# Chapter 3

# Jets and Differential Invariants

Transformation groups figure prominently in Lie's theory of symmetry groups of differential equations, which we discuss in Chapter 4. They will act on the basic space coordinatized by the independent and dependent variables relevant to the system of differential equations under consideration. Since we are dealing with *differential equations* we must be able to handle the derivatives of the dependent variables on the same footing as the independent and dependent variables themselves. This chapter is devoted to a detailed study of the proper geometric context for these purposes —the so-called "jet spaces" or "jet bundles", well known to nineteenth century practitioners, but first formally defined by Ehresmann, [**16**]. After presenting a simplified version of the basic construction, we then discuss how group transformations are "prolonged" so that the derivative coordinates are appropriately acted upon, and, in the case of infinitesimal generators, deduce the explicit prolongation formula.

A differential invariant is merely an invariant, in the standard sense, for a prolonged group of transformations acting on the jet space $\mathrm{J}^n$. Just as the ordinary invariants of a group action serve to characterize invariant equations, so differential invariants will completely characterize invariant systems of differential equations for the group, as well as invariant variational principles. As such they form the basic building blocks of many physical theories, where one begins by postulating the invariance of the differential equations, or the variational problem, under a prescribed symmetry group. Historically, the subject was initiated by Halphen, [**21**], and then developed in great detail, with numerous applications, by Lie, [**33**], and Tresse, [**53**]. In this chapter, we discuss the basic theory of differential invariants, and some fundamental methods for constructing them. Applications of these results to the study of differential equations and variational problems will be discussed in the following chapters.

*Transformations and Functions*

A general system of (partial) differential equations involves $p$ independent variables $x = (x^1, \ldots, x^p)$, which we can view as local coordinates on the Euclidean space $X \simeq \mathbb{R}^p$, and $q$ dependent variables $u = (u^1, \ldots, u^q)$, coordinates on $U \simeq \mathbb{R}^q$. The *total space* will be the Euclidean space $E = X \times U \simeq \mathbb{R}^{p+q}$ coordinatized by the independent and dependent variables.† The symmetries we will focus on are (locally defined) diffeomorphisms on the

† It should be remarked that all these considerations extend to the more general global context, in which the total space is replaced by a vector bundle $E$ over the base $X$ coordinatized by the independent variables, the fibers being coordinatized by the dependent variables. Since our

 

space of independent and dependent variables:

$$(\bar{x}, \bar{u}) = g \cdot (x, u) = (\chi(x, u), \psi(x, u)). \tag{3.1}$$

These are often referred to as *point transformations* since they act pointwise on the total space $E$. However, it *is* convenient to specialize, on occasion, to more restrictive classes of transformations. For example, *base transformations* are only allowed to act on the independent variables, and so have the form $\bar{x} = \chi(x)$, $\bar{u} = u$. If we wish to preserve the bundle structure of the space $E$, we must restrict to the class of *fiber-preserving transformations* in which the changes in the independent variable are unaffected by the dependent variable, and so take the form

$$(\bar{x}, \bar{u}) = g \cdot (x, u) = (\chi(x), \psi(x, u)). \tag{3.2}$$

Most, but not all, important group actions are fiber-preserving.

In the case of connected groups, the action of the group can be recovered from that of its associated infinitesimal generators. A general vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \tag{3.3}$$

on the space of independent and dependent variables generates a flow $\exp(t\mathbf{v})$, which is a local one-parameter group of point transformations on $E$. The vector field will generate a one-parameter group of fiber-preserving transformations if and only if the coefficients $\xi^i = \xi^i(x)$ do not depend on the dependent variables. The group consists of base transformations if and only if the vector field is *horizontal*, meaning that all the vertical coefficients vanish: $\varphi^\alpha = 0$. Vice versa, *vertical* vector fields, which are characterized by the vanishing of the horizontal coefficients, $\xi^i = 0$, generate groups of vertical transformations; an important physical example is provided by the gauge transformations.

A (classical) solution[‡] to a system of differential equations will be described by a smooth function $u = f(x)$. (In the more general bundle-theoretic framework, solutions are described by *sections* of the bundle.) The graph of the function, $\Gamma_f = \{(x, f(x))\}$, determines a regular $p$-dimensional submanifold of the total space $E$. However, not every regular $p$-dimensional submanifold of $E$ defines the graph of a smooth function. Globally, it can intersect each fiber $\{x_0\} \times U$ in at most one point. Locally, it must be *transverse*, meaning that its tangent space contains no vertical tangent directions. The Implicit Function Theorem demonstrates that, locally, the transversality condition is both necessary and sufficient for a submanifold to represent the graph of a smooth function.

---

primary considerations are local, we shall not lose anything by specializing to (open subsets of) Euclidean space, which has the added advantage of avoiding excessive abstraction. Moreover, the sophisticated reader can easily supply the necessary translations of the machinery if desired. Incidentally, an even more general setting for these techniques, which avoids any *a priori* distinction between independent and dependent variables, is provided by the extended jet bundles defined in [**43**; Chapter 3].

‡ To avoid technical complications, we will only consider smooth or analytic solutions, although extensions of these results to more general types of solutions are certainly possible, [**49**, **50**].

 

**Proposition 3.1.** *A regular p-dimensional submanifold $\Gamma \subset E$ which is transverse at a point $z_0 = (x_0, u_0) \in \Gamma$ coincides locally, i.e., in a neighborhood of $x_0$, with the graph of a single-valued smooth function $u = f(x)$.*

Any point transformation (3.1) will act on a function $u = f(x)$ by pointwise transforming its graph. In other words if $\Gamma_f = \{(x, f(x))\}$ denotes the graph of $f$, then the transformed function $\bar{f} = g \cdot f$ will have graph

$$\Gamma_{\bar{f}} = \{(\bar{x}, \bar{f}(\bar{x}))\} = g \cdot \Gamma_f = \{g \cdot (x, f(x))\}. \tag{3.4}$$

In general, we can only assert that the transformed graph is another $p$-dimensional submanifold of $E$, and so the transformed function will not be well defined unless $g \cdot \Gamma_f$ is (at least) transverse. Transversality will, however, be ensured if the transformation $g$ is sufficiently close to the identity transformation, and the domain of $f$ is compact.

**Example 3.2.** Consider the one-parameter group of rotations

$$g_t \cdot (x, u) = (x \cos t - u \sin t,\ x \sin t + u \cos t), \tag{3.5}$$

acting on the space $E \simeq \mathbb{R}^2$ consisting of one independent and one dependent variable. Such a rotation transforms a function $u = f(x)$ by rotating its graph; therefore, the transformed graph $g_t \cdot \Gamma_f$ will be the graph of a well-defined function only if the rotation angle $t$ is not too large. The equation for the transformed function $\bar{f} = g_t \cdot f$ is given in implicit form

$$\bar{x} = x \cos t - f(x) \sin t, \qquad \bar{u} = x \sin t + f(x) \cos t,$$

so that $\bar{u} = \bar{f}(\bar{x})$ is found by eliminating $x$ from these two equations. For example, if $u = ax + b$ is affine, then the transformed function is also affine, and given explicitly by

$$\bar{u} = \frac{\sin t + a \cos t}{\cos t - a \sin t}\, \bar{x} \; + \; \frac{b}{\cos t - a \sin t}, \tag{3.6}$$

which is defined provided $\cot t \neq a$, i.e., provided the graph of $f$ has not been rotated to be vertical.

In general, given a point transformation as in (3.1), the transformed function $\bar{u} = \bar{f}(\bar{x})$ is found by eliminating $x$ from the parametric equations $\bar{u} = \psi(x, f(x))$, $\bar{x} = \chi(x, f(x))$, provided this is possible. If the transformation is fiber-preserving, then $\bar{x} = \chi(x)$ is a local diffeomorphism, and so the transformed function is always well defined, being given by $\bar{u} = \psi(\chi^{-1}(\bar{x}), f(\chi^{-1}(\bar{x})))$.

*Invariant Functions*

Given a group of point transformations $G$ acting on $E \simeq X \times U$, the characterization of all $G$-invariant functions $u = f(x)$ is of great importance.

**Definition 3.3.** A function $u = f(x)$ is said to be *invariant* under the transformation group $G$ if its graph $\Gamma_f$ is a (locally) $G$-invariant subset.

For example, the graph of any invariant function for the rotation group SO(2) acting on $\mathbb{R}^2$ must be an arc of a circle centered at the origin, so $u = \pm\sqrt{c^2 - x^2}$. Note that there are no globally defined invariant functions in this case.

*Remark*: In this framework, it is important to distinguish between an "invariant function", which is a section $u = f(x)$ of the bundle $E$, and an "invariant", which, as in Definition 2.25, is a real-valued function $I(x,u)$ defined (locally) on $E$. It is hoped that this will not cause too much confusion in the sequel.

In general, since any invariant function's graph must, locally, be a union of orbits, the existence of invariant functions passing through a point $z = (x,u) \in E$ requires that the orbit $\mathcal{O}$ through $z$ be of dimension at most $p$, the number of independent variables, and, furthermore, be transverse. Since the tangent space to the orbit $T\mathcal{O}|_z = \mathfrak{g}|_z$ agrees with the space spanned by the infinitesimal generators of $G$, the transversality condition requires that, at each point, $\mathfrak{g}|_z$ contain no vertical tangent vectors. For example, the infinitesimal generator of the rotation group is $\mathbf{v} = -u\partial_x + x\partial_u$, which is vertical at $u = 0$. Thus, the rotation group fails the transversality criterion on the $x$-axis, and, as we saw, there are no smooth, rotationally invariant functions $u = f(x)$ passing through such points. In the case the group acts both (semi-)regularly and transversally, then we can characterize the invariant functions by the use of the functionally independent (real-valued) invariants of the group action. The following result is a direct corollary of Theorem 2.34.

**Theorem 3.4.** *Let $G$ be a transformation group acting semi-regularly and transversally on $E \simeq X \times U$ with $s$-dimensional orbits. Let*

$$I_1(x,u), \ldots, I_{p-s}(x,u), J_1(x,u), \ldots, J_q(x,u),$$

*be a complete set of functionally independent invariants for $G$. Then any $G$-invariant function $u = f(x)$, can, locally, be written in the implicit form*

$$w = h(y), \qquad \text{where} \qquad y = I(x,u), \quad w = J(x,u). \tag{3.7}$$

In the fiber-preserving case, if we assume that the orbits of the projected action of $G$ on $X$ also have dimension $s$, then we can take the first $p - s$ invariants $I_1(x), \ldots, I_{p-s}(x)$ to depend only on the independent variables, and forming a complete system of invariants on $X$.

**Example 3.5.** A "similarity solution" of a system of partial differential equations is just an invariant function for a group of scaling transformations. As a specific example, consider the one-parameter scaling group $(x,y,u) \mapsto (\lambda x, \lambda^\alpha y, \lambda^\beta u)$. The independent invariants are provided by the ratios $y = y/x^\alpha$, $w = u/x^\beta$, so any scale-invariant function can be written as $w = h(y)$, or, explicitly, $u = x^\beta h(y/x^\alpha)$.

As usual, the most convenient characterization of the invariant functions is based on an infinitesimal condition. Since the graph of a function is defined by the vanishing of its components $u^\alpha - f^\alpha(x)$, our general invariance Theorem 2.71 imposes the infinitesimal invariance conditions

$$0 = \mathbf{v}(u^\alpha - f^\alpha(x)) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial f^\alpha}{\partial x^i}\,, \quad \alpha = 1, \ldots, q,$$

which must hold whenever $u = f(x)$, for every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$, as in (3.3). These first order partial differential equations are known in the literature as the "invariant surface conditions" associated with the given transformation group, cf. [**9**].

**Definition 3.6.** The *characteristic* of the vector field $\mathbf{v}$ given by (2.7) is the $q$-tuple of functions $Q(x,u^{(1)})$, depending on $x$, $u$ and first order derivatives of $u$, defined by

$$Q^\alpha(x,u^{(1)}) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial u^\alpha}{\partial x^i}\,, \qquad \alpha = 1,\ldots,q. \tag{3.8}$$

**Theorem 3.7.** *A function $u = f(x)$ is invariant under a connected group of point transformations if and only if it is a solution to the first order system of quasi-linear partial differential equations*

$$Q^\alpha(x,u^{(1)}) = 0, \qquad \alpha = 1,\ldots,q, \tag{3.9}$$

*determined by all the characteristics $Q(x,u^{(1)})$ of the set of infinitesimal generators $\mathbf{v} \in \mathfrak{g}$.*

For example, the characteristic of the rotation vector field $-u\partial_x + x\partial_u$ is $Q = x + u u_x$. Any rotationally invariant function must satisfy the differential equation $x + u u_x = 0$. This equation can be readily integrated: $x^2 + u^2 = c$, and hence the graph is an arc of a circle. Similarly, the infinitesimal generator of the scaling group of Example 3.5 is the vector field $x\partial_x + \alpha y \partial_y + \beta u \partial_u$. The characteristic is $Q = \beta u - x u_x - \alpha y u_y$, and the scale-invariant functions constitute the general solution to the linear partial differential equation $x u_x + \alpha y u_y = \beta u$.

*Jets and Prolongations*

Since we are interested in studying the symmetries of differential equations, we need to know not only how the group transformations act on the independent and dependent variables, but also how they act on the derivatives of the dependent variables. In the last century, this was done automatically, without fretting about the precise mathematical foundations of the method; in more recent times, geometers have formalized this geometrical construction through the general definition of the "jet space" (or bundle) associated with the total space of independent and dependent variables. The jet space coordinates will represent the derivatives of the dependent variables. We describe a simple, direct formulation of these spaces using local coordinates.

A smooth, scalar-valued function $f(x^1,\ldots,x^p)$ depending on $p$ independent variables has $p_k = \binom{p+k-1}{k}$ different $k^{\text{th}}$ order partial derivatives $\partial_J f(x) = \dfrac{\partial^k f}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}}$, indexed by all unordered (symmetric) multi-indices $J = (j_1,\ldots,j_k)$, $1 \le j_\kappa \le p$, of order $k = \#J$. Therefore, if we have $q$ dependent variables $(u^1,\ldots,u^q)$, we require $q_k = q p_k$ different coordinates $u^\alpha_J$, $1 \le \alpha \le q$, $\#J = k$, to represent all the $k^{\text{th}}$ order derivatives $u^\alpha_J = \partial_J f^\alpha(x)$ of a function $u = f(x)$. For the total space $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$, the $n^{\text{th}}$ *jet space* $\mathrm{J}^n = \mathrm{J}^n E = X \times U^{(n)}$ is the Euclidean space of dimension $p + q^{(n)} \equiv p + q\binom{p+n}{n}$, whose coordinates consist of the $p$ independent variables $x^i$, the $q$ dependent variables $u^\alpha$, and the derivative coordinates $u^\alpha_J$, $\alpha = 1,\ldots,q$, of orders $1 \le \#J \le n$. The points in the vertical space (fiber) $U^{(n)}$ are denoted by $u^{(n)}$, and consist of all the dependent variables and their derivatives up to order $n$; thus the coordinates of a typical point $z \in \mathrm{J}^n$ are denoted by $(x, u^{(n)})$. Since the derivative coordinates $u^{(n)}$ form a subset of the derivative coordinates $u^{(n+k)}$, there is a natural projection $\pi_n^{n+k}\colon \mathrm{J}^{n+k} \to \mathrm{J}^n$ on the jet spaces, with $\pi_n^{n+k}(x, u^{(n+k)}) = (x, u^{(n)})$. In particular, $\pi_0^n(x, u^{(n)}) = (x, u)$ is the projection from $\mathrm{J}^n$ to $E = \mathrm{J}^0$. If $M \subset E$ is an open subset, then $\mathrm{J}^n M = (\pi_0^n)^{-1} M \subset \mathrm{J}^n E$ is the open subset of the $n^{\text{th}}$ jet space which projects back down to $M$.

A smooth function $u = f(x)$ from $X$ to $U$ has $n^{\text{th}}$ *prolongation* $u^{(n)} = f^{(n)}(x)$ (also known as the *n-jet* and denoted $\mathrm{j}_n f$), which is the function from $X$ to $U^{(n)}$ defined by evaluating all the partial derivatives of $f$ up to order $n$; thus the individual coordinate functions of $f^{(n)}$ are $u_J^\alpha = \partial_J f^\alpha(x)$. In particular, $f^{(0)} = f$. Note that the graph of the prolonged function $f^{(n)}$, namely $\Gamma_f^{(n)} = \{(x, f^{(n)}(x))\}$, will be a $p$-dimensional submanifold of $\mathrm{J}^n$. At a point $x \in X$, two functions have the same $n^{\text{th}}$ order prolongation, and so determine the same point of $\mathrm{J}^n$, if and only if they have $n^{\text{th}}$ *order contact*, meaning that they and their first $n$ derivatives agree at the point, which is the same as requiring that they have the same $n^{\text{th}}$ order Taylor polynomial at the point $x$. Thus, a more intrinsic way of defining the jet space $\mathrm{J}^n$ is to consider it as the set of equivalence classes of smooth functions using the equivalence relation of $n^{\text{th}}$ order contact. Note that the process of prolongation is compatible with the jet space projections, so $\pi_n^{n+k} \circ f^{(n+k)} = f^{(n)}$.

If $g$ is a (local) point transformation (3.1), then $g$ acts on functions by transforming their graphs, and hence also naturally acts on the derivatives of the functions. This allows us to define the induced *prolonged transformation* $(\bar{x}, \bar{u}^{(n)}) = g^{(n)} \cdot (x, u^{(n)})$ on the jet space $\mathrm{J}^n$. More specifically, given a point $z_0 = (x_0, u_0^{(n)})$, choose a representative smooth function $u = f(x)$ whose $n^{\text{th}}$ prolongation has the prescribed derivatives at $x_0$, so $z_0 = (x_0, f^{(n)}(x_0)) \in \mathrm{J}^n$. The transformed point $\bar{z}_0 = g^{(n)} \cdot z_0$ is found by evaluating the derivatives of the transformed function $\bar{f} = g \cdot f$ at the image point $\bar{x}_0$, defined so that $(\bar{x}_0, \bar{u}_0) = (\bar{x}_0, \bar{f}(\bar{x}_0)) = g \cdot (x_0, f(x_0))$; therefore $\bar{z}_0 = (\bar{x}_0, \bar{u}_0^{(n)}) = (x_0, \bar{f}(x_0))$. This definition assumes that $\bar{f}$ is smooth at $\bar{x}_0$ — otherwise the prolonged transformation is not defined at $(x_0, u_0^{(n)})$. Thus, the prolonged transformation $g^{(n)}$ maps the graph $\Gamma_f^{(n)}$ of the $n^{\text{th}}$ prolongation of a function $u = f(x)$ to the graph of the $n^{\text{th}}$ prolongation of its image $\bar{f} = g \cdot f$:

$$g^{(n)} \cdot \Gamma_f^{(n)} = \Gamma_{g \cdot f}^{(n)}. \tag{3.10}$$

A straightforward chain rule argument shows that the construction does not depend on the particular function $f$ used to represent the point of $\mathrm{J}^n$; in particular, in view of the identification of the points in $\mathrm{J}^n$ with Taylor polynomials of order $n$, it suffices to determine how the point transformations act on polynomials of degree at most $n$. Note that the prolongation process preserves compositions, $(g \circ h)^{(n)} = g^{(n)} \circ h^{(n)}$, and is compatible with the jet space projections, $\pi_n^{n+k} \circ g^{(n+k)} = g^{(n)}$. Given a (local) group of transformations acting on $E$, we define the prolonged group action, denoted by $G^{(n)}$, on the jet space $\mathrm{J}^n E$ by prolonging the individual transformations in $G$. In general, prolongation may only define a local action of the group $G$, although certain classes, e.g., global fiber-preserving actions, do have global prolongations.

 

**Example 3.8.** For a rotation in the one-parameter group considered in Example 3.2, the first prolongation $g_t^{(1)}$ will act on the space coordinatized by $(x, u, p)$, where, in accordance with classical notation, we use $p$ to represent the derivative coordinate $u_x$. Given a point $(x_0, u_0, p_0)$, we choose the linear polynomial $u = f(x) = p_0(x - x_0) + u_0$ as representative, noting that $f(x_0) = u_0$, $f'(x_0) = p_0$. The transformed function is given by (3.6), so

$$\bar{f}(\bar{x}) = \frac{\sin t + p_0 \cos t}{\cos t - p_0 \sin t}\, \bar{x} + \frac{u_0 - p_0 x_0}{\cos t - p_0 \sin t}.$$

Then, $\bar{x}_0 = x_0 \cos t - u_0 \sin t$, so $\bar{f}(\bar{x}_0) = \bar{u}_0 = x_0 \sin t + u_0 \cos t$, as we already knew, and $\bar{p}_0 = \bar{f}'(\bar{x}_0) = (\sin t + p_0 \cos t)/(\cos t - p_0 \sin t)$, which is defined provided $p_0 \neq \cot t$. Therefore, dropping the 0 subscripts, the first prolongation of the rotation group is explicitly given by

$$g_t^{(1)} \cdot (x, u, p) = \left( x \cos t - u \sin t,\ x \sin t + u \cos t,\ \frac{\sin t + p \cos t}{\cos t - p \sin t} \right), \tag{3.11}$$

defined for $p \neq \cot t$. Note, in particular, that even though the original group action is globally defined, its first prolongation is only a local transformation group.

**Example 3.9.** Example 3.8 is a particular case of a general point transformation

$$\bar{x} = \chi(x, u), \qquad \bar{u} = \psi(x, u), \tag{3.12}$$

on the space $E \simeq \mathbb{R} \times \mathbb{R}$ coordinatized by a single independent and a single dependent variable. In view of the ordinary calculus chain rule, the derivative coordinate $p = u_x$ on the jet space $\mathrm{J}^1$ will transform under the first prolongation of (3.12) according to a linear fractional transformation

$$\bar{p} = \frac{\alpha\, p + \beta}{\gamma\, p + \delta}, \tag{3.13}$$

whose coefficients

$$\alpha = \frac{\partial \psi}{\partial u}, \qquad \beta = \frac{\partial \psi}{\partial x}, \qquad \gamma = \frac{\partial \chi}{\partial u}, \qquad \delta = \frac{\partial \chi}{\partial x}, \tag{3.14}$$

are certain derivatives of the functions $\chi$, $\psi$ determining our change of variables. Further applications of the chain rule will yield the higher order prolongations in this case — see, for instance, (3.16) below.

**Exercise 3.10.** Before proceeding further, the reader should try to compute the second prolongation of the rotation group SO(2).

*Total Derivatives*

The chain rule computations used to compute prolongations are significantly simplified if we introduce the useful concept of a total derivative. We first define the functions that are to be differentiated.

**Definition 3.11.** A smooth, real-valued function $F\colon \mathrm{J}^n \to \mathbb{R}$, defined on an open subset of the $n^{\text{th}}$ jet space is called a *differential function* of *order* $n$.

 

Any $n^{\text{th}}$ order *differential equation* will be determined by the vanishing of a differential function of order $n$. For example, the planar Laplace equation $u_{xx}+u_{yy}=0$ is given by the second order differential function $F(x,u^{(2)})=u_{xx}+u_{yy}$ defined on $\mathrm{J}^2E$, where $E=\mathbb{R}^2\times\mathbb{R}$ has coordinates $x,y,u$. Note that any differential function of order $n$ automatically defines a differential function on any higher order jet space merely by treating the coordinates $(x,u^{(n)})$ of $\mathrm{J}^n$ as a subset of the coordinates $(x,u^{(n+k)})$ of $\mathrm{J}^{n+k}$ — this is the same as composing $F$ with the projection $\pi_n^{n+k}\colon \mathrm{J}^{n+k}\to\mathrm{J}^n$. In the sequel, we will not distinguish between $F$ and its compositions $F\circ\pi_n^{n+k}$. The *order* of a differential function will typically only refer to the minimal order jet space on which it is defined, which is the same as the maximal order derivative coordinate upon which it depends. Thus $u_{xx}+u_{yy}$ is a second order differential function, even though it also defines a function on any jet space $\mathrm{J}^k$ for $k\geq 2$.

**Definition 3.12.** Let $F(x,u^{(n)})$ be a differential function of order $n$. The *total derivative* $F$ with respect to $x^i$ is the $(n+1)^{\text{st}}$ order differential function $D_iF$ satisfying

$$D_iF(x,f^{(n+1)}(x))=\frac{\partial}{\partial x^i}\,F(x,f^{(n)}(x)),$$

for any smooth function $u=f(x)$.

For example, in the case of one independent variable $x$ and one dependent variable $u$, the total derivative of a given differential function $F(x,u^{(n)})$ with respect to $x$ has the general formula

$$D_xF=\frac{\partial F}{\partial x}+u_x\,\frac{\partial F}{\partial u}+u_{xx}\,\frac{\partial F}{\partial u_x}+u_{xxx}\,\frac{\partial F}{\partial u_{xx}}+\cdots. \tag{3.15}$$

For example, $D_x(xuu_{xx})=uu_{xx}+xuu_{xxx}+xu_xu_{xx}$.

As a first application, note that the chain rule prolongation formula (3.13) can now be simply written

$$\bar p=\frac{d\bar u}{d\bar x}=\frac{D_x\psi}{D_x\chi}.$$

Using this notation, the action of the second prolongation of the point transformation (3.12) on the second order derivative coordinate $q=u_{xx}$ can be compactly written as

$$\bar q=\frac{d^2\bar u}{d\bar x^2}=\frac{1}{D_x\chi}\,D_x\left(\frac{D_x\psi}{D_x\chi}\right)=\frac{D_x\chi\cdot D_x^2\psi-D_x\psi\cdot D_x^2\chi}{(D_x\chi)^3}. \tag{3.16}$$

For example, in the case of the rotation group $\mathrm{SO}(2)$ acting on $\mathbb{R}^2$, formula (3.16) implies the explicit form

$$\left(x\cos t-u\sin t,\; x\sin t+u\cos t,\; \frac{\sin t+p\cos t}{\cos t-p\sin t},\; \frac{q}{(\cos t-p\sin t)^3}\right), \tag{3.17}$$

for the second prolongation, thereby solving Exercise 3.10.

 

In the general framework, the total derivative with respect to the $i^{\text{th}}$ independent variable $x^i$ is the first order differential operator

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J}, \tag{3.18}$$

where $u^\alpha_{J,i} = D_i(u^\alpha_J) = u^\alpha_{j_1 \ldots j_k i}$. The sum in (3.18) is over all symmetric multi-indices $J$ of arbitrary order. Even though $D_i$ involves an infinite summation, when applying the total derivative to any particular differential function, only finitely many terms (namely, those for $\#J \leq n$, where $n$ is the order of $F$) are needed. In particular, each total derivative $D_i F$ of an $(n+1)^{\text{st}}$ order differential function depends linearly on the $(n+1)^{\text{st}}$ derivative coordinates. Higher order total derivatives are defined in the obvious manner, so that $D_J = D_{j_1} \cdot \ldots \cdot D_{j_k}$ for any multi-index $J = (j_1, \ldots, j_k)$, $1 \leq j_\nu \leq p$.

**Exercise 3.13.** Prove that $F(x, u^{(n)})$ is a differential function all of whose total derivatives vanish, $D_i F = 0$, $i = 1, \ldots, p$, if and only if $F$ is constant.

*Prolongation of Vector Fields*

Given a vector field $\mathbf{v}$ generating a one-parameter group of point transformations $\exp(t\mathbf{v})$ on $E \simeq X \times U$, the associated $n^{\text{th}}$ order prolonged vector field $\mathbf{v}^{(n)}$ is the vector field on the jet space $\mathrm{J}^n$ which is the infinitesimal generator of the prolonged one-parameter group $\exp(t\mathbf{v})^{(n)}$. Thus, according to (1.7), at any point $(x, u^{(n)}) \in \mathrm{J}^n$,

$$\mathbf{v}^{(n)}\big|_{(x,u^{(n)})} = \frac{d}{dt} \exp(t\mathbf{v})^{(n)} \cdot (x, u^{(n)}) \bigg|_{t=0}. \tag{3.19}$$

The explicit formula for the prolonged vector field is provided by the following "prolongation formula". Although the formula can be proved by direct computation based on the definition (3.19), cf. [**43**], we will wait in order to present an alternative useful proof based on the contact structure of the jet space later in this chapter.

**Theorem 3.14.** *Let* $\mathbf{v}$ *be a vector field given by* (3.3), *and let* $Q = (Q^1, \ldots, Q^q)$ *be its* characteristic, *as in* (3.8). *The* $n^{\text{th}}$ prolongation *of* $\mathbf{v}$ *is given explicitly by*

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=j=0}^{n} \varphi^\alpha_J(x, u^{(j)}) \frac{\partial}{\partial u^\alpha_J}, \tag{3.20}$$

*with coefficients*

$$\varphi^\alpha_J = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}. \tag{3.21}$$

**Example 3.15.** Suppose we have just one independent and dependent variable. A general vector field $\mathbf{v} = \xi(x,u)\partial_x + \varphi(x,u)\partial_u$ has characteristic

$$Q(x, u, u_x) = \varphi(x,u) - \xi(x,u) u_x. \tag{3.22}$$

The second prolongation of $\mathbf{v}$ is a vector field

$$\mathbf{v}^{(2)} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u} + \varphi^x(x,u^{(1)})\,\frac{\partial}{\partial u_x} + \varphi^{xx}(x,u^{(2)})\,\frac{\partial}{\partial u_{xx}}\,, \tag{3.23}$$

on $\mathrm{J}^2$, whose coefficients $\varphi^x, \varphi^{xx}$ are given by (3.21), hence

$$\begin{aligned} \varphi^x &= D_x Q + \xi u_{xx} = \varphi_x + (\varphi_u - \xi_x) u_x - \xi_u u_x^2, \\ \varphi^{xx} &= D_x^2 Q + \xi u_{xxx} = \varphi_{xx} + (2\varphi_{xu} - \xi_{xx}) u_x + (\varphi_{uu} - 2\xi_{xu}) u_x^2 - \\ &\qquad\qquad - \xi_{uu} u_x^3 + (\varphi_u - 2\xi_x) u_{xx} - 3\xi_u u_x u_{xx}. \end{aligned} \tag{3.24}$$

In (3.24) the subscripts on $\xi$ and $\varphi$ indicate partial derivatives. For example, the second prolongation of the generator $\mathbf{v} = -u\partial_x + x\partial_u$ of the rotation group is given by

$$\mathbf{v}^{(2)} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u} + (1 + u_x^2)\,\frac{\partial}{\partial u_x} + 3 u_x u_{xx}\,\frac{\partial}{\partial u_{xx}}\,. \tag{3.25}$$

The group transformations (3.17) can be readily recovered by integrating the system of ordinary differential equations governing the flow of $\mathbf{v}^{(2)}$, as in (1.5); these are

$$\frac{dx}{dt} = -u, \qquad \frac{du}{dt} = x, \qquad \frac{dp}{dt} = 1 + p^2, \qquad \frac{dq}{dt} = 3pq,$$

where we have used $p$ and $q$ to stand for $u_x$ and $u_{xx}$ to avoid confusing derivatives with jet space coordinates. Note also that the first prolongation of $\mathbf{v}$ is obtained by omitting the second derivative terms in (3.23), or, equivalently, projecting back to $\mathrm{J}^1$.

**Example 3.16.** As a second example, suppose we have two independent variables, $x$ and $t$, and one dependent variable $u$. A vector field

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$$

has characteristic

$$Q = \varphi - \xi u_x - \tau u_t.$$

The second prolongation of $\mathbf{v}$ is the vector field

$$\mathbf{v}^{(2)} = \xi\partial_x + \tau\partial_t + \varphi\partial_u + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{xx}\partial_{u_{xx}} + \varphi^{xt}\partial_{u_{xt}} + \varphi^{tt}\partial_{u_{tt}}, \tag{3.26}$$

where, for example,

$$\begin{aligned} \varphi^x &= D_x Q + \xi u_{xx} + \tau u_{xt} = \varphi_x + (\varphi_u - \xi_x) u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \varphi^t &= D_t Q + \xi u_{xt} + \tau u_{tt} \ = \varphi_t - \xi_t u_x + (\varphi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2, \\ \varphi^{xx} &= D_x^2 Q + \xi u_{xxx} + \tau u_{xxt} \\ &= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx}) u_x - \tau_{xx} u_t + (\varphi_{uu} - 2\xi_{xu}) u_x^2 - \\ &\qquad - 2\tau_{xu} u_x u_t - \xi_{uu} u_x^3 - \tau_{uu} u_x^2 u_t + (\varphi_u - 2\xi_x) u_{xx} - \\ &\qquad - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}. \end{aligned} \tag{3.27}$$

See Chapter 4 for applications of these formulae to the study of symmetries of differential equations.

 

**Exercise 3.17.** Prove that the coefficients (3.21) of the prolonged vector field satisfy the classical recursive formula

$$\varphi^\alpha_{J,i} = D_i \varphi^\alpha_J - \sum_{j=1}^p D_i \xi^j \, u^\alpha_{J,j}. \tag{3.28}$$

In particular, the coefficients of the first prolongation of $\mathbf{v}$ are given by

$$\varphi^\alpha_i = D_i \varphi^\alpha - \sum_{j=1}^p \bigl( D_i \xi^j \bigr) \, u^\alpha_j. \tag{3.29}$$

For instance, formulae (3.27) can also be written in the form

$$\varphi^x = D_x \varphi - (D_x \xi) u_x - (D_x \tau) u_t, \qquad \varphi^t = D_t \varphi - (D_t \xi) u_x - (D_t \tau) u_t,$$
$$\varphi^{xx} = D_x \varphi^x - (D_x \xi) u_{xx} - (D_x \tau) u_{xt}.$$

An alternative approach to the prolongation formula for vector fields is to introduce the *evolutionary vector field*

$$\mathbf{v}_Q = \sum_{\alpha=1}^q Q^\alpha(x, u^{(1)}) \, \frac{\partial}{\partial u^\alpha}\,, \tag{3.30}$$

based on the characteristic of our vector field $\mathbf{v}$, cf. (3.8); $\mathbf{v}_Q$ is an example of a "generalized" vector field in that it no longer depends on just the independent and dependent variables, but also on their derivatives, and so does not determine a well-defined geometrical transformation on the total space $E$; see [**4**, **43**], for a survey of the general theory. Let

$$\mathbf{v}_Q^{(n)} = \sum_{\alpha=1}^q \sum_{\#J \geq 0} D_J Q^\alpha(x, u^{(1)}) \, \frac{\partial}{\partial u^\alpha_J}\,, \tag{3.31}$$

be the corresponding formal prolongation of $\mathbf{v}_Q$. Then the $n^{\text{th}}$ prolongation of $\mathbf{v}$ can be written as

$$\mathbf{v}^{(n)} = \mathbf{v}_Q^{(n)} + \sum_{i=1}^p \xi^i D_i^{(n)}, \tag{3.32}$$

where $D_i^{(n)}$ denotes the order $n$ truncation of the total derivative operator (3.18), i.e., restricting the sum to $\#J \leq n$.

Since the prolongation process respects the composition of maps, the commutator formula (1.13) immediately proves that it preserves Lie brackets:

$$[\mathbf{v}, \mathbf{w}]^{(n)} = [\mathbf{v}^{(n)}, \mathbf{w}^{(n)}], \tag{3.33}$$

and therefore defines a Lie algebra homomorphism from the space of vector fields on $E$ to the space of prolonged vector fields on $\mathrm{J}^n E$. Thus, the $n^{\text{th}}$ prolongation $\mathfrak{g}^{(n)}$ of a Lie algebra of vector fields on $E$ defines an isomorphic Lie algebra of vector fields on $\mathrm{J}^n$, generating the $n^{\text{th}}$ prolongation of the associated transformation group.

 

**Example 3.18.** According to the classification tables in [**44**], there are three distinct, locally inequivalent actions of the unimodular Lie group SL(2) on a two-dimensional complex manifold. Interestingly, the process of prolongation can be used to relate all three actions. Consider first the usual linear fractional action

$$(x,u) \longmapsto \left( \frac{\alpha x+\beta}{\gamma x+\delta}\,,\ u \right),$$

whose infinitesimal generators

$$\mathbf{v}_- = \partial_x, \qquad \mathbf{v}_0 = x\partial_x, \qquad \mathbf{v}_+ = x^2\partial_x, \tag{3.34}$$

span the intransitive Lie algebra of Type 3.3; see also Example 2.60. Using $p = u_x$ to denote the derivative coordinate, we find that the first prolongation of this group action is generated by the vector fields

$$\mathbf{v}_-^{(1)} = \partial_x, \qquad \mathbf{v}_0^{(1)} = x\partial_x - p\partial_p, \qquad \mathbf{v}_+^{(1)} = x^2\partial_x - 2xp\partial_p, \tag{3.35}$$

which, in accordance with (3.33), form a Lie algebra having the same $\mathfrak{sl}(2)$ commutation relations. The Lie algebra (3.35) clearly projects to the $(x,p)$–plane, thereby defining the Lie algebra of vector fields of Type 1.1 in our tables. Further, setting $q = u_{xx}$, the second prolongation of the vector fields (3.34) yields the Lie algebra

$$\mathbf{v}_-^{(2)} = \partial_x, \qquad \mathbf{v}_0^{(2)} = x\partial_x - p\partial_p - 2q\partial_q, \qquad \mathbf{v}_+^{(2)} = x^2\partial_x - 2xp\partial_p - (4xq+2p)\partial_q, \tag{3.36}$$

again having the same $\mathfrak{sl}(2)$ commutation relations. Define

$$w = \frac{q}{2p} = \frac{p_x}{2p} = \frac{u_{xx}}{2u_x}, \tag{3.37}$$

and use $(x,u,p,w)$ instead of $(x,u,p,q)$ as coordinates on $\{p \neq 0\} \subset \mathrm{J}^2$. The vector fields (3.36) then have the form

$$\mathbf{v}_-^{(2)} = \partial_x, \qquad \mathbf{v}_0^{(2)} = x\partial_x - p\partial_p - w\partial_w, \qquad \mathbf{v}_+^{(2)} = x^2\partial_x - 2xp\partial_p - (2xw+1)\partial_w. \tag{3.38}$$

Again, we can project this action to the $(x,w)$–plane, on which the vector fields (3.38) span a Lie algebra of Type 1.2. The associated group action is

$$(x,w) \longmapsto \left( \frac{\alpha x+\beta}{\gamma x+\delta}\,,\ (\gamma x+\delta)^2 w + \gamma(\gamma x+\delta) \right).$$

*Contact Forms*

We have already remarked on how the $n^{\text{th}}$ order jet space can be abstractly constructed using the equivalence relation of $n^{\text{th}}$ order contact. We now investigate the contact structure of jet spaces in some detail, determining a remarkable system of differential forms — the contact forms — which serve to characterize precisely prolonged functions and transformations. One immediate consequence of this approach is a simple proof of the prolongation formula (3.20).

To begin with, recall that each (smooth) function $u = f(x)$ determines a prolonged function $u^{(n)} = f^{(n)}(x)$ from $X$ to the $n^{\text{th}}$ jet space $\mathrm{J}^n$. The (easy) inverse problem is to characterize those sections of the jet space, meaning functions $F: X \to \mathrm{J}^n$ given by $u^{(n)} = F(x)$, which come from prolonging ordinary functions. This is essentially the same as the problem of determining which $p$-dimensional submanifolds of $\mathrm{J}^n$ are the graphs of the prolongations of functions. Although in local coordinates the solution to both problems is completely trivial — one just checks that the derivative coordinates match up properly — there is a more interesting and useful solution to these problems based on an intrinsically defined system of differential forms.

**Definition 3.19.** A differential one-form $\theta$ on the jet space $\mathrm{J}^n$ is called a *contact form* if it is annihilated by all prolonged functions. In other words, if $u = f(x)$ is any smooth function with $n^{\text{th}}$ prolongation $f^{(n)}: X \to \mathrm{J}^n$, then the pull-back of $\theta$ to $X$ via $f^{(n)}$ must vanish: $(f^{(n)})^* \theta = 0$.

**Example 3.20.** Consider the case of one independent and one dependent variable. On the first jet space $\mathrm{J}^1$, with coordinates $x, u, p = u_x$, a general one-form takes the form $\theta = a\,dx + b\,du + c\,dp$, where $a, b, c$ are functions of $x, u, p$. A function $u = f(x)$ has first prolongation $p = f'(x)$, hence $(f^{(1)})^*\theta$ equals

$$\big[a(x, f(x), f'(x)) + b(x, f(x), f'(x)) f'(x) + c(x, f(x), f'(x)) f''(x)\big]\, dx.$$

This will vanish for all functions $f$ if and only if $c = 0$ and $a = -bp$; hence $\theta = b(x, u, p)\theta_0$ must necessarily be a multiple of the *basic contact form* $\theta_0 = du - p\,dx = du - u_x\,dx$. Proceeding to the second jet space $\mathrm{J}^2$, with additional coordinate $q = u_{xx}$, a similar calculation shows that a one-form $\theta = a\,dx + b\,du + c\,dp + e\,dq$ is a contact form if and only if $\theta = b\,\theta_0 + c\,\theta_1$, where $\theta_1 = dp - q\,dx = du_x - u_{xx}\,dx$ is the next basic contact form. (Here, as we did earlier with differential functions, we are identifying the form $\theta_0$ with its pull-back $(\pi_1^2)^*\theta_0$ to $J^2$.) In general, provided $x, u \in \mathbb{R}$, the general contact form can be written as a linear combination of the basic contact forms $\theta_k = du_k - u_{k+1}\,dx$, $k = 0, \ldots, n-1$, where $u_k = D_x^k u$ is the $k^{\text{th}}$ order derivative of $u$.

Similar elementary computations show that, in the case of two dependent variables $u, v$, every contact form is a linear combination of the basic contact forms for $u$ and analogous ones for $v$, i.e., $dv - v_x\,dx$, $dv_x - v_{xx}\,dx$, etc. On the other hand, for two independent and one dependent variable, there is one basic contact form $\theta_0 = du - u_x\,dx - u_y\,dy$ on $\mathrm{J}^1$, two basic contact forms $\theta_x = du_x - u_{xx}\,dx - u_{xy}\,dy$, $\theta_y = du_y - u_{xy}\,dx - u_{yy}\,dy$, on $\mathrm{J}^2$, and so on. A similar argument provides us with the complete characterization of all contact forms.

**Theorem 3.21.** *Every contact form on* $\mathrm{J}^n$ *can be written as a linear combination,* $\theta = \sum_{J,\alpha} P_J^\alpha \theta_J^\alpha$, *with smooth coefficient functions* $P_J^\alpha(x, u^{(n)})$, *of the* basic contact forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha\, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \le \#J < n. \tag{3.39}$$

In (3.39), we call $\#J$ the *order* of the contact form $\theta_J^\alpha$. Note especially that the contact forms on $\mathrm{J}^n$ have orders at most $n - 1$.

 

**Theorem 3.22.** *A section* $u^{(n)} = F(x)$ *of the jet space* $\mathrm{J}^n$ *is the* prolongation *of some function* $u = f(x)$, *meaning* $F = f^{(n)}$, *if and only if* $F$ *annihilates all the contact forms on* $\mathrm{J}^n$*:*

$$F^*\theta^\alpha_J = 0, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J < n. \tag{3.40}$$

*A* transverse *p-dimensional submanifold* $\Gamma^{(n)} \subset \mathrm{J}^n$ *is (locally) the* graph *of a prolonged function* $u = f^{(n)}(x)$ *if and only if all the contact forms vanish on it:* $\theta^\alpha_J \mid \Gamma^{(n)} = 0$.

The general argument will be clear from consideration of the following two special cases. Let $x, u \in \mathbb{R}$. A section $F$ of $\mathrm{J}^1$ is described by a pair of functions $u = f(x)$, $p = g(x)$. The section annihilates the single contact form $\theta_0 = du - p\,dx$ provided $0 = F^*\theta_0 = df - g\,dx = [f'(x) - g(x)]\,dx$, which vanishes if and only if $g(x) = f'(x)$, showing that $F$ is just the first prolongation of $f$. Similarly, a section $F$ of $\mathrm{J}^2$ is given by $u = f(x)$, $p = g(x)$, $q = h(x)$. It annihilates the two basic contact forms provided $0 = F^*(du - p\,dx) = df - g\,dx = [f'(x) - g(x)]\,dx$, so $g(x) = f'(x)$, and $0 = F^*(dp - q\,dx) = dg - h\,dx = [g'(x) - h(x)]\,dx$, which implies $h(x) = g'(x) = f''(x)$. Therefore $F = f^{(2)}$ as desired.

The contact forms also provide us with a nice intrinsic characterization of the total derivative operators defined earlier. A one-form $\omega$ is called *horizontal* if it annihilates all the vertical tangent directions in $\mathrm{J}^n$. Thus, in local coordinates, the horizontal one-forms are just linear combinations, $\omega = \sum Q_i(x, u^{(n)})dx^i$, of the coordinate one-forms on the base $X$. To any one-form $\omega$ on $\mathrm{J}^n$, there is an intrinsically defined horizontal form $\omega_H$ on $\mathrm{J}^{n+1}$, called the *horizontal component* of $\omega$, which is defined so that $\omega = \omega_H + \theta$ where $\theta$ is a contact form of order $n$. In coordinates, using the general formula (3.39), we find

$$\begin{aligned}
\omega &= \sum_{i=1}^{p} Q_i(x, u^{(n)})\,dx^i + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} P^J_\alpha(x, u^{(n)})\,du^\alpha_J \\
&= \sum_{i=1}^{p} \left\{ Q_i + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} u^\alpha_{J,i} P^J_\alpha \right\} dx^i + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} P^J_\alpha \theta^\alpha_J.
\end{aligned}$$

The second summand is a contact form, and hence

$$\omega_H = \sum_{i=1}^{p} \left\{ Q_i(x, u^{(n)}) + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} u^\alpha_{J,i} P^J_\alpha(x, u^{(n)}) \right\} dx^i. \tag{3.41}$$

*Warning*: The horizontal component $\omega_H$ of a form $\omega$ on $\mathrm{J}^n$ depends on $(n+1)^{\text{st}}$ order derivatives. This occurs because we are including the $n^{\text{th}}$ order contact forms, which are only defined on $\mathrm{J}^{n+1}$, in the decomposition $\omega = \omega_H + \theta$. For example, if $p = q = 1$, the horizontal component of $\omega = Q\,dx + P\,du$ is $\omega_H = (Q + u_x P)\,dx$.

**Definition 3.23.** Let $F \colon \mathrm{J}^n \to \mathbb{R}$ be a differential function of order $n$. Then the *total differential* of $F$ is the horizontal component of its ordinary differential $dF$, written $\mathrm{D}F = (dF)_H$.

To compute the total differential $\mathrm{D}F$, first note that

$$dF = \sum_{i=1}^{p} \frac{\partial F}{\partial x^i}\, dx^i + \sum_{\alpha=1}^{q} \sum_{\#J \le n} \frac{\partial F}{\partial u^\alpha_J}\, du^\alpha_J, \tag{3.42}$$

Thus, by (3.41) and (3.18), the total differential of $F$ is given in terms of the total derivatives of $F$ by the formula

$$\mathrm{D}F = \sum_{i=1}^{p} D_i F\, dx^i. \tag{3.43}$$

For example, if $F(x, u, p, q)$ is a second order differential function, with $x, u \in \mathbb{R}$, $p = u_x$, $q = u_{xx}$, then

$$\begin{aligned} dF = F_x\, dx + F_u\, du + F_p\, dp + F_q\, dq &= (F_x + pF_u + qF_p + rF_q)\, dx + \\ &\quad + F_u(du - p\, dx) + F_p(dp - q\, dx) + F_q(dq - r\, dx), \end{aligned}$$

where $r = u_{xxx}$. The total differential $\mathrm{D}F = D_x F\, dx$ is the first component. Formula (3.42) proves the invariance of the total differential (and hence the "covariance" of the total derivatives) under general point transformations.

The decomposition $dF = \mathrm{D}F + \theta$ of an $n^{\text{th}}$ order differential function into horizontal and contact components does *not* work on $\mathrm{J}^n$. However, on $\mathrm{J}^n$ we have the alternative decomposition

$$dF = \widehat{\mathrm{D}}F + d_n F + \vartheta. \tag{3.44}$$

Here $\vartheta$ is a contact form of order at most $n-1$, and

$$d_n F = \sum_{\alpha=1}^{q} \sum_{\#J = n} \frac{\partial F}{\partial u^\alpha_J}\, du^\alpha_J. \tag{3.45}$$

is the (intrinsically defined) $n^{\text{th}}$ order *differential* of $F$. Furthermore,

$$\widehat{\mathrm{D}}F = \sum_{i=1}^{p} \widehat{D}_i F\, dx^i \tag{3.46}$$

is the "truncated total differential" of $F$. In (3.46), the operator $\widehat{D}_i$ is the $n^{\text{th}}$ order truncation of the total derivative (3.18), obtained by restricting the multi-index summation to run only over $J$'s with $\#J \le n-1$. For example, if $F = uu_{xx}$, then

$$\begin{aligned} dF = u_{xx}\, du + u\, du_{xx} &= u_x u_{xx}\, dx + u\, du_{xx} + u_{xx}(du - u_x\, dx) \\ &= (uu_{xxx} + u_x u_{xx})\, dx + u_{xx}(du - u_x\, dx) + u(du_{xx} - u_{xxx}\, dx). \end{aligned}$$

Therefore, $\mathrm{D}F = D_x F\, dx = (uu_{xxx} + u_x u_{xx})\, dx$, while $\widehat{\mathrm{D}}F = u_x u_{xx}\, dx$ and $d_2 F = u\, du_{xx}$.

The $n^{\text{th}}$ prolongation $g^{(n)}\colon \mathrm{J}^n \to \mathrm{J}^n$ of any point transformation $g$ can be characterized by the property that it maps the prolonged graph of any function to the prolonged graph of the transformed function. Theorem 3.22 immediately implies that $g^{(n)}$ maps contact forms to contact forms. Indeed, this latter property, coupled with its compatibility with the projection $\pi^n_0\colon \mathrm{J}^n \to E$, serves to essentially define the prolongation of $g$.

 

*Remark*: Bäcklund's Theorem, [**4**, **44**], shows that, when the number of dependent variables $q > 1$, *any* transformation that preserves the contact forms is necessarily the prologation of a point transformation. On the other hand, when $q = 1$, there are first order contact transformations that preserve the contact forms and yet are not prolonged point transformations. On the other hand, any higher order contact transformation is the prolongaiton of a first order one.

The infinitesimal version of the preceding property is:

**Proposition 3.24.** *A prolonged vector field* $\mathbf{v}^{(n)}$ *has the property that the* Lie derivative $\mathbf{v}^{(n)}(\theta)$ *of any contact form is itself a contact form.*

Let us use this result to prove the prolongation formula in Theorem 3.14. Let

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x, u^{(n)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}\,, \tag{3.47}$$

be the prolonged vector field on $\mathrm{J}^n$. Taking the Lie derivative of the contact form (3.39) with respect to (3.47), we find

$$\mathbf{v}^{(n)}(\theta_J^\alpha) = d\varphi_J^\alpha - \sum_{i=1}^{p} \bigl[\varphi_{J,i}^\alpha\, dx^i + u_{J,i}^\alpha\, d\xi^i\bigr], \qquad \#J < n. \tag{3.48}$$

Now, if this is to be a contact form on $\mathrm{J}^n$, its horizontal component must vanish. Equations (3.41, 42) imply that the coefficients of $\mathbf{v}^{(n)}$ satisfy the inductive form of the prolongation formula (3.28). Since (3.28) serves to uniquely specify the higher order coefficients in terms of $\xi^i$ and $\varphi^\alpha$, we deduce that the coefficients of a general contact vector field on $\mathrm{J}^n$ are given by the same prolongation formula (3.21) as with a prolonged point transformation; in particular, this observation serves to complete the proof of the prolongation formula.

*Differential Invariants*

The basic problem to be addressed in this section is the classification of the differential invariants of a given group action. We will be considering general point transformation groups $G$ acting on the space of independent and dependent variables $E$. We let $G^{(n)}$ denote the corresponding prolonged action on the $n^{\text{th}}$ jet space $\mathrm{J}^n = \mathrm{J}^n E$. We use the notation $g^{(n)}$ to denote the (prolonged) action of the individual group elements $g \in G$ on $\mathrm{J}^n$, and $\mathbf{v}^{(n)}$ for the associated infinitesimal generators.

**Definition 3.25.** Let $G$ be a group of point transformations. A *differential invariant* for $G$ is a differential function $I\colon \mathrm{J}^n \to \mathbb{R}$ which satisfies $I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$ for all $g \in G$ and all $(x, u^{(n)}) \in \mathrm{J}^n$ where $g^{(n)} \cdot (x, u^{(n)})$ is defined.

As usual, the differential invariant $I$ may only be defined on an open subset of the jet space $\mathrm{J}^n$, although, in accordance with our usual convention, we shall still write $I\colon \mathrm{J}^n \to \mathbb{R}$. Note that, as with general differential functions, any lower order differential invariant $I(x, u^{(k)})$, $k < n$ can also be viewed as an $n^{\text{th}}$ order differential invariant.

 

**Example 3.26.** Consider the usual action of the rotation group SO(2) on $E \simeq \mathbb{R}^2$, cf. Example 3.8. The radius $r = \sqrt{x^2+u^2}$ is an (ordinary) invariant of SO(2) (and, *a fortiori*, a differential invariant too). The first prolongation $\mathrm{SO}(2)^{(1)}$ was given in (3.11), and has one-dimensional orbits at each point of $\mathrm{J}^1$; therefore, by Theorem 2.30, besides the radius $r$, there is one additional first order differential invariant, which can be taken to be the function $w = (xu_x - u)/(x + uu_x)$, provided $x \neq -uu_x$. (Alternative pairs of differential invariants must be used near the points on $\mathrm{J}^1$ where $x + uu_x = 0$.) Geometrically, $w = \tan\phi$, where $\phi$ is the angle between the line from the origin to the point $(x,u) = (x, f(x))$ and the tangent to the graph of $u = f(x)$ at that point. The second prolongation $\mathrm{SO}(2)^{(2)}$ is given by (3.17), and also has one-dimensional orbits. (Indeed, the dimension of the orbits can never exceed that of the group.) The radius $r$ and first order differential invariant $w$ still provide two independent differential invariants on $\mathrm{J}^2$, and the curvature $\kappa = (1+u_x^2)^{-3/2} u_{xx}$ is the additional second order differential invariant.

As with ordinary invariants, we shall classify differential invariants up to functional independence, since any function $H(I_1, \ldots, I_k)$ of a collection of differential invariants $I_1, \ldots, I_k$ is also a differential invariant. In accordance with Proposition 1.36, the differential functions $F_1, \ldots, F_k \colon \mathrm{J}^n \to \mathbb{R}$ are functionally independent if their differentials are pointwise independent: $dF_1 \wedge \cdots \wedge dF_k \neq 0$. Since, as noted above, lower order differential invariants are also considered as $n^{\text{th}}$ order differential invariants, it will be important to distinguish the differential invariants which genuinely depend on the $n^{\text{th}}$ order derivative coordinates. We will call a set of differential functions on $\mathrm{J}^n$ strictly independent if, as functions of the $n^{\text{th}}$ order derivative coordinates alone, they are functionally independent. More specifically, we make the following definition.

**Definition 3.27.** A collection of $n^{\text{th}}$ order differential functions

$$F_1(x, u^{(n)}), \ldots, F_k(x, u^{(n)})$$

is called *strictly independent* if they and the derivative coordinate functions $x$, $u^{(n-1)}$, of order less than $n$, are all functionally independent.

For example, in the case of the rotation group discussed above, the second order differential invariants $r, w, \kappa$ are functionally independent, but not strictly independent since $r$ and $w$ have order less than two. Indeed, there is only one strictly independent second order differential invariant — the curvature $\kappa$ (or any function thereof).

**Proposition 3.28.** *The differential functions $F_1, \ldots, F_k$ are strictly independent if their $n^{\text{th}}$ order* differentials, *as given by*

$$d_n F = \sum_{\alpha=1}^{q} \sum_{\#J=n} \frac{\partial F}{\partial u_J^\alpha}\, du_J^\alpha. \tag{3.49}$$

*are linearly independent at each point:* $d_n F_1 \wedge d_n F_2 \wedge \cdots \wedge d_n F_k \neq 0$.

In particular, strict independence implies that none of the $F_\nu$'s, or any function thereof, can be of order strictly less than $n$. In the regular case, then, the number of strictly independent functions in a collection $\{F_1, \ldots, F_k\}$ is equal to the rank of their $k \times q_n$ top order Jacobian matrix $\left(\partial F_\nu / \partial u^\alpha_J\right)$, $\#J = n$.

*Dimensional Considerations*

In order to study the differential invariants of a group of transformations, a more detailed knowledge of the structure of the prolonged group action is required. Of particular importance is an understanding of the dimension of the (generic) orbits of the different prolongations $G^{(n)}$ on $\mathrm{J}^n$. Recall first that the dimension of the jet space $\mathrm{J}^n$ is denoted by $p + q^{(n)}$, where $q^{(n)} = q\binom{p+n}{n}$. The number of derivative coordinates of order exactly $n$ is denoted by

$$q_n = \dim \mathrm{J}^n - \dim \mathrm{J}^{n-1} = q^{(n)} - q^{(n-1)} = q\binom{p+n-1}{n}. \tag{3.50}$$

We let $s_n$ denote the maximal (generic) orbit dimension of $G^{(n)}$, so that $G^{(n)}$ acts semi-regularly on the open subset $V^n \subset \mathrm{J}^n$ which consists of all points contained in the orbits of maximal dimension. (If $G$ is a group of point transformations, then $s_n$ is well defined for each $n \geq 0$, whereas for a group of contact transformations, $s_n$ is defined only for $n \geq 1$.) If $G$ acts analytically, then the subset $V^n$ is dense in $\mathrm{J}^n$. For the time being, we restrict our attention to the subset $V^n$, thereby avoiding more delicate questions concerning singularities of the prolonged group action. Note that $s_n$ equals the maximal dimension of the subspace $\mathfrak{g}^{(n)}|_z \subset T\mathrm{J}^n|_z$ spanned by the prolonged infinitesimal generators of the group action at points $z \in \mathrm{J}^n$. According to Proposition 2.63, the prolonged orbit dimensions can also be computed as $s_n = r - h_n$, where $h_n$ denotes the dimension of the isotropy subgroup $H^{(n)}_z \subset G$ at any point $z \in V^n$.

According to Theorem 2.30, there are

$$i_n = p + q^{(n)} - s_n = p + q^{(n)} - r + h_n \tag{3.51}$$

functionally independent differential invariants of order at most $n$ defined in a neighborhood of any point $z \in V^n$. Since each differential invariant of order less than $n$ is included in this count, the integers $i_n$ form a nondecreasing sequence: $i_0 \leq i_1 \leq i_2 \leq \cdots$. The difference

$$j_n = i_n - i_{n-1} = q_n - s_n + s_{n-1} = q_n + h_n - h_{n-1} \tag{3.52}$$

will count the number of strictly independent $n^{\text{th}}$ order differential invariants. For groups of point transformations, we set $j_0 = i_0$ to be the number of ordinary invariants. Note that $j_n$ cannot exceed the number of independent derivative coordinates of order $n$, so $j_n \leq q_n$, which implies the elementary inequalities

$$i_{n-1} \leq i_n \leq i_{n-1} + q_n. \tag{3.53}$$

For example, in the case of the rotation group discussed in Example 3.26, each prolongation has one-dimensional orbits (indeed, $\mathrm{SO}(2)$ acts regularly on all of $\mathrm{J}^n$ for $n \geq 1$), and hence $s_0 = s_1 = s_2 = \cdots = 1$. Equation (3.2) implies that $i_0 = 1$, so there is one ordinary invariant — the radius $r$. Furthermore, $i_1 = 2$, so there is $j_1 = i_1 - i_0 = 1$ additional first order differential invariant, which can be chosen to be the angle $\phi$. In general, $i_n = n + 1$, so there is precisely $j_n = i_n - i_{n-1} = 1$ additional differential invariant at each order $n$.

Beyond the curvature $\kappa$, the higher order differential invariants for the rotation group will be constructed in Example 3.37 below.

If $\mathcal{O}^{(n)} \subset \mathrm{J}^n$ is any orbit of $G^{(n)}$, then, for any $k < n$, its projection $\pi^n_k(\mathcal{O}^{(n)}) \subset \mathrm{J}^k$ is an orbit of the $k^{\text{th}}$ prolongation $G^{(k)}$. Therefore, the maximal orbit dimension $s_n$ of $G^{(n)}$ is also a *nondecreasing* function of $n$, bounded by $r$, the dimension of $G$ itself:

$$s_0 \leq s_1 \leq s_2 \leq \cdots \leq r. \tag{3.54}$$

On the other hand, since the orbits cannot increase in dimension any more than the increase in dimension of the jet spaces themselves, we have the the elementary inequalities

$$s_{n-1} \leq s_n \leq s_{n-1} + q_n, \tag{3.55}$$

governing the orbit dimensions. Note that, in view of equations (3.50) and (3.51), the inequalities (3.55) are equivalent to those in (3.53). Also note that (3.55) implies that the isotropy subgroups $H^{(n)}_z$, $z \in V^n$, have nonincreasing dimensions: $h_{n-1} \geq h_n = r - s_n$. This follows directly from the observation that the isotropy subgroup $H^{(k)}_w$ of the projection $w = \pi^n_k(z) \in \mathrm{J}^k$ of a point $z \in \mathrm{J}^n$ is always contained in the isotropy subgroup $H^{(n)}_z$.

The inequalities (3.54) imply that the maximal orbit dimension eventually stabilizes, so that there exists an integer $s$ such that $s_m = s$ for all $m$ sufficiently large. In particular, if the orbit dimension is ever the same as that of $G$, meaning $s_n = r$ for some $n$, then $s_m = r$ for all $m \geq n$. We shall call $s$ the *stable orbit dimension*, and the minimal order $n$ for which $s_n = s$ the *order of stabilization* of the group. Once we've reached the stabilization order, the number of higher order differential invariants is immediate.

**Proposition 3.29.** *Let $n$ denote the order of stabilization of the group $G$. Then, for every $m > n$ there are precisely $q_m$ strictly independent $m^{\text{th}}$ order differential invariants.*

Consequently, any (finite-dimensional) group of transformations has an infinite number of differential invariants of arbitrarily large order.

*Infinitesimal Methods*

As with ordinary invariants, it is easier to determine differential invariants (of connected groups) using an infinitesimal approach. The basic infinitesimal invariance condition for differential invariants is an immediate corollary of Theorem 2.66.

**Proposition 3.30.** *A function $I\colon \mathrm{J}^n \to \mathbb{R}$ is a* differential invariant *for a connected transformation group $G$ if and only if it is annihilated by all the prolonged infinitesimal generators*:

$$\mathbf{v}^{(n)}(I) = 0 \qquad \textit{for all} \qquad \mathbf{v} \in \mathfrak{g}. \tag{3.56}$$

**Example 3.31.** In the case of the rotation group $\mathrm{SO}(2)$, its second prolongation has infinitesimal generator $\mathbf{v}^{(2)}$ given by (3.25). Applying this vector field to the functions given in Example 3.26, we find $\mathbf{v}^{(2)}(r) = \mathbf{v}^{(2)}(w) = \mathbf{v}^{(2)}(\kappa) = 0$, re-proving the fact that $r, w, \kappa$ are differential invariants. Note that these differential invariants can be deduced directly from the form of $\mathbf{v}^{(2)}$ using the method of characteristics, as was done in Example 2.67.

 

**Example 3.32.** Consider the three-parameter similarity group

$$(x,u) \longmapsto (\lambda x + a, \lambda u + b), \qquad (x,u) \in E \simeq \mathbb{R}^2,$$

consisting of translations and scalings, and generated by the vector fields $\partial_x, \partial_u, x\partial_x + u\partial_u$. There are no ordinary invariants since the group acts transitively on $E = \mathbb{R}^2$. Furthermore, all three vector fields happen to coincide with their first prolongations, and hence there is one independent first order differential invariant, namely $u_x$. The second prolongations are $\partial_x, \partial_u, x\partial_x + u\partial_u - u_{xx}\partial_{u_{xx}}$, and hence there are no differential invariants of (strictly) second order. There is a single third order differential invariant, namely $u_{xx}^{-2}u_{xxx}$, a single fourth order invariant, $u_{xx}^{-3}u_{xxxx}$, and, in general, a single $n^{\text{th}}$ order differential invariant $u_{xx}^{1-n}D_x^n u$. Therefore, the number of strictly independent differential invariants is given by $j_0 = j_2 = 0$, $j_1 = j_3 = \cdots = j_n = 1$, $n \geq 3$. This implies that $i_0 = 0$, $i_1 = i_2 = 1$, $i_3 = 2, \ldots, i_n = n-1$, and hence the maximal orbit dimensions are $s_0 = s_1 = 2$, $s_2 = s_3 = \cdots = 3 = \dim G$, a fact that can also be deduced by looking at the dimension of the space spanned by the prolonged infinitesimal generators. Note, in particular, that the orbit dimensions "pseudo-stabilized" at order 0 since $s_0 = s_1$, but that the true order of stabilization is $n = 2$. Thus, the fact that $s_k = s_{k+1}$ does *not*, in general, imply that $k$ is the order of stabilization of a transformation group $G$. However, this pseudo-stabilization phenomenon is quite rare.

The preceding example corresponds to a particular case of our classification tables for Lie group actions in the plane — namely Case 1.7 with $\alpha = k = 1$. More generally, consider Case 1.7 with $\alpha = k = r - 2 \geq 1$, which is the $r$ parameter group generated by the vector fields

$$\partial_x, \quad \partial_u, \quad x\partial_u, \quad \ldots, \quad x^{r-3}\partial_u, \quad x\partial_x + (r-2)u\partial_u.$$

(What is the associated group action?) Using the prolongation formula (3.21), it is not hard to see that the prolonged orbit dimensions are given by $s_0 = 2$, $s_1 = 3, \ldots, s_{r-3} = s_{r-2} = r - 1$, $s_{r-1} = s_r = \cdots = r$. In this case, the orbit dimensions pseudo-stabilize at order $r - 3$ and stabilize at order $r - 1$. Consequently, the prolonged orbit dimensions of a transformation group can pseudo-stabilize at an arbitrarily high order.

**Exercise 3.33.** Find the differential invariants of the latter case.

**Exercise 3.34.** Prove that the orbit dimensions for Case 1.7 with $\alpha \neq k$ do *not* pseudo-stabilize.

*Invariant Differential Operators*

Since any transformation group action has differential invariants of arbitrarily high order, it is incumbent upon us to find a more systematic method for determining them all. The basic tool is the use of certain "invariant" differential operators, introduced by Lie, [**33**], and Tresse, [**53**], which have the property of mapping $n^{\text{th}}$ order differential invariants to $(n+1)^{\text{st}}$ order differential invariants, and thus, by iteration, produce hierarchies of differential invariants of arbitrarily large order. In fact, we can guarantee the existence of sufficiently many such differential operators and differential invariants so as to completely generate all the higher order independent differential invariants of the group by successively differentiating the lower order differential invariants. Thus, a complete description of all the differential invariants is provided by a collection of low order "fundamental" differential invariants along with the requisite invariant differential operators. To introduce the general method, we begin with the simplest case when there is only one independent variable, where the construction of higher order differential invariants is facilitated by the following result.

**Proposition 3.35.** *Suppose $X = \mathbb{R}$ and $G$ is a transformation group acting on $E \simeq X \times U$. Let $s = I(x, u^{(n)})$ and $v = J(x, u^{(n)})$ be functionally independent differential invariants, at least one of which has order exactly $n$. Then the derivative $dv/ds = (D_x J)/(D_x I)$ is an $(n+1)^{\text{st}}$ order differential invariant.*

*Proof*: The statement can be verified directly, but we provide a more generally applicable proof based on the contact structure of $\mathrm{J}^n$. According to Proposition 2.73, if $I(x, u^{(n)})$ is a differential invariant, its differential $dI$ is an invariant one-form on $\mathrm{J}^n$. As in Definition 3.23, we decompose $dI$ into its horizontal and contact components, $dI = D_x I\, dx + \theta_I$, where $\theta_I$ is a contact form on $\mathrm{J}^{n+1}$. Similarly, if $J$ is any other $n^{\text{th}}$ order differential invariant, then, on the open subset of $\mathrm{J}^{n+1}$ where $D_x I \neq 0$, we have $dJ = D_x J\, dx + \theta_J = [(D_x J)/(D_x I)]\, dI + \vartheta$ for some contact form $\vartheta$. The prolonged group transformations in $G^{(n+1)}$ map contact forms to contact forms. Therefore, the invariance of both $dI$ and $dJ$ immediately implies that the coefficient $D_x J/D_x I$ must be an invariant function for $G^{(n+1)}$. Finally, the functional independence of $I$ and $J$ is enough to guarantee that $(D_x J)/(D_x I)$ has order $n+1$. *Q.E.D.*

Let us reinterpret this result. If $s = I(x, u^{(n)})$ is any given differential invariant, then $\mathcal{D} = d/ds = (D_x I)^{-1} D_x$ is an *invariant differential operator* for the prolonged group actions, since if $J$ is any other differential invariant, so is $\mathcal{D}J$. Therefore, we can iterate $\mathcal{D}$, producing a sequence $\mathcal{D}^k J = d^k J/ds^k$, $k = 0, 1, 2, \ldots$, of higher and higher order differential invariants. Let us first apply this result when there is just one independent and one dependent variable. In this case, once we know two independent differential invariants, all higher order differential invariants can be calculated by successive differentiation with respect to the invariant differential operator $\mathcal{D}$.

**Theorem 3.36.** *Suppose $G$ is a connected group of point transformations acting on the jet spaces corresponding to $E \simeq \mathbb{R} \times \mathbb{R}$. Then, for some $n \geq 0$, there are precisely two functionally independent differential invariants $I$, $J$ of order $n$ (or less). Furthermore for any $k \geq 0$, a complete system of functionally independent differential invariants of order $n+k$ is provided by $I$, $J$, $\mathcal{D}J, \ldots, \mathcal{D}^k J$, where $\mathcal{D} = (D_x I)^{-1} D_x$ is the associated invariant differential operator.*

We will call the differential invariants $I$ and $J$ the *fundamental differential invariants* for the group $G$. Note that the differential invariants constructed by this method are well defined on the open subset of $\mathrm{J}^{n+k}$ where $I$ and $J$ are defined and where $D_x I \neq 0$. At the points where $D_x I = 0$, one must use an alternative differential invariant to avoid singularities.

**Example 3.37.** For the rotation group $\mathrm{SO}(2)$, as discussed in Example 3.26, we can apply Theorem 3.36 when $n = 1$, since $r$ and $w$ provide two independent first order differential invariants. The second order differential invariant resulting from Theorem 3.36, however, is not exactly the curvature, but the more complicated second order differential invariant

$$\frac{dw}{dr} = \frac{D_x w}{D_x r} = \frac{\sqrt{x^2+u^2}}{(x+u u_x)^3}\Big[(x^2+u^2)u_{xx} - (1+u_x^2)(x u_x - u)\Big].$$

However, since we know that there is only one independent second order differential invariant, we must be able to re-express the curvature in terms of this new differential invariant; we find

$$\kappa = (1+u_x^2)^{-3/2} u_{xx} = (1+w^2)^{-3/2}\big[w_r + r^{-1}(w+w^3)\big].$$

If we replace $w = \tan\phi$ by the angle $\phi$ described in Example 3.26, then we find the interesting SO(2)-invariant formula $\kappa = \phi_r \cos\phi + r^{-1}\sin\phi$ expressing the curvature of a curve in terms of the radial variation of the angle $\phi$. Higher order differential invariants are given by successive derivatives $d^k w/dr^k$ (or, alternatively, $d^k\phi/dr^k$). These can be rewritten explicitly in terms of $x$ and $u$ using the invariant differential operator $D_r = r(x+u u_x)^{-1} D_x$, or, more simply, the alternative invariant differential operator $r^{-1} D_r = (x+u u_x)^{-1} D_x$.

*Invariant Differential Forms*

The construction of differential invariants described above admits an additional important simplification, based on the following observation. Note that the proof of Proposition 3.35 relies on a simple fact: if $I$ is any differential invariant, its total differential $\mathrm{D}I = D_x I\, dx$, cf. Definition 3.23, is a "contact-invariant" one-form, in the following sense.

**Definition 3.38.** A differential one-form $\omega$ on $\mathrm{J}^n$ is called *contact-invariant* under a transformation group $G$ if and only if, for every $g \in G$, we have $(g^{(n)})^*\omega = \omega + \theta$ for some contact form $\theta = \theta_g$.

Contact forms are trivially contact-invariant, so only the horizontal contact-invariant forms are of interest. In the scalar case, if $\omega = P\,dx$ is a horizontal contact-invariant one-form (e.g., $\omega = \mathrm{D}I$ is the total differential of a differential invariant $I$, in which case $P = D_x I$), then every other horizontal contact-invariant one-form is of the form $J\,\omega = JP\,dx$, where $J$ is an arbitrary differential invariant. Thus, if we know two horizontal contact-invariant one-forms $P\,dx$, $\widetilde{P}\,dx$, their ratio $J = \widetilde{P}/P$ defines a differential invariant. A contact-invariant one-form serves to define an invariant differential operator.

**Proposition 3.39.** *Let $G$ be a group of point transformations, and let $\omega = P(x,u^{(n)})\,dx$ be a contact-invariant horizontal one-form on $\mathrm{J}^n$. Then the associated differential operator $\mathcal{D} = (1/P)D_x$ is $G$-invariant, so that whenever $I$ is a differential invariant, so is $\mathcal{D}I$.*

The infinitesimal criterion for contact-invariance is that the Lie derivative $\mathbf{v}^{(n)}(\omega)$ of the form with respect to the prolonged infinitesimal generators be a contact form for every infinitesimal generator $\mathbf{v} \in \mathfrak{g}$. If $\mathbf{v} = \xi\partial_x + \varphi\partial_u$, then the Lie derivative of a horizontal one-form $P(x,u^{(n)})\,dx$ with respect to the prolonged vector field is readily computed using the intrinsic formulation of the total derivative:

$$\mathbf{v}^{(n)}(P\,dx) = \mathbf{v}^{(n)}(P)\,dx + P\,d\xi = \big[\mathbf{v}^{(n)}(P) + P D_x\xi\big]\,dx + \theta, \tag{3.57}$$

for some contact form $\theta$. Therefore, $P\,dx$ is contact-invariant under the group $G$ if and only if $\mathbf{v}^{(n)}(P)+PD_x\xi=0$ for each infinitesimal generator. In other words, the differential function $P$ is a relative differential invariant corresponding to the infinitesimal divergence multiplier $\mathbf{v}^{(n)}+D_x\xi$.

**Theorem 3.40.** *If $n$ is the* stabilization order *of a transformation group, then there exists a nontrivial horizontal contact-invariant one-form $\omega = P(x,u^{(n)})\,dx$ of order at most* $\max\{1,n\}$.

Note that, in the ordinary cases, for an $r$-dimensional group action, the simplest differential invariant $I$ has order $r-1$, and produces the $r^{\text{th}}$ order contact-invariant one-form D$I$, whereas the stabilization order is $n=r-2$, and Theorem 3.40 shows that there is a contact-invariant of order $r-2$. The formula for the simplest contact-invariant one-form for each of the transformation groups in Lie's classification of complex group actions is provided in Table 5. Note that in roughly half of the cases (specifically 1.2, 1.3, 1.7, 1.8, 1.9, 1.11, 2.2, and the three intransitive cases 3.1, 3.2, 3.3) the invariant one-form is of lower order than the order of stabilization of the group. It is not clear, though, how to recognize this phenomenon in advance. In all the cases except the pseudo-stabilization cases, the invariant one-form is of order strictly less than the order of the fundamental differential invariant, and a complete system of differential invariants is provided by successively applying the invariant differential operator to the fundamental differential invariant. The pseudo-stabilization Case 1.7a is unusual, in that it is the only one whose fundamental contact-invariant one-form is the total differential of the lowest order differential invariant, and hence requires a second fundamental differential invariant to generate all the higher order ones.

**Example 3.41.** The Euclidean group $\mathrm{SE}(2)=\mathrm{SO}(2)\ltimes\mathbb{R}^2$ acts via rotations and translations on $E\simeq\mathbb{R}^2$. Every $(x,u)$-independent rotational differential invariant, as given in Example 3.37, will provide a Euclidean differential invariant. In particular, the fundamental Euclidean invariant is the curvature $\kappa=(1+u_x^2)^{-3/2}u_{xx}$. The simplest contact-invariant one-form is $\omega=\sqrt{1+u_x^2}\,dx$, which is the Euclidean arc length element, often denoted $ds$. Proposition 3.39 implies the classical result that every Euclidean differential invariant is a function of the curvature and its derivatives $d^k\kappa/ds^k$ with respect to arc length.

The geometrical interpretation of the fundamental invariant one-form and differential invariant in the Euclidean case extends to other transformation groups, such as the special affine group SA(2), which is Case 2.1, and the projective group SL(3), which is Case 2.3, of importance in both differential geometry, [**20**], and, more recently, computer vision, [**45**]. In both cases, the simplest invariant one-form is identified with the group-invariant arc length element, while the fundamental differential invariant is identified with the group-invariant curvature. One is tempted to make a similar definition in all other cases (with the possible exceptions of the intransitive and pseudo-stabilization cases), so that a complete system of $G$-invariant differential invariants is provided by the $G$-invariant curvature and its derivatives with respect to the $G$-invariant arc length.

**Example 3.42.** Consider the Euclidean group $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ acting on the plane. The stabilization order is $n = 1$, and SE(2) acts transitively on $\mathrm{J}^1$. Its first prolongation has infinitesimal generators $\partial_x,\ \partial_u,\ -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x}$. Fixing a point in $\mathrm{J}^1$, we can locally identify $\mathrm{J}^1$ with a neighborhood of the identity in SE(2), in such a way that the group acts on itself by left multiplication. As such, the Maurer-Cartan forms provide a $G$-invariant coframe, explicitly constructed (in two ways) in Example 2.81. Translating that result into the present notation, we find that a Euclidean invariant coframe is

$$\omega^1 = \frac{du - u_x\,dx}{\sqrt{1+u_x^2}}, \qquad \omega^2 = \frac{du_x}{1+u_x^2},$$
$$\omega^3 = \sqrt{1+u_x^2}\,dx + \frac{u_x}{\sqrt{1+u_x^2}}\,(du - u_x\,dx).$$

The first element is a contact form; the third is equivalent, modulo a contact form, to the contact-invariant Euclidean arc length form. The horizontal component of the second one-form is the second order contact-invariant one-form $\omega_H^2 = (1+u_x^2)^{-1}u_{xx}\,dx$. Dividing by the arc-length form produces the fundamental curvature invariant $\kappa = (1+u_x^2)^{-3/2}u_{xx}$ for the Euclidean group.

**Exercise 3.43.** Construct an invariant coframe and fundamental differential invariant for the similarity group, consisting of translations, rotations, and scalings $(x,u) \mapsto \lambda(x,u)$. A considerably more substantial exercise is to do this for the special affine, affine, and projective groups — Cases 2.1, 2.2, and 2.3 in the Tables; see [**20**].