_{xx}(\phi_{xu} - \xi_{xx}) + 3u_x u_{xx}\phi_{uu} - 3u_{xt}\tau_{xx} \\ &+ u_{xxx}(\phi_u - 3\xi_x) - u_{xxt}\tau_x\end{aligned}$$

since the vector field $\vec{v}$ is projectable by section III.3. Solving the resulting equations shows that the symmetry algebra of the Korteweg-deVries equation is spanned by the four vector fields

$$\begin{aligned}\vec{v}_1 &= \partial_x \\ \vec{v}_2 &= \partial_t \\ \vec{v}_3 &= t\partial_x + \partial_u \\ \vec{v}_4 &= x\partial_x + 3t\partial_t - 2u\partial_u\end{aligned} \qquad (2.17)$$

with commutator table

$$
\begin{array}{c|cccc}
 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\
\hline
\vec{v}_1 & 0 & 0 & 0 & \vec{v}_1 \\
\vec{v}_2 & 0 & 0 & \vec{v}_1 & 3\vec{v}_2 \\
\vec{v}_3 & 0 & -\vec{v}_1 & 0 & -2\vec{v}_3 \\
\vec{v}_4 & -\vec{v}_1 & -3\vec{v}_2 & 2\vec{v}_3 & 0
\end{array}
\tag{2.18}
$$

The corresponding one parameter groups are

$$
\begin{aligned}
G_1 &: (x,t,u) \mapsto (x+\lambda,t,u) \\
G_2 &: (x,t,u) \mapsto (x,t+\lambda,u) \\
G_3 &: (x,t,u) \mapsto (x+\lambda t,t,u+\lambda) \\
G_4 &: (x,t,u) \mapsto (e^{\lambda}x,e^{3\lambda}t,e^{-\lambda}u)
\end{aligned}
\qquad \lambda \in \mathbb{R} \tag{2.19}
$$

In all the examples so far, the groups have been projectable, meaning that the transformations in the independent variables do not depend on the dependent variables. To show that this isn't always the case, and in preparation for the next section, we consider an elementary example.

<u>Example</u> 2.5 Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,y,u) and consider the equation

$$\Delta_0 : u_{xx} = 0$$

Let $\vec{v} = \xi\partial_x + \eta\partial_y + \phi\partial_u$ be an arbitrary vector field on Z, then the criterion for Δ_0 to be invariant under $\vec{v}$ is just

$$0 = \phi_{xx} + u_x(2\phi_{xu}-\xi_{xx}) + u_x^2(\phi_{uu}-2\xi_{xu})$$

$$- u_x^3\xi_{uu} - u_t(\tau_{xx}+2u_x\tau_{xu} + u_x^2\tau_{uu})$$

$$+ u_{xx}(\phi_u-2\xi_x-u_x\xi_u-u_t\tau_u) - 2u_{xt}(\tau_x+u_x\tau_u)$$

whenever $u_{xx} = 0$. Therefore

$$\phi_{xx} = 0$$

$$2\phi_{xu} = \xi_{xx}$$

$$\phi_{uu} = 2\xi_{xu}$$

$$\xi_{uu} = 0$$

$$\tau_x = \tau_u = 0$$

so the infinitesimal symmetry group is given by all vector fields of the form

$$[c_1+c_4x+c_5u]\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial t} + [c_3+c_6x+(c_7+2c_4x)u]\frac{\partial}{\partial u}$$

where c_i , $i = 1,\ldots,7$ are arbitrary functions of t .

III.3 Applications to Linear Equations

In this section the preceding theory will be applied to derive some results on the symmetry group of a linear partial differential equation in one dependent variable. It will be shown that for an equation of order $n \geq 3$ there are no nonprojectable symmetries. In the course of deriving the defining equations for the infinitesimal generators of the symmetry group, it will be explicitly shown that the group is a subgroup of the group of conformal transformations of the symbol of the equation. For the case of a second order equation, invariance under a nonprojectable one-parameter group implies that the equation is equivalent under a change of coordinates to a parametrized ordinary differential equation, as was shown by Ovsjannikov, [O1;chapter 6]. These, however, can be invariant under nonprojectable groups, as was demonstrated in example 2.5. Many of the results of this section can be easily generalized to certain classes of quasi-linear and even nonlinear equations with quasilinear top order terms. Multi-index notation as given in definition II.2.1 will be used throughout.

Consider the linear partial differential equation

$$\Delta(u) = \sum_{k=0}^{n} \sum_{I \in S_k^p} a^I(x) \cdot \partial_I u = 0 \tag{3.1}$$

where $x=(x^1,\ldots,x^p)$ and for $I=(i_1,\ldots,i_p)$, ∂_I denotes the partial derivative $\partial_1^{i_1}\partial_2^{i_2}\cdots\partial_p^{i_p}$, where $\partial_j = \partial/\partial x^j$. More abstractly, (3.1) can be viewed as defining a closed subvariety $\Delta_0 \subset J_n^*(Z,p)$ where Z is a smooth manifold of dimension p+1 where equation (3.1) is the local coordinate expression for this subvariety. Here we are only concerned with the local symmetries of (3.1) so we are justified in restricting our attention to the case when Z is an open subset of $\mathbb{R}^p \times \mathbb{R}$, and looking only at the trivialized jet bundle $J_n(\mathbb{R}^p,\mathbb{R}) \approx \mathbb{R}^p \times \mathbb{R} \times \odot^n_*(\mathbb{R}^p,\mathbb{R})$. In any case, the important expression is equation (3.1), the abstract terminology being important only for theoretical considerations.

In order to derive the symmetry group of (3.1), the prolongation formula (1.1) for the infinitesimal generators will be used to derive the defining equations of the symmetry algebra, which will in turn be used to derive properties of the (local) symmetry group. Let

$$\vec{v} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \phi(x,u)\partial_u$$

be the local coordinate expression for a vector field on Z. The prolonged vector field on $J_n^*(Z,p)$ is given by

$$pr^{(n)}\vec{v} = \vec{v} + \sum_{k=1}^{n} \sum_{I \in S_k^p} \phi^I(x,u,u^{(k)})\partial_{u_I}$$

where the ϕ^I are given in theorem 1.3. The condition of invariance of Δ_0 under the one-parameter group generated by v is simply

$$pr^{(n)}\vec{v}[\Delta(u)] = \mu\Delta(u) \tag{3.2}$$

for some multiplying factor $\mu: J_n^*(Z,p) \to \mathbb{R}$, which may a priori depend on all the derivatives of u of order $\leq n$. However, since the expressions for the ϕ^I's are polynomials in the u_J's, μ must be a polynomial in the partial derivatives of u. Henceforth, we will use the symbol $\mathcal{L}(k)$ to denote any polynomial expression in the u_J's for $\Sigma J<k$ whose

explicit form is not needed in an equation.

The first observation is that if $\psi(x,u)$ is an arbitrary smooth function, then its total derivatives have the form

$$D_J\psi = \frac{\partial\psi}{\partial u} u_J + \mathcal{L}(k) \qquad k = \Sigma J \tag{3.3}$$

as can easily be verified using lemma II.4.3. (A more refined expression giving the (k-1)st order terms will be given in lemma 3.6 to follow.) As a corollary of this and the prolongation formula, for $\Sigma I=m$

$$\phi^I = \frac{\partial\phi}{\partial u} u_I - \sum_{k=1}^{p} \sum_{\sigma=1}^{p} \left[(i_k + \delta^k_\sigma)\frac{\partial \xi^\sigma}{\partial u} u_k + i_k\frac{\partial \xi^\sigma}{\partial x^k}\right] u_{I-\delta^k+\delta^\sigma} + \mathcal{L}(m) \tag{3.4}$$ †

where δ^k_σ is the Kronecker δ. To see this, it suffices to note that the only terms in the sum in (1.1) that contribute m-th order terms are when $K=I$ or $K=\delta^k$.

<u>Lemma 3.1</u> If Δ is invariant under the one parameter group generated by the vector field $\vec{v}$ with multiplying factor μ, then

$$\mu = \mu_0(x,u) + \sum_{k=1}^{p} \mu_k(x,u)u_k \tag{3.5}$$

for some functions $\mu_0,\mu_1,\ldots,\mu_p: Z \to \mathbb{R}$. Moreover, if $(x,u) \in Z$

,
$$\mu_k(x,u) = \alpha_k\frac{\partial \xi^k}{\partial u} \qquad k=1,\ldots,p \tag{3.6}$$

where $\alpha_k \in \mathbb{Z}$ is given by

$$\alpha_k = - \sup\{ j_k: J \in S^p_n,\ a^J(x) \neq 0 \} - 1. \tag{3.6}$$

---

†Here and elsewhere, any term in a sum involving a multi-index expression that results in a multi-index with a negative entry shall be ignored. Thus, in (3.4), if $i_k=0$, that term is ignored.

Proof

When written out in further detail, equation (3.2) is

$$\sum_I [\, a^I \phi^I + \sum_{\sigma=1}^{p} \xi^\sigma \frac{\partial a^I}{\partial x^\sigma} \cdot u_I \,] = \mu \sum_I a^I u_I .$$

Using (3.4) the left hand side is

$$\sum_{I \in S_n^p} \left[\left(a^I \frac{\partial \phi}{\partial u} + \sum_\sigma \xi^\sigma \frac{\partial a^I}{\partial x^\sigma} \right) u_I - \sum_{k,\sigma} \left((i_k + \delta_\sigma^k) \frac{\partial \xi^\sigma}{\partial u} u_k + i_k \frac{\partial \xi^\sigma}{\partial x^k} \right) u_{I-\delta^k+\delta^\sigma} \right] + \mathcal{L}(n). \quad (3.8)$$

This proves that μ must take the form (3.5). Equating the coefficients of $u_k u_J$ for $J \in S_n^p$ yields the following important equations:

$$- \sum_{\sigma=1}^{p} (j_k+1) a^{J-\delta^\sigma+\delta^k} \frac{\partial \xi^\sigma}{\partial u} = \mu_k a^J \qquad (3.9)$$

for all $1 \le k \le p$ and $J \in S_n^p$. Now given k, let $\ell = \sup\{ j_k : J \in S_n^p,\ a^J \neq 0 \}$. For this value of k and some J with $j_k = \ell$, $a^J \neq 0$, (3.9) reduces to

$$- (\ell+1)\, a^J \frac{\partial \xi^k}{\partial u} = \mu_k a^J$$

since $a^{J-\delta^\sigma+\delta^k} = 0$ for $\sigma \neq k$, but this immediately gives the lemma.

Q.E.D.

Combining equations (3.6) and (3.9) we have

$$\sum_{\sigma=1}^{p} (j_k+1+\alpha_k \delta_\sigma^k)\, a^{J-\delta^\sigma+\delta^k} \frac{\partial \xi^\sigma}{\partial u} = 0 \qquad (3.10)$$

for all $1 \le k \le p$, $J \in S_n^p$.

Lemma 3.2 For each $k = 1,\dots,p$ and each $(x,u) \in Z$, either $\frac{\partial \xi^k}{\partial u}(x,u) = 0$ or $a^{n\delta^k} \neq 0$.

Proof

Assume first that $\alpha_k \neq -1$, meaning that some derivatives in the x^k direction do actually appear in the highest order terms of Δ. Assume further that $a^{n\delta^k}(x) = 0$ and $\frac{\partial \xi^k}{\partial u} \neq 0$. Suppose $\alpha_k = -\ell-1$ with $0<\ell<n$ so that $a^{\ell\delta^k+I}(x) \neq 0$ for some $I \in S^p_{n-\ell}$ with $i_k=0$. Choose j so that $i_j \neq 0$ (this is possible since $\ell<n$) hence (3.10) in the case $J = (\ell+1)\delta^k + I - \delta^j$ and j replacing k gives

$$i_j a^{\ell\delta^k+I}(x) \frac{\partial \xi^k}{\partial u}(x,u) = 0$$

since $a^{J-\delta^\sigma+\delta^j}(x)=0$ unless $\sigma=k$, giving a contradiction. On the other hand, if $\alpha_k = -1$, then (3.10) in the case $J = \delta^k+I$ for $I \in S^p_{n-1}$ with $i_k=0$ gives

$$(i_j+1)a^{I+\delta^j}(x) \frac{\partial \xi^k}{\partial u}(x,u) = 0$$

for all $j\neq k$, showing that $a^J(x) = 0$ for all $J \in S^p_n$ with $j_k=0$. Finally we use the continuity of the coefficients a^J and the ξ^k to infer that Δ is not an n-th order differential operator in some open set. Q.E.D.

Definition 3.3 An n-th order linear partial differential operator Δ is partially degenerate if there is a coordinate system $(x^1,\ldots,x^p)$ on $\mathbb{R}^p$ such that

$$\Delta = a_n(x)\, \partial_1^n + \Delta'$$

where Δ' is a linear partial differential operator of order $n'\leq n-1$. Δ is strongly degenerate if there is a coordinate system such that

$$\Delta = a_n(x)\, \partial_1^n + a_{n-1}(x)\, \partial_1^{n-1} + \ldots + a_1(x)\, \partial_1 + a_0(x)$$

i.e. Δ is equivalent under a change of coordinates in the independent variables to a parametrized ordinary differential operator.

The next lemma gives a complete characterization of partially degenerate differential operators.

Lemma 3.4 An n-th order linear partial differential operator is partially degenerate iff its n-th order coefficient functions are of the form

$$a^I(x) = \frac{n!}{I!}\rho^I(x)\hat{a}(x) \qquad I \in S_n^p$$

for some real-valued functions $\rho^1,\dots,\rho^p,\hat{a}$.

Proof

Let $x = \phi(\hat{x})$ be an arbitrary change of coordinates. Let $\partial_i = \partial/\partial x^i$ and $\hat{\partial}_i = \partial/\partial\hat{x}^i$. Let

$$\rho^i(x) = \hat{\partial}_1\phi^i[\phi^{-1}(x)]$$

so that

$$\hat{\partial}_1 = \sum_{i=1}^{p}\rho^i(x)\partial_i \qquad \text{at } x = \phi(\hat{x}).$$

It is a straightforward exercise to check by induction that

$$\hat{\partial}_1^n = \sum_{I \in S_n^p}\frac{n!}{I!}\rho^I(x)\partial_I + \Delta'$$

for some operator Δ' of order n-1. Conversely, given $\rho^1,\dots,\rho^n$ which do not all simultaneously vanish, it is easy to construct a change of coordinates $x = \phi(\hat{x})$ with $\partial_1\phi^i[\phi^{-1}(x)] = \rho^i(x)$. Q.E.D.

We are now in a position to prove our first main result.

Theorem 3.5 If $\Delta(u) = 0$ is a linear partial differential equation of order $n \geq 2$, which is invariant under a nonprojectable one parameter group of transformations, then Δ is strongly degenerate.

Proof

First it will be shown via the criteria given in the previous lemma that Δ must be partially degenerate. Given (x,u), suppose that the vector field on Z generating the one parameter group is nonprojectable at (x,u), which means that $\frac{\partial \xi^k}{\partial u}(x,u) \neq 0$ for at least one integer k. Fix this k. It is now claimed that for $I \in S_n^p$,

$$a^I(x) = \frac{n!}{I!}\,\rho^I(x,u)\hat{a}(x,u) \qquad \text{where} \qquad \rho^j(x,u) = \frac{\partial \xi^j}{\partial u}(x,u) \tag{3.11}$$

which in particular proves the partial degeneracy of Δ by letting u assume any fixed value. Suppose $I = (n-\ell)\delta^j + J$ for $J \in S_\ell^p$ with $j_k=0$. Equation (3.11) will be proven by induction on ℓ. For $\ell=0$, let

$$a^{n\delta^k}(x) = [\rho^k(x,u)]^n \cdot \hat{a}(x,u).$$

By lemma 3.2 we know that $\alpha_k = -n-1$ (so in particular $\hat{a}(x,u) \neq 0$), hence (3.10) gives for $\ell \neq 0$

$$-\ell a^I(x)\frac{\partial \xi^k}{\partial u} + (n-\ell+1)\sum_{\sigma\neq k} a^{(n-\ell+1)\delta^k + J - \delta^\sigma}(x)\frac{\partial \xi^\sigma}{\partial u} = 0.$$

Therefore, by induction

$$a^I(x)\rho^k = (n-\ell+1)\sum_{\substack{\sigma\neq k\\ j_\sigma\neq 0}} \frac{n!}{(J-\delta^\sigma)!(n-\ell+1)!}\,\rho^I\rho^k$$

$$= (n-\ell+1)\frac{n!}{M!}\left[\sum_{\substack{\sigma\neq k\\ i_\sigma\neq 0}} \frac{1}{i_\sigma}\right]\rho^I\rho^k$$

$$= \frac{n!}{I!}\left[\sum_{\substack{\sigma\neq k\\ i_\sigma\neq 0}} i_\sigma\right]\rho^I\rho^k = \frac{n!}{I!}\,\ell\rho^I\rho^k$$

where M is the multi-index with components $m_\sigma = i_\sigma - 1$ for $i_\sigma \neq 0$, $\sigma \neq k$, $m_\sigma = 0$ for $i_\sigma = 0$ and $m_k = n-\ell+1$. This proves the partial degeneracy of Δ. To prove the strong degeneracy, we work in the coordinate system $(x^1,\ldots,x^p)$ so that

$$\Delta = a_n(x)\,\partial_1^n + a_{n-1}(x)\,\partial_1^{n-1} + \ldots + a_1(x)\,\partial_1 + \Delta'$$

where Δ' is a differential operator of order $n' \leq n-1$. By lemma 3.2, in this coordinate system $\frac{\partial \xi^k}{\partial u} = 0$ for $k=2,\ldots,n$, hence

$$\mu = \mu_0(x,u) + \mu_1(x,u)u_1 \quad \text{where} \quad \mu_1(x,u) = -(n+1)\frac{\partial \xi^1}{\partial u}(x,u).$$

Now it suffices to notice that in (3.2), the only terms containing $u_k u_J$ for $k \neq 1$ are

$$\sum a^J(x) \sum_{k=2}^{p} \frac{\partial \xi^1}{\partial u} u_k u_{J-\delta^k+\delta^1} = 0$$

where the sum extends over all $J \in S^p_{n'}$ except $n'\delta^1$. This immediately implies $a^J(x) = 0$ for all such J, showing that Δ' is in reality of order $n'-1$, proving the theorem.

Q.E.D.

Lemma 3.6 Let $\phi: \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be smooth and let $J \in S^p_n$ for $n>2$, then

$$D_J\phi = u_J\frac{\partial\phi}{\partial u} + \sum_{\sigma=1}^{p} j_\sigma u_{J-\delta^\sigma} D_\sigma \frac{\partial\phi}{\partial u} + \mathcal{L}(n-1)$$

Where $\mathcal{L}(n-1)$ denotes terms involving u_I for $\Sigma I < n-1$.

Proof

From lemma II.4.3 we have

$$D_k\phi = u_k\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial x^k}$$

$$D_\ell D_k\phi = u_{k\ell}\frac{\partial\phi}{\partial u} + u_k D_\ell\left(\frac{\partial\phi}{\partial u}\right) + D_\ell\left(\frac{\partial\phi}{\partial x^k}\right)$$

$$= u_{k\ell}\frac{\partial\phi}{\partial u} + u_k u_\ell \frac{\partial^2\phi}{\partial u^2} + u_k \frac{\partial^2\phi}{\partial x^\ell \partial u} + u_\ell \frac{\partial^2\phi}{\partial x^k \partial u} + \mathcal{L}(1)$$

$$D_m D_\ell D_k \phi = u_{k\ell m}\frac{\partial\phi}{\partial u} + u_{k\ell}u_m\frac{\partial^2\phi}{\partial u^2} + u_{km}u_\ell\frac{\partial^2\phi}{\partial u^2} + u_{\ell m}u_k\frac{\partial^2\phi}{\partial u^2} +$$

$$+ u_{k\ell}\frac{\partial^2\phi}{\partial x^m\partial u} + u_{km}\frac{\partial^2\phi}{\partial x^\ell\partial u} + u_{\ell m}\frac{\partial^2\phi}{\partial x^k\partial u} + \mathcal{L}(2)$$

$$= u_{k\ell m}\frac{\partial\phi}{\partial u} + u_{k\ell}D_m\left(\frac{\partial\phi}{\partial u}\right) + u_{km}D_\ell\left(\frac{\partial\phi}{\partial u}\right) + u_{\ell m}D_k\left(\frac{\partial\phi}{\partial u}\right) + \mathcal{L}(2)$$

proving the lemma for the case n=3. In general, by induction on n

$$D_k D_J \phi = u_{J+\delta^k}\frac{\partial\phi}{\partial u} + u_J D_k\left(\frac{\partial\phi}{\partial u}\right) + \sum_{\sigma=1}^{p} j_\sigma u_{J-\delta^\sigma+\delta^k} D_\sigma\left(\frac{\partial\phi}{\partial u}\right) + \mathcal{L}(n)$$

for $\Sigma J=n$, which proves the formula for $\Sigma J=n+1$. Q.E.D.

Theorem 3.7 The symmetry group of a linear partial differential equation of order $n\geq 3$ contains no nonprojectable symmetries.

Proof

By theorem 3.5, it only remains to consider the strongly degenerate equations. Suppose

$$\Delta(u) = \sum_{i=0}^{n} a^i(x)\partial_1^i u$$

and $v = \Sigma\xi^i\partial_i + \phi\partial_u$ is any infinitesimal symmetry of $\Delta(u)=0$. Using equation (1.1) and lemma 3.6, it can be seen that the only terms in (3.2) involving $u_{11}u_{(n-1)\delta^1}$ are

$$a^n(x)\left[-\binom{n}{2}u_{(n-1)\delta^1}\frac{\partial\xi^1}{\partial u}u_{11} - \binom{n}{n-1}u_{11}\frac{\partial\xi^1}{\partial u}u_{(n-1)\delta^1}\right] = 0.$$

(This relies heavily on the fact that n>2.) Therefore $\frac{\partial\xi^1}{\partial u} = 0$, hence $\vec{v}$ is projectable. Q.E.D.

The example $u_{xx}=0$ whose symmetry group was computed in section 3.2 shows that the previous theorem is not true for n=2.

Proposition 3.8 Suppose Δ_0 is an n-th order linear partial differential equation which is invariant under the one parameter projectable group with infinitesimal generator $\vec{v} = \Sigma\xi^i\partial_i + \phi\partial_u$, then $\phi(x,u) = \alpha(x) + \beta(x)u$ where α (and, if Δ has no zero order term, β) is a solution of Δ_o.

Proof

In the case n>2, lemma 3.6 implies that the only terms in the equation (3.2) involving $u_k u_J$ for $\Sigma J = n-1$ are

$$\sum_{J \in S_n^p} \sum_{\sigma=1}^{p} j_\sigma a^J(x)\frac{\partial^2\phi}{\partial u^2} u_\sigma u_{J-\delta^\sigma} = 0$$

since the multiplying factor μ depends only on (x,u) by theorem 3.4. This implies $\frac{\partial^2\phi}{\partial u^2} = 0$, hence $\phi(x,u) = \alpha(x) + \beta(x)u$. A similar argument works for the cases n=1,2. (See the first and second formulae in the proof of lemma 3.6.) The assertion about α and β follows from the following more general proposition. Q.E.D.

Proposition 3.9 If the linear partial differential equation $\Delta(u)=0$ is invariant under the one parameter group generated by the vector field $\vec{v} = \Sigma\xi^i\partial_i + \phi\partial_u$, then $\phi(\cdot,0)$ is a solution to Δ. If Δ contains no zero order term, then $\phi(\cdot,u)$ forms a one parameter family of solutions to Δ.

Proof

First consider the case where Δ has no zero order term. In terms of local coordinates on $J_n^*(Z,p)$, suppose $u^{(n)} \in J_n^*(Z,p;\mathcal{U})|_{(x,u)}$ has coordinates u_I, and let $u^{(n)}(\lambda)$ for $\lambda \in \mathbb{R}$ denote the jet over (x,u) with coordinates λu_I. Thus

$$\Delta(x,u,u^{(n)}(\lambda)) = \lambda\Delta(x,u,u^{(n)}).$$

The n-th prolongation of $\vec{v}$ at the point $(x,u,u^{(n)}(\lambda))$ will have coefficient functions $\phi^I(x,u,u^{(n)}(\lambda))$ and

$$pr^{(n)}\vec{v}[\Delta(x,u,u^{(n)}(\lambda))] = \sum a^I(x)\phi^I(x,u,u^{(n)}(\lambda)) + \sum\vec{v}(a^I(x))\lambda u_I = 0$$

whenever $\Delta(x,u,u^{(n)}) = 0$. Subtracting $\lambda pr^{(n)}\vec{v}[\Delta(x,u,u^{(n)})]$ from the above equation yields

$$\sum a^I(x)[\phi^I(x,u,u^{(n)}(\lambda)) - \lambda\phi^I(x,u,u^{(n)})] = 0$$

whenever $\Delta(x,u,u^{(n)}) = 0$. In particular, letting $\lambda=0$ and noting that $\phi^I(x,u,0) = \partial_I\phi(x,u)$ gives the result. The corresponding result for the case that Δ has a zero order term follows by a similar argument, replacing u by λu. Q.E.D.

The next stage is to give a geometric interpretation of the projectable symmetry group of a linear partial differential equation as a subgroup of the group of conformal transformations of its top order symbol. For simplicity assume that Z is an open subset of the trivial line bundle $\mathbb{1}_X$ over a p dimensional manifold X.

Definition 3.10 Let X be a smooth manifold and a: $X \to \bigodot_n TX$

a η-linear symmetric form on T^*X. A transformation $f\colon X \to X$ is called a conformal transformation of a if for any covectors $\omega_1,\ldots,\omega_k \in T^*X|_{f(x)}$

$$a[\delta f(\omega_1)\odot\ldots\odot\delta f(\omega_k),\ x) = \mu(x)\cdot a(\omega_1\odot\ldots\odot\omega_k,\ f(x))$$

for some real-valued function $\mu\colon X\to \mathbb{R}$.

A vector field $\vec{v}$ will be called an infinitesimal conformal transformation if the local transformations $\exp(t\vec{v})$ are conformal. The following lemma characterizes the infinitesimal conformal transformations of a smooth η-linear symmetric form.

Lemma 3.11 Let $x = (x^1,\ldots,x^p)$ be a local coordinate system on X. The vector field $\vec{v}= \Sigma\xi^i(x)\partial_i$ is an infinitesimal conformal transformation of $a = \Sigma a^I(x)\partial_I$ iff there is a function $\lambda\colon X \to \mathbb{R}$ such that†

$$\sum_{k=1}^{p}\Big[i_k a^I(x)\partial_k\xi^k(x) + \sum_{j\neq k}(i_j+1)a^{I-\delta^j+\delta^k}(x)\partial_j\xi^k(x)\Big] =$$

$$= \lambda(x)a^I(x) + \sum_{k=1}^{p}\partial_k a^I(x)\xi^k(x) \qquad I\in S^p_n.$$

Proof

Denote $f_t = \exp t\vec{v}$ so that $\frac{d}{dt}\big|_{t=0} = \vec{v}$. Now

$$\delta f_t(dx^i) = \sum_j \partial_j f^i_t(x)dx^j$$

hence, recalling that $\langle dx^I,\ \partial_I\rangle = I!$, for $I\in S^p_n$,

$$\frac{d}{dt}\Big|_{t=0} a(\delta f_t(dx^I),\ x) = \frac{d}{dt}\Big|_{t=0}\sum_{J\in S^p_n} J!\,\partial_J f^I_t(x)a^J(x)$$

---

†Here, as usual, the terms in the sum over j with $i_j=0$ are omitted.

$$= \sum_{j,k} (I-\delta^j+\delta^k)!\ i_j \partial_j \xi^k(x) a^{I-\delta^k-\delta^j}(x)$$

since $\partial_j f_0^k(x) = \delta_k^j \xi^k(x)$. On the other hand, this expression must equal

$$\frac{d}{dt}\Big|_{t=0} \mu_t(x) a(dx^I, f_t(x)) = I! \left[\lambda(x) a^I(x) + \sum_{k=1}^{p} \xi^k(x) \partial_k a^I(x)\right].$$

Dividing both equations by I! gives the result. The converse follows via exponentiation. Q.E.D.

Now let us look at the top order terms in the defining equations for the symmetry group of $\Delta(u)=0$. By lemma 3.6 and proposition 3.7, we have for $J \in S_n^p$, $n>2$

$$D_J \phi = \beta u_J + \sum_{\sigma=1}^{p} j_\sigma \partial_\sigma \beta u_{J-\delta^\sigma} + \mathcal{L}(n-1)$$

therefore

$$\phi^J = (\beta - \sum_\sigma j_\sigma \partial_\sigma \xi^\sigma) u_J - \sum_{\sigma \neq \tau} j_\sigma \partial_\sigma \xi^\tau u_{J-\delta^\sigma+\delta^\tau} + \mathcal{L}(n).$$

Now, since $\mu = \mu_0(x,u)$, the coefficient of u_J in (3.2) is

$$(\beta - \sum_\sigma j_\sigma \partial_\sigma \xi^\sigma) a^J(x) - \sum_{\sigma \neq \tau} (j_\tau + 1) \partial_\sigma \xi^\tau a^{J-\delta^\tau+\delta^\sigma}(x) + \sum_\sigma \partial_\sigma a^J(x) \xi^\sigma = \mu_0(x,u) a^J(x).$$

Note that this implies $\mu = \mu_0(x)$. Moreover, we have

<u>Proposition 3.12</u> The group of projectable symmetries of a linear partial differential equation in one dependent variable, when projected onto the space X of independent variables is a subgroup of the conformal group of the top order symbol of the equation.

III.4 Groups of Equivalent Systems

In previous sections, in examples I.3.1 and III.2.2, the symmetry groups for two equivalent forms of the heat equation, namely the first order system $u_x=v$; $v_x=u_t$ and the single second order equation $u_{xx}=u_t$ were calculated. Looking back on these examples, it can be noticed that the infinitesimal symmetries form isomorphic Lie algebras. In fact, more than this is true; the coefficient functions of ∂_v for all infinitesimal symmetries of the first order system are just the coefficient functions of ∂_{u_x} of the first prolongation of the corresponding infinitesimal symmetry of the second order equation when v is substituted for u_x. Thus, the symmetry group of the first order system can be considered as the first prolongation of the symmetry group of the second order equation. It is the aim of this section to investigate in what sense this phenomenon is true in general. It will be shown that barring the presence of "higher order symmetries" the symmetry group of a first order system is the prolongation of the symmetry group of an equivalent higher order equation (at least locally). The higher order symmetries are groups whose transformations depend on the derivatives as well as just the independent and dependent variables in the equation. This will all be made more precise in what is to follow. At the end of this section, an example of an equation which possesses higher order symmetries -- the wave equation -- will be considered.

We first need to describe what exactly is meant by the equivalent first order system to a partial differential equation in the language of extended jet bundles. Suppose $\Delta_0 \subset J_k^*(Z,p)$ is a k-th order system of partial differential equations. If $(u^1,\ldots,u^q)$ denote the dependent variables corresponding to some coordinate system (x,u) on Z, then the dependent variables in the equivalent first order system will be the induced coordinates on the jet bundle u_K^i for all multi-indices $K \in S^p$ with $0 \leq \Sigma K < k$. (Here we identify u_0^i with u^i.) In other words, we are considering Δ_0 as a first order system of equations over the new manifold $J_{k-1}^*(Z,p)$, i.e. as a subvariety of $J_1^*(J_{k-1}^*(Z,p),p)$. Using the embedding i_{k-1}^k given in corollary II.5.3 it is not hard to see that this first order system is nothing but

$$i_{k-1}^k(\Delta_0) \subset J_1^*(J_{k-1}^*(Z,p),p).$$

Example 4.1 Consider the manifold $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u) and let $k=2$. Suppose $\Delta_0 \subset J_2^*(Z,2)$ is given by the equations

$$\Delta^i(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \qquad i=1,\ldots,\alpha.$$

The local coordinates of $J_1^*(Z,2)$ will be denoted by (x,t,u,v,w), where v corrseponds to u_x and w to u_t. Then the local coordinates on $J_1^*(J_1^*(Z,2),2)$ are $(x,t,u,v,w,u_x,u_t,v_x,v_t,w_x,w_t)$ and the subbundle $J_2^*(Z,2)$ (which we will henceforth identify with $i_1^2(J_2^*(Z,2))$) is given by the equations

$$u_x = v \qquad u_t = w \qquad v_t = w_x. \tag{4.1}$$

Therefore the first order system corresponding to Δ_0 is given by equations (4.1) and the further equations

$$\Delta^i(x,t,u,v,w,v_x,w_x,w_t) = 0 \qquad i=1,\dots,\alpha.$$

For instance, in this set-up the first order system corresponding to the heat equation $u_{xx} = u_t$ is

$$u_x = v \qquad u_t = w \qquad v_t = w_x \qquad v_x = w. \tag{4.2}$$

This is not exactly the first order system previously considered. Some tedious calculations similar to those in example I.3.1 shows that the infinitesimal symmetry group of (4.2) is indeed the first prolongation of the infinitesimal symmetry group of the heat equation. It is not difficult to extend the results to be derived for equivalent systems of the form (4.2) to the more abbreviated systems of the form $u_x = v$; $v_x = u_t$; the details are left to the reader.

More generally, we can replace a system of k-th order partial differential equations by an equivalent system of $(k-\ell)$th order equations for some $1\leq\ell<k$. This is accomplished via the embedding i_ℓ^k from corollary II.5.3:

$$\Delta_0 \subset J_k^*(Z,p) \subset J_{k-\ell}^*(J_\ell^*(Z,p),p).$$

<u>Lemma 4.2</u> Suppose $\Delta_0 \subset J_k^*(Z,p)$ is a system of k-th order partial differential equations. Let $1\leq\ell<k$ and suppose $s' \subset J_\ell^*(Z,p)$ is a p-section such that $j_{k-\ell}^* s' \subset \Delta_0$, then locally $s' = j_\ell^* s$ for some p-section $s \subset Z$ which is a solution to Δ_0.

<u>Proof</u>

The proof of this lemma follows from the fact that if s' is a p-section of $J_\ell^*(Z,p)$ with $j_{k-\ell}^* s' \subset J_k^*(Z,p)$, then locally $s' = j_\ell^* s$, where $s = \pi_0^\ell(s')$, π_0^ℓ being the projection of $J_\ell^*(Z,p)$ onto Z. To prove this fact, we first use lemma II.5.9 to get that $\pi_0^\ell(s')$ is locally a p-section of Z. Then the local coordinate description of $j_{k-\ell}^* s'$ shows that $s' = j_\ell^* s$. Q.E.D.

Note that the projection $\pi_0^\ell(s')$ is not necessarily a global p-section of Z since there might be self-intersections. This lemma justifies the use of the word equivalent, since the smooth solutions of the higher and lower order equivalent systems are in one-to-one correspondence via the extended jet map. Now we are in a position to consider the symmetry groups of the equivalent systems. Suppose G is a local group of transformations acting on Z whose k-th prolongation leaves Δ_0 invariant. Using corollary II.5.4 we see that $J_k^*(Z,p)$ is an invariant subvariety of the transformation group $pr^{(k-\ell)}[pr^{(\ell)}G]$ acting on $J_{k-\ell}^*(J_\ell^*(Z,p),p)$ and

$$pr^{(k-\ell)}[pr^{(\ell)}G] \mid J_k^*(Z,p) = pr^{(k)}G.$$

We conclude that if G is a symmetry group of a k-th order system of partial differential equations, then $pr^{(\ell)}G$ is a symmetry group of the equivalent $(k-\ell)$-th order system.

Conversely, suppose G' is a local group of transformations acting on $J_\ell^*(Z,p)$ such that Δ_0 is an invariant subvariety of the prolonged action $pr^{(k-\ell)}G'$. An obvious necessary condition for G' to satisfy in order to be the prolongation of some group G acting on Z is that it be <u>projectable</u>, i.e. if $\pi_0^\ell(j) = \pi_0^\ell(j')$ for $j,j' \in J_\ell^*(Z,p)$ then $\pi_0^\ell(gj) = \pi_0^\ell(gj')$ for all $g \in G_j' \cap G_{j'}'$. The non-projectable groups will be called <u>higher order symmetries</u>, since while they do transform solutions of Δ_0 to solutions of Δ_0, the transformations depend on the derivatives of the solutions as well as the solution values themselves. They can be considered a special case of non-point transformations, cf. [O2]. We have already seen that the heat equation possesses no higher order symmetries (at least no first order symmetries); example 4.4 will discuss an equation that possesses some higher order symmetries.

A second criterion that the group G' must meet in order to be a prolongation is that Δ_0 is really a k-th order equation. For instance, if $\Delta_0 = (\pi_\ell^k)^{-1}[\Delta_0']$ for some ℓ-th order equation Δ_0', then any transformation of $J_\ell^*(Z,p)$ leaving Δ_0' invariant will preserve Δ_0, and the projectable ones are not expected to necessarily be prolonged group actions. What can be proven is summarized in the next theorem.

<u>Theorem 4.3</u> Let $1 \leq \ell < k$ be integers, and let

$$\Delta_0 \subset J_k^*(Z,p) \subset J_{k-\ell}^*(J_\ell^*(Z,p),p)$$

be a k-th order system of partial differential equations, with the equivalent $(k-\ell)$-th order system of equations. If G is a local group of transformations acting on Z such that Δ_0 is invariant under the prolonged group action $pr^{(k)}G$, then letting $G' = pr^{(\ell)}G$, Δ_0 is invariant under $pr^{(k-\ell)}G'$. Conversely, if G' is a local group of transformations acting projectably on $J_\ell^*(Z,p)$ such that Δ_0 is invariant under $pr^{(k-\ell)}G'$, then letting G be the projected group action on Z, $pr^{(\ell)}G$ agrees locally with G' on $\pi_\ell^k[\Delta_0]$, which is a G' invariant subvariety of $J_\ell^*(Z,p)$.

<u>Proof</u>

The first statement has already been demonstrated in the remarks preceding the statement of the theorem. To prove the converse, note that it suffices to show $\mathfrak{g}'$, the algebra of infinitesimal generators of G', agrees with $pr^{(\ell)}\mathfrak{g}$, the ℓ-th prolongation of the algebra of infinitesimal generators of G. To do this, it suffices to check that if $\vec{v}'$ is any projectable vector field on $J_\ell^*(Z,p)$ with projection $\vec{v}$ on Z such that Δ_0 is invariant under $pr^{(k-\ell)}\vec{v}'$, then $pr^{(\ell)}\vec{v} = \vec{v}'$ on $\pi_\ell^k[\Delta_0]$.

Choose local coordinates (x,u) on Z with induced local coordinates $(x,u,u^{(k)})$ on $J_k^*(Z,p)$ and $(x,u,u^{(\ell)})$ on $J_\ell^*(Z,p)$ so that $\pi_\ell^k(x,u,u^{(k)}) = (x,u,u^{(\ell)})$. The induced local coordinates on $J_{k-\ell}^*(J_\ell^*(Z,p),p)$ are given by $(x,u,u^{(\ell)},u^{(k-\ell)},(u^{(\ell)})^{(k-\ell)})$, the individual matrix entries given by $(u_J^i)_K$ for all $1\le i\le q$, $J,K \in S^p$, with $0\le\Sigma J\le\ell$, $0\le\Sigma K\le k-\ell$, where as usual we identify u_0^i with u^i and $(u_J^i)_0$ with u_J^i. By corollary II.5.3, $J_k^*(Z,p)$ is given by the equations

$$(u_J^i)_K = (u_{J'}^i)_{K'} \qquad i=1,\ldots,q, \quad J+K = J'+K'.$$

Now let $\vec{v}'$ be given in local coordinates by

$$\vec{v}' = \sum \xi^j \frac{\partial}{\partial x^j} + \sum \phi_i \frac{\partial}{\partial u^i} + \sum \phi_i^J \frac{\partial}{\partial u_J^i}$$

so that

$$\vec{v} = \sum \xi^j \frac{\partial}{\partial x^j} + \sum \phi_i \frac{\partial}{\partial u^i}$$

and

$$\mathrm{pr}^{(k-\ell)}\vec{v}' = \vec{v}' + \sum (\phi_i^J)^K \frac{\partial}{\partial (u_J^i)_K}$$

the $(\phi_i^J)^K$ being given by the prolongation formula. Note that since $\vec{v}'$ is projectable, ξ^j and ϕ_i are functions of (x,u) alone.

Now applying the infinitesimal criterion of invariance to Δ_0 considered as a $(k-\ell)$-th order system, we must have

$$(\phi_i^J)^K(\hat{j}) = (\phi_i^{J'})^{K'}(\hat{j}) \qquad i=1,\ldots,q, \quad J+K = J'+K'$$

whenever $\hat{j} \in \Delta_0$. In particular, letting $K = \delta^\sigma$, $K' = 0$ for some $1 \leq \sigma \leq p$, then the prolongation formula implies that

$$D_\sigma \phi_i^J - \sum_\tau (u_J^i)_\tau D_\sigma \xi^\tau = \phi_i^{J+\delta^\sigma}$$

on Δ_0 for all $0 \leq \Sigma J \leq \ell-1$. Note that these are precisely the recursion relations for the prolongation of vector fields as given in corollary 1.3 since $\Delta_0 \subset J_k^*(Z,p)$, which proves the theorem. Q.E.D.

Note that the theorem does not imply that G' and $\mathrm{pr}^{(\ell)}G$ agree everywhere on $J_\ell^*(Z,p)$ for instance, in the coordinates given in example 4.1, we could have an equation invariant under the transformation $(x,t,u,v,w) \to (x,t,u,w,v)$, but this projects to the identity transformation on $Z = \mathbb{R}^2 \times \mathbb{R}$, so is not a prolongation.

<u>Example 4.4</u> The Wave Equation.

Let $Z = \mathbb{R}^2 \times \mathbb{R}$ with coordinates (x,t,u) and consider the second order equation $\Delta_0 \subset J_2^*(Z,2)$ given by

$$u_{xx} = u_{tt}. \tag{4.3}$$

The equivalent first order system for Δ_0 is given by

$$
\begin{aligned}
u_x &= v \\
u_t &= w \\
v_t &= w_x \\
v_x &= w_t
\end{aligned}
\tag{4.4}
$$

where (x,t,u,v,w) are local coordinates on $J_1^*(Z,2)$. In this example the symmetry group of the first order system (4.4) will be computed. Let $\vec{v}' = \xi\partial_x + \tau\partial_t + \phi\partial_u + \psi\partial_v + \chi\partial_w$ be a vector field on $J_1^*(Z,2)$ with first prolongation

$$pr^{(1)}\vec{v}' = \vec{v}' + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \psi^x\partial_{v_x} + \psi^t\partial_{v_t} + \chi^x\partial_{w_x} + \chi^t\partial_{w_t}.$$

The defining equations for the symmetry group of the wave equation are therefore

$$
\begin{aligned}
\phi^x &= \psi \\
\phi^t &= \chi \\
\psi^t &= \chi^x \\
\psi^x &= \chi^t
\end{aligned}
\tag{4.5}
$$

which hold whenever (4.4) holds. Using the prolongation fromula and substituting (4.4) into (4.5) gives rise to the following system of equations for the coefficient functions:

$$\phi_x + v\phi_u - v\xi_x - v^2\xi_u - w\tau_x - vw\tau_u = \psi \tag{4.6}$$

$$\phi_t + w\phi_u - v\xi_t - vw\xi_u - w\tau_t - w^2\tau_u = \chi \tag{4.7}$$

$$\phi_v - v\xi_v - w\tau_v = 0 \tag{4.8}$$

$$\phi_w - v\xi_w - w\tau_w = 0 \tag{4.9}$$

$$\psi_x + v\psi_u = \chi_t + w\chi_u \tag{4.10}$$

$$\psi_t + w\psi_u = \chi_x + v\chi_u \tag{4.11}$$

$$\xi_w = \tau_v \tag{4.12}$$

$$\xi_v = \tau_w \tag{4.13}$$

$$\psi_v - \xi_x - v\xi_u = \chi_w - \tau_t - w\tau_u \tag{4.14}$$

$$\psi_w - \tau_x - v\tau_u = \chi_v - \xi_t - w\xi_u \tag{4.15}$$

$$\psi_w - \xi_t - w\xi_u = \chi_v - \tau_x - v\tau_u \tag{4.16}$$

$$\psi_v - \tau_t - w\tau_u = \chi_w - \xi_x - v\xi_u. \tag{4.17}$$

Now (4.14-17) imply

$$\psi_v = \chi_w \tag{4.18}$$

$$\psi_w = \chi_v \tag{4.19}$$

$$\xi_x + v\xi_u = \tau_t + w\tau_u \tag{4.20}$$

$$\xi_t + w\xi_u = \tau_x + v\tau_u. \tag{4.21}$$

Let $\alpha = v+w$, $\beta = v-w$, then by (4.12-13) we can represent

$$\xi = f'(\alpha) + g'(\beta) \tag{4.22}$$

$$\tau = f'(\alpha) - g'(\beta) + 2h(x,t,u) \tag{4.23}$$

where f and g further depend on (x,t,u) and the primes indicate derivatives with respect to α or β as the case may be. Next, (4.6-9) show that

$$\phi = \alpha f'(\alpha) - f(\alpha) + \beta g'(\beta) - g(\beta) + k(x,t,u) \tag{4.24}$$

$$\psi = -(f_x + g_x) - v(f_u + g_u) + k_x + vk_u - 2wh_x - 2vwh_u \tag{4.25}$$

$$\chi = -(f_t + g_t) - w(f_u + g_u) + k_t + wk_u - 2wh_t - 2w^2h_u. \tag{4.26}$$

Substituting these expressions into (4.18-19) and taking derivatives with respect to α and β gives

$$f_x'''(\alpha) - f_t'''(\alpha) + \beta f_u'''(\alpha) = 0$$
$$g_x'''(\beta) + g_t'''(\beta) + \alpha g_u'''(\beta) = 0$$

so that f and g are of the form

$$f = f^*(\eta,\alpha) + f^2(x,t,u)\alpha^2 + f^1(x,t,u)\alpha + f^0(x,t,u) \tag{4.27}$$
$$g = g^*(\zeta,\beta) + g^2(x,t,u)\beta^2 + g^1(x,t,u)\beta + g^0(x,t,u) \tag{4.28}$$

where $\eta = x+t$ and $\zeta = x-t$. Resubstituting into (4.18-19), after a little algebra, we get

$$f_u^2 = g_u^2 = 0 \qquad f_x^2 = f_t^2 \qquad g_x^2 = -g_t^2 \tag{4.29}$$
$$f_u^1 = -h_u \qquad f_\zeta^1 = -h_\zeta \tag{4.30}$$
$$g_u^1 = h_u \qquad g_\eta^1 = h_\eta . \tag{4.31}$$

Now (4.29) imply that the f^2 and g^2 terms in (4.27-28) can be incorporated into the f* and g* terms respectively. Moreover, by (4.30-31) we can assume, again by incorporating excess terms into f* and g*, that $f^1 = -h$ and $g^1 = h$. Therefore f and g are of the form

$$f = f^*(\eta,\alpha) - h(x,t,u)\alpha + f^0(x,t,u) \tag{4.32}$$
$$g = g^*(\zeta,\beta) + h(x,t,u)\beta + g^0(x,t,u). \tag{4.33}$$

It is an easy matter to check that equations (4.11) and (4.20-21) are now satisfied identically. Equation (4.10) gives

$$h_{uu} = 0 \tag{4.34}$$

$$\begin{array}{ll} f_{uu}^0 + g_{uu}^0 = k_{uu} & f_{\zeta u}^0 + g_{\zeta u}^0 = k_{\zeta u} \\ f_{\eta u}^0 + g_{\eta u}^0 = k_{\eta u} & f_{\eta\zeta}^0 + g_{\eta\zeta}^0 = k_{\eta\zeta} . \end{array} \tag{4.35}$$

By (4.34) we can represent

$$h = c(x,t)u + d(x,t) \tag{4.36}$$

and (4.35) shows that

$$f^0 + g^0 = k - c_0 u - a(\eta) - b(\zeta) \tag{4.37}$$

where c_0 is a constant. Therefore, using (4.6-7,22-24) it can be seen the the infinitesimal symmetry algebra of (4.4) is the space of all vector fields $\xi\partial_x + \tau\partial_t + \phi\partial_u + \psi\partial_v + \chi\partial_w$ with

$$\begin{aligned}
\xi &= f_1(v+w,x+t) + g_1(v-w,x-t) \\
\tau &= f_1(v+w,x+t) - g_1(v-w,x-t) \\
\phi &= (v+w)f_1(v+w,x+t) - f(v+w,x+t) + (v-w)g_1(v-w,x-t) - \\
&\quad - g(v-w,x-t) + cu + a(v+w) + b(v-w) \qquad (4.38) \\
\psi &= - f_2(v+w,x+t) - g_2(v-w,x-t) + cv \\
\chi &= - f_2(v+w,x+t) + g_2(v-w,x-t) + cw
\end{aligned}$$

where f and g are arbitrary functions of two variables (the subscripts indicating partial derivatives with respect to the variables), a and b arbitrary functions of a single variable and c an arbitrary constant.

To see what is going on with these higher order symmetries, let us consider a specific example. Let

$$f(\alpha,\eta) = \frac{1}{4}\alpha^2 \qquad g(\beta,\zeta) = \frac{1}{4}\beta^2 \qquad a = b = c = 0$$

so the vector field under consideration is

$$\vec{v}_0' = v\partial_x + w\partial_t + \frac{1}{2}(v^2 + w^2)\partial_u .$$

The one parameter group generated by $\vec{v}_0'$ is

$$G_0': \quad (x,t,u,v,w) \to (x + \lambda t, t + \lambda w,\ u + \frac{\lambda}{2}(v^2+w^2), v,w) \quad \lambda \in \mathbb{R}.$$

What this means for the solutions of the wave equation is that if $u = F(x,t)$ is a solution to the wave equation $u_{xx} - u_{tt} = 0$, if we solve the implicit equations

$$v = F_x(x + \lambda v,\ t + \lambda w) \qquad w = F_t(x + \lambda v,\ t + \lambda w)$$

for v,w then the invariance under G_0' implies that

$$u = F(x + \lambda v,\ t + \lambda w) - \frac{\lambda}{2}(v^2 + w^2)$$

gives a one parameter family of solutions. This may be verified directly by taking derivatives.

IV. Group Invariant Solutions

In the last chapter of this thesis, we prove and make applications of theorem 1.6 on the construction of the group invariant solutions to higher order systems of partial differential equations. Let Z be a manifold, G be a regular group of transformations acting on Z and Δ_0 a k-th order system of partial differential equations on Z, which is invariant under the prolonged action of G. The first section of this chapter gives a characterization of the subbundle of $J_k^*(Z,p)$ given by the extended k-jets of G invariant p-sections of Z. It is then shown that fiberwise this subbundle is isomorphic to the extended jet bundle $J_k^*(Z/G,p-\ell)$, where ℓ is the dimension of the orbits of G. This result is immediately applied to prove the existence of a k-th order system of partial differential equations Δ_0/G for $(p-\ell)$-sections of Z/G, whose solutions provide the G invariant solutions of Δ_0 when pulled back to Z. In section 2 we apply this theorem to construct interesting group invariant solutions to the Korteweg-deVries equation and Burgers' equation. Section 3 takes up the problem of explicit solutions in the context of symmetry groups. Its purpose is to provide readily verifiable conditions on the group action in the case Z is a vector bundle which a) take explicit sections of the bundle to explicit sections, and b) ensure that the group invariant sections are explicit sections of the bundle. Criterion a) is just the projectibility (or compatibility) criterion; criterion b) is implied by the transversality of the group action to the fibers.

IV.1 <u>The Fundamental Theorem</u>

Suppose Z is a smooth manifold of dimension p+q and that G is a local group of transformations acting regularly on Z. If G has ℓ dimensional orbits then the quotient space Z/G has the structure of a smooth manifold of dimension $p+q-\ell$. The appendix should be consulted for more detail on this construction. Let $\pi_G: Z \to Z/G$ denote the standard projection which associates to each point of Z the G orbit that it lies in.

<u>Definition 1.1</u> A <u>locally G-invariant p-section</u> of Z is a p dimensional submanifold $s \subset Z$ such that for each point $z \in s$ there is a nucleus of G, $N_z \subset G_z$ with the property that any transformation $g \in N_z$ satisfies $g \cdot z \in s$.

A global G-invariant p-section is a p dimensional submanifold $s \subset Z$ that is left invariant by all the transformations in G. If $N_z = G_z$ for all points $z \in s$, then s is a globally G-invariant section. Since G acts regularly, any locally G-invariant p-section can be extended to a global G-invariant p-section simply by taking its saturation. Note that a necessary condition for G to admit invariant p-sections is that $p \geq \ell$ and in this case, the global invariant p-sections are in one-to-one corrsepondence via the projection π_G with the $p-\ell$ dimensional submanifolds of Z/G.

The main goal of this section is to provide an easy characterization of the subbundle of $J_k^*(Z,p)$ given by the extended k-jets of G-invariant p-sections of Z and to show that it is isomorphic to the inverse image of the extended k-jet bundle $J_k^*(Z/G, p-\ell)$ under the projection π_G. This in turn will yield as a direct corollary the fundamental theorem on the existence of G-invariant solutions to a system of partial differential equations on Z that is invariant under the prolonged action of G. The first step in this program is the following elementary lemma.

Lemma 1.2 If $s \subset Z$ is a global G-invariant p-section, then its projection $s/G = \pi_G(s) \subset Z/G$ is a $(p-\ell)$-section of Z/G. Conversely, if $s/G \subset Z/G$ is a $(p-\ell)$-section of Z/G, then $\pi_G^{-1}(s/G)$ is a global G-invariant p-section of Z. Moreover, $d^k\pi_G$ maps $\mathfrak{J}_k s|_z$ onto $\mathfrak{J}_k(s/G)|_{\pi_G z}$.

Given a section ω of the k-th order tangent bundle $\mathfrak{J}_k Z$, then for each $z \in Z$ there is a well-defined map

$$\cdot \odot \omega|_z : \mathfrak{J}_\ell Z|_z \to \mathfrak{J}_{k+\ell} Z|_z$$

depending smoothly on z and given by the formula

$$\nu \odot \omega|_z(f) = \nu[\omega(f(z))] \qquad \nu \in \mathfrak{J}_\ell Z|_z, \quad f \in C^\infty(Z, \mathbb{R}).$$

In future, the dependence of this map on z will be suppressed, so the above formula is more succinctly written $\nu \odot \omega(f) = \nu[\omega(f)]$. If Ω is a k-prolonged differential system, i.e. a vector subbundle of $\mathfrak{J}_k Z$, and Λ a vector subspace of $\mathfrak{J}_\ell Z|_z$, then $\Lambda \odot \Omega$ will denote the vector subspace of $\mathfrak{J}_{k+\ell} Z|_z$ spanned by all $\lambda \odot \omega$ for $\lambda \in \Lambda$ and ω a section of Ω. If Ω' is an ℓ-prolonged differential system, then $\Omega \odot \Omega'$ is a

$(k+\ell)$-prolonged differential system. Note that $\Omega \odot \Omega' \neq \Omega' \odot \Omega$. Let $\mathfrak{g}$ denote the involutive differential system on Z spanned by the infinitesimal generators of G. There is a simple characterization of the k-th prolongation of $\mathfrak{g}$ in the sense of [SS], namely

$$\begin{aligned} \mathfrak{g}^{(1)} &= \mathfrak{g} \\ \mathfrak{g}^{(k)} &= \mathfrak{g} + \mathfrak{J}_{k-1}Z \odot \mathfrak{g} \\ &= \mathfrak{g}^{(k-1)} + TZ \odot \mathfrak{g}^{(k-1)}. \end{aligned}$$

<u>Definition 1.3</u> The <u>G-invariant</u> <u>k-jet</u> <u>subbundle</u> of p-sections of Z is defined by

$$\begin{aligned} \mathrm{Inv}^{(k)}(G,p) &= \{\Lambda \in \mathrm{Grass}^{(k)}(\mathfrak{J}_k Z,p):\ \Lambda_1 \supset \mathfrak{g},\ \Lambda_{i+1} \supset \Lambda_i \odot \mathfrak{g} \quad i=1,\dots,k-1\} \\ &\subset \mathrm{Grass}^{(k)}(\mathfrak{J}_k Z,p) \simeq J_k^*(Z,p). \end{aligned}$$

<u>Lemma 1.4</u> Suppose $\Lambda \in \mathrm{Grass}^{(k)}(\mathfrak{J}_k Z,p)|_z$ and M is a submanifold of Z passing through z with $\mathfrak{J}_k M|_z \subset \Lambda$. Then there exists a submanifold s of Z passing through z with $M \subset s$ in some small neighborhood of z and $\mathfrak{J}_k s|_z = \Lambda$.

<u>Proof</u>

Choose local coordinates $(z^1,\dots,z^{p+q})$ centered at z such that Λ is spanned by $\{\partial_I : \Sigma I \le k,\ i_{p+1}=\dots=i_{p+q}=0\}$ and $\mathfrak{J}_k M|_z$ by $\{\partial_I : \Sigma I \le k,\ i_{\ell+1}=\dots=i_{p+q}=0\}$. Let $\pi_\ell(z)=(z^1,\dots,z^\ell,0,\dots,0,z^{p+1},\dots,z^{p+q})$ and define $s = \{(z^1,\dots,z^{p+q}) : \pi_\ell(z^1,\dots,z^{p+q}) \in M\}$. Q.E.D.

The next theorem shows that $\mathrm{Inv}^{(k)}(G,p)$ actually does represent the subbundle of G-invariant p-sections and gives its "identification" with the extended k-jet bundle of $(p-\ell)$-sections of Z/G.

Theorem 1.5 There is a natural map

$$\pi_G^{(k)}: \mathrm{Inv}^{(k)}(G,p) \to J_k^*(Z/G,p-\ell)$$

with the following properties:

i) $\pi_G^{(k)}: \mathrm{Inv}^{(k)}(G,p)|_z \xrightarrow{\sim} J_k^*(Z/G,p-\ell)|_{\pi_G z}$ gives an isomorphism of fibers.

ii) Given a p-section $s \subset Z$, then there exists a $(p-\ell)$-section $s/G \subset Z/G$ with $\pi_G(s) = s/G$ iff $j_k^* s \subset \mathrm{Inv}^{(k)}(G,p)$, in which case $\pi_G^{(k)}(j_k^* s) = j_k^*(s/G)$.

Proof

First suppose that $j_k^* s \subset \mathrm{Inv}^{(k)}(G,p)$. In particular this implies that for each $z \in s$, $Ts|_z \supset \mathfrak{g}|_z$, which implies that s is locally G-invariant.

Conversely, suppose s is a G-invariant p-section of Z with image $\pi_G(s) = s/G$ a $(p-\ell)$-section of Z/G. Choose local coordinates $(z^1,\ldots,z^{p+q})$ centered at $z \in s$ so that $\mathfrak{g}$ is spanned by $\{\partial_1,\ldots,\partial_\ell\}$ and s is given by $\{z: z^{p+1}=\ldots=z^{p+q}=0\}$. Thus $\mathfrak{J}_i s|_z$ is spanned by $\{\partial_I|_z: \Sigma I \le i,\ i_{p+1}=\ldots=i_{p+q}=0\}$ and hence $\mathfrak{J}_{i+1} s|_z \supset \mathfrak{J}_i s|_z \otimes \mathfrak{g}$ for all i. Finally, given $\Lambda \in \mathrm{Inv}^{(k)}(G,p)|_z$, by lemma 1.4 there exists a locally G-invariant p-section s of Z with $\mathfrak{J}_k s|_z = \Lambda$. Define $\pi_G^{(k)}(\Lambda) = d^k\pi_G(\Lambda)$ so that by lemma 1.2, $\pi_G^{(k)}(j_k^* s|_z) = j_k^*(s/G)|_{\pi_G z}$. Q.E.D.

Theorem 1.6 Let Δ be a k-th order system of partial differential equations on Z so that $\Delta' = \Delta \cap \mathrm{Inv}^{(k)}(G,p)$ is the corresponding system of partial differential equations for G-invariant solutions to Δ. If

Δ' is invariant under the prolonged group action $pr^{(k)}G$, then a p-section s of Z is a G-invariant solution to Δ iff s/G is a solution to the reduced differential equation $\Delta/G = \pi_G^{(k)}(\Delta')$. In particular, if Δ is G-invariant, then Δ' is also G-invariant.

Proof

If s is a G-invariant solution to Δ, then $j_k^* s \subset \Delta \cap \mathrm{Inv}^{(k)}(G,p)$ by the previous theorem, and therefore $\pi_G^{(k)}(j_k^* s) = j_k^*(s/G) \subset \Delta/G$. Conversely, given s/G, a solution to Δ/G, then let s be the corresponding G-invariant p-section of Z. By the isomorphism given in i) of theorem 1.5, $j_k^*(s/G)|_{\pi_G Z} \in \Delta/G|_{\pi_G Z}$ implies $j_k^* s|_Z \in \Delta'|_Z \subset \Delta|_Z$, giving the result. Finally, the last statement of the theorem follows from the fact that $\mathrm{Inv}^{(k)}(G,p)$ is a $pr^{(k)}G$ invariant subbundle of $J_k^*(Z,p)$. Q.E.D.

To show that this theorem is the optimum result on the existence of a system of partial differential equations Δ/G on Z/G whose solutions give the G-invariant solutions to Δ, we briefly consider a few elementary examples. On the manifold $Z = \mathbb{R}^3$ with coordinates (x,y,u) consider the first order equation $\Delta = \{xu_x + u_y = 0\}$. Let G be the one parameter group of translations in the x-coordinate. The equation Δ is not G-invariant, but $\Delta' = \{u_x=0,\ u_y=0\}$ is and admits the G-invariant solutions u=constant. With G and Z the same, consider the equation $\Delta = \{u_y - xu - x^2u_x = 0\}$, which is again not G-invariant. Δ' is also not G-invariant, but it admits the solution $u=0$. In this case $\Delta'|\{u=0\}$ is G-invariant. However, even this is not necessarily true as the example $\Delta = \{u_y - xu = xu_z\}$ on $\mathbb{R}^4$ with G being translation in the x-coordinate shows. Again u=0 is a G-invariant solution to Δ, but $\Delta'|\{u=0\}$ is not G-invariant.

IV.2 Examples of Invariant Solutions

Given a differential equation $\Delta_0 \subset J_k^*(Z,p)$ and a group G acting regularly on Z with ℓ-dimensional leaves leaving Δ_0 invariant, the construction of the differential equation $\Delta_0/G \subset J_k^*(Z/G, p-\ell)$ will be illustrated in this section with a couple of practical examples: the Korteweg-deVries equation and Burgers' equation. Further interesting examples may be found in [BC2] and [O1]. The first step in this construction is to find suitable local coordinates on the quotient manifold Z/G . These will be provided by what are classically known as "a complete set of functionally independent invariants" of the group G. An invariant is a smooth (local) function $\bar{f} : Z \to \mathbb{R}$ such that $\bar{f}(gz) = \bar{f}(z)$ for all $z \in Z$, $g \in G_z$. The invariance property of such a function f means that it factors through Z/G :

$$\begin{tikzcd} Z \arrow[rr, "\bar{f}"] \arrow[dr, "\pi_G"'] & & \mathbb{R} \\ & Z/G \arrow[ur, "f"'] & \end{tikzcd}$$

It is well known that for a group acting regularly with ℓ dimensional leaves on an n-dimensional manifold, then locally there are $n-\ell$ functionally independent invariants, which form a local coordinate system on Z/G .

Let $(\xi^1,\ldots,\xi^{p-\ell},\zeta^1,\ldots,\zeta^q) = (\xi,\zeta)$ local coordinates on Z/G , so that $\bar{\xi}^i = \xi^i \circ \pi_G$ and $\bar{\zeta}^i = \zeta^i \circ \pi_G$ are the invariants on Z .

Let (x,u) be local coordinates on Z. It may be assumed without loss of generality that G acts transversally to the fibers $\{x = \text{const}\}$ since otherwise there are no G-invariant sections of the form $u = u(x)$. Now suppose s/G is a $(p-\ell)$ - section of Z/G transversal to the fibers $\{\xi = \text{const}\}$ with corresponding G-invariant p-section $s \subset Z$ which is assumed to be transversal to the fibers $\{x = \text{const}\}$. Therefore we can write $s = \{(x,u(x))\}$ and $s/G = \{(\xi,\zeta(\xi))\}$ for smooth functions u and ζ, so that

$$j_k^* s = \{(x,u(x),\partial_*^k u(x))\}$$

$$j_k^*(s/G) = \{(\xi,\zeta(\xi),\partial_*^k \zeta(\xi))\} \quad .$$

Now the projection π_G gives $\xi = \xi(x,u)$, $\zeta = \zeta(x,u)$, so substituting in the equation $\zeta = \zeta(\xi)$ gives upon differentiation

$$D_*^k \zeta(x,u,u^{(k)}) = \partial_*^k \zeta(\xi) \cdot \varepsilon_k [D_*^k \xi(x,u,u^{(k)})] \; .$$

Now, the transversality condition assures that this equation can be solved for

$$u^{(k)} = \omega_*^k(x,u,\xi,\zeta,\zeta^{(k)}) \tag{2.1}$$

Now suppose the G-invariant equation Δ_o is given by

$$\Delta^i(x,u,u^{(k)}) = 0 \qquad i = 1,\ldots,\alpha \quad .$$

The equation Δ_o/G is found by substituting the expressions (2.1) for the derivatives of u, giving

$$\tilde{\Delta}^i(x,u,\xi,\zeta,\zeta^{(k)}) = 0 \qquad i = 1,\dots,\alpha .$$

Now theorem 1.1 assures us that these equations can be written in the form

$$c_i(x) \cdot \hat{\Delta}^i(\xi,\zeta,\zeta^{(k)}) = 0 \qquad i = 1,\dots,\alpha .$$

hence the equation Δ_o/G is given by

$$\hat{\Delta}^i(\xi,\zeta,\zeta^{(k)}) = 0 \quad . \qquad i = 1,\dots,\alpha$$

This process will become clearer in the following examples.

Example 2.1 The Korteweg-deVries Equation

We again consider the equation

$$\Delta_o : u_t + uu_x + u_{xxx} = 0 \tag{2.2}$$

of example III. 2.4, the symmetry group of which is given by (III.2.16). We shall investigate some of the one parameter subgroups of the symmetry group and construct the group invariant solutions corresponding to these subgroups by solving the resulting ordinary differential equations. As a first example consider the vector field $c\partial_x + \partial_t$ where c is a real constant. The corresponding group is given by

$$G_c : (x,t,u) \mapsto (x+\lambda c, t+\lambda, u) \qquad \lambda \in \mathbb{R}$$

which has independent invariants u and $\xi = x - ct$. The invariant solutions will be travelling waves moving with velocity c. We have

$$u_t = -cu' \qquad u_x = u' \qquad u_{xxx} = u'''$$

where the primes mean derivatives with respect to ξ. The equation Δ_0/G_c on $\mathbb{R}^3/G_c \simeq \mathbb{R}^2$ for the G_c-invariant solutions is then

$$u''' + (u-c)u' = 0$$

which integrates to

$$u'' + \frac{1}{2}u^2 - cu + \frac{1}{2}k_0 = 0 .$$

Multiplying by $2u'$ and integrating again gives

$$(u')^2 + \frac{1}{3}u^3 - cu^2 + k_0 u + k_1 = 0$$

for constants k_0, k_1. Thus the general G_c-solution will be an elliptic function of the variable ξ. In the special case that u, u_x, $u_{xx} \to 0$ as $|x| \to \infty$, then $k_0 = k_1 = 0$ then we obtain the soliton solution for $c > 0$:

$$u(x,t) = 3c \cdot \operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct) + \delta\right]$$

for δ an arbitrary phase shift. More generally, if we require only that u be bounded, then the cnoidal wave solution results:

$$u(x,t) = 3c \cdot \operatorname{cn}^2\left[\frac{\sqrt{c}}{2}(x-ct) + \delta\right]$$

where the modulus of the Jacobian elliptic function is 1 (see [WH; §13.12]). The general G_c-invariant solution will be a more complicated elliptic function.

Next consider the vector field $\vec{v}_3 = t\partial_x + \partial_u$ whose one parameter group is given in (III.2.17). Coordinates on $\mathbb{R}^3/G_3 \simeq \mathbb{R}^2$ are given by the invariants t and $\zeta = u - x/t$. Then

$$u_t = \zeta' - \frac{x}{t^2} \qquad u_x = \frac{1}{t} \qquad u_{xxx} = 0$$

so the equation Δ_0/G_3 is just

$$t\zeta' + \zeta = 0 .$$

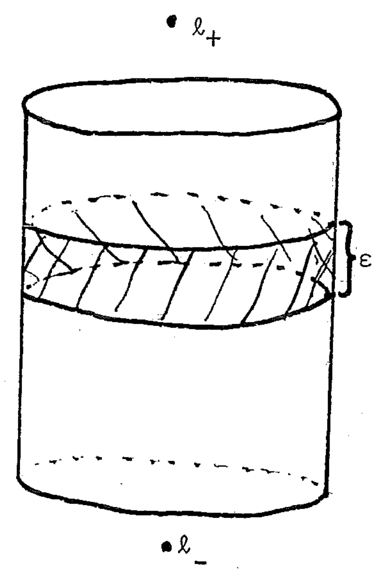
which is of first order. Therefore the general G_3-invariant solution to Δ_0 is

$$u(x,t) = \frac{x+k_0}{t}$$

for some constant k_0 . Similarly it can be shown that the invariant solutions for the infinitesimal operator $(a+t)\partial_x + \partial_u$ are

$$u(x,t) = \frac{x+k_0}{t+a} .$$

As an illustration of the unfortunate fact that while the equation for a symmetric solution will involve fewer independent variables, it does not necessarily have to be "simpler" than the original equation, we attempt to construct the scale invariant solutions corresponding to the vector field $\vec{v}_4 = x\partial_x + 3t\partial_t - 2u\partial_u$. The one parameter group is denoted by G_4 - (III.2.17). To let G_4 act regularly, we must restrict to the submanifold $Z' = \mathbb{R}^3 \sim \{0\}$. The quotient manifold Z'/G_4 is non-Hausdorff; it can be realized as a cylinder $S^1 \times \mathbb{R}$ together with two exceptional points corresponding to the vertical leaves $\ell_+ = \{x=t=0, u>0\}$ and $\ell_- = \{x=t=0, u<0\}$. If (θ,h) are the coordinates on $S^1 \times \mathbb{R}$, then a neighborhood basis of ℓ_+ (resp. ℓ_-) is given by $\{\ell_+\} \cup \{(\theta,h) : 0 < h < \varepsilon\}$ (resp. $\{\ell_-\} \cup \{(\theta,h) : -\varepsilon < h < 0\}$) for all positive numbers ε. A G_4-invariant solution of the Korteweg-deVries equation corresponds to a curve in Z'/G_4 that is a solution to Δ_0/G_4. Note that if the curve passes through either of the exceptional points, the corresponding G_4-invariant solution is not an explicit function of (x,t), so that we may safely ignore the exceptional points and concentrate on the Hausdorff submanifold $S^1 \times \mathbb{R} \subset Z'/G_4$. The equation Δ_0/G_4, however, becomes a highly non-linear third order ordinary differential equation. For instance, in the local coordinates



$$\zeta = x^2 u \qquad \xi = x^3 t^{-1}$$

treating ξ as the independent variable

$$u_t = -xt^{-2}\zeta'$$

$$u_x = -2x^{-3}\zeta + 3t^{-1}\zeta'$$

$$u_{xxx} = -24x^{-5}\zeta + 24x^{-2}t^{-1}\zeta' + 27x^4t^{-3}\zeta'''$$

hence in these coordinates Δ_o/G_4 is

$$27\xi^3\zeta'''+(24-\xi+3\zeta)\xi\zeta' - 2\zeta(\zeta+12) = 0$$

which is in some sense a considerably more complicated equation than the Korteweg-deVries equation, even though it is an ordinary differential equation. Choosing other coordinates on Z'/G_4 only seems to make matters worse. It would be interesting to try to solve this equation numerically to get some idea of how the G_4-invariant solutions of Δ_o evolve in time.

Example 2.2 Burgers' Equation

Consider the equation

$$\Delta_o : u_t + uu_x + u_{xx} = 0$$

whose symmetry group was calculated in example III.2.1, and given in (III.2.7). Various one-parameter subgroups of this group will be considered and the invariant solutions corresponding to them will be derived. As a first example, the travelling wave solutions will be found. These correspond to the vector field $c\partial_x + \partial_t$, where c is the wave velocity. This exponentiates to the group action

$$G_c : (x,t,u) \longmapsto (x+\lambda c, t+\lambda, u) \qquad \lambda \in \mathbb{R}$$

and has invariants u and $\xi = x - ct$. The equation Δ_0/G_c is then

$$u'' + (u-c)u' = 0$$

which has a first integral

$$u' + \tfrac{1}{2}u^2 - cu + k_0 = 0 \ .$$

Let $d = 2k_0 - c^2$, then we get the G_c-invariant solutions

$$u(x,t) = \begin{cases} \sqrt{d}\,\tan[\tfrac{1}{2}\sqrt{d}(ct-x+\delta)] + c & d > 0 \\ 2(x-ct+\delta)^{-1} + c & d = 0 \\ \sqrt{-d}\,\tanh[\tfrac{\sqrt{-d}}{2}(x-ct+\delta)] + c & d < 0 \end{cases}$$

where δ is a constant. For the vector field $(a+t)\partial_x + \partial_u$, an argument similar to the one given for the Korteweg-deVries equation gives the invariant solutions

$$u(x,t) = \frac{x+k}{t+a} \qquad k \in \mathbb{R} \ .$$

A much more interesting case arises when the scale invariant solutions are considered. The vector field is $\vec{v}_3 = x\partial_x + 2t\partial_t - u\partial_u$ which corresponds to the group G_3 given in (III.2.9b). As in the Korteweg-deVries equation, the action of G_3 is regular on $Z' = \mathbb{R}^3 \sim \{0\}$ and Z'/G_3 is the same non-Hausdorff manifold. Using local coordinates

$$\xi = t^{-1}x^2 \qquad w = xu$$

it is checked through computation that the equation for the G_3-invariant solutions is

$$4\xi^2 w'' + \xi(2w-2-\xi)w' + w(2-w) = 0$$

which on the surface looks intractable. However, substituting $w = 4\xi\phi'/\phi$ where ϕ is a smooth positive function of ξ, the equation reduces to

$$4\xi^2 \left[\frac{4\xi\phi'' - \xi\phi' + 2\phi'}{\phi}\right]' = 0$$

or, upon integration

$$4\xi\phi'' + (2-\xi)\phi' + k\phi = 0$$

for some constant k. In the special case that $k = 0$, this equation is readily integrable to

$$\phi(\xi) = 4[c\, E(\tfrac{1}{2}\sqrt{\xi}) + k']$$

where k' is a constant and

$$E(x) = \int_0^x e^{x^2} dx \; .$$

Thus the G_3-invariant solution to Burgers' equation in this special case is

$$u(x,t) = \frac{c\ x\ e^{x^2/4t}}{\sqrt{t}[cE(x/2\sqrt{t})+k']} \quad .$$

Finally, consider the vector field $\vec{v}_5 = xt\partial_x + t^2\partial_t + (x-tu)\partial_u$ with one parameter group G_5 which acts regularly on $Z'' = \mathbb{R}^3 \sim \{x=t=0\}$. Then $Z''/G_5 \simeq S^1 \times \mathbb{R}$, a cross section of the foliation given by the line bundle over the circle $\{u=0, x^2+t^2=1\}$ that twists twice as the circle is traversed, and is hence diffeomorphic to a cylinder. Convenient local coordinates are given by the invariants

$$\xi = \frac{x}{t} \qquad w = tu - x$$

and the resulting equation Δ_o/G_5 is

$$w'' + ww' = 0 \ .$$

This gives the G_5 invariant solutions

$$u(x,t) = \frac{-k}{t}\left[\tan\left(\frac{kx+k't}{2t}\right) + x\right]$$

$$u(x,t) = \frac{x+k}{t}$$

where k and k' are arbitrary real constants. Other G_5 - invariant solutions can be found by using different coordinate patches on Z''/G_5.

IV.3 Explicit Solutions

Given a local group of transformations acting on a smooth manifold Z whose k-th prolongation leaves a k-th order system of partial differential equations $\Delta_0 \subset J_k^*(Z,p)$ invariant, we have seen that G transforms solutions of Δ_0 to other solutions of Δ_0 and, in the case that G acts regularly on Z, we have given a procedure for the construction of G invariant solutions to Δ_0. In most concrete applications of this theory, Z will be an open subset of $\mathbb{R}^p \times \mathbb{R}^q$ (or a bit more generally an open subset of a vector bundle over a p-dimensional base manifold X). In these cases, the solutions of Δ_0 that warrant the most interest are the "explicit" solutions, i.e. p-sections of Z which are honest sections with respect to the bundle structure of Z. In this section some conditions on the action of the transformation group G will be found which ensure that explicit sections are transformed into explicit sections, and others which ensure that the group invariant sections are explicit sections. Most of these conditions can be readily verified in practice by looking at the infinitesimal generators of the group action.

These results will be proven in a bit more general context for convenience. Namely, we shall assume that Z is a smooth manifold of dimension p+q together with a smooth q dimensional involutive differential system $\mathcal{U} \subset TZ$. In the vector bundle case, $\mathcal{U}$ denotes the tangent spaces to the q dimensional fibers of the bundle. Note that since $\mathcal{U}$ is involutive, it generates a foliation of Z with q dimensional leaves. Given a point $z \in Z$, let $\mathfrak{g}|_z$ denote the subspace of $TZ|_z$ spanned by the infinitesimal generators of G.

Definition 3.1 The transformation group G acts transversally to $\mathcal{U}$ if $\mathfrak{g}|_z \cap \mathcal{U}|_z = \{0\}$ for all $z \in Z$. G acts compatibly with $\mathcal{U}$ if the action of G preserves the leaves of $\mathcal{U}$; in other words, for all $g \in G$ and any leaf U of $\mathcal{U}$, if $z, z' \in Z_g \cap U$, then gz and gz' are on the same leaf gU of $\mathcal{U}$. G acts completely compatibly with $\mathcal{U}$ if G acts compatibly and the intersection of any orbit of G and any leaf of $\mathcal{U}$ consists of at most one point.

Lemma 3.2 The following are equivalent

i) G acts compatibly with $\mathcal{U}$.

ii) For all $g \in G$ and $z \in Z_g$, $dg(\mathcal{U}|_z) = \mathcal{U}|_{gz}$.

iii) $[\mathfrak{g}, \mathcal{U}] \subset \mathcal{U}$.

The proof of this lemma is a straightforward exercise in the theory of transformation groups. In practice, the third criterion is the most easy to verify. The next definition generalizes the concept of an explicit section of a vector bundle.

Definition 3.3 A p-section $s \subset Z$ is called a local explicit section of Z with respect to $\mathcal{U}$ if for all $z \in s$, $Ts|_z \cap \mathcal{U}|_z = \{0\}$, i.e. s is transversal to $\mathcal{U}$. A local explicit section s is called an explicit section if s intersects each leaf of $\mathcal{U}$ in at most one point. An explicit section s is called a global explicit section if s intersects each leaf of $\mathcal{U}$ in exactly one point.

Proposition 3.4 Suppose G acts compatibly with $\mathcal{U}$, $g \in G$ and $s \subset Z_g$ is a (local) explicit p-section, then $gs = \{gz: z \in s\}$ is also a (local) explicit p-section of Z.

Proof

First suppose that s is just a local explicit section so that $Ts|_z \cap \mathcal{U}|_z = \{0\}$ for all $z \in s$. Then using condition ii) of lemma 3.2

$$\begin{aligned} T(gs)|_{gz} \cap \mathcal{U}|_{gz} &= dg[Ts|_z] \cap dg[\mathcal{U}|_z] \\ &= dg[Ts|_z \cap \mathcal{U}|_z] = \{0\}, \end{aligned}$$

hence gs is a local explicit section. Furthermore, since the transformation g preserves the leaves of $\mathcal{U}$, the statement about global explicit sections follows immediately. Q.E.D.

A corresponding result is true about global explicit sections providing the group action is globally defined. In the case of vector bundles, the condition of compatibility is the same as the previously discussed condition of projectability. Thus, what we have shown in proposition 3.4 is that a projectable group acting on a vector bundle transforms sections to sections.

Example 3.5 This demonstrates that transversality of the group action is not sufficient to transform explicit sections to explicit sections. Let $f: \mathbb{R} \to \mathbb{R}$ be any nonzero smooth function and consider the additive group $G = \mathbb{R}$ acting on $\mathbb{R}^2$ by

$$G: \quad (x,u) \mapsto (\,x+\lambda f(x-u),\ u+\lambda f(x-u)) \quad \lambda \in \mathbb{R},\ (x,u) \in \mathbb{R}^2.$$

The infinitesimal generator of this group action is

$$\vec{v} = f(x-u)\cdot\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\right]$$

which is transversal to the one dimensional differential system spanned by $\partial/\partial u$, but is not compatible with this differential system for f nonconstant. Consider the explicit 1-section $s = \{(x,0)\}$. Then the transformed section $\lambda s = \{(x+\lambda f(x), \lambda f(x))\}$ is not explicit if

$$x + \lambda f(x) = \hat{x} + \lambda f(\hat{x}) \qquad f(x) \neq f(\hat{x})$$

which can easily be arranged by suitable choices of f and λ. Moreover, λs will not even be locally explicit if $1+\lambda f'(x) = 0$ for some real x.

Note that if G acts regularly on Z and transversally to $\mathcal{U}$, then $\mathcal{U}/G = d\pi_G(\mathcal{U})$ is an involutive q dimensional differential system on the quotient manifold Z/G and that $d\pi_G|\mathcal{U}$ has rank q.

<u>Proposition 3.6</u> Suppose G acts regularly on Z with ℓ dimensional leaves. If G acts transversally to $\mathcal{U}$ and s is a G invariant p-section of Z such that $s/G = \pi_G(s)$ is a local $\mathcal{U}/G$ explicit $(p-\ell)$-section of Z/G, then s is a local $\mathcal{U}$ explicit section of Z. Conversely, if s is a local $\mathcal{U}$ explicit G invariant p-section of Z, then s/G is a local $\mathcal{U}/G$ explicit $(p-\ell)$-section of Z/G and G acts transversally to $\mathcal{U}$ on some open neighborhood of s.

<u>Proof</u>

Let $z \in s$ with $z' = \pi_G(z) \in s/G$. Since $T(s/G)|_{z'} = d\pi_G(Ts|_z)$ we have

$$0 = T(s/G)|_{z'} \cap \mathcal{U}/G|_{z'} = d\pi_G[Ts|_z \cap \mathcal{U}|_z].$$

Since $d\pi_G$ has maximal rank on $\mathcal{U}$, we conclude that s is a local G invariant $\mathcal{U}$ explicit p-section of Z. Conversely, since s is G invariant, $\mathcal{G}|_z \subset Ts|_z$ for all $z \in s$, hence $\mathcal{G}|_z \cap \mathcal{U}|_z = \{0\}$ for all $z \in s$. This is an open condition on $\mathcal{G}$ which proves the proposition.

Q.E.D.

Corollary 3.7 If G acts transversally, regularly and completely compatibly, then s is a G invariant (global) $\mathcal{U}$ explicit p-section of Z iff s/G is a (global) $\mathcal{U}/G$ explicit $(p-\ell)$-section of Z/G.

Suppose $\xi = (\xi^1,\ldots,\xi^m)$ where $m=p+q-\ell$ is a system of local coordinates on Z/G and $(x,u) = (x^1,\ldots,x^p,u^1,\ldots,u^q)$ is a system of local coordinates on Z that is flat with respect to $\mathcal{U}$, so that the connected components of the leaves of $\mathcal{U}$ are given by {x=constant} in these coordinates. (Equivalently, we prescribe that $\mathcal{U}$ is spanned by $\{\partial/\partial u^1,\ldots,\partial/\partial u^q\}$.) The projection map π_G is then given by the equations

$$\xi^i = I^i(x,u) \qquad i=1,\ldots,m.$$

The smooth functions I^i are classically called a complete set of functionally independent invariants of the group action on Z. The preceding proposition then states that a necessary condition for the existence of G invariant explicit sections is that the rank of the Jacobian matrix

$$\left(\frac{\partial I^i}{\partial u^j}\right)$$

which is classically called the "power of completeness" of these invariants with respect to the dependent variables u, must equal q, the number of these variables. See [01; page 59] for a statement and proof of this result in a classical context.

<u>Example 3.8</u> Let $Z = S^1 \times \mathbb{R}$ with coordinates (θ, u) and let be the differential system spanned by the vector field $\partial/\partial u$. Let $G=\mathbb{R}$ act on Z via

$$G: \quad (\theta,u) \mapsto (\theta + \lambda \bmod 2\pi,\ e^{\lambda}u) \qquad \lambda \in \mathbb{R}.$$

The infinitesimal generator of G is $\frac{\partial}{\partial\theta} + u\frac{\partial}{\partial u}$ so that G acts transversally and compatibly with $\mathcal{U}$. However, it can easily be seen that any nonzero global G-invariant 1-section must intersect each leaf of $\mathcal{U}$ in a countably infinite number of points of the form

$$\{\ldots,\ e^{-2\pi}u_0,\ u_0,\ e^{2\pi}u_0,\ e^{4\pi}u_0, \ldots\}$$

and hence cannot be a global explicit section of Z.

Appendix: Local Transformation Groups

For the sake of completeness, the fundamental results on local groups of transformations that have been referred to in the body of the thesis have been collected together in an appendix. This material will be of an expository nature. The main references are the monograph of Palais [Pl] and the book of Pontryagin [PO], which should be consulted for the details of the proofs.

Definition A.1 A topological space G is a local group if there exist open subsets $U_0 \subset G \times G$ and $V_0 \subset G$ and continuous maps $m: U_0 \to G$ and $i: V_0 \to G$ called multiplication and inversion respectively that satisfy

i) If $g_1, g_2, g_3 \in G$, $(g_1, g_2) \in U_0$, $(g_2, g_3) \in U_0$, $(m(g_1, g_2), g_3) \in U_0$, $(g_1, m(g_2, g_3)) \in U_0$, then

$$m(m(g_1,g_2),g_3) = m(g_1,m(g_2,g_3)).$$

(This is just the associative law for the group multiplication.)

ii) There is a distinguished element $e \in G$ called the identity that satisfies $\{e\} \times G \subset U_0$, $G \times \{e\} \subset U_0$ and $m(e,g) = m(g,e) = g$ for every $g \in G$.

iii) The open set V_0 satisfies $V_0 \times i(V_0) \subset U_0$ and $i(V_0) \times V_0 \subset U_0$ and $m(g,i(g)) = m(i(g),g) = e$ for every $g \in V_0$.

Definition A.2 If G is a local group, then a neighborhood of the identity of G is called a nucleus of G.

From now on the multiplication $m(g_1,g_2)$ will be denoted simply by g_1g_2 and the inversion $i(g)$ by g^{-1}. For any integer n there is a nucleus $U_n \subset V_0 \subset G$ such that for any $g_1,\ldots,g_n \in U_n$ the product $g_1g_2\cdots g_n$ is well defined and does not depend upon the order in which the multiplications are performed. Subsequently, if any product of n elements in a local group is displayed, it will always be tacitly assumed that all these elements belong to the nucleus U_n. Note that any nucleus of a local group is itself a local group.

<u>Definition A.3</u> A local group G is called a <u>local</u> <u>(m-parameter)</u> <u>Lie group</u> if there exists a nucleus of G homeomorphic to an open subset of Euclidean space R^m such that the multiplication and inversion maps are smooth maps under this homeomorphism.

It is a celebrated result, cf. [MZ], that continuity of the multiplication and inversion maps is enough to ensure their differentiability.

<u>Theorem A.4</u> [PO; page 435] If G is a local Lie group, then G can be smoothly enbedded in a global Lie group G' of the same dimension.

The Lie algebra $\mathfrak{g}$ of a local Lie group G is the algebra of all left invariant vector fields on G with the usual Lie bracket $[\cdot,\cdot]$, cf. [W; chapter 3] for the global case. Let Z be a smooth n dimensional manifold.

<u>Definition A.5</u> A local Lie group G is a <u>local group of transformations</u> acting on the manifold Z when there is an open set

U_0 with $\{e\}\times Z \subset U_0 \subset G\times Z$ and a smooth map $\Phi: U_0 \to Z$ which is consistent with the group structure on G.

In other words, let us define

$$Z_g = \{z \in Z: (g,z) \in U_0\} \qquad \text{for } g \in G$$

$$G_z = \{g \in G: (g,z) \in U_0\} \qquad \text{for } z \in Z$$

$$\Phi_g: Z_g \to Z \qquad \Phi_g(z) = \Phi(g,z)$$

$$\Phi_z: G_z \to Z \qquad \Phi_z(g) = \Phi(g,z).$$

Suppose $g_1, g_2 \in G$ and g_1g_2 is well defined, then we prescribe that $\Phi_{g_1g_2} = \Phi_{g_1} \circ \Phi_{g_2}$ on their common domains of definition. Furthermore prescribe that $\Phi_e = 1\!\!1_Z$, the identity map of Z, which implies that $\Phi_{g^{-1}} = \Phi_g^{-1}$ on $\Phi_g[Z_g] \cap Z_{g^{-1}}$. Since we are mainly interested in the local behavior of G, we can assume without loss of generality that $Z_{g^{-1}} = \Phi_g[Z_g]$ for all $g \in G$. Often the map Φ will be suppressed, so $\Phi(g,x)$ will be denoted simply by gx.

Given a local group of transformations acting on a manifold Z, there is a corresponding infinitesimal action of the Lie algebra given by

$$\phi: \mathfrak{g}\times Z \to TZ$$
$$\phi(\alpha,z) = d\Phi_z(\alpha|_e) \in TZ|_z.$$

Thus, given a left invariant vector field α on G, there is an induced vector field $\phi(\alpha)$ on Z, where $\phi(\alpha)|_z = \phi(\alpha,z)$. Alternately, we can define $\phi(\alpha)$ by the formula

$$\phi(\alpha)|_z = \frac{d}{dt}\Big|_{t=0} \Phi(\exp(t\alpha),z)$$

where exp denotes the exponential mapping of G. The Lie bracket is preserved under the map ϕ,

$$\phi[\alpha,\beta] = [\phi(\alpha),\phi(\beta)] \qquad \alpha,\beta \in \mathfrak{g}.$$

Definition A.6 The transformation group G acts nonsingularly on Z if the dimension of the subspace of $TZ|_z$ spanned by all infinitesimal generators of G, $\phi(\alpha)$, $\alpha \in \mathfrak{g}$, is independent of the point $z \in Z$.

Definition A.7 A subset $S \subset Z$ is invariant under the action of a transformation group G iff $\Phi(g,z) \in S$ for all $z \in S$, $g \in G_z$. S is locally invariant iff for each $z \in S$ there is a nucleus $G_z' \subset G_z$ with $\Phi(g,z) \in S$ for all $g \in G_z'$.

Theorem A.8 Given a local group G, with Lie algebra $\mathfrak{g}$, acting on a smooth manifold Z, for each integer k let

$$Z_k = \{z \in Z: \dim \phi(\mathfrak{g})|_z = k\}$$

then Z_k is invariant under G.

Proof

Suppose $\alpha_0 \in \mathfrak{g}$ and $\vec{v}_0 = \phi(\alpha_0)$ is the corresponding vector field on Z. Choose $\alpha_1,\ldots,\alpha_{m-1}$, which together with α_0 span $\mathfrak{g}$, and let $\vec{v}_i = \phi(\alpha_i)$. Now it suffices to show that if $\sigma: [0,1] \to Z$ is an integral curve of $\vec{v}_0$ and $\dim \phi(\mathfrak{g})|_{z_0} = \ell$ for some $z_0 \in \operatorname{im} \sigma$, then

$\dim \phi(\mathfrak{g})|_z \le \ell$ for all $z \in \operatorname{im} \sigma$, since using this, it is readily seen that $\operatorname{im} \sigma$ is wholly contained in some Z_ℓ.

To prove this fact, assume $\vec{v}_0|_{z_0} \ne 0$, since otherwise the proof is trivial. Choose local coordinates $(z^1,\ldots,z^n)$ near z_0 such that in terms of these coordinates $\vec{v}_0 = \partial/\partial z^n$. Let

$$\vec{v}_i = \sum_{j=1}^{n} \eta_{ij}(z) \frac{\partial}{\partial z^j}$$

in these coordinates, so that the functions $\eta_{ij}(z)$ are smooth. Since the map ϕ preserves the Lie bracket,

$$[\vec{v}_0,\vec{v}_i] = \sum_{j=0}^{m-1} c_{ij}(z)\, \vec{v}_j = \sum_{j,k} c_{ij}(z)\eta_{jk}(z)\frac{\partial}{\partial z^k}$$

for each $i=1,\ldots,m-1$, where the $c_{ij}(z)$ are smooth real valued functions. Evaluating the Lie bracket in the local coordinates gives

$$\frac{\partial}{\partial z^n} \eta_{ik}(z) = \sum_{j=0}^{m-1} c_{ij}(z)\eta_{jk}(z) \qquad \begin{array}{l} i=i,\ldots,m-1 \\ k=1,\ldots,n-1 \end{array}$$

Let $\eta(t) = \eta(0,\ldots,0,t)$ be the $(m-1)\times(n-1)$ matrix with entries $\eta_{ij}(0,\ldots,0,t)$ where we are restricting our attention just to $\operatorname{im} \sigma$. Thus $\eta(t)$ is a matrix solution to the ordinary differential equation

$$\frac{d}{dt} \eta(t) = c(t)\cdot\eta(t)$$

where $c(t)$ denotes the $(m-1)\times(m-1)$ matrix with entries $c_{ij}(0,\ldots,0,t)$. It is a well-known fact that the rank of the matrix $\eta(t)$ is nonincreasing, cf. [HA]. But we know that

$$\dim \phi(\mathfrak{g})|_{(0,\ldots,0,t)} = 1 + \operatorname{rank} \eta(t)$$

which proves the theorem. Q.E.D.

The last theorem shows that for most purposes G can be assumed to have nonsingular action on Z, simply by just considering the invariant sets Z_i for each i separately. Using this theorem, and the Frobenius theorem [W;§1.60,64] it makes sense to define the orbit of a point $z \in Z$ to be the maximal connected integral manifold of the involutive quasi-differential system $\phi(\mathfrak{g})$. (Note that since the dimension of $\phi(\mathfrak{g})|_z$ may vary from point to point, $\phi(\mathfrak{g})$ is not technically a differential system (distribution), hence the use of the adjective"quasi.") The next theorem gives the infinitesimal criterion for the local invariance of submanifolds of Z under the action of G.

Theorem A.9 A submanifold $S \subset Z$ is locally invariant under the group action of G iff $TS|_z \supset \phi(\mathfrak{g})|_z$ for all $z \in S$. A closed submanifold $S \subset Z$ is invariant iff it is locally invariant.

Usually we shall be interested in the local invariance of subvarieties of Z that are given by the vanishing of a smooth function $F: Z \to \mathbb{R}^k$. Recall that F is a submersion if dF has maximal rank everywhere.

Theorem A.10 Let $S = F^{-1}\{0\}$ for $F: Z \to \mathbb{R}^k$ a smooth submersion. Then S is invariant under G iff

$$dF[\phi(\mathfrak{g})|_z] = 0$$

for all $z \in S$.

This theorem is usually what is meant by the infinitesimal criterion for the invariance of a subvariety. In local coordinates $(z^1,\ldots,z^n)$ on Z, suppose $\phi(\mathfrak{g})$ is spanned by the vector fields

$$\vec{v}_i = \sum_{j=1}^{n} \xi_i^j(z) \frac{\partial}{\partial z^j} \qquad i=1,\ldots,\ell$$

If $F(z) = (F^1(z),\ldots,F^k(z))$, then $S = F^{-1}\{0\}$ is invariant under G iff

$$\vec{v}_i F^j = \sum \xi_i^k(z) \frac{\partial F^j}{\partial z^k}(z) = 0 \qquad \begin{matrix} i=1,\ldots,\ell \\ j=1,\ldots,k \end{matrix}$$

whenever $F^1(z) = \ldots = F^k(z) = 0$.

Definition A.11 Let O be an orbit of G, then O is regular iff for each $z \in O$ there exist arbitrarily small neighborhoods V containing z with the property that for any orbit O' of G, $O' \cap V$ is connected. The group G acts regularly on Z iff it acts nonsingularly and every orbit is regular.

Definition A.12 Given a subset $S \subset Z$, the saturation of S is the union of all the orbits passing through S.

Given a local group of transformations acting on a smooth manifold Z, let Z/G denote the quotient set of all orbits of G, and let $\pi_G : Z \to Z/G$ be the projection that associates to each point in Z the orbit of G passing through that point. There is a natural topology on Z/G given by the images of saturated open subsets of Z under the projection π_G.

Theorem A.13 [P1; page 19] If G acts regularly on the smooth manifold Z, then the quotient space Z/G can be endowed with the structure of a smooth manifold such that the projection $\pi_G: Z \to Z/G$ is a smooth map between manifolds. The null space of $d\pi_G|_Z$ is $\mathfrak{g}|_Z$ and the range is $T(Z/G)|_{\pi_G(z)}$.

To find the coordinate charts on Z/G for this construction, recall that since G acts regularly, its orbits have constant dimension ℓ, and there exist regular coordinate systems $\chi: V \to \mathbb{R}^n$ for $V \subset Z$ open, such that any orbit of G intersects V in only one slice $\chi^{-1}\{z^{\ell+1}=c_{\ell+1}, \ldots, z^n=c_n\}$, the c_i's being constants. The induced coordinate chart $V/G = \pi_G(V)$ is given by

$$\chi/G: V/G \to \mathbb{R}^{n-\ell} \qquad \chi/G[\pi_G(z)] = (z^{\ell+1}, \ldots, z^n) \quad z \in V.$$

Let us look at this construction from a more classical viewpoint. Suppose $(z^1, \ldots, z^n)$ is any local coordinate system on Z. A real valued function $F: Z \to \mathbb{R}$ is called an invariant of G if $F(gz)=F(z)$ for all $z \in Z$, $g \in G_z$. By the existence of regular coordinate systems on Z, we know that locally there always exist $n-\ell$ functionally independent invariants of G, say $F^1, \ldots, F^{n-\ell}$. Let $F = (F^1, \ldots, F^{n-\ell}): Z \to \mathbb{R}^{n-\ell}$. The functional independence of these invariants is another way of saying that the Jacobian map $dF: TZ \to T\mathbb{R}^{n-\ell}$ has maximal rank. We conclude that these invariants provide local coordinates on the quotient manifold Z/G. The reader should consult Ovsjannikov [O1; chapter 3] for an exposition of the subject from this viewpoint, although no explicit reference is made to the quotient manifold.

If $I: Z \to W$ is any smooth G-invariant function, then there is a corresponding smooth function $I/G: Z/G \to W$ such that

$$I(z) = I/G[\pi_G(z)] \qquad z \in Z.$$

If $S \subset Z$ is any smooth (locally) G invariant submanifold of Z, then $S/G = \pi_G[S]$ is a smooth submanifold of Z/G. Conversely, if S/G is any smooth submanifold of Z/G, then $S = \pi_G^{-1}[S/G]$ is a smooth G invariant submanifold of Z. Note that if $\dim S = p$, and G has ℓ dimensional orbits, then $\dim S/G = p-\ell$.

It should be remarked that the quotient manifold Z/G does not necessarily satisfy the Hausdorff topological axiom. For instance, consider the case $Z = \mathbb{R}^2 - \{0\}$ and let $G = \mathbb{R}$ be the group of translations in the first coordinate:

$$G: \quad (x,u) \mapsto (x+\lambda,u) \qquad \lambda \in \mathbb{R},$$

which is a local, regular group action on Z. The quotient manifold Z/G can be realized a copy of the real line with two infinitely close origins, which are given by the orbits $\{(x,0): x>0\}$ and $\{(x,0): x<0\}$. It is, however, entirely possible to develop a theory of smooth manifolds that does not use the Hausdorff separation axiom, with little change in the relevant results in the local theory. It is assumed implicitly throughout this thesis that non-Hausdorff manifolds might arise. The interested reader should consult [P1] for details on this point.

Symbol Index

symbol		page
$\partial_*^k f$	$= \partial f + \partial^2 f + \ldots + \partial^k f$	81
ε_k	Faa-di-Bruno injection	80
π_k	projection inverse to ε_k	80
$d^k f$	i) $\varepsilon_k[\partial_*^k f]$	81
	ii) induced map on k-th order tangent bundles	73
$\partial_j^i f$	matrix blocks of $d^k f$	82
D_k	total derivative in x^k direction	98
D_I	total derivative corresponding to multi-index I	98
$D^k f$	k-th order total differential of function f	96
$D_*^k f$	$= Df + D^2 f + \ldots + D^k f$	96
$\mathbb{1}_X$	identity map of a space X	
$\mathbb{1}_p$	identity map of $\mathbb{R}^p$	
$\mathbb{1}_M^p$	trivial bundle $\mathbb{R}^p \times M$ over manifold M	
$GL^{(k)}(V)$	k-th order prolonged general linear group of V	82
$GL^{(k)}(n)$	k-th order prolonged linear group of $\mathbb{R}^n$	82
$PGL^{(k)}(n)$	k-th order prolonged projective linear group of $\mathbb{R}^n$	91
$Grass^{(k)}(V,p)$	k-th order prolonged Grassmannian of prolonged p-planes for vector space V	85
$Grass^{(k)}(V,p;W)$	trivialized prolonged Grassmannian	90
$C^\infty(M,p)\vert_m$	space of germs of smooth p-sections of M passing through m	74
$J_k^*(M,p)$	k-th order extended jet bundle of p-sections of M	74
$J_k^*(M,p;\mathfrak{U})$	trivialized k-th order extended jet bundle	93
$j_k^* s$	k-th order extended jet of a p-section	75

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