Lectures on Moving Frames

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Abstract. The goal of these lectures is to survey the equivariant method of moving frames developed by the author and a large number of collaborators and other researchers over the past fifteen years. A variety of applications in geometry, differential equations, computer vision, classical invariant theory, the calculus of variations, and numerical analysis are discussed.

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1. Introduction.

According to Akivis, [**1**], the idea of moving frames can be traced back to the method of moving trihedrons introduced by the Estonian mathematician Martin Bartels (1769–1836), a teacher of both Gauß and Lobachevsky. The modern method of moving frames or repères mobiles[†] was primarily developed by Élie Cartan, [**32**, **33**], who forged earlier contributions by Cotton, Darboux, Frenet and Serret into a powerful tool for analyzing the geometric properties of submanifolds and their invariants under the action of transformation groups.

In the 1970's, several researchers, cf. [**40**, **58**, **59**, **77**], began the attempt to place Cartan's intuitive constructions on a firm theoretical foundation. I've been fascinated by the power of the method since my student days, but, for many years, could not see how to release it from its rather narrow geometrical confines, e.g. Euclidean or equiaffine actions on submanifolds of Euclidean space. The crucial conceptual leap is to decouple the moving frame theory from reliance on any form of frame bundle or connection, and define a moving frame as an equivariant map from the manifold or jet bundle back to the transformation group. In other words,

$$\text{Moving frames} \ \neq \ \text{Frames}\,!$$

A careful study of Cartan's analysis of the case of projective curves, [**32**], reveals that Cartan was well aware of this viewpoint; however, this important and instructive example did not receive the attention it deserved. Once freed from the confining fetters of frames, Mark Fels and I, [**51**, **52**], were able to formulate a new, powerful, constructive approach to the equivariant moving frame theory that can be systematically applied to general transformation groups. All classical moving frames can be reinterpreted in this manner, but the equivariant approach applies in far broader generality.

Cartan's construction of the moving frame through the normalization process is interpreted with the choice of a cross-section to the group orbits. Building on these two simple ideas, one may algorithmically construct equivariant moving frames and, as a result, complete systems of invariants for completely general group actions. The existence of a moving frame requires freeness of the underlying group action. Classically, non-free actions are made free by prolonging to jet space, leading to differential invariants and the solution to equivalence and symmetry problems via the differential invariant signature. More recently, the moving frame method was also applied to Cartesian product actions, leading to classification of joint invariants and joint differential invariants, [**120**]. Afterwards, a seamless amalgamation of jet and Cartesian product actions dubbed *multi-space* was proposed in [**121**] to serve as the basis for the geometric analysis of numerical approximations, and, via the application of the moving frame method, to the systematic construction of invariant numerical algorithms, [**84**].

With the basic moving frame machinery in hand, a plethora of new, unexpected, and significant applications soon appeared. In [**118**, **8**, **85**, **86**], the theory was applied to produce new algorithms for solving the basic symmetry and equivalence problems of polynomials that form the foundation of classical invariant theory. The moving frame

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[†] In French, the term "repère mobile" refers to a temporary mark made during building or interior design, and so a more accurate English translation might be "movable landmarks".

method provides a direct route to the classification of joint invariants and joint differential invariants, [**52, 120, 16**], establishing a geometric counterpart of what Weyl, [**154**], in the algebraic framework, calls the first main theorem for the transformation group. In [**29, 15, 4, 7, 139, 114**], the characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection, [**21, 22, 24, 50, 132**]. Applications to the classification of joint invariants and joint differential invariants appear in [**52, 120, 16**]. In computer vision, joint differential invariants have been proposed as noise-resistant alternatives to the standard differential invariant signatures, [**23, 31, 44, 111, 151, 152**]. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes, [**28, 29, 15, 121, 84**]. The all-important recurrence formulae lead to a complete characterization of the differential invariant algebra of group actions, and lead to new results on minimal generating invariants, even in very classical geometries, [**124, 69, 125, 73, 70**]. The general problem from the calculus of variations of directly constructing the invariant Euler-Lagrange equations from their invariant Lagrangians was solved in [**87**]. Applications to the evolution of differential invariants under invariant submanifold flows, leading to integrable soliton equations and signature evolution in computer vision, can be found in [**126, 79**].

Applications of equivariant moving frames that are being developed by other research groups include the computation of symmetry groups and classification of partial differential equations [**96, 112**]; geometry of curves and surfaces in homogeneous spaces, with applications to integrable systems, [**98, 99, 100, 101, 125, 73**]; symmetry and equivalence of polygons and point configurations, [**17, 78**], recognition of DNA supercoils, [**138**], recovering structure of three-dimensional objects from motion, [**7**], classification of projective curves in visual recognition, [**63**]; construction of integral invariant signatures for object recognition in 2D and 3D images, [**53**]; determination of invariants and covariants of Killing tensors, with applications to general relativity, separation of variables, and Hamiltonian systems, [**43, 105, 106**]; further developments in classical invariant theory, [**8, 85, 86**]; computation of Casimir invariants of Lie algebras and the classification of subalgebras, with applications in quantum mechanics, [**18, 19**]. A rigorous, algebraically-based reformulation of the method, suitable for symbolic computations, has been proposed by Hubert and Kogan, [**71, 72**]. Mansfield's recent text, [**97**]. gives a good introduction to the basic ideas and some of the important applications.

Finally, in recent work with Pohjanpelto, [**128, 129, 130**], the theory and algorithms have recently been extended to the vastly more complicated case of infinite-dimensional Lie pseudo-groups. Applications to infinite-dimensional symmetry groups of partial differential equations can be found in [**38, 39, 113, 148**], to the classification of Laplace invariants and factorization of linear partial differential operators in [**140**], to climate and turbulence modeling in [**9**], and to general Cartan equivalence problems in [**149**].

2. Lie Groups and Lie Algebras.

We will be interested in the action of both finite-dimensional Lie groups and, later, infinite-dimensional Lie pseudo-groups on an m-dimensional manifold M. All manifolds, functions, etc., will be assumed to be at least smooth, meaning C^∞, or even analytic when

necessary. Since our considerations are primarily local, the reader will not lose much by assuming that M is an open subset of the Euclidean space $\mathbb{R}^m$. One can equally well work in the complex category if desired. We will assume the reader is familiar with the basic notions of tangent space, vector field, flow, Lie bracket, cotangent space, differential form, wedge product, pull-back, and the exterior derivative d. See [**116**; Chapter 1] for a painless introduction to the main concepts.

A *Lie group* is, by definition, a group G that also has the structure of a smooth manifold that makes the group multiplication and inversion smooth maps. We let r denote the dimension of G. Familiar examples include the general linear group $\mathrm{GL}(m)$ of $m \times m$ invertible matrices, the special Euclidean group $\mathrm{SE}(m)$ of rigid motions (translations and rotations) of $\mathbb{R}^m$ and the group $\mathrm{A}(m)$ consisting of affine transformations $z \mapsto A\,z + b$ of $\mathbb{R}^m$. In fact, any subgroup of $\mathrm{GL}(m)$ which is topologically closed forms a Lie group; most (but not all) Lie groups arise as such matrix Lie groups.

The Lie algebra $\mathfrak{g}$ is the space of right-invariant vector fields[†] on G. Since each such vector field is uniquely determined by its value at the identity $e \in G$, we can identify $\mathfrak{g} \simeq TG|_e$ with the r-dimensional tangent space to the group at the identity, and hence $\mathfrak{g}$ is an r-dimensional vector space. We fix a basis $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_r$ of $\mathfrak{g}$, which we refer to as the *infinitesimal generators* of the Lie group. The nonzero Lie algebra element $\mathbf{0} \neq \widehat{\mathbf{v}} \in \mathfrak{g}$ are in one-to-one correspondence with the connected one-parameter (or one-dimensional) subgroups of G, identified as its flow $\exp(t\,\widehat{\mathbf{v}})e$ through the identity.

The Lie algebra is also equipped with a Lie bracket operation $[\widehat{\mathbf{v}}, \widehat{\mathbf{w}}]$, since the Lie bracket between vector fields preserves right-invariance. The Lie bracket is bilinear, skew symmetric, and satisfies the Jacobi identity:

$$[\mathbf{v}, \mathbf{w}] = -[\widehat{\mathbf{v}}, \widehat{\mathbf{w}}], \qquad [\widehat{\mathbf{u}}, [\widehat{\mathbf{v}}, \widehat{\mathbf{w}}]] + [\widehat{\mathbf{v}}, [\widehat{\mathbf{w}}, \widehat{\mathbf{u}}]] + [\widehat{\mathbf{w}}, [\widehat{\mathbf{u}}, \widehat{\mathbf{v}}]] = 0, \qquad (2.1)$$

for any $\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \widehat{\mathbf{w}} \in \mathfrak{g}$.

In particular,

$$[\widehat{\mathbf{v}}_i, \widehat{\mathbf{v}}_j] = \sum_{k=1}^r C_{ij}^k \, \widehat{\mathbf{v}}_k, \qquad (2.2)$$

where the coefficients C_{ij}^k are known as the *structure constants* of the Lie algebra. Note that the structure constants depend on the selection of a basis; their behavior under a change of basis is easily found. Interestingly, a recent application of moving frames, [**18**], has been to calculate the *structure invariants*, meaning combinations of structure constants that do not depend on the basis, a question of importance in the classification of Lie algebras and quantum mechanics.

The right-invariant one-forms on a Lie group are known as the *Maurer–Cartan forms*. By the same reasoning, they form an r-dimensional vector space dual to the Lie algebra, and denoted $\mathfrak{g}^*$. The dual basis to the space of Maurer–Cartan forms is denoted by $\mu^1, \ldots, \mu^r$,

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[†] One can also use left-invariant vector fields here — the only differences are some changes in signs. The only reason to prefer right-invariant is that they generalize more readily to the infinite-dimensional case.

where under the natural pairing between vector fields and one-forms $\langle \widehat{\mathbf{v}}_i ; \mu^j \rangle = \delta_i^j$ is the Kronecker delta, equal to 1 for $i = j$ and 0 otherwise. The *structure equations* for the Lie group are dual to the commutation relations (2.2), and take the form

$$d\mu^k = - \sum_{i<j} C_{ij}^k \, \mu^i \wedge \mu^j. \tag{2.3}$$

The r-dimensional Lie group G acts on the m-dimensional manifold M, meaning that $(g, z) \mapsto g \cdot z$ is a smooth map from $G \times M$ to M. We will also allow the possibility that the action is only *local*, meaning that $g \cdot z$ may only be defined fro group elements sufficiently near the identity. An example of a local action is the projective action of $G = \mathrm{GL}(2)$ on $M = \mathbb{R}$:

$$g \cdot z = \frac{\alpha \, z + \beta}{\gamma \, z + \delta}, \qquad z \in \mathbb{R}, \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2), \tag{2.4}$$

which is not defined when the denominator vanishes.

Definition 2.1. The *orbits* of a group action are the minimal invariant subsets. In other words (assuming the action is global) the orbit through $z \in M$ is $\mathcal{O}_z = \{ g \cdot z \,|\, g \in G \}$.

Definition 2.2. Given $z \in M$, the isotropy subgroup $G_z = \{ g \,|\, g \cdot z = z \}$ consists of all elements that fix it. More generally, the isotropy or symmetry subgroup of a subset $S \subset M$ is $G_S = \{ g \,|\, g \cdot S = S \}$.

Definition 2.3. A group action is *free* $G_z = \{e\}$ for all $z \in M$; i.e., the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity. The action is *locally free* if G_z is a discrete subgroup of G.

Lemma 2.4. *A group acts locally freely if and only if its orbits all have the same dimension as G.*

The action is *semi-regular* if all the orbits have the same dimension. A semi-regular action is *regular* if the orbits intersect sufficiently small coordinate charts only once, i.e., they form a regular foliation. The most familiar example of an irregular action is the irrational flow on the torus, in which every orbit is dense. The action is *effective* if the only group element which fixes *every* point in M is the identity; in other words, $g \cdot z = z$ for all $z \in M$ if and only if $g = e$. If the group G does not act effectively, one can, without any loss of generality, replace G by the effectively acting quotient group G/G_M^*, where $G_M^* = \{ g \in G \,|\, g \cdot z = z \text{ for all } z \in M \}$ is the *global isotropy subgroup*.

For each Lie algebra element $\widehat{\mathbf{v}} \in \mathfrak{g}$, let $\mathbf{v}$ denote the vector field on M whose flow $\exp(t\,\mathbf{v})$ coincides with the action of the one-parameter subgroup generated by $\widehat{\mathbf{v}}$. We call the resulting vector fields the *infinitesimal generators* of the action of G on M; these form a Lie algebra of vector fields on M isomorphic to $\mathfrak{g}$. If G is a connected Lie group, its action can be completely reconstructed by exponentiating its infinitesimal generators. In particular, the vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_r$ corresponding to our chosen basis $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_r$ of $\mathfrak{g}$ satisfy the same Lie algebra commutation relations (2.2).

At each point $z \in M$, the space $\mathfrak{g}|_z = \{ \mathbf{v}|_z \,|\, \widehat{\mathbf{v}} \in \mathfrak{g} \}$ spanned by the infinitesimal generators can be identified with the tangent space to the orbit through z. Thus, G acts

locally freely at z if and only if $\dim \mathfrak{g}|_z = r = \dim G$, and so local freeness can be checked infinitesimally. On the other hand, freeness is a global condition, that requires knowing the complete group action.

Definition 2.5. An *invariant* of the action of G in M is a real-valued function $I: M \to \mathbb{R}$ such that $I(g \cdot z) = I(z)$ for all $g \in G$ and all $z \in M$.

Observe that I is an invariant if and only if it is constant on the orbits of G. We allow the possibility that I is only defined on an open subset of M, in which case the invariance condition is only imposed when both z and $g \cdot z$ lie in the domain of I. A *local invariant* is defined so that the invariance condition only need hold for g sufficiently close to the identity.

Clearly, if $I_1, \ldots, I_k$ are invariants, so is any function thereof $I = H(I_1, \ldots, I_k)$. We therefore only need to classify invariants up to functional dependence.

Theorem 2.6. *If G acts regularly on the m-dimensional manifold M with s-dimensional orbits, then, locally, there exist precisely $m - s$ functionally independent invariants $I_1, \ldots, I_{m-s}$ with the property that any other invariant can be written as a function thereof.*

If the action is semi-regular, then the same result holds for local invariants.

The infinitesimal criterion for invariance is established by differentiating the invariance formula

$$I(\exp(t\,\mathbf{v})z) = I(z) \qquad \text{for} \qquad \mathbf{v} \in \mathfrak{g}$$

with respect to t and setting $t = 0$.

Theorem 2.7. *Let G be connected. A function $I: M \to \mathbb{R}$ is an invariant if and only if*

$$\mathbf{v}_i(I) = 0 \qquad \text{for all} \qquad i = 1, \ldots, r. \tag{2.5}$$

Similarly, invariance of a submanifold $N \subset M$ given implicitly by the vanishing of functions has an associated infinitesimal invariance criterion.

Theorem 2.8. *Let G be connected. Let $N \subset M$ be a submanifold defined implicitly by the vanishing of one or more functions $F_\nu(z) = 0$ where $\nu = 1, \ldots, k$. Assume that the Jacobian matrix $\left(\partial F_\nu / \partial z^i \right)$ has rank k for all $z \in N$. Then N is an invariant submanifold — and so G is a symmetry group of N — if and only if*

$$\mathbf{v}_i(F_\nu) = 0 \qquad \text{whenever} \qquad F(z) = 0 \tag{2.6}$$

for all $i = 1, \ldots, r$ and $\nu = 1, \ldots, k$.

3. Jets.

In this section, we introduce the so-called "jet spaces" or "jet bundles", well known to nineteenth century practitioners, but first formally defined by Ehresmann, [**48**], in his seminal paper on the subject of infinite-dimensional Lie pseudo-groups.

Our basic arena is an m-dimensional manifold M. We let $\mathrm{J}^n = \mathrm{J}^n(M, p)$ denote the nth order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds

$S \subset M$ under the equivalence relation of nth order contact. In particular, $J^0 = M$. We let $j_n S \subset J^n$ denote the n–jet of the submanifold S, which forms a p-dimensional submanifold of the jet space.

When we introduce local coordinates $z = (x, u)$ on M, we consider the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. In these coordinates, a (transverse) p-dimensional submanifold is realized as the graph of a function $u = f(x)$. Two such submanifolds have n^{th} order contact at a point $(x_0, u_0) = (x_0, f(x_0))$ if and only if they have the same n^{th} order Taylor polynomials at x_0. Thus, the induced coordinates on the jet bundle J^n are denoted by $z^{(n)} = (x, u^{(n)})$, consisting of independent variables x^i, dependent variables u^α, and their derivatives u_J^α, $\alpha = 1, \ldots, q$, of order $\#J \leq n$. Here $J = (j_1, \ldots, j_k)$, with $1 \leq j_\nu \leq p$, is a symmetric multi-index of order $k = \#J$. We will also write $j_n f(x)$ for the n–jet or Taylor polynomial of f at the point x. There is an evident projection $\pi_n^k \colon J^k \to J^n$ whenever $k > n$, given by $\pi_n^k(x, u^{(k)}) = (x, u^{(n)})$ — In other words, omit all derivative coordinates of order $> n$.

A real-valued function $F \colon J^n \to \mathbb{R}$, defined on an open subset of the jet space, is known as a *differential function*, written $F(x, u^{(n)})$. We will can evaluate F on any higher order jet by composition with the project, so $F \circ \pi_n^k \colon J^k \to \mathbb{R}$. The *order* of a differential function is the highest order derivative coordinate it explicitly depends on, i.e.,

$$\operatorname{ord} F = \max \left\{ \#J \ \middle| \ \frac{\partial F}{\partial u_J^\alpha} \not\equiv 0 \quad \text{for some} \quad \alpha \right\}.$$

A general system of nth order (partial) differential equations in p independent variables $x = (x^1, \ldots, x^p)$, and q dependent variables $u = (u^1, \ldots, u^q)$ is defined by the vanishing of one or more differential functions of order $\leq n$:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l. \tag{3.1}$$

The jets $(x, u^{(n)})$ that satisfy the equations (3.1) define a subvariety $\mathcal{S}_\Delta \subset J^n$. In applications, we assume that the Jacobian matrix of the system with respect to to all the jet variables has *maximal rank l*, and hence, by the implicit function theorem, $\mathcal{S}_\Delta$ is, in fact, a submanifold. A (classical) *solution* to the system is a smooth function $u = f(x)$, or, equivalently, a submanifold, whose n–jet belongs to the subvariety: $j_n S \subset \mathcal{S}_\Delta$. This is merely a restatement, in jet language, of the usual criterion for a classical solution to a system of differential equations.

Given an r-dimensional Lie group G act smoothly on the manifold M, we let $G^{(n)}$ denote the nth prolongation of G to the jet bundle $J^n = J^n(M, p)$ induced by the action of G on p-dimensional submanifolds. In practical examples, for n sufficiently large, the prolonged action $G^{(n)}$ becomes regular and free on a dense open subset $\mathcal{V}^n \subset J^n$, the set of *regular jets*. It has been rigorously proved that, if G acts (locally) effectively on each open subset of M, then, for $n \gg 0$ sufficiently large, its nth prolongation $G^{(n)}$ acts locally freely on an open subset $\mathcal{V}^n \subset J^n$, [**119**].

4. Symmetries of Differential Equations.

We will begin by reviewing a few relevant points from Lie's theory of symmetry groups

of differential equations as presented, for instance, in the textbook [**116**]. In general, by a *symmetry* of the system (3.1) we mean a transformation which takes solutions to solutions. The most basic type of symmetry is a (locally defined) invertible map on the space of independent and dependent variables:

$$(\bar{x}, \bar{u}) = g \cdot (x, u) = (\Xi(x, u), \Phi(x, u)).$$

Such transformations act on solutions $u = f(x)$ by pointwise transforming their graphs; in other words if $\Gamma_f = \{(x, f(x))\}$ denotes the graph of f, then the transformed function $\bar{f} = g \cdot f$ will have graph

$$\Gamma_{\bar{f}} = \{(\bar{x}, \bar{f}(\bar{x}))\} = g \cdot \Gamma_f \equiv \{g \cdot (x, f(x))\}. \tag{4.1}$$

Definition 4.1. A local Lie group of transformations G is called a *symmetry group* of the system of partial differential equations (3.1) if $\bar{f} = g \cdot f$ is a solution whenever f is.

We will always assume that the transformation group G is connected, thereby excluding discrete symmetry groups, which, while also of great interest for differential equations, are unfortunately not amenable to infinitesimal, constructive techniques. Connectivity implies that it suffices to work with the associated infinitesimal generators, which form a Lie algebra of vector fields

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \tag{4.2}$$

on the space of independent and dependent variables. The group transformations in G are recovered from the infinitesimal generators by the usual process of exponentiation. Thus, the one-parameter group $G = \{g_\varepsilon | \varepsilon \in \mathbb{R}\}$ generated by the vector field (4.2) is the solution $g_\varepsilon \cdot (x_0, u_0) = (x(\varepsilon), u(\varepsilon))$ to the first order system of ordinary differential equations

$$\frac{dx^i}{d\varepsilon} = \xi^i(x, u), \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x, u), \tag{4.3}$$

with initial conditions (x_0, u_0) at $\varepsilon = 0$.

For example, the vector field

$$\mathbf{v} = -u \, \partial_x + x \, \partial_u$$

generates the rotation group

$$x(\varepsilon) = x \cos \varepsilon - u \sin \varepsilon, \qquad u(\varepsilon) = x \sin \varepsilon + u \cos \varepsilon,$$

which transforms a function $u = f(x)$ by rotating its graph.

Since the transformations in G act on functions $u = f(x)$, they also act on their derivatives, and so induce "prolonged transformations" $(\bar{x}, \bar{u}^{(n)}) = \mathrm{pr}^{(n)} g \cdot (x, u^{(n)})$. The explicit formula for the prolonged group transformations is rather complicated, and so it is easier to work with the prolonged infinitesimal generators, which are vector fields

$$\mathrm{pr}^{(n)} \mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \varphi_J^\alpha(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}, \tag{4.4}$$

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on the space of independent and dependent variables and their derivatives up to order n, which are denoted by $u_J^\alpha = \partial^J u^\alpha / \partial x^J$, where $J = (j_1, \ldots, j_n)$, $1 \le j_\nu \le p$. The coefficients φ_J^α of $\mathrm{pr}^{(n)}\, \mathbf{v}$ are given by the explicit formula

$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha, \tag{4.5}$$

in terms of the coefficients ξ^i, φ^α of the original vector field (4.2). Here D_i denotes the total derivative with respect to x^i (treating the u's as functions of the x's), and $D_J = D_{j_1} \cdot \ldots \cdot D_{j_n}$ the corresponding higher order total derivative. Furthermore, the q-tuple $Q = (Q^1, \ldots, Q^q)$ of functions of x's, u's and first order derivatives of the u's defined by

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha(x, u) - \sum_{i=1}^{p} \xi^i(x, u)\, \frac{\partial u^\alpha}{\partial x^i}, \qquad \alpha = 1, \ldots, q, \tag{4.6}$$

is known as the *characteristic* of the vector field (4.2), and plays a significant role in our subsequent discussion. The main point the reader should glean from this paragraph is not the particular complicated expressions in (4.4, 5, 6) (although, of course, these *are* required when performing any particular calculation), but rather that there *are* known, explicit formulas which can, in a relatively straightforward manner, be computed. See [**116**] for details.

Theorem 4.2. *A connected group of transformations G is a symmetry group of the (nondegenerate) system of differential equations (3.1) if and only if the classical infinitesimal symmetry criterion*

$$\mathrm{pr}^{(n)}\, \mathbf{v}(\Delta_\nu) = 0, \qquad \nu = 1, \ldots, r, \qquad \textit{whenever} \quad \Delta = 0. \tag{4.7}$$

holds for every infinitesimal generator $\mathbf{v}$ of G.

The equations (4.7) are known as the *determining equations* of the symmetry group for the system. They form a large over-determined linear system of partial differential equations for the coefficients ξ^i, φ^α of $\mathbf{v}$, and can, in practice, be explicitly solved to determine the complete (connected) symmetry group of the system (3.1). There are now a wide variety of computer algebra packages available which will automate most of the routine steps in the calculation of the symmetry group of a given system of partial differential equations. See [**65**] for a survey of the different packages available, and a discussion of their strengths and weaknesses.

Example 4.3. The classic example illustrating the basic techniques is the linear heat equation

$$u_t = u_{xx}. \tag{4.8}$$

An infinitesimal symmetry of the heat equation will be a vector field $\mathbf{v} = \xi\, \partial_x + \tau\, \partial_t + \varphi\, \partial_u$, where ξ, τ, φ are functions of x, t, u. To determine which coefficient functions ξ, τ, φ yield genuine symmetries, we need to solve the symmetry criterion (4.7), which, in this case, is

$$\varphi^t = \varphi^{xx} \qquad \textit{whenever} \qquad u_t = u_{xx}. \tag{4.9}$$

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Here, utilizing the characteristic $Q = \varphi - \xi u_x - \tau u_t$ given by (4.6),

$$\varphi^t = D_t Q + \xi u_{xt} + \tau u_{tt}, \qquad \varphi^{xx} = D_x^2 Q + \xi u_{xxx} + \tau u_{xxt}, \qquad (4.10)$$

are the coefficients of the terms $\partial_{u_t}, \partial_{u_{xx}}$ in the second prolongation of $\mathbf{v}$, cf. (4.5). Substituting the formulas (4.10) into (4.9), and replacing u_t by u_{xx} wherever it occurs, we are left with a polynomial equation involving the various derivatives of u whose coefficients are certain derivatives of ξ, τ, φ. Since ξ, τ, φ only depend on x, t, u we can equate the individual coefficients to zero, leading to the complete set of *determining equations*:

Coefficient	*Monomial*
$0 = -2\tau_u$	$u_x u_{xt}$
$0 = -2\tau_x$	u_{xt}
$0 = -\tau_{uu}$	$u_x^2 u_{xx}$
$-\xi_u = -2\tau_{xu} - 3\xi_u$	$u_x u_{xx}$
$\varphi_u - \tau_t = -\tau_{xx} + \varphi_u - 2\xi_x$	u_{xx}
$0 = -\xi_{uu}$	u_x^3
$0 = \varphi_{uu} - 2\xi_{xu}$	u_x^2
$-\xi_t = 2\varphi_{xu} - \xi_{xx}$	u_x
$\varphi_t = \varphi_{xx}$	1

The general solution to these elementary differential equations is readily found:

$$\xi = c_1 + c_4 x + 2c_5 t + 4c_6 xt,$$
$$\tau = c_2 + 2c_4 t + 4c_6 t^2,$$
$$\varphi = (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x,t),$$

where c_i are arbitrary constants and $\alpha_t = \alpha_{xx}$ is an arbitrary solution to the heat equation. Therefore, the symmetry algebra of the heat equation is spanned by the vector fields

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = u\partial_u, \qquad \mathbf{v}_4 = x\partial_x + 2t\partial_t,$$
$$\mathbf{v}_5 = 2t\partial_x - xu\partial_u, \qquad \mathbf{v}_6 = 4xt\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u,$$
$$\mathbf{v}_\alpha = \alpha(x,t)\partial_u, \qquad \text{where} \qquad \alpha_t = \alpha_{xx}.$$

The corresponding one-parameter groups are, respectively, x and t translations, scaling in u, the combined scaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$, Galilean boosts, an "inversional symmetry", and the addition of solutions stemming from the linearity of the equation. See [**116**] for more details.

Example 4.4. The celebrated Korteweg–deVries (KdV) equation, [**47, 116**], is

$$u_t + u_{xxx} + uu_x = 0. \qquad (4.11)$$

A vector field $\mathbf{v}$ is an infinitesimal symmetry of the KdV equation if and only if

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + uu_x) = \widehat{\varphi}^t + \widehat{\varphi}^{xxx} + u\,\widehat{\varphi}^x + u_x\,\widehat{\varphi} = 0 \qquad \text{whenever} \qquad u_t + u_{xxx} + uu_x = 0.$$

Substituting the prolongation formulas, and equating the coefficients of the independent derivative monomials to zero, leads to the infinitesimal determining equations which together with their differential consequences reduce to the system

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0, \qquad \varphi = \xi_t - \tfrac{2}{3}\,u\tau_t, \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -2\,\xi_x, \quad (4.12)$$

while all the derivatives of the components of order two or higher vanish. The general solution

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2\,c_4 u,$$

defines the four-dimensional KdV symmetry algebra with the basis given by

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x, \qquad \mathbf{v}_3 = t\,\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3t\,\partial_t + x\,\partial_x - 2u\,\partial_u. \qquad (4.13)$$

In this example, the classical symmetry group is disappointingly trivial, consisting of easily guessed translations and scaling symmetries. The action of the KdV symmetry group on M, which can be obtained by composing the flows of the symmetry algebra basis and is given by

$$\begin{aligned}
(T, X, U) &= \exp(\lambda_4 \mathbf{v}_4) \circ \exp(\lambda_3 \mathbf{v}_3) \circ \exp(\lambda_2 \mathbf{v}_2) \circ \exp(\lambda_1 \mathbf{v}_1)(t, x, u) \\
&= \left(e^{3\lambda_4}(t + \lambda_1),\ e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2),\ e^{-2\lambda_4}(u + \lambda_3) \right),
\end{aligned} \qquad (4.14)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the group parameters.

Theorem 4.2 guarantees that these are the only continuous classical symmetries of the equation. (There are, however, higher order generalized symmetries, cf. [116], which account for the infinity of conservation laws of this equation.) Sometimes the complicated calculation of the symmetry group of a system of differential equations yields only rather trivial symmetries; however, there are numerous examples where this is not the case and new and physically and/or mathematically important symmetries have arisen from a complete group analysis.

A wide range of applications of symmetry groups, including the construction of explicit solutions, integration of ordinary differential equations, determination of conservation laws, linearization of nonlinear partial differential equations, and so on, can be found in [12, 30, 74, 116, 117].

5. Equivariant Moving Frames.

We begin by outlining the basic moving frame construction in [52]. Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

Definition 5.1. A *moving frame* is a smooth, G-equivariant map $\rho \colon M \to G$.

There are two principal types of equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases} \qquad (5.1)$$

If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. All classical moving frames are left equivariant, but, in many cases, the right

versions are easier to compute. In many geometrical situations, one can identify our left moving frames with the usual frame-based versions, but these identifications break down for more general transformation groups.

Theorem 5.2. *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Proof: To see the necessity of freeness, suppose $z \in M$, and let $g \in G_z$ belong to its isotropy subgroup. Let $\rho : M \to G$ be a left moving frame. Then, by left equivariance of ρ,

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

To prove regularity, suppose that $z \in M$ and that there exist points $z_\kappa = g_\kappa \cdot z$ belonging to the orbit of z such that $z_\kappa \to z$ as $\kappa \to \infty$. Thus, by continuity,

$$\rho(z_\kappa) = \rho(g_\kappa \cdot z) = g_\kappa \cdot \rho(z) \longrightarrow \rho(z) \qquad \text{as} \qquad \kappa \longrightarrow \infty,$$

which implies that $g_\kappa \to e$ in G. This suffices to ensure regularity of the orbit through z.

The sufficiency of these conditions will follow from the direct construction of the moving frame, which we describe next.

The practical construction of a moving frame is based on Cartan's method of *normalization*, [**81, 32, 52**], which requires the choice of a (local) *cross-section* to the group orbits.

Definition 5.3. Suppose G acts semi-regularly on the m-dimensional manifold M with s-dimensional orbits. A (local) *cross-section* is an $(m - s)$-dimensional submanifold $\mathcal{K} \subset M$ such that $\mathcal{K}$ intersects each orbit transversally, meaning that

$$T\mathcal{K}|_k \cap T\mathcal{O}|_k = T\mathcal{K}|_k \cap \mathfrak{g}|_k = \{0\} \qquad \text{for all} \qquad k \in \mathcal{K}. \tag{5.2}$$

The cross-section is *regular* if $\mathcal{K}$ intersects each orbit at most once.

The transversality condition (5.2) can thus be checked infinitesimally. Indeed, the (non-empty) subset $\mathcal{K}$ defined by the s equations

$$F_1(z) = c_1, \qquad \ldots \qquad F_s(z) = c_s, \tag{5.3}$$

forms a cross-section if and only if the $s \times s$ matrix

$$\mathbf{v}(F) = \left(\mathbf{v}_\kappa(F_i) \right), \tag{5.4}$$

obtained by applying the basis infinitesimal generators to the functions, is invertible on each point of $\mathcal{K}$, i.e., each solution to (5.3). In particular, a *coordinate cross-section* is defined by setting s of the coordinates to constants,

$$z_{i_1}(z) = c_1, \qquad \ldots \qquad z_{i_s}(z) = c_s, \tag{5.5}$$

subject to the requirement that

$$\det\left(\mathbf{v}_\kappa(z_{i_\nu}) \right) = \det\left(\zeta_\kappa^{i_\nu}(z) \right) \neq 0,$$

at all points satisfying (5.5). At any point, one can always choose a local coordinate cross-section if desired. So let us, for simplicity, concentrate on these from now on, and (by possibly relabeling the coordinates) assume that the first s coordinates are set equal to constants.

Theorem 5.4. *Let G act freely and regularly on M, and let $\mathcal{K} \subset M$ be a regular cross-section. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in \mathcal{K}$. Then $\rho \colon M \to G$ is a right moving frame for the group action.*

Proof: Given a point $\widehat{z} = h \cdot z$, if $g \cdot z = k \in \mathcal{K}$, then $\widehat{g} = g \cdot h^{-1}$ satisfies $\widehat{g} \cdot \widehat{z} = g \cdot h^{-1} \cdot h \cdot z = k \in \mathcal{K}$ also, and hence

$$\rho(h \cdot z) = \rho(\widehat{z}) = \widehat{g} = g \cdot h^{-1} = \rho(z) \cdot h^{-1},$$

proving right equivariance. $\hspace{8cm}$ *Q.E.D.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right[†] moving frame $g = \rho(z)$ associated with the *coordinate cross-section*

$$\mathcal{K} = \{\, z_1 = c_1, \ldots, z_r = c_r \,\}$$

is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{5.6}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Substituting the moving frame formulae into the remaining transformation rules leads to a complete system of invariants for the group action.

Theorem 5.5. *If $g = \rho(z)$ is the moving frame solution to the normalization equations (5.6), then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \ldots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{5.7}$$

form a complete system of functionally independent invariants.

Definition 5.6. The *invariantization* of a scalar function $F \colon M \to \mathbb{R}$ with respect to a right moving frame ρ is the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

Invariantization amounts to restricting F to the cross-section, $I \,|\, \mathcal{K} = F \,|\, \mathcal{K}$, and then requiring that I be constant along the orbits. In particular, if $I(z)$ is an invariant, then $\iota(I) = I$, so invariantization defines a projection, depending on the moving frame, from functions to invariants. In general, invariantization maps

$$F(z_1, \ldots, z_n) \;\longmapsto\; \iota(F) = F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z)). \tag{5.8}$$

[†] The left version can be obtained directly by replacing g by g^{-1} throughout the construction.

In particular, if $J(z)$ is any invariant, then we deduce

$$J(z_1, \ldots, z_n) = J(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z)). \tag{5.9}$$

This result is known as the *Replacement Rule*, and provides a simple means of immediately rewriting any invariant in terms of the fundamental invariants.

Example 5.7. Consider the standard action

$$y = x \cos \phi - u \sin \phi, \qquad v = x \sin \phi + u \cos \phi, \tag{5.10}$$

of the rotation group $G = \mathrm{SO}(2)$ on $M = \mathbb{R}^2$. The orbits are the circles centered at the origin and the origin itself. The action is free on the punctured plane $\widetilde{M} = \mathbb{R}^2 \setminus \{0\}$. Let us choose the cross-section

$$\mathcal{K} = \{\, u = 0, \ x > 0 \,\}.$$

Solving the normalization equation

$$v = x \sin \phi + u \cos \phi = 0$$

leads to the right moving frame:

$$\phi = -\tan^{-1} \frac{u}{x}, \tag{5.11}$$

which defines a right-equivariant map $\rho \colon \widetilde{M} \to \mathrm{SO}(2)$. The fundamental invariant is obtained by substituting the moving frame formula (5.11) into the unnormalized coordinate $y = x \cos \phi - u \sin \phi$, leading to

$$r = \iota(x) = \sqrt{x^2 + u^2}\,.$$

Finally, the invariantization of a function $F(x, y)$ is given by

$$\iota\big[\, F(x, u) \,\big] = F(r, 0).$$

In particular, if $J(x, y) = x^2 + y^2$ is an invariant, then the Replacement Rule

$$\iota(J) = r^2 + 0^2 = r^2 = J$$

gives us its formula in terms of the fundamental invariant. Of course, this example is too elementary on its own, but helps clarify the more complicated calculations seen later on.

Remark: Hubert and Kogan, [**71**, **72**], have formulated a completely algebraic version of the preceding construction, valid for polynomial and algebraic group actions, and shown its effectiveness for determining rational and algebraic invariants. In particular, the algebraic implementation of the Replacement Theorem leads to a rewrite rule for expressing other invariants in terms of the generating invariants.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 5.1. There are two basic methods for converting a non-free (but effective) action into a free action. The first is to look at the product action of G on several copies of M, leading to joint invariants. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants. Combining the two methods of prolongation and product will lead to joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a common framework, called multispace, [**84**, **121**].

6. Moving Frames on Jet Space and Differential Invariants.

Traditional moving frames are obtained by prolonging the group action to the nth order submanifold jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$. Given the prolonged group action $G^{(n)}$ on J^n, by an nth *order moving frame* $\rho^{(n)} \colon \mathrm{J}^n \to G$, we mean an equivariant map defined on an open subset of the jet space.

Theorem 6.1. *An nth order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

Our normalization construction will produce a moving frame and a complete system of differential invariants in the neighborhood of any regular jet. Local coordinates $z = (x, u)$ on M — considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables — induce local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n with components u_J^α representing the partial derivatives of the dependent variables with respect to the independent variables, [**116**, **117**]. We compute the prolonged transformation formulae

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}, \qquad \text{or} \qquad (y, v^{(n)}) = g^{(n)} \cdot (x, u^{(n)}),$$

by implicit differentiation of the v's with respect to the y's. For simplicity, we restrict to a coordinate cross-section by choosing $r = \dim G$ components of $w^{(n)}$ to normalize to constants:

$$w_1(g, z^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r. \tag{6.1}$$

Solving the normalization equations (6.1) for the group transformations leads to the explicit formulae $g = \rho^{(n)}(z^{(n)})$ for the right moving frame. As in Theorem 5.5, substituting the moving frame formulae into the unnormalized components of $w^{(n)}$ leads to the *fundamental n^{th} order differential invariants*

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{6.2}$$

Once the moving frame is established, the *invariantization* process will map general differential functions $F(x, u^{(n)})$ to differential invariants $I = \iota(F) = F \circ I^{(n)}$. As before, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants. The fundamental differential

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invariants $I^{(n)}$ are obtained by invariantization of the coordinate functions

$$
\begin{aligned}
H^i(x, u^{(n)}) &= \iota(x^i) = y^i(\rho^{(n)}(x, u^{(n)}), x, u), \\
I_K^\alpha(x, u^{(k)}) &= \iota(u_J^\alpha) = v_K^\alpha(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}).
\end{aligned}
\tag{6.3}
$$

In particular, those corresponding to the normalization components (6.1) of $w^{(n)}$ will be constant, and are known as the *phantom differential invariants*.

Theorem 6.2. *Let $\rho^{(n)} \colon \mathrm{J}^n \to G$ be a moving frame of order $\leq n$. Every n^{th} order differential invariant can be locally written as a function $J = \Phi(I^{(n)})$ of the fundamental n^{th} order differential invariants (6.3). The function Φ is unique provided it does not depend on the phantom invariants.*

Example 6.3. The paradigmatic example is the action of the orientation-preserving Euclidean group $\mathrm{SE}(2)$ on plane curves $C \subset M = \mathbb{R}^2$. The group transformation $g \in \mathrm{SE}(2)$ maps the point $z = (x, u)$ to the point $w = (y, v) = g \cdot z$, given by

$$
y = x \cos \phi - u \sin \phi + a, \qquad v = x \sin \phi + u \cos \phi + b.
\tag{6.4}
$$

For simplicity let us assume our curve is given (locally) by the graph of a function $u = f(x)$. (Extensions to general parametrized curves are straightforward.) The prolonged group transformations

$$
v_y = \frac{\sin \phi + u_x \cos \phi}{\cos \phi - u_x \sin \phi}, \qquad v_{yy} = \frac{u_{xx}}{(\cos \phi - u_x \sin \phi)^3},
$$
$$
v_{yyy} = \frac{(\cos \phi - u_x \sin \phi) u_{xxx} - 3 u_{xx}^2 \sin \phi}{(\cos \phi - u_x \sin \phi)^5}, \qquad \cdots
\tag{6.5}
$$

and so on, are found by successively applying the implicit differentiation operator

$$
\frac{d}{dy} = \frac{1}{\cos \phi - u_x \sin \phi} \frac{d}{dx}
\tag{6.6}
$$

to v as given in (6.4). Choosing the cross-section normalizations

$$
y = 0, \qquad v = 0, \qquad v_y = 0,
\tag{6.7}
$$

we solve for the group parameters

$$
\phi = -\tan^{-1} u_x, \qquad a = -\frac{x + u u_x}{\sqrt{1 + u_x^2}}, \qquad b = \frac{x u_x - u}{\sqrt{1 + u_x^2}},
\tag{6.8}
$$

which defines the right-equivariant moving frame $\rho \colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$. The corresponding left-equivariant moving frame is obtained by inversion:

$$
a = x, \qquad b = u, \qquad \phi = \tan^{-1} u_x
\tag{6.9}
$$

This can be identified with the classical left moving frame, [**32**, **61**], as follows: the translation component $(a, b) = (x, u) = z$ is the point on the curve, while the columns of the normalized rotation matrix

$$
R = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \quad \longmapsto \quad \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{t}, \mathbf{n})
$$

are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (6.8) into the prolonged transformation formulae (6.5), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}},$$

$$v_{yyy} \longmapsto \frac{d\kappa}{ds} = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3}, \qquad (6.10)$$

$$v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3,$$

where

$$\frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}} \frac{d}{dx} \qquad (6.11)$$

is the arc length derivative — which is itself found by substituting the moving frame formulae (6.8) into the implicit differentiation operator (6.6). A complete system of differential invariants for the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order. See [**120**] for complete details.

Example 6.4. Let $n \neq 0, 1$. In classical invariant theory, the planar actions

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad v = (\gamma x + \delta)^{-n} u, \qquad (6.12)$$

of $G = \mathrm{GL}(2)$ play a key role in the equivalence and symmetry properties of binary forms, when $u = q(x)$ is a polynomial of degree $\leq n$, [**66**, **118**, **8**]. We identify the graph of the function $u = q(x)$ as a plane curve. The prolonged action on such graphs is found by implicit differentiation:

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta \sigma^{n-1}}, \qquad v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma \sigma u_x + n(n-1)\gamma^2 u}{\Delta^2 \sigma^{n-2}},$$

$$v_{yyy} = \frac{\sigma^3 u_{xxx} - 3(n-2)\gamma \sigma^2 u_{xx} + 3(n-1)(n-2)\gamma^2 \sigma u_x - n(n-1)(n-2)\gamma^3 u}{\Delta^3 \sigma^{n-3}},$$

and so on, where $\sigma = \gamma p + \delta$, $\Delta = \alpha\delta - \beta\gamma \neq 0$. On the regular subdomain

$$\mathcal{V}^2 = \{uH \neq 0\} \subset J^2, \qquad \text{where} \qquad H = uu_{xx} - \frac{n-1}{n} u_x^2$$

is the classical Hessian covariant of u, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1.$$

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Solving for the group parameters gives the right moving frame formulae[†]

$$\alpha = u^{(1-n)/n}\sqrt{H}, \qquad\qquad \beta = -x\,u^{(1-n)/n}\sqrt{H},$$
$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n}u_x, \qquad\qquad \delta = u^{1/n} - \tfrac{1}{n}\,x\,u^{(1-n)/n}u_x. \tag{6.13}$$

Substituting the normalizations (6.13) into the higher order transformation rules gives us the differential invariants, the first two of which are

$$v_{yyy} \;\longmapsto\; J = \frac{T}{H^{3/2}}, \qquad\qquad v_{yyyy} \;\longmapsto\; K = \frac{V}{H^2}, \tag{6.14}$$

where

$$T = u^2 u_{xxx} - 3\,\frac{n-2}{n}\,uu_x u_{xx} + 2\,\frac{(n-1)(n-2)}{n^2}\,u_x^3,$$
$$V = u^3 u_{xxxx} - 4\,\frac{n-3}{n}\,u^2 u_x u_{xx} + 6\,\frac{(n-2)(n-3)}{n^2}\,uu_x{}^2 u_{xx} -$$
$$\qquad - 3\frac{(n-1)(n-2)(n-3)}{n^3}\,u_x^4,$$

and can be identified with classical covariants, which may be constructed using the basic transvectant process of classical invariant theory, cf. [66, 118]. Using $J^2 = T^2/H^3$ as the fundamental differential invariant will remove the ambiguity caused by the square root. As in the Euclidean case, higher order differential invariants are found by successive application of the normalized implicit differentiation operator $D_s = uH^{-1/2}D_x$ to the fundamental invariant J.

A general cross-section $\mathcal{K}^n \subset \mathrm{J}^n$ is prescribed implicitly by setting $r = \dim G$ differential functions $Z = (Z_1, \dots, Z_r)$ to constants:

$$Z_1(x, u^{(n)}) = c_1, \;\; \dots \;\; Z_r(x, u^{(n)}) = c_r. \tag{6.15}$$

Usually — but not always, [96, 125] — the functions are selected from the jet space coordinates x^i, u_J^α, resulting in a *coordinate cross-section*. The corresponding value of the right moving frame at a jet $z^{(n)} \in \mathrm{J}^n$ is the unique group element $g = \rho^{(n)}(z^{(n)}) \in G$ that maps it to the cross-section:

$$\rho^{(n)}(z^{(n)}) \cdot z^{(n)} = g^{(n)} \cdot z^{(n)} \in \mathcal{K}^n. \tag{6.16}$$

The moving frame $\rho^{(n)}$ clearly depends on the choice of cross-section, which is usually designed so as to simplify the required computations as much as possible.

Once the cross-section has been fixed, the induced moving frame engenders an invariantization process, that effectively maps functions to invariants, differential forms to invariant differential forms, and so on, [52, 123]. Geometrically, the *invariantization* of any object is defined as the unique invariant object that coincides with its progenitor when restricted to the cross-section. In particular, invariantization does not affect invariants,

---

[†] See [8] for a detailed discussion of how to resolve the square root ambiguities.

and hence defines a morphism that projects the algebra (or, more correctly, sheaf) of differential functions onto the algebra of differential invariants.

Computationally, the invariantization of a differential function is constructed by first writing out how it is transformed by the prolonged group action: $F(z^{(n)}) \mapsto F(g^{(n)} \cdot z^{(n)})$. One then replaces all the group parameters by their *right* moving frame formulae $g = \rho^{(n)}(z^{(n)})$, resulting in the differential invariant

$$\iota\big[\, F(z^{(n)}) \,\big] = F\big(\, \rho^{(n)}(z^{(n)}) \cdot z^{(n)}\, \big). \tag{6.17}$$

Differential forms and differential operators are handled in an analogous fashion — see [**52**, **87**] for complete details.

In particular, the *normalized differential invariants* induced by the moving frame are obtained by invariantization of the basic jet coordinates:

$$H^i = \iota(x^i), \qquad I_J^\alpha = \iota(u_J^\alpha), \tag{6.18}$$

which we collectively denote by $(H, I^{(n)}) = (\ \ldots\ H^i\ \ldots\ I_J^\alpha\ \ldots\ )$ for $\#J \leq n$. These naturally split into two classes: Those corresponding to the cross-section functions Z_κ are constant, and known as the *phantom differential invariants*. The remainder, known as the *basic differential invariants*, form a complete system of functionally independent differential invariants.

Once the normalized differential invariants are known, the invariantization process (6.17) is implemented by simply replacing each jet coordinate by the corresponding normalized differential invariant (6.18), so that

$$\iota\big[\, F(x, u^{(n)}) \,\big] = \iota\big[\, F(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ ) \,\big] = F(\ \ldots\ H^i\ \ldots\ I_J^\alpha\ \ldots\ ) = F(H, I^{(n)}). \tag{6.19}$$

In particular, a differential invariant is not affected by invariantization, leading to the very useful *Replacement Theorem*:

$$J(x, u^{(n)}) = J(H, I^{(n)}) \quad \text{whenever } J \text{ is a differential invariant.} \tag{6.20}$$

This permits one to straightforwardly rewrite any known differential invariant in terms the normalized invariants, and thereby establishes their completeness.

In a similar manner, the invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are obtained by invariantization of the total derivatives:

$$\mathcal{D}_i = \iota(D_i), \qquad i = 1, \ldots, p. \tag{6.21}$$

Equivalently, they can be defined as the dual differential operators arising from the invariant horizontal forms

$$\omega^i = \iota(dx^i), \qquad i = 1, \ldots, p, \tag{6.22}$$

obtained by (horizontal, [**87**]) invariantization of the horizontal one-forms $dx^1, \ldots, dx^p$. The horizontal forms $\omega^1, \ldots, \omega^p$ are only invariant modulo contact forms (as defined below), and so, in the language of [**117**], form a *contact-invariant coframe*.

The invariant differential operators do not commute in general, but are subject to the commutation formulae

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i, \tag{6.23}$$

where the coefficients $Y_{jk}^i = -Y_{kj}^i$ are certain differential invariants known as the *commutator invariants*. Their explicit formmulas in terms of the fundamental differential invariants will be found below.

7. Equivalence and Signatures.

The moving frame method was developed by Cartan expressly for the solution to problems of equivalence and symmetry of submanifolds under group actions. Two submanifolds $S, \overline{S} \subset M$ are said to be *equivalent* if $\overline{S} = g \cdot S$ for some $g \in G$. A *symmetry* of a submanifold is a group transformation that maps S to itself, and so is an element $g \in G_S$. As emphasized by Cartan, [**32**], the solution to the equivalence and symmetry problems for submanifolds is based on the functional interrelationships among the fundamental differential invariants restricted to the submanifold.

Suppose we have constructed an nth order moving frame $\rho^{(n)} \colon \mathrm{J}^n \to G$ defined on an open subset of jet space. A submanifold S is called *regular* if its n-jet $\mathrm{j}_n S$ lies in the domain of definition of the moving frame. For any $k \geq n$, we use $J^{(k)} = I^{(k)} \,|\, S = I^{(k)} \circ \mathrm{j}_k S$ to denote the kth order *restricted differential invariants*. The kth order *signature* $\mathcal{S}^{(k)} = \mathcal{S}^{(k)}(S)$ is the set parametrized by the restricted differential invariants; S is called *fully regular* if $J^{(k)}$ has constant rank $0 \leq t_k \leq p = \dim S$ for all $k \geq n$. In this case, $\mathcal{S}^{(k)}$ forms a submanifold of dimension t_k — perhaps with self-intersections. In the fully regular case,

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p,$$

where t is the *differential invariant rank* and s the *differential invariant order* of S.

Theorem 7.1. *Two fully regular p-dimensional submanifolds $S, \overline{S} \subset M$ are (locally) equivalent, $\overline{S} = g \cdot S$, if and only if they have the same differential invariant order s and their signature manifolds of order $s + 1$ are identical: $\mathcal{S}^{(s+1)}(\overline{S}) = \mathcal{S}^{(s+1)}(S)$.*

Since symmetries are the same as self-equivalences, the signature also determines the symmetry group of the submanifold.

Theorem 7.2. *If $S \subset M$ is a fully regular p-dimensional submanifold of differential invariant rank t, then its symmetry group G_S is an $(r-t)$–dimensional subgroup of G that acts locally freely on S.*

A submanifold with maximal differential invariant rank $t = p$, and hence only a discrete symmetry group, is called *nonsingular*. The number of symmetries is determined by the *index* of the submanifold, defined as the number of points in S map to a single generic point of its signature:

$$\operatorname{ind} S = \min \left\{ \# (J^{(s+1)})^{-1}\{\zeta\} \;\middle|\; \zeta \in \mathcal{S}^{(s+1)} \right\}.$$

Theorem 7.3. *If S is a nonsingular submanifold, then its symmetry group is a discrete subgroup of cardinality $\# G_S = \operatorname{ind} S$.*

At the other extreme, a rank 0 or *maximally symmetric* submanifold has all constant differential invariants, and so its signature degenerates to a single point.

Theorem 7.4. *A regular p-dimensional submanifold S has differential invariant rank 0 if and only if its symmetry group is a p-dimensional subgroup $H = G_S \subset G$ and an H–orbit: $S = H \cdot z_0$.*

Remark: "Totally singular" submanifolds may have even larger, non-free symmetry groups, but these are not covered by the preceding results. See [**119**] for details and precise characterization of such submanifolds.

Example 7.5. The *Euclidean signature* for a curve in the Euclidean plane is the planar curve $\mathcal{S}(C) = \{(\kappa, \kappa_s)\}$ parametrized by the curvature invariant κ and its first derivative with respect to arc length. Two planar curves are equivalent under oriented rigid motions if and only if they have the same signature curves. The maximally symmetric curves have constant Euclidean curvature, and so their signature curve degenerates to a single point. These are the circles and straight lines, and, in accordance with Theorem 7.4, each is the orbit of its one-parameter symmetry subgroup of SE(2). The number of Euclidean symmetries of a curve is equal to its index — the number of times the Euclidean signature is retraced as we go around the curve.

An example of a Euclidean signature curve is displayed in Figure 1. The first figure shows the curve, and the second its Euclidean signature; the axes are κ and κ_s in the signature plot. Note in particular the approximate three-fold symmetry of the curve is reflected in the fact that its signature has winding number three. If the symmetries were exact, the signature would be exactly retraced three times on top of itself. The final figure gives a discrete approximation to the signature which is based on the invariant numerical algorithms to be discussed below.

In Figure 3 we display some signature curves computed from an actual medical image — a 70×70, 8-bit gray-scale image of a cross section of a canine heart, obtained from an MRI scan. We then display an enlargement of the left ventricle. The boundary of the ventricle has been automatically segmented through use of the conformally Riemannian moving contour or snake flow that was proposed in [**80**] and successfully applied to a wide variety of 2D and 3D medical imagery, including MRI, ultrasound and CT data, [**156**]. Underneath these images, we display the ventricle boundary curve along with two successive smoothed versions obtained application of the standard Euclidean-invariant curve shortening procedure. Below each curve is the associated spline-interpolated discrete signature curves for the smoothed boundary, as computed using the invariant numerical approximations to κ and κ_s discussed below. As the evolving curves approach circularity the signature curves exhibit less variation in curvature and appear to be winding more and more tightly around a single point, which is the signature of a circle of area equal to the area inside the evolving curve. Despite the rather extensive smoothing involved, except for an overall shrinking as the contour approaches circularity, the basic qualitative features of the different signature curves, and particularly their winding behavior, appear to be remarkably robust.
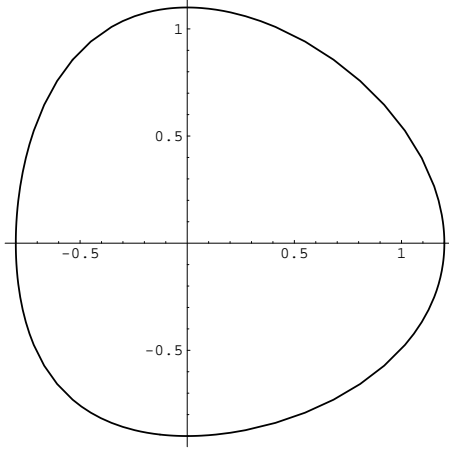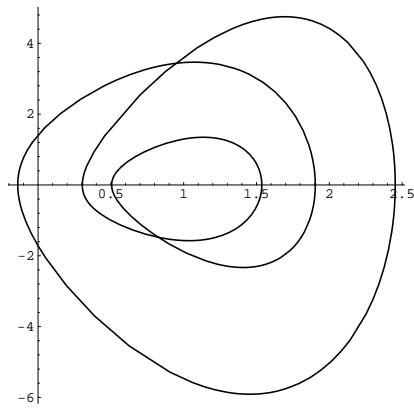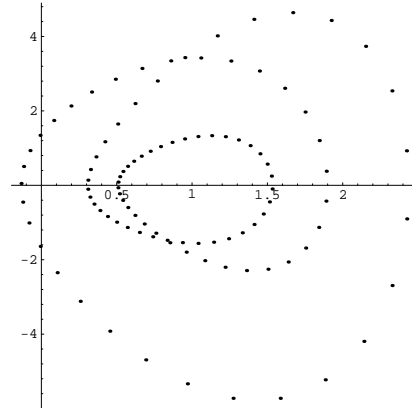
The Original Curve
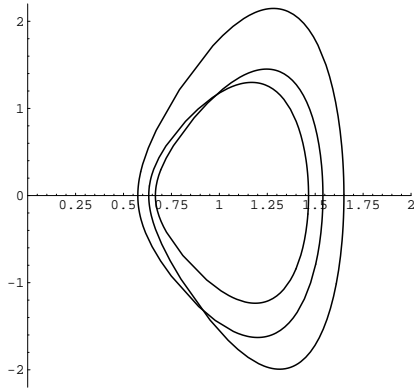
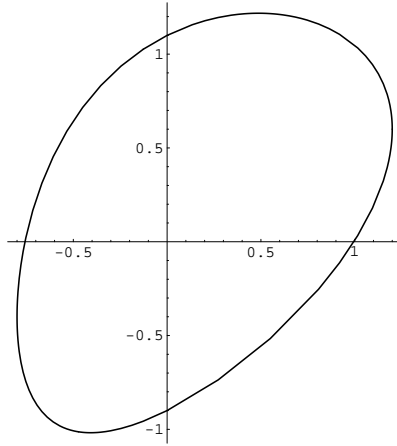Euclidean Signature Curve
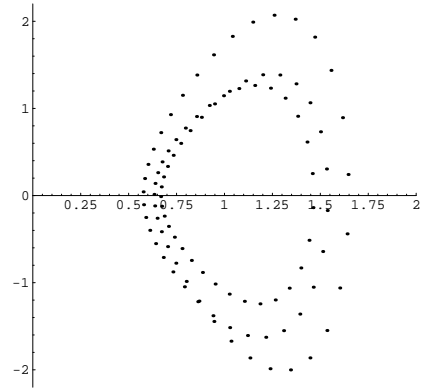
Discrete Euclidean Signature

Affine Signature Curve
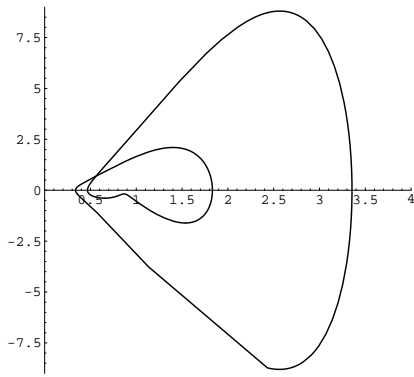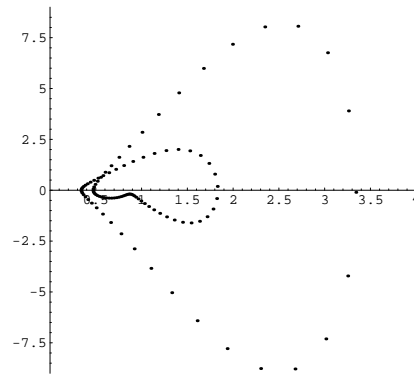

Discrete Affine Signature

Figure 1.   The Curve $x = \cos t + \frac{1}{5}\cos^2 t$, $y = \sin t + \frac{1}{10}\sin^2 t$.
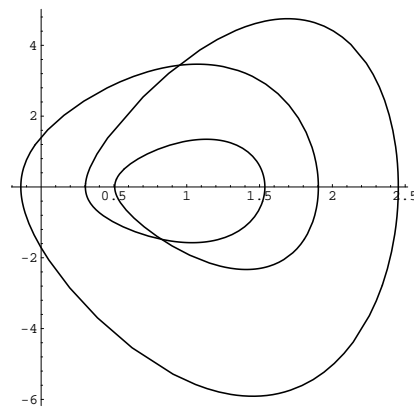
The Original Curve

Euclidean Signature Curve

Discrete Euclidean Signature

Affine Signature Curve
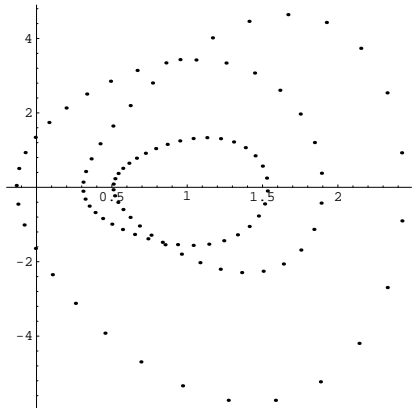
Discrete Affine Signature

Figure 2.   The Curve $x = \cos t + \frac{1}{5}\cos^2 t$, $y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$.
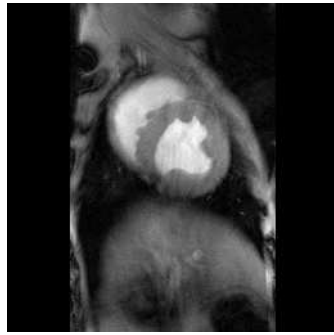
Original Canine Heart
MRI Image

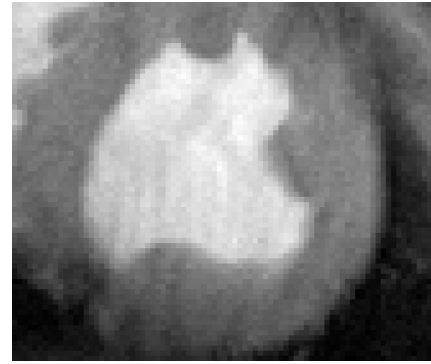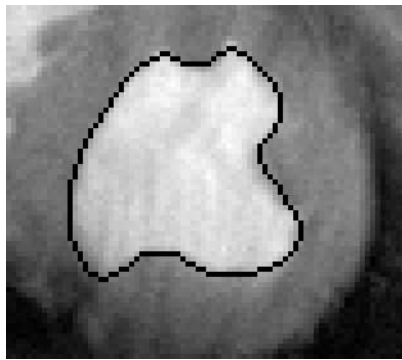Blow Up of the Left Ventricle

Boundary of Left Ventricle
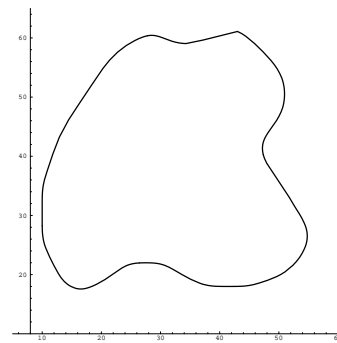
Original Contour
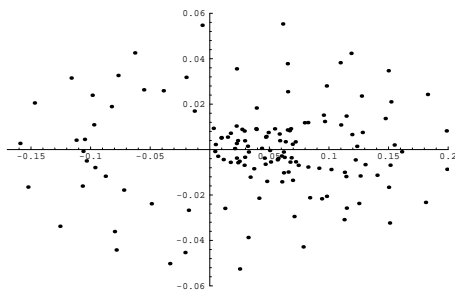
Discrete Euclidean Signature
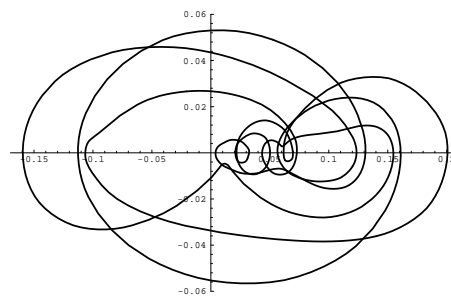
Smoothly Connected
Euclidean Signature

Figure 3.   Canine Left Ventricle Signature.
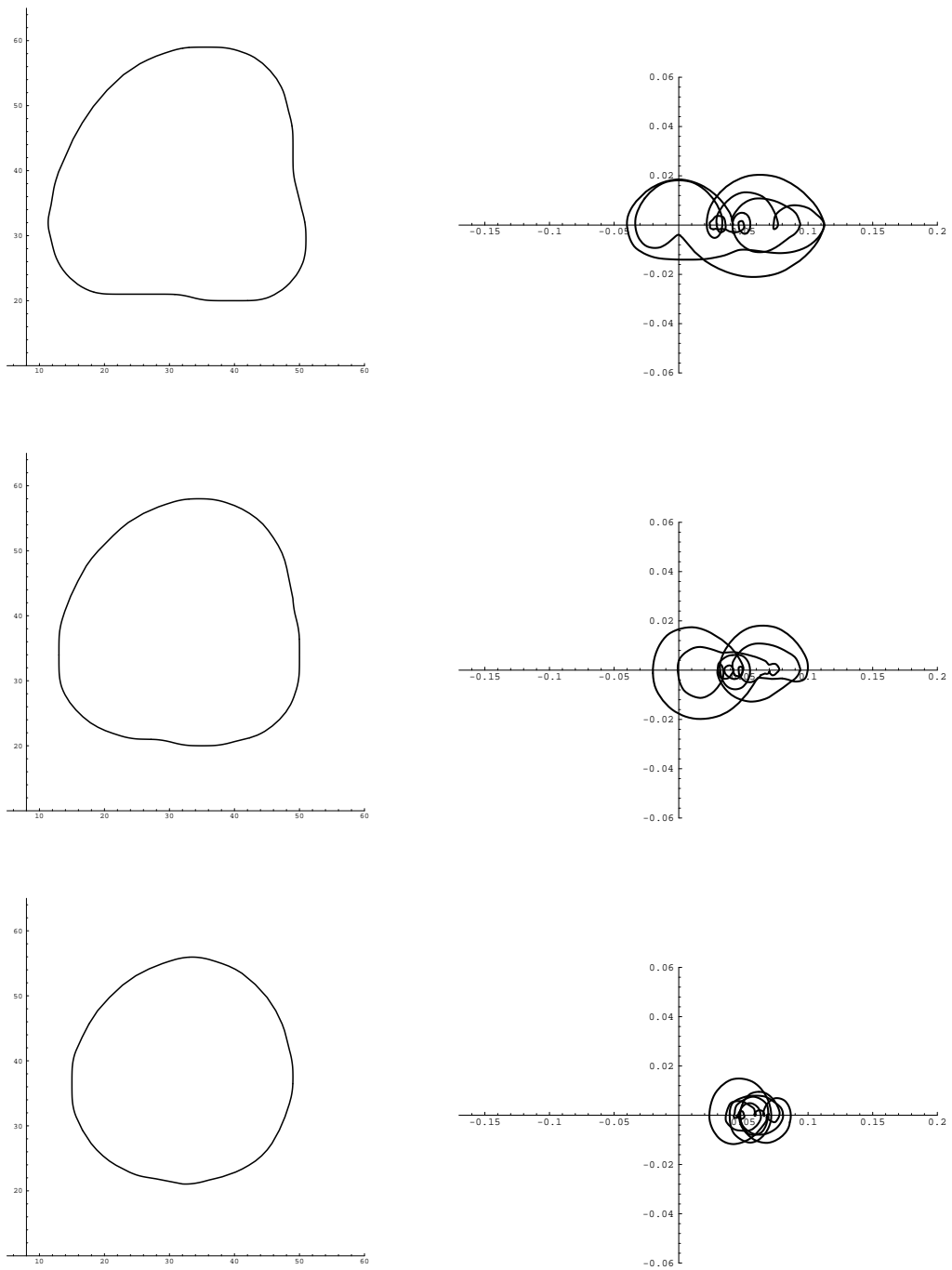
Figure 4.   Smoothed Canine Left Ventricle Signatures.

Thus, the signature curve method has the potential to be of practical use in the general problem of object recognition and symmetry classification. It offer several advantages over more traditional approaches. First, it is purely local, and therefore immediately applicable to occluded objects. Second, it provides a mechanism for recognizing symmetries and approximate symmetries of the object. The design of a suitably robust "signature metric" for practical comparison of signatures is the subject of ongoing research. See the paper by Shakiban and Lloyd, [**139**], for recent developments in this direction. In [**67**, **68**], the Euclidean-invariant signature is applied to design a program that automatically assembles jigsaw puzzles. An example appears in the following figure.
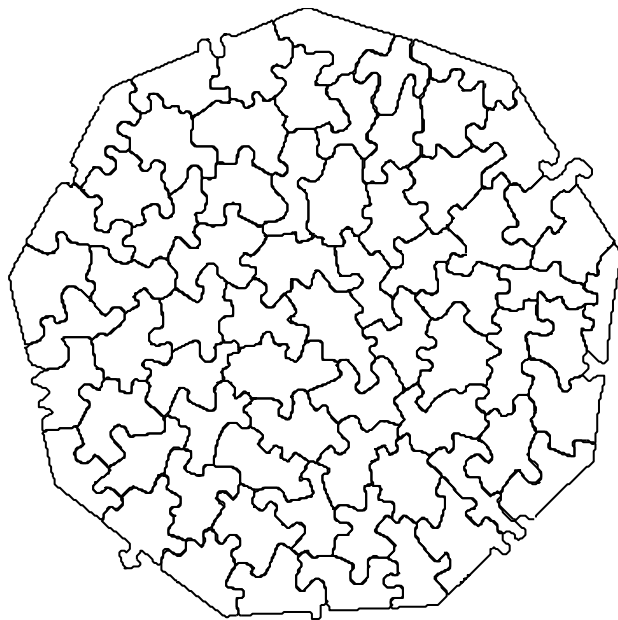
Figure 5.   The Baffler Jigsaw Puzzle.

Example 7.6. Let us next consider the equivalence and symmetry problems for binary forms. According to the general moving frame construction in Example 6.4, the signature curve $\mathcal{S} = \mathcal{S}(q)$ of a function (polynomial) $u = q(x)$ is parametrized by the covariants J^2 and K, as given in (6.14). The following solution to the equivalence problem for complex-valued binary forms, [**8, 115, 118**], is an immediate consequence of the general equivalence Theorem 7.1.

Theorem 7.7. *Two nondegenerate complex-valued forms $q(x)$ and $\overline{q}(x)$ are equivalent if and only if their signature curves are identical: $\mathcal{S}(q) = \mathcal{S}(\overline{q})$.*

All equivalence maps $\overline{x} = \varphi(x)$ solve the two rational equations

$$J(x)^2 = \overline{J}(\overline{x})^2, \qquad K(x) = \overline{K}(\overline{x}). \tag{7.1}$$

In particular, the theory guarantees φ is necessarily a linear fractional transformation!

Theorem 7.8. *A nondegenerate binary form $q(x)$ is maximally symmetric if and only if it satisfies the following equivalent conditions:*
- *q is complex-equivalent to a monomial x^k, with $k \neq 0, n$.*
- *The covariant T^2 is a constant multiple of $H^3 \not\equiv 0$.*
- *The signature is just a single point.*
- *q admits a one-parameter symmetry group.*
- *The graph of q coincides with the orbit of a one-parameter subgroup of $\mathrm{GL}(2)$.*

A binary form $q(x)$ is nonsingular if and only if it is not complex-equivalent to a monomial if and only if it has a finite symmetry group.

The symmetries of a nonsingular form can be explicitly determined by solving the rational equations (7.1) with $\overline{J} = J$, $\overline{K} = K$. See [**8**] for a MAPLE implementation of this method for computing discrete symmetries and classification of univariate polynomials. In particular, we obtain the following useful bounds on the number of symmetries.

Theorem 7.9. *If $q(x)$ is a binary form of degree n which is not complex-equivalent to a monomial, then its projective symmetry group has cardinality*

$$k \leq \begin{cases} 6n - 12 & \text{if } V = cH^2 \text{ for some constant } c, \text{ or} \\ 4n - 8 & \text{in all other cases.} \end{cases}$$

In her thesis, Kogan, [**85**], extends these results to forms in several variables. In particular, a complete signature for ternary forms, [**86**], leads to a practical algorithm for computing discrete symmetries of, among other cases, elliptic curves.

8. Joint Invariants and Joint Differential Invariants.

One practical difficulty with the differential invariant signature is its dependence upon high order derivatives, which makes it very sensitive to data noise. For this reason, a new signature paradigm, based on joint invariants, was proposed in [**120**]. We consider now the joint action

$$g \cdot (z_0, \ldots, z_n) = (g \cdot z_0, \ldots, g \cdot z_n), \qquad g \in G, \quad z_0, \ldots, z_n \in M. \tag{8.1}$$
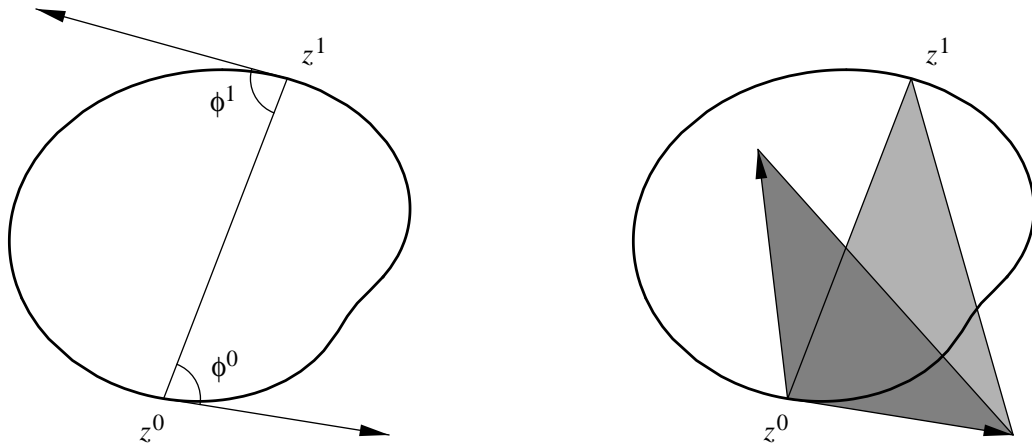
Figure 6.   First and Second Order Joint Euclidean Differential Invariants.

of the group G on the $(n+1)$-fold Cartesian product $M^{\times(n+1)} = M \times \cdots \times M$. An invariant $I(z_0, \ldots, z_n)$ of (8.1) is an $(n+1)$-*point joint invariant* of the original transformation group. In most cases of interest, although not in general, if G acts effectively on M, then, for $n \gg 0$ sufficiently large, the product action is free and regular on an open subset of $M^{\times(n+1)}$. Consequently, the moving frame method outlined in Section 1 can be applied to such joint actions, and thereby establish complete classifications of joint invariants and, via prolongation to Cartesian products of jet spaces, joint differential invariants. We will discuss two particular examples — planar curves in Euclidean geometry and projective geometry, referring to [**120**] for details.

Example 8.1. *Euclidean joint differential invariants.* Consider the proper Euclidean group SE(2) acting on oriented curves in the plane $M = \mathbb{R}^2$. We begin with the Cartesian product action on $M^{\times 2} \simeq \mathbb{R}^4$. Taking the simplest cross-section $x_0 = u_0 = x_1 = 0, u_1 > 0$ leads to the normalization equations

$$y_0 = x_0 \cos \theta - u_0 \sin \theta + a = 0, \qquad v_0 = x_0 \sin \theta + u_0 \cos \theta + b = 0,$$
$$y_1 = x_1 \cos \theta - u_1 \sin \theta + a = 0. \tag{8.2}$$

Solving, we obtain a right moving frame

$$\theta = \tan^{-1}\left(\frac{x_1 - x_0}{u_1 - u_0}\right), \qquad a = -x_0 \cos \theta + u_0 \sin \theta, \qquad b = -x_0 \sin \theta - u_0 \cos \theta, \tag{8.3}$$

along with the fundamental interpoint distance invariant

$$v_1 = x_1 \sin \theta + u_1 \cos \theta + b \quad \longmapsto \quad I = \| z_1 - z_0 \|. \tag{8.4}$$

Substituting (8.3) into the prolongation formulae (6.5) leads to the the normalized first

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and second order joint differential invariants

$$
\begin{aligned}
\frac{dv_k}{dy} &\longmapsto \quad J_k = -\,\frac{(z_1 - z_0) \cdot \dot{z}_k}{(z_1 - z_0) \wedge \dot{z}_k}\,, \\
\frac{d^2 v_k}{dy^2} &\longmapsto \quad K_k = -\,\frac{\|\, z_1 - z_0\,\|^3\,(\dot{z}_k \wedge \ddot{z}_k)}{\big[\,(z_1 - z_0) \wedge \dot{z}_0\,\big]^3}\,,
\end{aligned}
\tag{8.5}
$$

for $k = 0, 1$. Note that

$$
J_0 = -\cot\phi_0, \qquad J_1 = +\cot\phi_1,
\tag{8.6}
$$

where $\phi_k = \sphericalangle(z_1 - z_0, \dot{z}_k)$ denotes the angle between the chord connecting z_0, z_1 and the tangent vector at z_k, as illustrated in Figure 6. The modified second order joint differential invariant

$$
\widehat{K}_0 = -\|\, z_1 - z_0\,\|^{-3}\, K_0 = \frac{\dot{z}_0 \wedge \ddot{z}_0}{\big[\,(z_1 - z_0) \wedge \dot{z}_0\,\big]^3}
\tag{8.7}
$$

equals the ratio of the area of triangle whose sides are the first and second derivative vectors $\dot{z}_0$, $\ddot{z}_0$ at the point z_0 over the *cube* of the area of triangle whose sides are the chord from z_0 to z_1 and the tangent vector at z_0; see Figure 6.

On the other hand, we can construct the joint differential invariants by invariant differentiation of the basic distance invariant (8.4). The normalized invariant differential operators are

$$
D_{y_k} \quad \longmapsto \quad \mathcal{D}_k = -\,\frac{\|\, z_1 - z_0\,\|}{(z_1 - z_0) \wedge \dot{z}_k}\, D_{t_k}\,.
\tag{8.8}
$$

Proposition 8.2. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance $I = \|\, z_1 - z_0\,\|$ and its invariant derivatives with respect to (8.8).*

A generic product curve $\mathbf{C} = C_0 \times C_1 \subset M^{\times 2}$ has joint differential invariant rank $2 = \dim \mathbf{C}$, and its joint signature $\mathcal{S}^{(2)}(\mathbf{C})$ will be a two-dimensional submanifold parametrized by the joint differential invariants I, J_0, J_1, K_0, K_1 of order ≤ 2. There will exist a (local) syzygy $\Phi(I, J_0, J_1) = 0$ among the three first order joint differential invariants.

Theorem 8.3. *A curve C or, more generally, a pair of curves $C_0, C_1 \subset \mathbb{R}^2$, is uniquely determined up to a Euclidean transformation by its reduced joint signature, which is parametrized by the first order joint differential invariants I, J_0, J_1. The curve(s) have a one-dimensional symmetry group if and only if their signature is a one-dimensional curve if and only if they are orbits of a common one-parameter subgroup (i.e., concentric circles or parallel straight lines); otherwise the signature is a two-dimensional surface, and the curve(s) have only discrete symmetries.*

For $n > 2$ points, we can use the two-point moving frame (8.3) to construct the additional joint invariants

$$
y_k \quad \longmapsto \quad H_k = \|\, z_k - z_0\,\|\,\cos\psi_k, \qquad\qquad v_k \quad \longmapsto \quad I_k = \|\, z_k - z_0\,\|\,\sin\psi_k,
$$

where $\psi_k = \sphericalangle(z_k - z_0, z_1 - z_0)$. Therefore, a complete system of joint invariants for SE(2) consists of the angles ψ_k, $k \geq 2$, and distances $\|\, z_k - z_0\,\|$, $k \geq 1$. The other interpoint
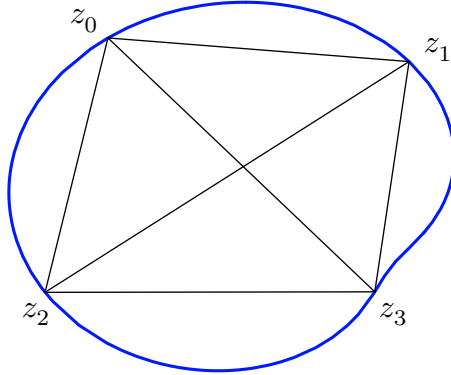
Figure 7.   Four-Point Euclidean Curve Invariants.

distances can all be recovered from these angles; vice versa, given the distances, and the sign of one angle, one can recover all other angles. In this manner, we establish a "First Main Theorem" for joint Euclidean differential invariants.

Theorem 8.4.   *If $n \geq 2$, then every n-point joint E(2) differential invariant is a function of the interpoint distances $\| z_i - z_j \|$ and their invariant derivatives with respect to (8.8). For the proper Euclidean group SE(2), one must also include the sign of one of the angles, say $\psi_2 = \sphericalangle(z_2 - z_0, z_1 - z_0)$.*

Generic three-pointed Euclidean curves still require first order signature invariants. To create a Euclidean signature based entirely on joint invariants, we take four points z_0, z_1, z_2, z_3 on our curve $C \subset \mathbb{R}^2$. As illustrated in Figure 7, there are six different interpoint distance invariants

$$
\begin{aligned}
a = \| z_1 - z_0 \|, \qquad b = \| z_2 - z_0 \|, \qquad c = \| z_3 - z_0 \|, \\
d = \| z_2 - z_1 \|, \qquad e = \| z_3 - z_1 \|, \qquad f = \| z_3 - z_2 \|,
\end{aligned}
\tag{8.9}
$$

which parametrize the joint signature $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(C)$ that uniquely characterizes the curve C up to Euclidean motion. This signature has the advantage of requiring no differentiation, and so is not sensitive to noisy image data. There are two local syzygies

$$
\Phi_1(a, b, c, d, e, f) = 0, \qquad\qquad \Phi_2(a, b, c, d, e, f) = 0, \tag{8.10}
$$

among the the six interpoint distances. One of these is the universal *Cayley–Menger syzygy* which is valid for all possible configurations of the four points, and is a consequence of their coplanarity, cf. [**13**, **108**]. The second syzygy in (8.10) is curve-dependent and serves to effectively characterize the joint invariant signature. Euclidean symmetries of the curve, both continuous and discrete, are characterized by this joint signature. For example, the

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number of discrete symmetries equals the signature index — the number of points in the original curve that map to a single, generic point in $\mathcal{S}$.

A wide variety of additional cases, including curves and surfaces in two and three-dimensional space under the Euclidean, equi-affine, affine and projective groups, are investigated in detail in [**120**].

9.   Multi-Space for Curves.

In modern numerical analysis, the development of numerical schemes that incorporate additional structure enjoyed by the problem being approximated have become quite popular in recent years. The first instances of such schemes are the symplectic integrators arising in Hamiltonian mechanics, and the related energy conserving methods, [**37**, **92**, **150**]. The design of symmetry-based numerical approximation schemes for differential equations has been studied by various authors, including Shokin, [**141**], Dorodnitsyn, [**45**, **46**], Axford and Jaegers, [**76**], and Budd and Collins, [**26**]. These methods are closely related to the active area of geometric integration of differential equations, [**27**, **62**, **103**]. In practical applications of invariant theory to computer vision, group-invariant numerical schemes to approximate differential invariants have been applied to the problem of symmetry-based object recognition, [**15**, **29**, **28**].

In this section, we outline the basic construction of multi-space that forms the foundation for the study of the geometry of discrete approximations to derivatives and numerical solutions to differential equations; see [**121**] for more details. We will only discuss the case of curves, which correspond to functions of a single independent variable, and hence satisfy ordinary differential equations. The more difficult case of higher dimensional submanifolds, corresponding to functions of several variables that satisfy partial differential equations, relies on a new approach to multi-dimensional interpolation theory, [**122**].

Numerical finite difference approximations to the derivatives of a function $u = f(x)$ rely on its values $u_0 = f(x_0), \ldots, u_n = f(x_n)$ at several distinct points $z_i = (x_i, u_i) = (x_i, f(x_i))$ on the curve. Thus, discrete approximations to jet coordinates on J^n are functions $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)} = M \times \cdots \times M$. In order to seamlessly connect the jet coordinates with their discrete approximations, then, we need to relate the (extended) jet space for curves, $\mathrm{J}^n = \mathrm{J}^n(M, 1)$, to the Cartesian product space $M^{\times(n+1)}$. Now, as the points $z_0, \ldots, z_n$ coalesce, the approximation $F(z_0, \ldots, z_n)$ will not be well-defined unless we specify the "direction" of convergence. Thus, strictly speaking, F is not defined on all of $M^{\times(n+1)}$, but, rather, on the "off-diagonal" part, by which we mean the subset

$$M^{\diamond(n+1)} = \left\{ (z_0, \ldots, z_n) \mid z_i \neq z_j \text{ for all } i \neq j \right\} \subset M^{\times(n+1)}$$

consisting of all *distinct* $(n+1)$-tuples of points. As two or more points come together, the limiting value of $F(z_0, \ldots, z_n)$ will be governed by the derivatives (or jet) of the appropriate order governing the direction of convergence. This observation serves to motivate our construction of the nth *order multi-space* $M^{(n)}$, which shall contain both the jet space J^n and the off-diagonal Cartesian product space $M^{\diamond(n+1)}$ in a consistent manner.

Definition 9.1. An $(n+1)$-*pointed curve* $\mathbf{C} = (z_0, \ldots, z_n; C)$ consists of a smooth curve C and $n+1$ not necessarily distinct points $z_0, \ldots, z_n \in C$ thereon. Given $\mathbf{C}$, we let $\#i = \#\{\, j \mid z_j = z_i \,\}$. Two $(n+1)$-pointed curves $\mathbf{C} = (z_0, \ldots, z_n; C)$, $\widetilde{\mathbf{C}} = (\widetilde{z}_0, \ldots, \widetilde{z}_n; \widetilde{C})$, have n^{th} *order multi-contact* if and only if

$$z_i = \widetilde{z}_i, \qquad \text{and} \qquad \mathrm{j}_{\#i-1}C|_{z_i} = \mathrm{j}_{\#i-1}\widetilde{C}|_{z_i}, \qquad \text{for each} \qquad i = 0, \ldots, n.$$

Definition 9.2. The nth order *multi-space*, denoted $M^{(n)}$ is the set of equivalence classes of $(n+1)$-pointed curves in M under the equivalence relation of nth order multi-contact. The equivalence class of an $(n+1)$-pointed curves $\mathbf{C}$ is called its nth order *multi-jet*, and denoted $\mathbf{j}_n\mathbf{C} \in M^{(n)}$.

In particular, if the points on $\mathbf{C} = (z_0, \ldots, z_n; C)$ are all distinct, then $\mathbf{j}_n\mathbf{C} = \mathbf{j}_n\widetilde{\mathbf{C}}$ if and only if $z_i = \widetilde{z}_i$ for all i, which means that $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ have all $n+1$ points in common. Therefore, we can identify the subset of multi-jets of multi-pointed curves having distinct points with the off-diagonal Cartesian product space $M^{\diamond(n+1)} \subset \mathrm{J}^n$. On the other hand, if all $n+1$ points coincide, $z_0 = \cdots = z_n$, then $\mathbf{j}_n\mathbf{C} = \mathbf{j}_n\widetilde{\mathbf{C}}$ if and only if $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ have n^{th} order contact at their common point $z_0 = \widetilde{z}_0$. Therefore, the multi-space equivalence relation reduces to the ordinary jet space equivalence relation on the set of coincident multi-pointed curves, and in this way $\mathrm{J}^n \subset M^{(n)}$. These two extremes do not exhaust the possibilities, since one can have some but not all points coincide. Intermediate cases correspond to "off-diagonal" Cartesian products of jet spaces

$$\mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_i} \equiv \left\{ (z_0^{(k_1)}, \ldots, z_i^{(k_i)}) \in \mathrm{J}^{k_1} \times \cdots \times \mathrm{J}^{k_i} \;\middle|\; \pi(z_\nu^{(k_\nu)}) \text{ are distinct} \right\}, \qquad (9.1)$$

where $\sum k_\nu = n$ and $\pi\colon \mathrm{J}^k \to M$ is the usual jet space projection. These *multi-jet spaces* appear in the work of Dhooghe, [**44**], on the theory of "semi-differential invariants" in computer vision.

Theorem 9.3. *If M is a smooth m-dimensional manifold, then its nth order multi-space $M^{(n)}$ is a smooth manifold of dimension $(n+1)m$, which contains the off-diagonal part $M^{\diamond(n+1)}$ of the Cartesian product space as an open, dense submanifold, and the nth order jet space J^n as a smooth submanifold.*

The proof of Theorem 9.3 requires the introduction of coordinate charts on $M^{(n)}$. Just as the local coordinates on J^n are provided by the coefficients of Taylor polynomials, the local coordinates on $M^{(n)}$ are provided by the coefficients of interpolating polynomials, which are the classical divided differences of numerical interpolation theory, [**110, 133**].

Definition 9.4. Given an $(n+1)$-pointed graph $\mathbf{C} = (z_0, \ldots, z_n; C)$, its divided differences are defined by $[\, z_j \,]_C = f(x_j)$, and

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,]_C = \lim_{z \to z_k} \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z \,]_C - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]_C}{x - x_{k-1}}. \qquad (9.2)$$

When taking the limit, the point $z = (x, f(x))$ must lie on the curve C, and take limiting values $x \to x_k$ and $f(x) \to f(x_k)$.

In the non-confluent case $z_k \neq z_{k-1}$ we can replace z by z_k directly in the difference quotient (9.2) and so ignore the limit. On the other hand, when all $k+1$ points coincide, the kth order confluent divided difference converges to

$$[\, z_0 \ldots z_0 \,]_C = \frac{f^{(k)}(x_0)}{k!}. \tag{9.3}$$

Remark: Classically, one employs the simpler notation $[\, u_0 u_1 \ldots u_k \,]$ for the divided difference $[\, z_0 z_1 \ldots z_k \,]_C$. However, the classical notation is ambiguous since it assumes that the mesh $x_0, \ldots, x_n$ is fixed throughout. Because we are regarding the independent and dependent variables on the same footing — and, indeed, are allowing changes of variables that scramble the two — it is important to adopt an unambiguous divided difference notation here.

Theorem 9.5. *Two $(n+1)$-pointed graphs $\mathbf{C}, \widetilde{\mathbf{C}}$ have n^{th} order multi-contact if and only if they have the same divided differences:*

$$[\, z_0 z_1 \ldots z_k \,]_C = [\, z_0 z_1 \ldots z_k \,]_{\widetilde{C}}, \qquad k = 0, \ldots, n.$$

The required local coordinates on multi-space $M^{(n)}$ consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n, \qquad \begin{aligned} u^{(0)} &= u_0 = [\, z_0 \,]_C, & u^{(1)} &= [\, z_0 z_1 \,]_C, \\ u^{(2)} &= 2 \,[\, z_0 z_1 z_2 \,]_C & \ldots & u^{(n)} = n! \,[\, z_0 z_1 \ldots z_n \,]_C, \end{aligned} \tag{9.4}$$

prescribed by $(n+1)$-pointed graphs $\mathbf{C} = (z_0, \ldots, z_n; C)$. The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n, cf. (9.3).

10. Invariant Numerical Methods.

To implement a numerical solution to a system of differential equations

$$\Delta_1(x, u^{(n)}) = \cdots = \Delta_k(x, u^{(n)}) = 0. \tag{10.1}$$

by finite difference methods, one relies on suitable discrete approximations to each of its defining differential functions Δ_ν, and this requires extending the differential functions from the jet space to the associated multi-space, in accordance with the following definition.

Definition 10.1. An $(n+1)$-*point numerical approximation of order k* to a differential function $\Delta \colon \mathrm{J}^n \to \mathbb{R}$ is an function $F \colon M^{(n)} \to \mathbb{R}$ that, when restricted to the jet space, agrees with Δ to order k.

The simplest illustration of Definition 10.1 is provided by the divided difference coordinates (9.4). Each divided difference $u^{(n)}$ forms an $(n+1)$-point numerical approximation to the nth order derivative coordinate on J^n. According to the usual Taylor expansion, the order of the approximation is $k = 1$. More generally, any differential function $\Delta(x, u, u^{(1)}, \ldots, u^{(n)})$ can immediately be assigned an $(n+1)$-point numerical approximation $F = \Delta(x_0, u^{(0)}, u^{(1)}, \ldots, u^{(n)})$ by replacing each derivative by its divided

difference coordinate approximation. However, these are by no means the only numerical approximations possible.

Now let us consider an r-dimensional Lie group G which acts smoothly on M. Since G evidently maps multi-pointed curves to multi-pointed curves while preserving the multi-contact equivalence relation, it induces an action on the multi-space $M^{(n)}$ that will be called the nth *multi-prolongation* of G and denoted by $G^{(n)}$. On the jet subset $\mathrm{J}^n \subset M^{(n)}$ the multi-prolonged action reduced to the usual jet space prolongation. On the other hand, on the off-diagonal part $M^{\diamond(n+1)} \subset M^{(n)}$ the action coincides with the $(n+1)$-fold Cartesian product action of G on $M^{\times(n+1)}$.

We define a *multi-invariant* to be a function $K \colon M^{(n)} \to \mathbb{R}$ on multi-space which is invariant under the multi-prolonged action of $G^{(n)}$. The restriction of a multi-invariant K to jet space will be a differential invariant, $I = K \mid \mathrm{J}^n$, while restriction to $M^{\diamond(n+1)}$ will define a joint invariant $J = K \mid M^{\diamond(n+1)}$. Smoothness of K will imply that the joint invariant J is an *invariant n^{th} order numerical approximation to the differential invariant I*. Moreover, every invariant finite difference numerical approximation arises in this manner. Thus, the theory of multi-invariants *is* the theory of invariant numerical approximations!

Furthermore, the restriction of a multi-invariant to an intermediate multi-jet subspace, as in (9.1), will define a joint differential invariant, [**120**] — also known as a semi-differential invariant in the computer vision literature, [**44, 111**]. The approximation of differential invariants by joint differential invariants is, therefore, based on the extension of the differential invariant from the jet space to a suitable multi-jet subspace (9.1). The invariant numerical approximations to joint differential invariants are, in turn, obtained by extending them from the multi-jet subspace to the entire multi-space. Thus, multi-invariants also include invariant semi-differential approximations to differential invariants as well as joint invariant numerical approximations to differential invariants and semi-differential invariants — all in one seamless geometric framework.

Effectiveness of the group action on M implies, typically, freeness and regularity of the multi-prolonged action on an open subset of $M^{(n)}$. Thus, we can apply the basic moving frame construction. The resulting *multi-frame* $\rho^{(n)} \colon M^{(n)} \to G$ will lead us immediately to the required multi-invariants and hence a general, systematic construction for invariant numerical approximations to differential invariants. Any multi-frame will evidently restrict to a classical moving frame $\rho^{(n)} \colon \mathrm{J}^n \to G$ on the jet space along with a suitably compatible product frame $\rho^{\diamond(n+1)} \colon M^{\diamond(n+1)} \to G$.

In local coordinates, we use $w_k = (y_k, v_k) = g \cdot z_k$ to denote the transformation formulae for the individual points on a multi-pointed curve. The multi-prolonged action on the divided difference coordinates gives

$$y_0, \ldots, y_n, \qquad \begin{aligned} v^{(0)} &= v_0 = [\, w_0 \,], & v^{(1)} &= [\, w_0 w_1 \,], \\ v^{(2)} &= [\, w_0 w_1 w_2 \,], & \ldots & v^{(n)} &= n! \, [\, w_0, \ldots, w_n \,], \end{aligned} \tag{10.2}$$

where the formulae are most easily computed via the difference quotients

$$[\, w_0 w_1 \ldots w_{k-1} w_k \,] = \frac{[\, w_0 w_1 w_2 \ldots w_{k-2} w_k \,] - [\, w_0 w_1 w_2 \ldots w_{k-2} w_{k-1} \,]}{y_k - y_{k-1}}, \tag{10.3}$$

$$[\, w_j \,] = v_j,$$

and then taking appropriate limits to cover the case of coalescing points. Inspired by the constructions in [**52**], we will refer to (10.2) as the *lifted divided difference invariants*.

To construct a multi-frame, we need to normalize by choosing a cross-section to the group orbits in $M^{(n)}$, which amounts to setting $r = \dim G$ of the lifted divided difference invariants (10.2) equal to suitably chosen constants. An important observation is that in order to obtain the limiting differential invariants, we must require our local cross-section to pass through the jet space, and define, by intersection, a cross-section for the prolonged action on J^n. This compatibility constraint implies that we are only allowed to normalize the first lifted independent variable $y_0 = c_0$.

With the aid of the multi-frame, the most direct construction of the requisite multi-invariants and associated invariant numerical differentiation formulae is through the invariantization of the original finite difference quotients (9.2). Substituting the multi-frame formulae for the group parameters into the lifted coordinates (10.2) provides a complete system of multi-invariants on $M^{(n)}$; this follows immediately from Theorem 5.5. We denote the fundamental multi-invariants by

$$ y_i \longmapsto H_i = \iota(x_i), \qquad v^{(n)} \longmapsto K^{(n)} = \iota(u^{(n)}), \qquad (10.4) $$

where ι denotes the invariantization map associated with the multi-frame. The fundamental differential invariants for the prolonged action of G on J^n can all be obtained by restriction, so that $I^{(n)} = K^{(n)} \,|\, \mathrm{J}^n$. On the jet space, the points are coincident, and so the multi-invariants H_i will all restrict to the *same* differential invariant $c_0 = H = H_i \,|\, \mathrm{J}^n$ — the normalization value of y_0. On the other hand, the fundamental joint invariants on $M^{\diamond(n+1)}$ are obtained by restricting the multi-invariants $H_i = \iota(x_i)$ and $K_i = \iota(u_i)$. The multi-invariants can computed by using a multi-invariant divided difference recursion

$$ [\, I_j \,] = K_j = \iota(u_j) $$

$$ [\, I_0 \ldots I_k \,] = \iota(\,[\, z_0 z_1 \ldots z_k \,]\,) = \frac{[\, I_0 \ldots I_{k-2} I_k \,] - [\, I_0 \ldots I_{k-2} I_{k-1} \,]}{H_k - H_{k-1}}, \qquad (10.5) $$

and then relying on continuity to extend the formulae to coincident points. The multi-invariants

$$ K^{(n)} = n! \,[\, I_0 \ldots I_n \,] = \iota(\, u^{(n)} \,) \qquad (10.6) $$

define the fundamental first order invariant numerical approximations to the differential invariants $I^{(n)}$. Higher order invariant approximations can be obtained by invariantization of the higher order divided difference approximations. The moving frame construction has a significant advantage over the infinitesimal approach used by Dorodnitsyn, [**45, 46**], in that it does not require the solution of partial differential equations in order to construct the multi-invariants.

Given a regular G-invariant differential equation

$$ \Delta(x, u, u^{(1)}, \ldots, u^{(n)}) = 0, \qquad (10.7) $$

we can invariantize the left hand side to rewrite the differential equation in terms of the fundamental differential invariants:

$$ \iota\big(\, \Delta(x, u, u^{(1)}, \ldots, u^{(n)})\big) = \Delta(H, I^{(0)}, I^{(1)}, \ldots, I^{(n)}) = 0. $$

The invariant finite difference approximation to the differential equation is then obtained by replacing the differential invariants $I^{(k)}$ by their multi-invariant counterparts $K^{(k)}$:

$$\Delta(c_0, K^{(0)}, \ldots, K^{(n)}) = 0. \tag{10.8}$$

Example 10.2. Consider the elementary action

$$(x, u) \quad \longmapsto \quad (\lambda^{-1}x + a, \lambda u + b)$$

of the three-parameter similarity group $G = \mathbb{R}^2 \ltimes \mathbb{R}$ on $M = \mathbb{R}^2$. To obtain the multi-prolonged action, we compute the divided differences (10.2) of the basic lifted invariants

$$y_k = \lambda^{-1}x_k + a, \qquad v_k = \lambda u_k + b.$$

We find

$$v^{(1)} = [w_0 w_1] = \frac{v_1 - v_0}{y_1 - y_0} = \lambda^2 \frac{u_1 - u_0}{x_1 - x_0} = \lambda^2 [z_0 z_1] = \lambda^2 u^{(1)}.$$

More generally,

$$v^{(n)} = \lambda^{n+1} u^{(n)}, \qquad n \geq 1. \tag{10.9}$$

Note that we may compute the multi-space transformation formulae assuming initially that the points are distinct, and then extending to coincident cases by continuity. (In fact, this gives an alternative method for computing the standard jet space prolongations of group actions!) In particular, when all the points coincide, each $u^{(n)}$ reduces to the nth order derivative coordinate, and (10.9) reduces to the prolonged action of G on J^n. We choose the normalization cross-section defined by

$$y_0 = 0, \qquad v_0 = 0, \qquad v^{(1)} = 1,$$

which, upon solving for the group parameters, leads to the basic moving frame

$$a = -\sqrt{u^{(1)}} \, x_0, \qquad b = -\frac{u_0}{\sqrt{u^{(1)}}}, \qquad \lambda = \frac{1}{\sqrt{u^{(1)}}}, \tag{10.10}$$

where, for simplicity, we restrict to the subset where $u^{(1)} = [z_0 z_1] > 0$. The fundamental joint similarity invariants are obtained by substituting these formulae into

$$y_k \longmapsto H_k = (x_k - x_0)\sqrt{u^{(1)}} = (x_k - x_0)\sqrt{\frac{u_1 - u_0}{x_1 - x_0}},$$

$$v_k \longmapsto K_k = \frac{u_k - u_0}{\sqrt{u^{(1)}}} = (u_k - u_0)\sqrt{\frac{x_1 - x_0}{u_1 - u_0}},$$

both of which reduce to the trivial zero differential invariant on J^n. Higher order multi-invariants are obtained by substituting (10.10) into the lifted invariants (10.9), leading to

$$K^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}} = \frac{n! \, [z_0 z_1 \ldots z_n]}{[z_0 z_1 z_2]^{(n+1)/2}}.$$

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In the limit, these reduce to the differential invariants $I^{(n)} = (u^{(1)})^{-(n+1)/2} u^{(n)}$, and so $K^{(n)}$ give the desired similarity-invariant, first order numerical approximations. To construct an invariant numerical scheme for any similarity-invariant ordinary differential equation

$$\Delta(x, u, u^{(1)}, u^{(2)}, \ldots u^{(n)}) = 0,$$

we merely invariantize the defining differential function, leading to the general similarity–invariant numerical approximation

$$\Delta(0, 0, 1, K^{(2)}, \ldots, K^{(n)}) = 0.$$

Example 10.3. For the action (6.4) of the proper Euclidean group of SE(2) on $M = \mathbb{R}^2$, the multi-prolonged action is free on $M^{(n)}$ for $n \geq 1$. We can thereby determine a first order multi-frame and use it to completely classify Euclidean multi-invariants. The first order transformation formulae are

$$y_0 = x_0 \cos\theta - u_0 \sin\theta + a, \qquad v_0 = x_0 \sin\theta + u_0 \cos\theta + b,$$

$$y_1 = x_1 \cos\theta - u_1 \sin\theta + a, \qquad v^{(1)} = \frac{\sin\theta + u^{(1)} \cos\theta}{\cos\theta - u^{(1)} \sin\theta}, \qquad (10.11)$$

where $u^{(1)} = [\, z_0 z_1 \,]$. Normalization based on the cross-section $y_0 = v_0 = v^{(1)} = 0$ results in the right moving frame

$$a = -x_0 \cos\theta + u_0 \sin\theta = -\frac{x_0 + u^{(1)} u_0}{\sqrt{1 + (u^{(1)})^2}},$$

$$\tan\theta = -u^{(1)}. \qquad (10.12)$$

$$b = -x_0 \sin\theta - u_0 \cos\theta = \frac{x_0 u^{(1)} - u_0}{\sqrt{1 + (u^{(1)})^2}},$$

Substituting the moving frame formulae (10.12) into the lifted divided differences results in a complete system of (oriented) Euclidean multi-invariants. These are easily computed by beginning with the fundamental joint invariants $I_k = (H_k, K_k) = \iota(x_k, u_k)$, where

$$y_k \longmapsto H_k = \frac{(x_k - x_0) + u^{(1)} (u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0) \frac{1 + [\, z_0 z_1 \,][\, z_0 z_k \,]}{\sqrt{1 + [\, z_0 z_1 \,]^2}},$$

$$v_k \longmapsto K_k = \frac{(u_k - u_0) - u^{(1)} (x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0) \frac{[\, z_0 z_k \,] - [\, z_0 z_1 \,]}{\sqrt{1 + [\, z_0 z_1 \,]^2}}.$$

The multi-invariants are obtained by forming divided difference quotients

$$[\, I_0 I_k \,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\, z_0 z_1 z_k \,]}{1 + [\, z_0 z_k \,][\, z_0 z_1 \,]},$$

where, in particular, $I^{(1)} = [\, I_0 I_1 \,] = 0$. The second order multi-invariant

$$I^{(2)} = 2 [\, I_0 I_1 I_2 \,] = 2 \frac{[\, I_0 I_2 \,] - [\, I_0 I_1 \,]}{H_2 - H_1} = \frac{2 [\, z_0 z_1 z_2 \,] \sqrt{1 + [\, z_0 z_1 \,]^2}}{\left(1 + [\, z_0 z_1 \,][\, z_1 z_2 \,] \right)\left(1 + [\, z_0 z_1 \,][\, z_0 z_2 \,] \right)}$$

$$= \frac{u^{(2)} \sqrt{1 + (u^{(1)})^2}}{\left[\, 1 + (u^{(1)})^2 + \tfrac{1}{2} u^{(1)} u^{(2)} (x_2 - x_0) \,\right]\left[\, 1 + (u^{(1)})^2 + \tfrac{1}{2} u^{(1)} u^{(2)} (x_2 - x_1) \,\right]}$$

provides a Euclidean–invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \to z_0} I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}} \, .$$

Similarly, the third order multi-invariant

$$I^{(3)} = 6 \, [\, I_0 I_1 I_2 I_3 \,] = 6 \, \frac{[\, I_0 I_1 I_3 \,] - [\, I_0 I_1 I_2 \,]}{H_3 - H_2}$$

will form a Euclidean–invariant approximation for the normalized differential invariant $\kappa_s = \iota(u_{xxx})$, the derivative of curvature with respect to arc length, [**29**, **52**].

To compare these with the invariant numerical approximations proposed in [**28**, **29**], we reformulate the divided difference formulae in terms of the geometrical configurations of the four distinct points z_0, z_1, z_2, z_3 on our curve. We find

$$H_k = \frac{(z_1 - z_0) \cdot (z_k - z_0)}{\| z_1 - z_0 \|} = r_k \cos \phi_k,$$

$$[\, I_0 I_k \,] = \tan \phi_k,$$

$$K_k = \frac{(z_1 - z_0) \wedge (z_k - z_0)}{\| z_1 - z_0 \|} = r_k \sin \phi_k,$$

where

$$r_k = \| z_k - z_0 \|, \qquad \phi_k = \sphericalangle(z_k - z_0, z_1 - z_0),$$

denotes the distance and the angle between the indicated vectors. Therefore,

$$I^{(2)} = \frac{2 \tan \phi_2}{r_2 \cos \phi_2 - r_1} \, ,$$

$$I^{(3)} = \frac{6(r_2 \cos \phi_2 - r_1) \tan \phi_3 - (r_3 \cos \phi_3 - r_1) \tan \phi_2}{(r_2 \cos \phi_2 - r_1)(r_3 \cos \phi_3 - r_1)(r_3 \cos \phi_3 - r_2 \cos \phi_2)} \, .$$

(10.13)

Interestingly, $I^{(2)}$ is *not* the same Euclidean approximation to the curvature that was used in [**29**, **28**]. The latter was based on the Heron formula for the radius of a circle through three points:

$$I^\star = \frac{4\Delta}{abc} = \frac{2 \sin \phi_2}{\| z_1 - z_2 \|} \, .$$

(10.14)

Here Δ denotes the area of the triangle connecting z_0, z_1, z_2 and

$$a = r_1 = \| z_1 - z_0 \|, \qquad b = r_2 = \| z_2 - z_0 \|, \qquad c = \| z_2 - z_1 \|,$$

are its side lengths. The ratio tends to a limit $I^\star / I^{(2)} \to 1$ as the points coalesce. The geometrical approximation (10.14) has the advantage that it is symmetric under permutations of the points; one can achieve the same thing by symmetrizing the divided difference version $I^{(2)}$. Furthermore, $I^{(3)}$ is an invariant approximation for the differential invariant κ_s, that, like the approximations constructed by Boutin, [**15**], converges properly for arbitrary spacings of the points on the curve.

In his thesis, [82, 84], Pilwon Kim developed the invariantization techniques to a variety of numerical integrators, e.g., Euler and Runge–Kutta, for ordinary differential equations with symmetry, with sometimes striking results. Given a symmetry group of an ordinary differential equation, we can apply the invariantization procedure to standard numerical integration schemes such as the Euler and the Runge–Kutta methods to derive invariantized numerical schemes that respect the symmetries. Invariantization under a well-chosen group has the effect of transforming the points at each step along the orbits of the symmetry group to the proper place where the numerical scheme works better. Since it is the symmetry group that acts on the points, the numerical scheme remains valid after the transformation. In this way we invariantize existing numerical schemes, not necessarily changing the mesh. The invariantization also can be applied to numerical methods for both ordinary differential equations and partial differential equations. Moreover this method works efficiently with symmetry groups that are more complicated than the similarity or scaling group.

In general, suppose $N_\Delta(z_1, \ldots, z_k)$ defines a numerical integration scheme for a differential equation (10.7). Given a group transformation g, we define the *g-transformed numerical scheme* as

$$N_\Delta^g(z_1, \ldots, z_k) = N_\Delta(g \cdot z_1, \ldots, g \cdot z_n).$$

If g defines a symmetry of the differential equation, in the sense that it maps solutions to solutions, [116], then it is not hard to see that N_Δ^g is also a numerical scheme for the differential equation.

Example 10.4. The elementary Euler method for the first order differential equation

$$\Delta(x, u, u_x) = u_x - f(x, u) = 0$$

is given by the function

$$N_\Delta(z_0, z_1) = u_1 - u_0 + (x_1 - x_0)f(x_0, u_0), \tag{10.15}$$

which is defined on the joint space $(\mathbb{R}^2)^{\diamond 2}$. Consider the previous one-parameter group

$$(\widetilde{x}, \widetilde{y}) = \varepsilon \cdot (x, y) = (x, y + \varepsilon e^x) \qquad \text{for all} \qquad \varepsilon \in \mathbb{R}. \tag{10.16}$$

The transformed Euler scheme is

$$\begin{aligned}
N_\Delta^\varepsilon(z_0, z_1) &= N_\Delta(\varepsilon \cdot z_0, \varepsilon \cdot z_1)) = N_\Delta(x_0, u_0 + \varepsilon e_0^x, x_1, u_1 + \varepsilon e_1^x) \\
&= (u_1 + \varepsilon e^{x_1}) - (u_0 + \varepsilon e^{x_0}) - (x_1 - x_0)f(x_0, u_0).
\end{aligned} \tag{10.17}$$

Suppose G is a symmetry group for a differential equation Δ, and let ρ be a moving frame for G. The invariantization of the numerical scheme N_Δ with respect to the moving frame ρ is given by

$$I_\Delta(z_1, \ldots, z_k) \equiv N_\Delta^{\rho(z)}(z_1, \ldots, z_k) = N_\Delta(\rho(z) \cdot z_1, \ldots, \rho(z) \cdot z_k).$$

This implies that at each step, we apply the numerical scheme after shifting the points to a fixed cross-section and map the result back to the original location. Note that to invariantize a k-step method, a moving frame is extended and applied to the joint space

$M^{\diamond(k+1)}$. In particular, the invariantization of (10.17) using the moving frame $\varepsilon = \rho(z) = -y_0 e^{-x_0}$ is

$$I_\Delta(z) = u_1 - u_0 e^{x_1 - x_0} - (x_1 - x_0) f(x_0, u_0).$$

The key to the success of the invariantized numerical scheme lies in the intelligent choice of cross-section for the moving frame. We usually set the dependent variables and/or some of their derivatives to zero. Even though the associated computations can become complicated, the more the symmetry group is prolonged, the more choices we have for a cross-section.

Unfortunately, invariantization by elementary symmetry groups has no effect. Every standard numerical scheme is already invariant with respect to the affine symmetry group $\widetilde{z} = A z + b$. However, as we will see below, affine symmetries can be still used to enhance the numerical scheme when combined with other nontrivial symmetry groups.

In the following examples, we concentrate on the fourth order Runge–Kutta method (RK) since is the most widely used single-step numerical scheme for ordinary differential equations. Implementation of the resulting invariantized Runge–Kutta schemes (IRK) is straightforward, and requires only a small number of lines to be added to existing numerical codes.

Example 10.5. The logistic equation

$$u_x = u \left(1 - \frac{u}{100} \right)$$

has the one-parameter symmetry group with infinitesimal generator $\mathbf{v} = e^{-x} u^2 \partial_u$. The corresponding prolonged group transformations are

$$(\widetilde{x}, \widetilde{u}, \widetilde{u}') = \left(x, \ \frac{u}{1 - \varepsilon e^{-x} u}, \ \frac{u_x - \varepsilon e^{-x} u^2}{(1 - \varepsilon e^{-x} u)^2} \right).$$

Setting $\widetilde{u_x} = 0$ gives the moving frame $\rho(x, u, u_x) = e^x u^{-2} u_x$ and therefore

$$\rho(x, u, u_x) \cdot (x, u, u_x) = \left(x, \ \frac{u^2}{u - u_x}, \ 0 \right).$$

Since the standard RK scheme involves $z_0 = (x_0, u_0, u_{x,0})$ and $z_1 = (x_1, u_1, u_{x,1})$, it is defined on the joint space $(\mathrm{J}^1)^{\diamond 2} \simeq (\mathbb{R}^3)^{\diamond 2}$. The previous moving frame is now extended and defined on the joint space as $\rho(z_0, z_1) = \rho(z_0)$, i.e., it depends only on the first point. The invariantized numerical scheme $\iota[N_\Delta]$ can be obtained by substitution

$$(x_0, \ u_0, \ u_{x,0}; \ x_1, \ u_1, \ u_{x,1}) \ \longmapsto$$
$$\left(x_0, \ \frac{u_0^2}{u_0 - u_{x,0}}, 0; \ x_1, \ \frac{u_0^2 u_1}{u_0^2 - e^{x_0 - x_1} u_1 u_{x,0}}, \ \frac{u_0^4 u_{x,1} - e^{x_0 - x_1} u_0^2 u_1^2 u_{x,0}}{(u_0^2 - e^{x_0 - x_1} u_1 u_{x,0})^2} \right).$$

As Figure 8 shows, the performance of invariantized RK is considerably better than that of the standard RK.
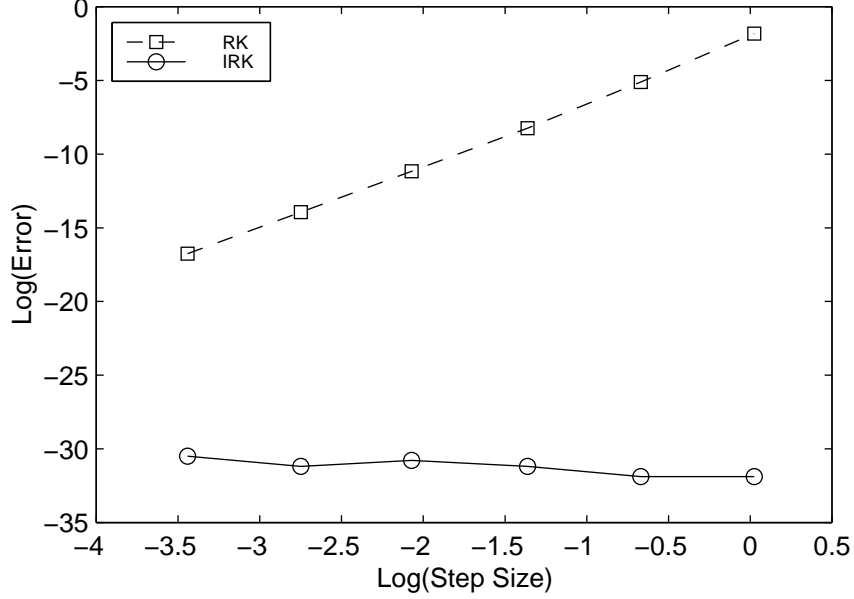
Figure 8.   Invariantized Runge–Kutta Schemes for the Logistic Equation.

Example 10.6. Ames's equation

$$u_{xx} = -\frac{u_x}{x} - e^u$$

is a stiff equation that arises in a wide range of fields, including kinetics and heat transfer, vortex motion of incompressible fluids, and the mass distribution of gaseous interstellar material under influence of its own gravitational fields, [**2**]; it is also known as the Frank-Kaminetskii equation, the Gel'fand equation, and the Barenblatt equation. The infinitesimal generators

$$\mathbf{v}_1 = -x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}, \qquad \mathbf{v}_2 = -\frac{1}{2}x\ln x\frac{\partial}{\partial x} + (1 + \ln x)\frac{\partial}{\partial u},$$

induce the prolonged one-parameter symmetry groups

$$(x, u, u_x) \longmapsto \begin{cases} (e^{\varepsilon_1}x, u + 2\varepsilon_1, e^{\varepsilon_1}u_x), \\ \left(e^{\ln xe^{-\frac{1}{2}\varepsilon_2}}, u + 2\ln x(1 - e^{-\frac{1}{2}\varepsilon_2}) + \varepsilon_2, \dfrac{-2e^{-\frac{1}{2}\varepsilon_2} + xu_x + 2}{e^{\ln xe^{-\frac{1}{2}\varepsilon_2 - \frac{1}{2}\varepsilon_2}}} \right). \end{cases}$$

The first is a scaling transformation group, which does not change the performance of the original scheme as mentioned above. The difficulty with the second one is that we cannot set $\widetilde{y}$ or $\widetilde{y}'$ zero. However, we can build a better transformation by proper combination of the two groups. Let $\rho_1(z_0; z_1) = \ln x_0$ and $\rho_2(z_0; z_1) = -u_0$. Through the successive applications of the two moving frames $\rho_1\rho_2$, every point (x, y) is projected to the cross-section $\widetilde{y} = 0$. The corresponding invariantized numerical scheme is written

$$I_\Delta(z) = (N_\Delta^{\rho_1})^{\rho_2}(z) = N_\Delta(\, \rho_2(\rho_1(z)\cdot z)\cdot(\rho_1(z)\cdot z)\,).$$

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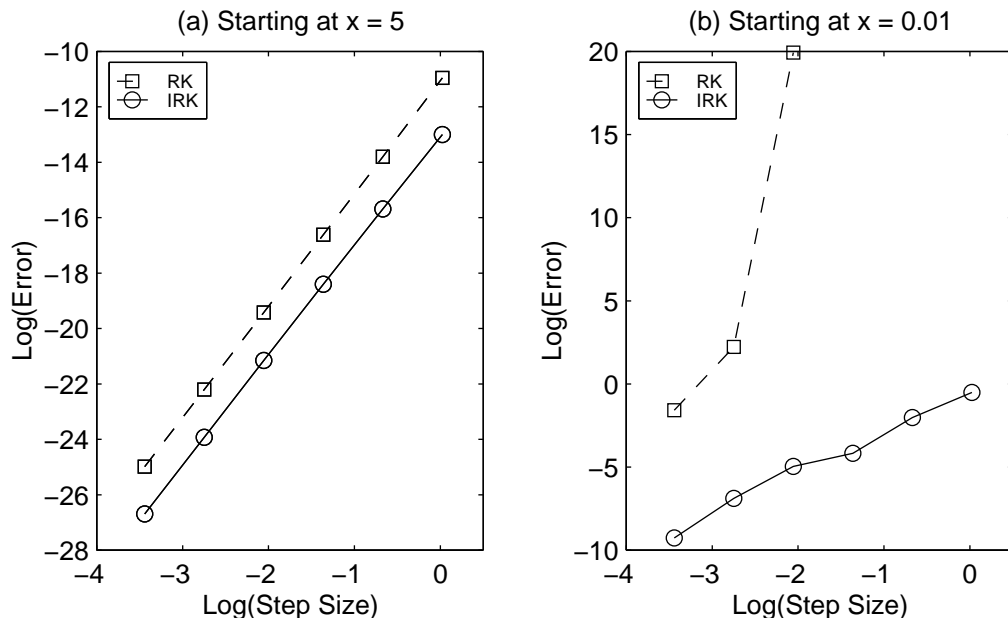
Figure 9.   Invariantized Runge–Kutta Schemes for Ames' Equation.

Figure 9 is the comparison between the RK and the IRK scheme when they start at $x = 5$. Even in this domain the performance of IRK exceeds RK, but more dramatic difference appears when they apply around $x = 0$. The invariantized Runge–Kutta method successfully avoids the equation's stiffness by preserving the equation's geometric structure.

Extensions to partial differential equations are under development. In [**83**], Kim develops an invariantized Crank-Nicolson scheme for Burgers' equation that avoids problems with numerical oscillations near sharp transition regions. In [**155**], the authors develop invariant schemes for nonlinear partial differential equations of use in image processing, including the Hamilton–Jacobi equation.

11.   The Invariant Bicomplex.

Let us return to the case of prolonged group actions on jet space and develop some further machinery required in the more advanced applications of moving frames to differential invariants, differential equations, and the calculus of variations. The full power of the equivariant construction becomes evident once we incorporate the contact structure and induced variational bicomplex on the infinite order jet bundle $J^\infty = J^\infty(M, p)$, which we now review, [**3, 117**].

Separating the local coordinates $(x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ on M into independent and dependent variables naturally splits† the differential one-forms on J^∞ into *horizontal forms*, spanned by $dx^1, \ldots, dx^p$, and *vertical forms*, spanned by the *basic contact*

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† The splitting, which depends on the choice of local coordinates, only works at infinite order, which is the reason we work on J^∞.

one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{11.1}$$

Let π_H and π_V denote the projections mapping one-forms on J^∞ to their horizontal and vertical (contact) components, respectively. We accordingly decompose the differential $d = \pi_H \circ d + \pi_V \circ d = d_H + d_V$, which results in the *variational bicomplex* on J^∞. If $F(x, u^{(n)})$ is any differential function, its horizontal differential is

$$d_H F = \sum_{i=1}^{p} (D_i F) \, dx^i, \tag{11.2}$$

in which $D_i = D_{x^i}$ denote the usual total derivatives with respect to the independent variables. Thus, $d_H F$ can be identified with the "total gradient" of F. Similarly, its vertical differential is

$$d_V F = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha} \theta_J^\alpha = \sum_{\alpha, J} \frac{\partial F}{\partial u_J^\alpha} D_J \theta^\alpha = \mathrm{D}_F(\theta), \tag{11.3}$$

in which the total derivatives act as Lie derivatives on the contact forms $\theta = (\theta^1, \ldots, \theta^q)^T$, and D_F denotes the *formal linearization operator* or *Fréchet derivative* of the differential function F. Thus, the vertical differential $d_V F$ can be identified[†] with the (first) *variation*, hence the name "variational bicomplex".

Let $\pi_n : \mathrm{J}^\infty \to \mathrm{J}^n$ be the natural jet space projections. Choosing a cross-section $K^n \subset \mathcal{V}^n \subset \mathrm{J}^n$, we extend the induced n^{th} order moving frame $\rho^{(n)}$ to the infinite jet bundle by setting $\rho(x, u^{(\infty)}) = \rho^{(n)}(x, u^{(n)})$ whenever $(x, u^{(n)}) = \pi_n(x, u^{(\infty)})$ lies in the domain of definition of $\rho^{(n)}$. We will employ our moving frame to *invariantize* the variational bicomplex. As before, the invariantization of a differential form is the unique invariant differential form that agrees with its progenitor on the cross-section. In particular, the invariantization process does not affect invariant differential forms. In practice, one determines the invariantization by first transforming the differential form by the prolonged group action and then substituting the moving frame formulae for the group parameters.

As in (6.18), the fundamental differential invariants are obtained by invariantizing the jet coordinates: $H^i = \iota(x^i)$, $I_J^* {}_\alpha = \iota(u_J^\alpha)$. Let

$$\varpi^i = \iota(dx^i) = \omega^i + \eta^i, \qquad \text{where} \qquad \omega^i = \pi_H(\varpi^i), \quad \eta^i = \pi_V(\varpi^i), \tag{11.4}$$

denote the *invariantized horizontal one-forms*. Their horizontal components $\omega^1, \ldots, \omega^p$ prescribe, in the language of [**117**], a contact-invariant coframe for the prolonged group action, while the contact forms $\eta^1, \ldots, \eta^p$ are required to make $\varpi^1, \ldots, \varpi^p$ fully G-invariant. Finally, the *invariantized basis contact forms* are denoted by

$$\vartheta_J^\alpha = \iota(\theta_J^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{11.5}$$

[†] This becomes clearer when you rewrite $\theta_J^\alpha = \delta u_J^\alpha$.

Invariantization of more general differential forms relies on the fact that it preserves the exterior algebra structure, and so

$$\iota(\Omega + \Psi) = \iota(\Omega) + \iota(\Psi), \qquad \iota(\Omega \wedge \Psi) = \iota(\Omega) \wedge \iota(\Psi), \tag{11.6}$$

for any differential forms (or functions) Ω, Ψ on J^∞.

As in the ordinary bicomplex construction, the decomposition of invariant one-forms on J^∞ into invariant horizontal and invariant contact components induces a decomposition of the differential. However, now $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$ splits into three constituents, where $d_{\mathcal{H}}$ adds an invariant horizontal form, $d_{\mathcal{V}}$ adds a invariant contact form, while $d_{\mathcal{W}}$ replaces an invariant horizontal one-form with a combination of wedge products of two invariant contact forms. They satisfy the "quasi-tricomplex" identities

$$\begin{aligned} d_{\mathcal{H}}^2 &= 0, & d_{\mathcal{H}} d_{\mathcal{V}} + d_{\mathcal{V}} d_{\mathcal{H}} &= 0, \\ d_{\mathcal{W}}^2 &= 0, & d_{\mathcal{V}} d_{\mathcal{W}} + d_{\mathcal{W}} d_{\mathcal{V}} &= 0, \end{aligned} \qquad d_{\mathcal{V}}^2 + d_{\mathcal{H}} d_{\mathcal{W}} + d_{\mathcal{W}} d_{\mathcal{H}} = 0. \tag{11.7}$$

Interestingly, this same structure also arises in the study of the topology of foliations, [**146**]. Fortunately, the third, anomalous component $d_{\mathcal{W}}$ plays no role (to date) in the applications; in particular, $d_{\mathcal{W}} F = 0$ for any differential function F. Even better, if the group acts projectably, $d_{\mathcal{W}} \equiv 0$. The corresponding dual invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are then defined so that

$$d_{\mathcal{H}} F = \sum_{i=1}^p (\mathcal{D}_i F) \, \varpi^i, \qquad d_{\mathcal{H}} \Omega = \sum_{i=1}^p \varpi^i \wedge \mathcal{D}_i \, \Omega, \tag{11.8}$$

for any differential function F and, more generally, differential form Ω, on which the $\mathcal{D}_i$ act via Lie differentiation. Keep in mind that, in general, the invariant differential operators do not commute; see (6.23) below.

The most important fact underlying the moving frame construction is that, while it does preserve algebraic structure, the invariantization map ι does *not* respect the differential. The *recurrence formulae*, [**52**, **87**], which we now review, provide the missing "correction terms", i.e., $d\iota(\Omega) - \iota(d\Omega)$. Remarkably, they can be explicitly and algorithmically constructed using merely linear differential algebra — without knowing the explicit formulas for either the differential invariants or invariant differential forms, the invariant differential operators, or even the moving frame!

With this in hand, we can formulate the *universal recurrence formula*.

Theorem 11.1. *If Ω is any differential function or form on J^∞, then*

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^r \nu^\kappa \wedge \iota\,[\mathbf{v}_\kappa(\Omega)], \tag{11.9}$$

where $\nu^1, \ldots, \nu^r$ are the invariantized Maurer–Cartan forms *dual to the infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_r$, while $\mathbf{v}_\kappa(\Omega)$ denotes the corresponding Lie derivative of Ω.*

In general, the invariantized Maurer–Cartan forms are obtained by pulling back the dual Maurer–Cartan forms $\mu^1, \ldots, \mu^r$ on G via the moving frame map: $\nu^\kappa = \rho^* \mu^\kappa$. The full details, [**87**], are, fortunately, not required thanks to the following marvelous result that allows us to compute them directly without reference to their underlying definition:

Proposition 11.2. *Let* $\mathcal{K} = \{ Z_1(x, u^{(n)}) = c_1, \ldots, Z_r(x, u^{(n)}) = c_r \}$ *be the cross-section defining our moving frame, so that* $c_\lambda = \iota(Z_\lambda)$ *are the phantom differential invariants. Then the corresponding* phantom recurrence formulae

$$0 = d\iota(Z_\lambda) = \iota(dZ_\lambda) + \sum_{\kappa=1}^r \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(Z_\lambda)], \qquad \lambda = 1, \ldots, r, \tag{11.10}$$

can be uniquely solved for the invariantized Maurer–Cartan forms:

$$\nu^\kappa = \sum_{i=1}^p R_i^\kappa \, \varpi^i \ + \ \sum_{\alpha,J} S_\alpha^{\kappa,J} \, \vartheta_J^\alpha, \tag{11.11}$$

where $R_i^\kappa, S_\alpha^{\kappa,J}$ *are certain differential invariants.*

The R_i^κ are called the *Maurer–Cartan invariants*, [**70**, **124**]. In the case of curves, $p = 1$, they are the entries of the Frenet–Serret matrix $\mathcal{D}\rho^{(n)}(x, u^{(n)}) \cdot \rho^{(n)}(x, u^{(n)})^{-1}$, cf. [**61**].

Substituting (11.11) into the universal formula (11.9) produces a complete system of explicit recurrence relations for all the differentiated invariants and invariant differential forms. In particular, taking Ω to be any one of the individual jet coordinate functions x^i, u_J^α, results in the recurrence formulae for the fundamental differential invariants (6.18):

$$\mathcal{D}_i H^j = \delta_i^j + \sum_{\kappa=1}^r R_i^\kappa \, \iota(\xi_\kappa^i), \qquad \mathcal{D}_i I_J^\alpha = I_{Ji}^\alpha + \sum_{\kappa=1}^r R_i^\kappa \, \iota(\varphi_{J,\kappa}^\alpha), \tag{11.12}$$

where δ_i^j is the usual Kronecker delta, and $\xi_\kappa^i, \varphi_{J,\kappa}^\alpha$ are the coefficients of the prolonged infinitesimal generators (4.4). Owing to the functional independence of the non-phantom differential invariants, these formulae, in fact, serve to completely prescribe the structure of the non-commutative differential invariant algebra engendered by G, [**52**, **69**, **124**].

Similarly, the recurrence formulae (11.9) for the invariant horizontal forms are

$$d\varpi^i = d[\iota(dx^i)] = \iota(d^2 x^i) + \sum_{\kappa=1}^r \nu^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= \sum_{\kappa=1}^r \sum_{k=1}^p \iota\left(D_k \xi_\kappa^i\right) \nu^\kappa \wedge \varpi^k + \sum_{\kappa=1}^r \sum_{\alpha=1}^q \iota\left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha}\right) \nu^\kappa \wedge \vartheta^\alpha. \tag{11.13}$$

The terms in (11.13) involving wedge products of two horizontal forms are

$$d_\mathcal{H} \, \varpi^i = - \sum_{j<k} Y_{jk}^i \, \varpi^j \wedge \varpi^k,$$

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where

$$Y_{jk}^i = - Y_{kj}^i = \sum_{\kappa=1}^r \sum_{j=1}^p \left[R_k^\kappa \, \iota(D_j \xi_\kappa^i) - R_j^\kappa \, \iota(D_k \xi_\kappa^i) \right] \tag{11.14}$$

are called the *commutator invariants*, since they prescribe the commutators of the invariant differential operators, cf. (6.23) The terms in (11.13) involving wedge products of a horizontal and a contact form yield

$$d_{\mathcal{V}} \, \varpi^i = \sum_{\kappa=1}^r \left[\sum_{\alpha=1}^q \iota \left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha} \right) R_j^\kappa \, \varpi^i \wedge \vartheta^\alpha - \sum_{k=1}^p \iota(D_k \xi_\kappa^i) \, S_\alpha^{\kappa,J} \, \varpi^k \wedge \vartheta_J^\alpha \right]. \tag{11.15}$$

Finally, the remaining terms, involving wedge products of two contact forms, provide the formulas for the anomalous third component of the differential:

$$d_{\mathcal{W}} \, \varpi^i = \sum_{\kappa=1}^r \sum_{\alpha=1}^q \iota \left(\frac{\partial \xi_\kappa^i}{\partial u^\alpha} \right) S_\alpha^{\kappa,J} \, \vartheta_J^\alpha \wedge \vartheta^\alpha. \tag{11.16}$$

In a similar fashion, we derive the recurrence formulae (11.9) for the differentiated invariant contact forms: In particular, the horizontal components

$$\mathcal{D}_i \vartheta_J^\alpha = \vartheta_{Ji}^\alpha + \sum_{\kappa=1}^r R_i^\kappa \, \iota\big(\mathbf{v}_\kappa(\theta_J^\alpha) \big). \tag{11.17}$$

can be inductively solved to express the higher order invariantized contact forms as certain invariant derivatives of those of order 0:

$$\vartheta_J^\alpha = \mathcal{E}_J^\alpha(\vartheta) = \sum_{\beta=1}^q \mathcal{E}_{J,\beta}^\alpha(\vartheta^\beta), \tag{11.18}$$

in which $\vartheta = (\vartheta^1, \ldots, \vartheta^q)^T$ denotes the column vector containing the order zero invariantized contact forms, while $\mathcal{E}_J^\alpha = (\mathcal{E}_{J,1}^\alpha, \ldots, \mathcal{E}_{J,q}^\alpha)$ is a row vector of invariant differential operators, i.e., each $\mathcal{E}_{J,\alpha} = \sum A_{J,\alpha}^K \mathcal{D}^K$ for certain differential invariants $A_{J,\alpha}^K$.

Combining these formulae allows us to express the invariant vertical derivative or *invariant variation* of any differential invariant K in the form

$$d_{\mathcal{V}} \, K = \mathcal{A}_K(\vartheta), \tag{11.19}$$

in which $\mathcal{A}_K$ is a row vector of invariant differential operators. Formula (11.19) can be viewed as the invariant version of the vertical differentiation formula (11.3), and so will refer to $\mathcal{A}_K$ as the *invariant linearization operator* associated with the differential invariant K. Similarly, we derive the recurrence formulae for the vertical differentials of the invariant horizontal forms:

$$d_{\mathcal{V}} \, \varpi^i = \sum_{j=1}^p \sum_{\alpha=1}^q \mathcal{B}_{j\alpha}^i(\vartheta^\alpha) \wedge \varpi^j = \sum_{j=1}^p \mathcal{B}_j^i(\vartheta) \wedge \varpi^j \tag{11.20}$$

in which $\mathcal{B}_j^i = (\mathcal{B}_{j1}^i, \ldots, \mathcal{B}_{jq}^i)$ is a family of p^2 row-vector-valued invariant differential operators, known, collectively, as the *invariant Hamiltonian operator complex*, stemming from its role in the calculus of variations, cf. [**87**, **135**].

Example 11.3. Let us return to the Euclidean group acting on plane curves initiated in Example 6.3. The basic invariant horizontal one-form $\varpi = \iota(dx)$ is obtained by first transforming dx by a general group element:

$$dx \ \longmapsto \ dy = (\cos\phi - u_x \sin\phi)\, dx + (\sin\phi)\theta, \tag{11.21}$$

where

$$\theta = du - u_x\, dx, \qquad \theta_x = du_x - u_{xx}\, dx, \qquad \ldots, \tag{11.22}$$

are the ordinary basis contact forms. Substituting the moving frame formulae (6.8) for the group parameters into (11.21) yields the basic invariant horizontal one-form

$$\varpi = \iota(dx) = \frac{dx + u_x\, du}{\sqrt{1 + u_x^2}} = \sqrt{1 + u_x^2}\; dx + \frac{u_x}{\sqrt{1 + u_x^2}}\, \theta. \tag{11.23}$$

Its (non-invariant) horizontal component is the contact-invariant arc length form

$$\omega = \pi_H(\varpi) = ds = \sqrt{1 + u_x^2}\; dx,$$

and so the corresponding invariant differential operator is the usual arc length derivative $\mathcal{D} = D_s$. In the same manner we obtain the basis invariant contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{\sqrt{1 + u_x^2}}, \qquad \vartheta_1 = \iota(\theta_x) = \frac{(1 + u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1 + u_x^2)^2}, \qquad \ldots\,. \tag{11.24}$$

To construct the recurrence formulae for the differentiated functions and forms, we begin with the prolonged infinitesimal generators of $SE(2)$:

$$\mathbf{v}_1 = \partial_x, \qquad\qquad \mathbf{v}_2 = \partial_u,$$
$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + (4u_x u_{xxx} + 3u_{xx}^2)\,\partial_{u_{xxx}} + \cdots.$$

The pulled back dual Maurer–Cartan forms ν^1, ν^2, ν^3 are found by applying the universal recurrence formulae (11.9) to the phantom invariants:

$$0 = dH = \iota(dx) + \iota(\mathbf{v}_1(x))\,\nu^1 + \iota(\mathbf{v}_2(x))\,\nu^2 + \iota(\mathbf{v}_3(x))\,\nu^3 = \varpi + \nu^1,$$
$$0 = dI_0 = \iota(du) + \iota(\mathbf{v}_1(u))\,\nu^1 + \iota(\mathbf{v}_2(u))\,\nu^2 + \iota(\mathbf{v}_3(u))\,\nu^3 = \vartheta + \nu^2,$$
$$0 = dI_1 = \iota(du_x) + \iota(\mathbf{v}_1(u_x))\,\nu^1 + \iota(\mathbf{v}_2(u_x))\,\nu^2 + \iota(\mathbf{v}_3(u_x))\,\nu^3 = \kappa\,\varpi + \vartheta_1 + \nu^3,$$

since $du = u_x\, dx + \theta$, $du_x = u_{xx}\, dx + \theta_x$. Therefore,

$$\nu^1 = -\varpi, \qquad \nu^2 = -\vartheta, \qquad \nu^3 = -\kappa\,\varpi - \vartheta_1. \tag{11.25}$$

We are now ready to substitute the non-phantom invariants into (11.9):

$$dκ = d\iota(u_{xx}) = \iota(du_{xx}) + \iota(\mathbf{v}_1(u_{xx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xx}))\,\nu^3$$
$$= \iota(u_{xxx}\, dx + \theta_{xx}) - \iota(3u_x u_{xx})\,(\kappa\,\varpi + \vartheta_1) = I_3\varpi + \vartheta_2,$$
$$dI_3 = d\iota(u_{xxx}) = \iota(du_{xxx}) + \iota(\mathbf{v}_1(u_{xxx}))\,\nu^1 + \iota(\mathbf{v}_2(u_{xxx}))\,\nu^2 + \iota(\mathbf{v}_3(u_{xxx}))\,\nu^3$$
$$= \iota(u_{xxxx}\, dx + \theta_{xxx}) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,(\kappa\,\varpi + \vartheta_1) = (I_4 - 3\kappa^3)\varpi + \vartheta_3 - 3\kappa^2\vartheta_1,$$

and so on. Breaking these formulas into their horizontal and vertical components yields

$$I_3 = \mathcal{D}\kappa = \kappa_s, \qquad\qquad d_{\mathcal{V}}\kappa = \vartheta_2,$$
$$I_4 = \mathcal{D}I_3 + 3\kappa^3 = \kappa_{ss} + 3\kappa^3, \qquad d_{\mathcal{V}}I_3 = d_{\mathcal{V}}\kappa_s = \vartheta_3 - 3\kappa^2\vartheta_1. \tag{11.26}$$

To proceed further, we compute the differentials of the invariant contact forms, again using (11.9, 25):

$$d\vartheta = \iota(d\theta) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta)) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta)) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta))$$
$$= \iota(dx \wedge \theta_x) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,\theta) = \varpi \wedge \vartheta_1,$$
$$d\vartheta_1 = \iota(d\theta_x) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta_x)) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta_x)) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta_x))$$
$$= \iota(dx \wedge \theta_{xx}) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(2\,u_x\theta_x + u_{xx}\theta) = \varpi \wedge (\vartheta_2 - \kappa^2\,\vartheta) - \kappa\,\vartheta_1 \wedge \vartheta,$$
$$d\vartheta_2 = \iota(d\theta_{xx}) + \nu^1 \wedge \iota(\mathbf{v}_1(\theta_{xx})) + \nu^2 \wedge \iota(\mathbf{v}_2(\theta_{xx})) + \nu^3 \wedge \iota(\mathbf{v}_3(\theta_{xx}))$$
$$= \iota(dx \wedge \theta_{xxx}) - (\kappa\,\varpi + \vartheta_1) \wedge \iota(3\,u_x\theta_{xx} + 3\,u_{xx}\theta_x + u_{xxx}\theta)$$
$$= \varpi \wedge (\vartheta_3 - 3\kappa^2\,\vartheta_1 - \kappa\,\kappa_s\vartheta) - \kappa_s\vartheta_1 \wedge \vartheta,$$

and so on. Concentrating on the terms involving the invariant horizontal form and comparing with (11.8), we deduce

$$\vartheta_1 = \mathcal{D}\vartheta, \qquad \vartheta_2 = \mathcal{D}\vartheta_1 + \kappa^2\,\vartheta = (\mathcal{D}^2 + \kappa^2)\,\vartheta,$$
$$\vartheta_3 = \mathcal{D}\vartheta_2 + 3\kappa^2\vartheta_1 + \kappa\,\kappa_s\vartheta = (\mathcal{D}^3 + 4\kappa^2\mathcal{D} + 3\kappa\,\kappa_s)\,\vartheta.$$

Substituting back into (11.26), we find

$$d_{\mathcal{V}}\kappa = (\mathcal{D}^2 + \kappa^2)\,\vartheta, \qquad d_{\mathcal{V}}\kappa_s = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\kappa\,\kappa_s)\,\vartheta.$$

Thus, the invariant linearization operators for the curvature and its arc length derivative are

$$\mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{A}_{\kappa_s} = \mathcal{D}^3 + \kappa^2\mathcal{D} + 3\kappa\,\kappa_s. \tag{11.27}$$

Finally, applying (11.9) and (11.25) to the invariant arc length form $\varpi = \iota(dx)$ yields

$$d\varpi = \iota(d^2x) + \nu^1 \wedge \iota(\mathbf{v}_1(dx)) + \nu^2 \wedge \iota(\mathbf{v}_2(dx)) + \nu^3 \wedge \iota(\mathbf{v}_3(dx))$$
$$= (\kappa\,\varpi + \vartheta_1) \wedge \iota(u_x\,dx + \theta) = \kappa\,\varpi \wedge \vartheta + \vartheta_1 \wedge \vartheta.$$

Therefore,

$$d_{\mathcal{V}}\varpi = -\kappa\,\vartheta \wedge \varpi, \qquad \text{and so} \qquad \mathcal{B} = -\kappa \tag{11.28}$$

is the invariant Hamiltonian operator.

12. Generating Differential Invariants.

Let us now apply the recurrence formulae to study the structure of the differential invariant algebra associated with the prolonged group action. A set of differential invariants $\mathcal{I} = \{I_1, \ldots, I_k\}$ is said to be *generating* if, locally, every differential invariant can be expressed as a function of the generators and their iterated invariant derivatives $\mathcal{D}_J I_\nu$. Let

$$\mathcal{I}^{(n)} = \{H^1, \ldots, H^p\} \cup \{\, I^*\alpha_J \mid \alpha = 1, \ldots, q,\ \#J \le n \,\} \tag{12.1}$$

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denote the entire set of fundamental differential invariants (6.18) of order $\leq n$. In particular, assuming we choose a cross-section that projects to a cross-section on M, then $\mathcal{I}^{(0)} = \{ H^1, \ldots, H^p, I^1, \ldots I^q \}$ are the ordinary invariants for the action of G on M. If, as in the examples treated here, G acts transitively on M, the normalized order 0 invariants are all constant, and hence are superfluous in any generating systems.

The first result is a direct consequence of the recurrence formulae (11.12) for the fundamental differential invariants and the fact that the Maurer–Cartan invariants, being solutions to the phantom recurrence relations, have order bounded by that of the moving frame.

Theorem 12.1. *If the moving frame has order n, then the set of normalized differential invariants $\mathcal{I}^{(n+1)}$ of order $\leq n + 1$ forms a generating set.*

Almost all applications rely on a cross-section $K^n \subset \mathrm{J}^n$ of *minimal order*, which means that its projections $K^k = \pi^n_k(K^n) \subset \mathrm{J}^k$ form cross-sections for all $0 \leq k < n$. In this case, one can significantly reduce the set of required generators, [**69**, **124**]:

Theorem 12.2. *If $K^n = \{ Z_1(x, u^{(n)}) = c_1, \ldots, Z_r(x, u^{(n)}) = c_r \}$ is a minimal order cross-section, then $\mathcal{I}^{(0)} \cup \mathcal{Z}^{(1)}$, where $\mathcal{Z}^{(1)} = \{ \iota(\mathcal{D}_i(Z_j)) \,|\, 1 \leq i \leq p, \ 1 \leq j \leq r \}$, form a generating set of differential invariants.*

The result is false in general if the cross-section is not minimal, [**124**]. An alternative interesting generating system was found in [**70**]; again, the proof is entirely based on the recurrence formulae.

Theorem 12.3. *Let $\mathcal{R} = \{ R^i_a \,|\, 1 \leq i \leq p, \ 1 \leq a \leq r \}$ be the Maurer–Cartan invariants. Then $\mathcal{I}^{(0)} \cup \mathcal{R}$ form a generating system.*

In both cases, the $\mathcal{I}^{(0)}$ constituent can be omitted if G acts transitively on M. The preceding generating sets are rarely minimal. For curves, where $p = 1$, under mild restrictions on the group action (specifically transitivity and no pseudo-stabilization under prolongation), there are exactly $m - 1$ independent generating differential invariants, and any other differential invariant is a function of the generating invariants and their successive derivatives with respect to the G-invariant arc length element. Thus, for instance, the differential invariants of a space curve $C \subset \mathbb{R}^3$ under the standard action of the Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ are generated by $m - 1 = 2$ differential invariants, namely its curvature and torsion.

For higher dimensional submanifolds, the minimal number of generating differential invariants cannot be fixed a priori, but depends the particularities of the group action and, in fact, can be arbitrarily large, even for surfaces in three-dimensional space, [**124**]. Even in very well-studied, classical situations, there are interesting subtleties that have not been noted before, [**73**, **125**].

Example 12.4. Consider the standard action of the special Euclidean group $\mathrm{SE}(3)$ on surfaces $S \subset \mathbb{R}^3$. The classical moving frame construction, [**61**; Chapter 10], or its equivariant reformulation, [**87**; Example 9.9], relies on the cross-section

$$ x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} \neq u_{yy}. \tag{12.2} $$

The two basic differential invariants are the principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy}), \qquad (12.3)$$

or, equivalently, the *mean curvature* and *Gauss curvature*

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1\kappa_2. \qquad (12.4)$$

The surface admits a classical moving frame provided we are at a non-umbilic point, where $\kappa_1 \neq \kappa_2$. (At a non-degenerate umbilic, one could, in principle, employ a higher order moving frame.) The corresponding invariant horizontal coframe $\varpi^1 = \iota(dx)$, $\varpi^2 = \iota(dy)$, can be identified with the diagonalizing Frenet frame on the surface, [**61**]. We let $\mathcal{D}_1, \mathcal{D}_2$ denote the dual invariant differential operators.

Let $I_{jk} = \iota(u_{jk})$ denote the higher order normalized differential invariants, so $I_{20} = \kappa_1$, $I_{11} = 0$, $I_{02} = \kappa_2$. The third order recurrence relations are readily found:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2}. \quad (12.5)$$

The two fourth order recurrence relations for

$$I_{22} = \mathcal{D}_2 I_{21} + \frac{I_{30}I_{12} - 2I_{12}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \mathcal{D}_1 I_{12} - \frac{I_{21}I_{03} - 2I_{21}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2$$

imply the celebrated *Codazzi syzygy*

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\kappa_{2,1} + \kappa_{1,2}\kappa_{2,2} - 2\kappa_{2,1}^2 - 2\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\kappa_2(\kappa_1 - \kappa_2) = 0. \qquad (12.6)$$

The well-known fact that the principal curvatures κ_1, κ_2, or, equivalently, the Gauss and mean curvatures H, K, form a generating system follows from Theorem 12.1 combined with (12.5). Remarkably, as we now show, neither is a minimal generating set!

Applying the moving frame machinery, the recurrence relations for the invariant horizontal forms are found to be

$$d_{\mathcal{H}}\varpi^1 = Y_2\,\varpi^1 \wedge \varpi^2, \qquad \text{where} \qquad Y_1 = \frac{\kappa_{2,1}}{\kappa_1 - \kappa_2}, \qquad Y_2 = \frac{\kappa_{1,2}}{\kappa_2 - \kappa_1}, \qquad (12.7)$$
$$d_{\mathcal{H}}\varpi^2 = Y_1\,\varpi^1 \wedge \varpi^2,$$

are the commutator invariants, which appear in Guggenheimer's proof of the fundamental existence theorem for Euclidean surfaces, [**61**; p. 234]. Note that the denominator in (12.7) vanishes at umbilic points on the surface, where the principal curvatures coincide $\kappa^1 = \kappa^2$, and the moving frame is not valid. The invariant differential operators therefore satisfy the commutation relation

$$\left[\mathcal{D}_1, \mathcal{D}_2\right] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1 = Y_2\mathcal{D}_1 - Y_1\mathcal{D}_2. \qquad (12.8)$$

An easy computation shows that the Codazzi syzygy (12.6) can be written compactly as

$$K = \kappa_1\kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2. \qquad (12.9)$$

which is the key identity employed by Guggenheimer, [**61**], for a short proof of Gauss' Theorema Egregium.

Let us now show how, for suitably nondegenerate surfaces, we can write the Gauss curvature K as a universal rational combination of the invariant derivatives of the mean curvature H. In view of the Codazzi formula (12.9), it suffices to write the commutator invariants Y_1, Y_2 in terms of the mean curvature. To this end, we note that the commutator identity (12.8) can be applied to any differential invariant. In particular,

$$\mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H = Y_2 \, \mathcal{D}_1 H - Y_1 \, \mathcal{D}_2 H, \tag{12.10}$$

and, furthermore, for $j = 1$ or 2,

$$\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_j H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_j H = Y_2 \, \mathcal{D}_1 \mathcal{D}_j H - Y_1 \, \mathcal{D}_2 \mathcal{D}_j H. \tag{12.11}$$

Provided the nondegeneracy condition

$$(\mathcal{D}_1 H)(\mathcal{D}_2 \mathcal{D}_j H) \neq (\mathcal{D}_2 H)(\mathcal{D}_1 \mathcal{D}_j H), \qquad \text{for } j = 1 \text{ or } 2, \tag{12.12}$$

holds, we can solve (12.10–11) to write the commutator invariants Y_1, Y_2 as rational functions of invariant derivatives of H. Plugging these expressions into the right hand side of the Codazzi identity (12.9) produces an explicit formula for the Gauss curvature as a rational function of the invariant derivatives, of order ≤ 4, of the mean curvature, valid for all surfaces satisfying the nondegeneracy condition (12.12).

In [**125**] it was also proved that, for suitably generic surfaces in $\mathbb{R}^3$, the algebra of equi-affine differential invariants is generated by the third order Pick invariant alone through invariant differentiation. In [**73**] it was proved that the algebras of conformal and projective differential invariants are also both generated by a single differential invariant.

13.   Invariant Variational Problems.

As first recognized by Sophus Lie, [**94**], every invariant variational problem can be written in terms of the differential invariants of the symmetry group. The associated Euler-Lagrange equations automatically inherit the symmetry group of the variational problem, and so can also be written in terms of the differential invariants, [**116**]. The formula for directly constructing the differential invariant form of the Euler–Lagrange equations from that of the variational problem was only known in a handful of particular cases, [**3**, **60**], until, applying the invariant variational bicomplex machinery, the general version was established in [**87**]. Recent applications to the equilibrium configurations of flexible Möbius bands can be found in [**145**].

Let us begin by recalling how variational problems $\mathcal{L}[u] = \int L(x, u^{(n)}) \, d\mathbf{x}$ appear in the variational bicomplex, [**3**]. The integrand or *Lagrangian form*

$$\lambda = L(x, u^{(n)}) \, d\mathbf{x} = L(x, u^{(n)}) \, dx^1 \wedge \ \cdots \ \wedge dx^p, \tag{13.1}$$

is a differential form on J^∞ of type $(p, 0)$, meaning that it involves p horizontal forms and no contact forms. Classically, to compute the associated Euler–Lagrange equations, one begins with the first variation, followed by an integration by parts. According to (11.3), we identify the first variation with the vertical differential $d_V \lambda = d_V L \wedge d\mathbf{x}$ of the Lagrangian form, which is a form of type $(p, 1)$. Integration by parts can be viewed as quotienting

out by the image of the horizontal differential, so $\omega \equiv \widetilde{\omega}$ whenever $\omega - \widetilde{\omega} = d_H \psi$ for some differential form ψ. The induced equivalence classes are represented by *source forms*

$$\omega = \sum_{\alpha=1}^{q} \Delta_\alpha(x, u^{(n)})\, \theta^\alpha \wedge d\mathbf{x}, \tag{13.2}$$

whose vanishing defines a system of differential equations: $\Delta_\alpha(x, u^{(n)}) = 0$. In the case of a variational problem, $\Delta_\alpha = E_\alpha(L) = 0$ are the classical Euler–Lagrange equations.

The Lagrangian of a G-invariant variational problem can be written in the invariant form

$$\lambda = \widetilde{L}(I^{(n)})\, \omega^1 \wedge \cdots \wedge \omega^p,$$

where $\omega^1, \ldots, \omega^p$ denote the contact invariant coframe induced by the moving frame, (11.4), while $\widetilde{L}(I^{(n)})$ is a function of the generating differential invariants $I = (I^1, \ldots, I^l)$ and their invariant derivatives $\mathcal{D}_J I^\kappa$ up to some finite order $\#J \leq k$. Since they differ by contact forms (which vanish when evaluated on submanifold jets), we do not affect anything by replacing the ω^i by their fully invariant counterparts ϖ^i, and so will use the fully *invariant Lagrangian form*

$$\widetilde{\lambda} = \widetilde{L}(I^{(n)})\, \varpi^1 \wedge \cdots \wedge \varpi^p \tag{13.3}$$

in our subsequent computations. To find the invariant form of the Euler–Lagrange equations, we first compute the invariant variation $d_\mathcal{V} \widetilde{\lambda}$, followed by an invariant integration by parts. Two new complications arise: first, whereas the ordinary vertical derivative does not affect the basis horizontal forms dx^i, formula (11.15) shows that this is not true for the invariant vertical derivatives of the invariant horizontal forms ϖ^i. Secondly, invariant integration by parts, which amounts to working modulo the image of the invariant horizontal differential $d_\mathcal{H}$, also introduces new terms owing to (11.14). As a result, the invariant Euler–Lagrange equation expressions are considerably more complicated.

For simplicity, let's just work out the case of curves, so we have only $p = 1$ independent variable, and $q \geq 1$ dependent variables. (The higher dimensional case has some extra twists; see [**87**] for details.) Consider an invariant Lagrangian form $\widetilde{\lambda} = \widetilde{L}(I^{(n)})\, \varpi$ depending on the generating differential invariants $I = (I^1, \ldots, I^l)$, their invariant derivatives $I_{,i}^\alpha = \mathcal{D}^i I^\alpha$, and the fully G-invariant arc length form $\varpi = \iota(dx)$. Its first variation is computed as follows:

$$d_\mathcal{V} \widetilde{\lambda} = d_\mathcal{V} (\widetilde{L}\, \varpi) = d_\mathcal{V} \widetilde{L} \wedge \varpi + \widetilde{L}\, d_\mathcal{V} \varpi = \sum_{i,\alpha} \frac{\partial \widetilde{L}}{\partial I_{,i}^\alpha}\, d_\mathcal{V} I_{,i}^\alpha \wedge \varpi + \widetilde{L}\, d_\mathcal{V} \varpi. \tag{13.4}$$

We then invariantly integrate by parts by applying the basic identity

$$F\, d_\mathcal{V} (\mathcal{D}H) \wedge \varpi \equiv -\,\mathcal{D}F\, d_\mathcal{V} H \wedge \varpi - F(\mathcal{D}H)\, d_\mathcal{V} \varpi, \tag{13.5}$$

where we work modulo the image of $d_\mathcal{H}$. We eventually arrive at the formula

$$d_\mathcal{V} \widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L})\, d_\mathcal{V} I \wedge \varpi - \mathcal{H}(\widetilde{L})\, d_\mathcal{V} \varpi, \tag{13.6}$$

where $\mathcal{E}(\widetilde{L})$, the *invariantized Eulerian* of $\widetilde{L}$, has components

$$\mathcal{E}_\alpha(\widetilde{L}) = \sum_{i=0}^{\infty} (-\mathcal{D})^i \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}} \, , \qquad \alpha = 1, \ldots, l, \tag{13.7}$$

while

$$\mathcal{H}(\widetilde{L}) = \sum_{\alpha=1}^{m} \sum_{i>j} I^\alpha_{,i-j} (-\mathcal{D})^j \, \frac{\partial \widetilde{L}}{\partial I^\alpha_{,i}} - \widetilde{L} \tag{13.8}$$

is known as the *invariantized Hamiltonian*, being the invariant counterpart of the usual Hamiltonian associated with a higher order Lagrangian $L(x, u^{(n)})$, cf. [**3**, **135**].

In the second phase of the computation, we use the recurrence formulae $(11.19, 20)$ to compute the vertical differentials

$$d_\mathcal{V} I = \mathcal{A}(\vartheta), \qquad d_\mathcal{V} \varpi = \mathcal{B}(\vartheta) \wedge \varpi, \tag{13.9}$$

of the differential invariants $I = (I^1, \ldots, I^l)$ and the invariant horizontal (arc length) form in terms of invariant derivatives of the zero$^{\text{th}}$ order invariant contact forms $\vartheta = (\vartheta^1, \ldots, \vartheta^q)$. Substituting (13.9) into (13.6) and performing one last integration by parts, we arrive at the key formula

$$d_\mathcal{V} \widetilde{\lambda} \equiv \mathcal{E}(\widetilde{L}) \, \mathcal{A}(\vartheta) \wedge \varpi - \mathcal{H}(\widetilde{L}) \, \mathcal{B}(\vartheta) \wedge \varpi \equiv \big[\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) \big] \vartheta \wedge \varpi,$$

where the * denotes the *formal invariant adjoint* of an invariant differential operator, so if

$$\mathcal{P} = \sum_n P_k \, \mathcal{D}^k, \qquad \text{then} \qquad \mathcal{P}^* = \sum_k (-\mathcal{D})^k \cdot P_k.$$

We conclude that the Euler-Lagrange equations for our invariant variational problem are equivalent to the invariant system of differential equations

$$\mathcal{A}^* \mathcal{E}(\widetilde{L}) - \mathcal{B}^* \mathcal{H}(\widetilde{L}) = 0. \tag{13.10}$$

Example 13.1. Any Euclidean-invariant variational problem corresponds to an invariant Lagrangian $\widetilde{\lambda} = \widetilde{L}(\kappa, \kappa_s, \kappa_{ss}, \ldots) \varpi$ depending on the arc length derivatives of the curvature, and the invariant arc length form (11.23). According to $(11.27, 28)$, $\mathcal{A} = \mathcal{D}^2 + \kappa^2 = \mathcal{A}^*$, while $\mathcal{B} = -\kappa = \mathcal{B}^*$. The invariant Euler-Lagrange formula (13.10) reduces to the known formula

$$(\mathcal{D}^2 + \kappa^2) \, \mathcal{E}(\widetilde{L}) + \kappa \, \mathcal{H}(\widetilde{L}) = 0 \tag{13.11}$$

for the Euclidean-invariant Euler-Lagrange equation, [**3**, **60**].

Additional, more intricate examples can be found in [**87**], as well as extensions to multiple integrals, i.e., higher dimensional submanifolds.

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14. Invariant Curve Flows.

Let us next discuss applications of the invariant variational bicomplex construction to invariant curve flows. (Extensions to higher dimensional invariant submanifold flows can be found in [**126**].) Setting $p = 1$, let us single out the $m = 1 + q$ invariant one-forms

$$\varpi, \ \vartheta^1, \ldots, \vartheta^q \tag{14.1}$$

consisting of the invariant arc length form $\varpi = \iota(dx)$ and the order 0 invariant contact forms $\vartheta^\alpha = \iota(\theta^\alpha)$. Let $C \subset M$ be a curve. Evaluating the coefficients of (14.1) on the curve jet $(x, u^{(n)}) = \mathrm{j}_n C|_z$ produces a G-equivariant coframe, i.e., a basis for the cotangent space $T^*M|_z$ at $z = (x, u) \in C$. Let $\mathbf{t}, \mathbf{n}_1, \ldots, \mathbf{n}_q$, denote the corresponding dual G–equivariant *frame* on C, with $\mathbf{t}$ tangent, while $\mathbf{n}_1, \ldots, \mathbf{n}_q$ form a basis for the complementary G–invariant *normal bundle* $N \to C$ induced by the moving frame.

In general, let

$$\mathbf{V} = \mathbf{V}_T + \mathbf{V}_N = I \, \mathbf{t} + \sum_{\alpha=1}^{q} J^\alpha \, \mathbf{n}_\alpha \tag{14.2}$$

be a G-equivariant section of $TM \to C$, where $\mathbf{V}_T, \mathbf{V}_N$ denote, respectively, its tangential and normal components, while $I, J^1, \ldots, J^q$ are differential invariants. We will, somewhat imprecisely, refer to $\mathbf{V}$ as a *vector field*, even though it depends on the underlying curve jet. Any $\mathbf{V}$ generates a G-invariant curve flow:

$$\frac{\partial C}{\partial t} = \mathbf{V}|_{C(t)}. \tag{14.3}$$

The tangential component $\mathbf{V}_T$ only affects the curve's internal parametrization, and hence can be ignored as far as the external curve geometry goes. For example, if we set $\mathbf{V}_T = 0$, the resulting vector field $\mathbf{V}_N$ is said to generate a *normal flow*, since each point on the curve moves in the G-invariant normal direction.

Example 14.1. The most well-studied are the Euclidean-invariant plane curve flows. The dual frame vectors to the invariant one-forms (11.23, 24) are the usual Euclidean frame vectors[†] — the unit tangent and unit normal:

$$\mathbf{t} = \frac{1}{\sqrt{1 + u_x^2}} \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right), \qquad \mathbf{n} = \frac{1}{\sqrt{1 + u_x^2}} \left(-u_x \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right). \tag{14.4}$$

A Euclidean-invariant normal flow is generated by a vector field of the form $\mathbf{V} = \mathbf{V}_N = J \, \mathbf{n}$, in which $J(\kappa, \kappa_s, \ldots)$ is any differential invariant. Particular cases include:

- $\mathbf{V} = \mathbf{n}$: the geometric optics or grassfire flow, [**14**, **137**];
- $\mathbf{V} = \kappa \, \mathbf{n}$: the celebrated curve shortening flow, [**55**, **57**], also used to great effect in image processing, [**131**, **137**];

[†] For simplicity, we are assuming the curve is represented as the graph of a function $u = u(x)$; generalizing the formulas to arbitrarily parametrized curves is straightforward, [**126**].

- $\mathbf{V} = \kappa^{1/3}\,\mathbf{n}$: the induced flow is equivalent, modulo reparametrization, to the equi-affine invariant curve shortening flow, also used in image processing, [6, 131, 137];

- $\mathbf{V} = \kappa_s\,\mathbf{n}$: this flow induces the modified Korteweg–deVries equation for the curvature evolution, and is the simplest example of a soliton equation arising in a geometric curve flow, [41, 56, 102];

- $\mathbf{V} = \kappa_{ss}\,\mathbf{n}$: this flow models thermal grooving of metals, [20].

A key question is how the differential invariants evolve under an invariant curve flow.

Theorem 14.2. *Let* $\mathbf{V}_N = \sum J^\alpha\,\mathbf{n}_\alpha$ *generate an invariant normal curve flow. If K is any differential invariant, then*

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J), \tag{14.5}$$

where $\mathcal{A}_K$ is the corresponding invariant linearization operator .

Example 14.3. For any of the Euclidean invariant normal plane curve flows $C_t = J\,\mathbf{n}$ listed in Example 14.1, we have, according to (11.27),

$$\frac{\partial \kappa}{\partial t} = (\mathcal{D}^2 + \kappa^2)\,J, \qquad \frac{\partial \kappa_s}{\partial t} = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\kappa\,\kappa_s)\,J. \tag{14.6}$$

For instance, for the grassfire flow $J = 1$, and so

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\kappa\,\kappa_s. \tag{14.7}$$

The first equation immediately implies finite time blow-up at a caustic for a convex initial curve segment, where $\kappa > 0$. For the curve shortening flow, $J = \kappa$, and

$$\frac{\partial \kappa}{\partial t} = \kappa_{ss} + \kappa^3, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{sss} + 4\kappa^2\kappa_s, \tag{14.8}$$

thereby recovering formulas used in Gage and Hamilton's analysis, [55]; see also [109]. Finally, for the modified Korteweg-deVries flow, $J = \kappa_s$,

$$\frac{\partial \kappa}{\partial t} = \kappa_{sss} + \kappa^2\kappa_s, \qquad \frac{\partial \kappa_s}{\partial t} = \kappa_{ssss} + \kappa^2\kappa_{ss} + 3\kappa\,\kappa_s^2. \tag{14.9}$$

Warning: Normal flows *do not preserve arc length*, and so the arc length parameter s will vary in time. Or, to phrase it another way, time differentiation ∂_t and arc length differentiation $\mathcal{D} = D_s$ *do not commute* — as can easily be seen in the preceding examples. Thus, one must be very careful *not* to interpret the resulting evolutions (14.7–9) as partial differential equations in the usual sense. Rather, one should regard the differential invariants $\kappa, \kappa_s, \kappa_{ss}, \dots$ as satisfying an infinite dimensional dynamical system of coupled ordinary differential equations.

A second important class are the invariant curve flows that preserve arc length, which requires $\left[\mathbf{V}, \mathcal{D}\right] = 0$, or, equivalently that the Lie derivative $\mathbf{V}(\varpi) \equiv 0$ is a contact form. Applying the Cartan formula and (11.20) to the latter characterization, we conclude that arc length preservation under (14.2) requires

$$\mathcal{D}I = \mathcal{B}(J) = \sum_{\alpha=1}^{q} \mathcal{B}_{\alpha}(J^{\alpha}), \qquad (14.10)$$

where $\mathcal{D}$ is the arc length derivative, while $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_q)$ is the *invariant Hamiltonian operator* (11.20).

Theorem 14.4. *Under an arc-length preserving flow,*

$$\kappa_t = \mathcal{R}_{\kappa}(J) \qquad \text{where} \qquad \mathcal{R}_{\kappa} = \mathcal{A}_{\kappa} - \kappa_s \mathcal{D}^{-1}\mathcal{B}. \qquad (14.11)$$

More generally, the time evolution of $\kappa_n = \mathcal{D}^n \kappa$ is given by arc length differentiation: $\partial \kappa_n / \partial t = \mathcal{D}^n \mathcal{R}_{\kappa}(J)$.

Here, the arc length and time derivatives commute, and hence the arc-length preserving flow (14.11) is an ordinary evolution equation — albeit possibly with nonlocal terms. Moreover, when (14.11) is a local evolution equation, it often turns out to be integrable, with $\mathcal{R}_{\kappa}$ the associated recursion operator, [**116**]. However, as yet, there is no general explanation for this phenomenon.

Example 14.5. For the Euclidean action on plane curves, the condition (14.10) that a curve flow generated by the vector field $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ preserve arc length is that

$$\mathcal{D}I = -\,\kappa\,J. \qquad (14.12)$$

Most of the curve flows listed in Example 14.1 have *non-local* arc length preserving counterparts owing to the non-invertibility of the arc length derivative operator on $\kappa\,J$. An exception is the modified Korteweg-deVries flow, where $J = \kappa_s$, and so $I = -\frac{1}{2}\kappa^2$. For such flows, the evolution of the curvature is given by (14.11), where

$$\mathcal{R}_{\kappa} = \mathcal{A}_{\kappa} - \kappa_s \mathcal{D}^{-1}\mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa = D_s^2 + \kappa^2 + \kappa_s D_s^{-1} \cdot \kappa \qquad (14.13)$$

is the modified Korteweg-deVries recursion operator, [**116**]. In particular, when $J = \kappa_s$, (14.11) is the modified Korteweg-deVries equation

$$\kappa_t = \mathcal{R}_{\kappa}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2 \kappa_s.$$

Example 14.6. In the case of space curves $C \subset \mathbb{R}^3$, under the usual action of the Euclidean group $G = \mathrm{SE}(3)$, the coordinate cross-section

$$K^2 = \left\{\, x = u = v = u_x = v_x = v_{xx} = 0 \,\right\}$$

produces the classical moving frame, [**61**, **87**]. There are two generating differential invariants: the curvature $\kappa = \iota(u_{xx})$ and the torsion $\tau = \iota(v_{xxx}/u_{xx})$. According to [**87**], the relevant moving frame formulae are

$$d_{\mathcal{V}}\,\kappa = \mathcal{A}_{\kappa}(\vartheta), \qquad d_{\mathcal{V}}\,\tau = \mathcal{A}_{\tau}(\vartheta), \qquad d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi,$$

where $\vartheta = (\vartheta_1, \vartheta_2)^T$ are the order 0 invariant contact forms, while

$$\mathcal{A}_\kappa = \left(\mathcal{D}^2 + (\kappa^2 - \tau^2), \; -2\tau\mathcal{D} - \tau_s \right),$$

$$\mathcal{A}_\tau = \left(\frac{2\tau}{\kappa}\mathcal{D}^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2}\mathcal{D} + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} , \right. \qquad\qquad \mathcal{B} = \left(-\kappa, \; 0 \right).$$

$$\left. \frac{1}{\kappa}\mathcal{D}^3 - \frac{\kappa_s}{\kappa^2}\mathcal{D}^2 + \frac{\kappa^2 - \tau^2}{\kappa}\mathcal{D} + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \right),$$

Thus, under an arc length preserving flow with normal component $\mathbf{V}_N = J\,\mathbf{n}_1 + K\,\mathbf{n}_2$, the curvature and torsion evolve according to

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R}\begin{pmatrix} J \\ K \end{pmatrix}, \qquad \text{where} \qquad \mathcal{R} = \begin{pmatrix} \mathcal{R}_\kappa \\ \mathcal{R}_\tau \end{pmatrix} = \begin{pmatrix} \mathcal{A}_\kappa \\ \mathcal{A}_\tau \end{pmatrix} - \begin{pmatrix} \kappa_s\mathcal{D}^{-1}\kappa & 0 \\ \tau_s\mathcal{D}^{-1}\kappa & 0 \end{pmatrix}$$

is the recursion operator for the integrable vortex filament flow, which corresponds to the choice $J = \kappa_s$, $K = \tau_s$. The latter flow can be mapped to the nonlinear Schrödinger equation via the Hasimoto transformation, [**64**, **91**].

15. Structure of Lie Pseudo–Groups.

With the moving frame constructions for finite-dimensional Lie group actions taking more or less final form, my attention has shifted to developing a comparably powerful theory that can be applied to infinite-dimensional Lie pseudo-groups. The subject is classical: Lie, [**93**], and Medolaghi, [**107**], classified all planar pseudo-groups, and gave applications to Darboux integrable partial differential equations, [**5**, **143**]. Cartan's famous classification of transitive simple pseudo-groups, [**34**], remains a milestone in the subject. Remarkably, despite numerous investigations, there is still no entirely satisfactory abstract object that will properly represent a Lie pseudo-group, cf. [**90**, **142**, **144**, **134**].

Pseudo-groups appear in a broad range of physical and geometrical contexts, including gauge theories in physics, [**11**, **75**]; canonical and area-preserving transformations in Hamiltonian mechanics, [**116**]; conformal symmetry groups on two-dimensional surfaces, [**49**]; foliation-preserving groups of transformations, with the associated characteristic classes defined by certain invariant forms, [**54**]; symmetry groups of both linear and nonlinear partial differential equations appearing in fluid and plasma mechanics, such as the Euler, Navier-Stokes and boundary layer equations, [**30**, **116**], in meteorology and turbulence modeling, [**9**, **10**, **136**], and in integrable (soliton) equations in more than one space dimension such as the Kadomtsev–Petviashvili (KP) equation, [**42**]. Applications of pseudo-groups to the design of geometric numerical integrators are being emphasized in recent work of McLachlan and Quispel, [**103**, **104**].

Juha Pohjanpelto and I, [**128**, **129**, **127**], have developed a practical moving frame theory for general Lie pseudo-group actions. Just as in the finite-dimensional theory, the new methods lead to general computational algorithms for

(i) determining complete systems of differential invariants, invariant differential operators, and invariant differential forms,

(ii) complete classifications of syzygies and recurrence formulae relating the differentiated invariants and invariant forms,

(iii) a general algorithm for computing the Euler–Lagrange equations associated with an invariant variational problem.

In [**38**, **39**], these algorithms were applied to the symmetry groups of the Korteweg–deVries and KP equations arising in soliton theory. More substantial examples, arising as symmetry pseudo-groups of nonlinear partial differential equations in fluid mechanics, meteorology, and gauge theories, are in the process of being investigated. Further extensions — pseudo-group algorithms for joint invariants and joint differential invariants, invariant numerical approximations, and so on — are also evident.

Let M be a smooth m-dimensional manifold. Let $\mathcal{D} = \mathcal{D}(M)$ denote the pseudo-group of all local diffeomorphisms $\varphi \colon M \to M$. For each $0 \leq n \leq \infty$, let $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$ denote the nth order diffeomorphism jet groupoid, [**95**, **153**], with source map $\boldsymbol{\sigma}^{(n)}\big(\mathrm{j}_n\varphi|_z\big) = z$ and target map $\boldsymbol{\tau}^{(n)}\big(\mathrm{j}_n\varphi|_z\big) = \varphi(z) = Z$. The groupoid multiplication is induced by composition of diffeomorphisms. Following Cartan, [**35**, **36**], we will consistently use lower case letters, $z, x, u, \ldots$ for the source coordinates and the corresponding upper case letters $Z, X, U, \ldots$ for the target coordinates of our diffeomorphisms φ. Given local coordinates $(z, Z) = (z^1, \ldots, z^m, Z^1, \ldots, Z^m)$ on an open subset of $M \times M$, the induced local coordinates of $g^{(n)} = \mathrm{j}_n\varphi|_z \in \mathcal{D}^{(n)}$ are denoted $(z, Z^{(n)})$, where the components Z_B^a of $Z^{(n)}$, for $a = 1, \ldots, m$, $\#B \leq n$, represent the partial derivatives $\partial^B \varphi^a / \partial z^B$ of φ at the source point $z = \boldsymbol{\sigma}^{(n)}(g^{(n)})$.

Since $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$, the inherited variational bicomplex structure, [**3**, **147**], provides a natural splitting of the cotangent bundle $T^*\mathcal{D}^{(\infty)}$ into horizontal and vertical (contact) components, [**3**, **117**], and we use $d = d_M + d_G$ to denote the induced splitting of the differential. In terms of local coordinates $g^{(\infty)} = (z, Z^{(\infty)})$, the horizontal subbundle of $T^*\mathcal{D}^{(\infty)}$ is spanned by the one-forms $dz^a = d_M z^a$, $a = 1, \ldots, m$, while the vertical subbundle is spanned by the basic *contact forms*

$$\Upsilon_B^a = d_G Z_B^a = dZ_B^a - \sum_{c=1}^m Z_{B,c}^a \, dz^c, \qquad a = 1, \ldots, m, \qquad \#B \geq 0. \qquad (15.1)$$

Composition of local diffeomorphisms induces an action of $\psi \in \mathcal{D}$ by right multiplication on diffeomorphism jets: $\mathrm{R}_\psi(\mathrm{j}_n\varphi|_z) = \mathrm{j}_n(\varphi \circ \psi^{-1})|_{\psi(z)}$. A differential form μ on $\mathcal{D}^{(n)}$ is *right-invariant* if $\mathrm{R}_\psi^* \mu = \mu$, where defined, for every $\psi \in \mathcal{D}$. Since the splitting of forms on $\mathcal{D}^{(\infty)}$ is invariant under this action, if μ is any right-invariant differential form, so are $d_M \mu$ and $d_G \mu$. The target coordinate functions $Z^a \colon \mathcal{D}^{(0)} \to \mathbb{R}$ are obviously right-invariant, and hence their horizontal differentials

$$\sigma^a = d_M Z^a = \sum_{b=1}^m Z_b^a \, dz^b, \qquad a = 1, \ldots, m, \qquad (15.2)$$

form an invariant horizontal coframe, while their vertical differentials

$$\mu^a = d_G Z^a = \phi = dZ^a - \sum_{b=1}^m Z_b^a \, dz^b, \qquad a = 1, \ldots, m, \qquad (15.3)$$

are the zero[th] order invariant contact forms. Let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ be the total derivative operators dual to the horizontal forms (15.2), so that

$$d_M F = \sum_{a=1}^{m} \mathbb{D}_{z^a} F \; dz^a \qquad \text{for any} \qquad F : \mathcal{D}^{(\infty)} \to \mathbb{R}. \tag{15.4}$$

Then the higher-order invariant contact forms are obtained by successively Lie differentiating the invariant contact forms (15.3):

$$\mu_B^a = \mathbb{D}_Z^B \mu^a = \mathbb{D}_Z^B \phi, \qquad \text{where} \qquad \mathbb{D}_Z^B = \mathbb{D}_{Z^{b_1}} \cdots \mathbb{D}_{Z^{b_k}}, \qquad \begin{matrix} a = 1, \ldots, m, \\ k = \#B \geq 0. \end{matrix} \tag{15.5}$$

As explained in [**128**], the right-invariant contact forms $\mu^{(\infty)} = (\; \ldots \; \mu_B^a \; \ldots \;)$ are to be viewed as the *Maurer–Cartan forms* for the diffeomorphism pseudo-group.

The next step in our program is to establish the structure equations for the diffeomorphism groupoid $\mathcal{D}^{(\infty)}$. Let $\mu[\![\, H \,]\!]$ denote the column vector whose components are the invariant contact form-valued formal power series

$$\mu^a[\![\, H \,]\!] = \sum_{\#B \geq 0} \frac{1}{B!} \mu_B^a H^B, \qquad a = 1, \ldots, m, \tag{15.6}$$

depending on the formal parameters $H = (H^1, \ldots, H^m)$. Further, let $dZ = \mu[\![\, 0 \,]\!] + \sigma$ denote column vectors of one-forms whose entries are $dZ^a = \mu^a + \sigma^a$ for $a = 1, \ldots, m$.

Theorem 15.1. *The complete structure equations for the diffeomorphism pseudo-group are obtained by equating coefficients in the power series identity*

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge \big(\mu[\![\, H \,]\!] - dZ \big), \qquad d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla_H \mu[\![\, 0 \,]\!] \wedge \sigma, \tag{15.7}$$

where $\nabla_H \mu[\![\, H \,]\!] = \left(\dfrac{\partial \mu^a}{\partial H^b} [\![\, H \,]\!] \right)$ *denotes the* $m \times m$ *formal power series Jacobian matrix.*

The key to analyzing pseudo-group actions is to work infinitesimally, using the generating Lie algebra[†] of vector fields. Let $\mathcal{X}(M)$ denote the space of locally defined vector fields on M, which we write in local coordinates as

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} \; . \tag{15.8}$$

Let $\mathrm{J}^n TM$, for $0 \leq n \leq \infty$, denote the tangent n-jet bundle. Local coordinates on $\mathrm{J}^n TM$ are indicated by $(z, \zeta^{(n)}) = (\; \ldots \; z^a \; \ldots \; \zeta_B^a \; \ldots \;)$, $a = 1, \ldots, m, \#B \leq n$, where the fiber coordinate ζ_B^a represents the partial derivative $\partial^B \zeta^a / \partial z^B$.

The literature contains several variants of the precise technical definition of a Lie pseudo-group. Ours is:

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† Here, we are using the term "Lie algebra" imprecisely, since, technically, the vector fields may only be locally defined, and so their Lie brackets only make sense on their common domains of definition.

Definition 15.2. A sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$ will be called a *Lie pseudo-group* if there exists $n_0 \geq 1$ such that for all finite $n \geq n_0$:

(a) the corresponding sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle,
(b) every smooth local solution $Z = \varphi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,
(c) $\mathcal{G}^{(n)} = \mathrm{pr}^{(n-n_0)} \mathcal{G}^{(n_0)}$ is obtained by prolongation.

The minimal value of n_0 is called the *order* of the pseudo-group.

Thus on account of conditions (a) and (c), for $n \geq n_0$, the pseudo-group jet sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ is defined in local coordinates by a formally integrable system of nth order nonlinear partial differential equations

$$F^{(n)}(z, Z^{(n)}) = 0, \tag{15.9}$$

known as the *determining equations* for the pseudo-group. Condition (b) says that the local solutions $Z = \varphi(z)$ to the determining equations are precisely the pseudo-group transformations.

Let $\mathfrak{g} \subset \mathcal{X}$ denote the Lie algebra of infinitesimal generators of the pseudo-group, i.e., the set of locally defined vector fields (15.8) whose flows belong to $\mathcal{G}$. In local coordinates, we can view $\mathrm{J}^n \mathfrak{g} \subset \mathrm{J}^n TM$ as defining a formally integrable linear system of partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \tag{15.10}$$

for the vector field coefficients (15.8), called the *linearized* or *infinitesimal determining equations* for the pseudo-group. They can be obtained by linearizing the nth order determining equations (15.9) at the identity jet. If $\mathcal{G}$ is the symmetry group of a system of differential equations, then the linearized determining equations (15.10) are (the involutive completion of) the usual determining equations for its infinitesimal generators obtained via Lie's algorithm, [**116**].

As with finite-dimensional Lie groups, the structure of a pseudo-group is described by its invariant Maurer–Cartan forms. A complete system of right-invariant one-forms on $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$ is obtained by restricting (or pulling back) the Maurer–Cartan forms (15.2–5). For simplicity, we continue to denote these forms by σ^a, μ^a_B. The restricted Maurer–Cartan forms are, of course, no longer linearly independent, but are subject to certain constraints prescribed by the pseudo-group. Remarkably, these constraints can be explicitly characterized by an invariant version of the linearized determining equations (15.10), which is formally obtained by replacing the source coordinates z^a by the corresponding target coordinates Z^a and the vector field jet coordinates ζ^a_B by the corresponding Maurer–Cartan form μ^a_B.

Theorem 15.3. *The linear system*

$$L^{(n)}(Z, \mu^{(n)}) = 0 \tag{15.11}$$

serves to define the complete set of dependencies among the right-invariant Maurer–Cartan forms $\mu^{(n)}$ on $\mathcal{G}^{(n)}$. Therefore, the structure equations for the pseudo-group $\mathcal{G}$ are obtained by restriction of the diffeomorphism structure equations (15.7) to the kernel of the linearized involutive system (15.11).

In this way, we effectively and efficiently bypass Cartan's more complicated prolongation procedure, [**25**, **36**], for accessing the pseudo-group structure equations. Examples of this procedure can be found in [**38**, **128**]; see also [**113**] for a comparison with other approaches.

Example 15.4. Let us consider the pseudo-group

$$X = f(x), \qquad Y = e(x, y) \equiv f'(x)\, y + g(x), \qquad U = u + \frac{e_x(x, y)}{f'(x)} = u + \frac{f''(x)\, y + g'(x)}{f'(x)},$$
$$(15.12)$$

acting on $M = \mathbb{R}^3$, with local coordaintes (x, y, u). Here $f(x) \in \mathcal{D}(\mathbb{R})$, while $g(x) \in \mathrm{C}^\infty(\mathbb{R})$. The determining equations are the first order involutive system

$$X_y = X_u = 0, \qquad Y_y = X_x \neq 0, \qquad Y_u = 0, \qquad Y_x = (U - u)X_x, \qquad U_u = 1. \quad (15.13)$$

The infinitesimal generators of the pseudo-group have the form

$$\mathbf{v} = \xi\, \frac{\partial}{\partial x} + \eta\, \frac{\partial}{\partial y} + \varphi\, \frac{\partial}{\partial u} = \phi(x)\, \frac{\partial}{\partial x} + \left[\, \phi'(x)\, y + \psi(x)\,\right] \frac{\partial}{\partial y} + \left[\, \phi''(x)\, y + \psi'(x)\,\right] \frac{\partial}{\partial u},$$
$$(15.14)$$

where $\phi(x), \psi(x)$ are arbitrary smooth functions. The infinitesimal generators (15.14) form the general solution to the first order involutive infinitesimal determining system

$$\xi_x = \eta_y, \qquad \xi_y = \xi_u = \eta_u = \varphi_u = 0, \qquad \eta_x = \varphi, \qquad (15.15)$$

obtained by linearizing (15.13) at the identity.

The Maurer–Cartan forms are obtained by repeatedly differentiating $\mu = d_G X$, $\widetilde{\mu} = d_G Y$ and $\nu = d_G U$, so that $\mu_{j,k,l} = \mathbb{D}_X^j \mathbb{D}_Y^k \mathbb{D}_U^l \mu$, etc. According to Theorem 15.3, they are subject to the linear relations

$$\mu_X = \widetilde{\mu}_Y, \qquad \mu_Y = \mu_U = \widetilde{\mu}_U = \nu_U = 0, \qquad \widetilde{\mu}_X = \nu, \qquad (15.16)$$

along with their "differential" consequences. Writing out (15.7), we are led to the following structure equations

$$d\mu_n = \sigma \wedge \mu_{n+1} - \sum_{j=1}^{[(n+1)/2]} \frac{n - 2j + 1}{n + 1} \binom{n+1}{j} \mu_j \wedge \mu_{n+1-j},$$

$$d\widetilde{\mu}_n = \sigma \wedge \widetilde{\mu}_{n+1} + \widetilde{\sigma} \wedge \mu_{n+1} - \sum_{j=0}^{n-1} \frac{n - 2j - 1}{n + 1} \binom{n+1}{j+1} \widetilde{\mu}_{j+1} \wedge \mu_{n-j}, \qquad (15.17)$$

$$d\sigma = -\, d\mu = -\, \sigma \wedge \mu_X,$$
$$d\widetilde{\sigma} = -\, d\widetilde{\mu} = -\, \sigma \wedge \widetilde{\mu}_X - \widetilde{\sigma} \wedge \mu_X,$$
$$d\tau = -\, d\nu = -\, d\widetilde{\mu}_X = -\, \sigma \wedge \widetilde{\mu}_{XX} - \widetilde{\sigma} \wedge \mu_{XX},$$

in which $\sigma = d_M X, \widetilde{\sigma} = d_M Y, \tau = d_M U$, and $\mu_n = \mu_{n,0,0}$, $\widetilde{\mu}_n = \widetilde{\mu}_{n,0,0}$, for $n = 0, 1, 2, \ldots$, form a basis for the Maurer–Cartan forms of the pseudo-group.

Example 15.5. Let $\mathcal{G}$ denote the symmetry group of the KdV equation (4.11), calculated in Example 4.4. We begin by completing the determining equations (4.12) to involution by cross-differentiation:

$$\xi_u = 0, \qquad 3\xi_x - \tau_t = 0, \qquad \varphi - \xi_t + \tfrac{2}{3}u\tau_t = 0, \qquad \tau_u = 0,$$
$$\tau_x = 0, \qquad \varphi_{uu} = 0, \qquad \varphi_{xu} = 0, \qquad \varphi_t + u\varphi_x + \varphi_{xxx} = 0,$$

and so on. The corresponding linear relations among the diffeomorphism Maurer–Cartan forms on $M = \mathbb{R}^3$ are formally obtained by substituting $(x, t, u) \longmapsto (X, T, U)$ and $\xi, \tau, \varphi) \longmapsto (\mu^x, \mu^t, \mu^u)$, resulting in the linear relations

$$\mu_U^x = 0, \qquad 3\mu_X^x - \mu_T^t = 0, \qquad \mu^u - \mu_T^x + \tfrac{2}{3}U\mu_T^t = 0, \qquad \mu_U^t = 0,$$
$$\mu_X^t = 0, \qquad \mu_{UU}^u = 0, \qquad \mu_{XU}^u = 0, \qquad \mu_T^u + U\mu_X^u + \mu_{XXX}^u = 0,$$

and so on. Solving this system by, say, Gaussian elimination, we find that there are precisely 4 independent invariant contact forms:

$$\omega^1 := \mu^t, \qquad \omega^2 := \mu^x, \qquad \omega^3 := \mu^u, \qquad \omega^4 := \mu_T^t,$$

which reflects the fact that the symmetry group of the KdV equation is a four-dimensional Lie group. The structure equations of the coframe are

$$d\sigma^t = \mu^4 \wedge \sigma^t,$$
$$d\sigma^x = \mu^3 \wedge \sigma^t + \tfrac{2}{3}U\mu^4 \wedge \sigma^t + \tfrac{1}{3}\mu^4 \wedge \sigma^x,$$
$$d\sigma^u = -\tfrac{2}{3}\mu^4 \wedge \sigma^u,$$
$$d\mu^1 = -\mu^4 \wedge \sigma^t,$$
$$d\mu^2 = -\mu^3 \wedge \sigma^t - \tfrac{2}{3}U\mu^4 \wedge \sigma^t - \tfrac{1}{3}\mu^4 \wedge \sigma^x,$$
$$d\mu^3 = \tfrac{2}{3}\mu^4 \wedge \sigma^u,$$
$$d\mu^4 = 0,$$

where $\sigma^t, \sigma^x, \sigma^u$ are the invariant horizontal forms. The Maurer–Cartan equations for the Lie symmetry pseudo-group $\mathcal{G}$ are obtained by restricting to a target fiber where T, X, U are fixed, whence

$$d\mu^1 = -\mu^1 \wedge \mu^4,$$
$$d\mu^2 = -\mu^1 \wedge \mu^3 - \tfrac{2}{3}U\mu^1 \wedge \mu^4 - \tfrac{1}{3}\mu^2 \wedge \mu^4,$$
$$d\mu^3 = \tfrac{2}{3}\mu^3 \wedge \mu^4,$$
$$d\mu^4 = 0.$$

16. Differential Invariants of Lie Pseudo–groups.

Our primary focus is to study the induced action of pseudo-groups on submanifolds. For $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M, p)$ denote the nth order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of nth order contact, cf. [**117**]. We employ the standard local coordinates

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u_J^\alpha \ \ldots \ ) \tag{16.1}$$

on J^n induced by a splitting of the local coordinates $z = (x, u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ on M into p independent and $q = m - p$ dependent variables, [116, 117]. The choice of independent and dependent variables induces the variational bicomplex structure on J^∞, [3, 147]. The basis *horizontal forms* are the differentials $dx^1, \ldots, dx^p$ of the independent variables, while the basis *contact forms* are denoted by

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^{p} u_{J,i}^\alpha \, dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{16.2}$$

This decomposition splits the differential $d = d_H + d_V$ on J^∞ into horizontal and vertical (or contact) components, and endows the space of differential forms with the structure of a variational bicomplex, [3, 87, 147].

Local diffeomorphisms $\varphi \in \mathcal{D}$ preserve the contact equivalence relation between submanifolds, and thus induce an action on the jet bundle $J^n = J^n(M, p)$, known as the nth prolonged action, which, by the chain rule, factors through the diffeomorphism jet groupoid $\mathcal{D}^{(n)}$. Let $\mathcal{H}^{(n)}$ denote the groupoid obtained by pulling back the pseudo-group jet groupoid $\mathcal{G}^{(n)} \to M$ via the projection $\widetilde{\pi}_0^n \colon J^n \to M$. Local coordinates on $\mathcal{H}^{(n)}$ are written $(x, u^{(n)}, g^{(n)})$, where $(x, u^{(n)})$ are the coordinates and $J^n(M, p)$, while the fiber coordinates $g^{(n)}$ serve to parametrize the pseudo-group jets.

Definition 16.1. A *moving frame* $\rho^{(n)}$ of *order* n is a $\mathcal{G}^{(n)}$ equivariant local section of the bundle $\mathcal{H}^{(n)} \to J^n$.

Thus, in local coordinates, the moving frame section has the form

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, \gamma^{(n)}(x, u^{(n)})), \qquad \text{where} \qquad g^{(n)} = \gamma^{(n)}(x, u^{(n)}) \tag{16.3}$$

defines a right equivariant map to the pseudo-group jets. A moving frame $\rho^{(k)} \colon J^k \to \mathcal{H}^{(k)}$ of order $k > n$ is *compatible* with $\rho^{(n)}$ provided $\widehat{\pi}_n^k \circ \rho^{(k)} = \rho^{(n)} \circ \widetilde{\pi}_n^k$ where defined, $\widehat{\pi}_n^k \colon \mathcal{H}^{(k)} \to \mathcal{H}^{(n)}$ and $\widetilde{\pi}_n^k \colon J^k \to J^n$ denoting the evident projections. A *complete moving frame* is provided by a mutually compatible collection of moving frames of all orders $k \geq n$.

As in the finite-dimensional construction, [52], the (local) existence of a moving frame requires that the prolonged pseudo-group action be free and regular.

Definition 16.2. The pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)} \in J^n$ if its *isotropy subgroup* is trivial, $\mathcal{G}_{z^{(n)}}^{(n)} = \{ \mathbf{g}^{(n)} \in \mathcal{G}^{(n)} \, | \, \mathbf{g}^{(n)} \cdot z^{(n)} = z^{(n)} \} = \{ \mathbb{1}_z^{(n)} \}$, and *locally freely* if $\mathcal{G}_{z^{(n)}}^{(n)}$ is discrete.

Warning: According to the standard definition, [52], any (locally) free action of a finite-dimensional Lie group satisfies the (local) freeness condition of Definition 16.2, but *not* necessarily conversely.

The pseudo-group acts locally freely at $z^{(n)}$ if and only if the prolonged pseudo-group orbit through $z^{(n)}$ has dimension $r_n = \dim \mathcal{G}^{(n)}|_z$. Thus, freeness of the pseudo-group at order n requires, at the very least, that

$$r_n = \dim \mathcal{G}^{(n)}|_z \leq \dim J^n = p + (m - p) \binom{p + n}{p}. \tag{16.4}$$

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Freeness thus provides an alternative and simpler means of quantifying the Spencer co-homological growth conditions imposed on the pseudo-group in [**88**, **89**]. Pseudo-groups having too large a fiber dimension r_n will, typically, act transitively on (a dense open subset of) J^n, and thus possess no non-constant differential invariants. A key result of [**130**], generalizing the finite-dimensional case, is the persistence of local freeness.

Theorem 16.3. *Let $\mathcal{G}$ be a Lie pseudo-group acting on an m-dimensional manifold M. If $\mathcal{G}$ acts locally freely at $z^{(n)} \in J^n$ for some $n > 0$, then it acts locally freely at any $z^{(k)} \in J^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$, for $k \geq n$.*

As in the finite-dimensional version, [**52**], moving frames are constructed through a normalization procedure based on a choice of *cross-section* to the pseudo-group orbits, i.e., a transverse submanifold of the complementary dimension.

Theorem 16.4. *Suppose $\mathcal{G}^{(n)}$ acts freely on an open subset $\mathcal{V}^n \subset J^n$, with its orbits forming a regular foliation. Let $K^n \subset \mathcal{V}^n$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^n$, define $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$ to be the unique pseudo-group jet such that $\widetilde{\boldsymbol{\sigma}}^{(n)}(\rho^{(n)}(z^{(n)})) = z^{(n)}$ and $\widetilde{\boldsymbol{\tau}}^{(n)}(\rho^{(n)}(z^{(n)})) \in K^n$ (when such exists). Then $\rho^{(n)} : J^n \to \mathcal{H}^{(n)}$ is a moving frame for $\mathcal{G}$ defined on an open subset of $\mathcal{V}^n$ containing K^n.*

Usually — and, to simplify the development, from here on — we select a coordinate cross-section of minimal order, defined by fixing the values of r_n of the individual submanifold jet coordinates $(x, u^{(n)})$. We write out the explicit formulae $(X, U^{(n)}) = F^{(n)}(x, u^{(n)}, g^{(n)})$ for the prolonged pseudo-group action in terms of a convenient system of group parameters $g^{(n)} = (g_1, \ldots, g_{r_n})$. The r_n components corresponding to our choice of cross-section variables serve to define the *normalization equations*

$$F_1(x, u^{(n)}, g^{(n)}) = c_1, \qquad \ldots \qquad F_{r_n}(x, u^{(n)}, g^{(n)}) = c_{r_n}, \qquad (16.5)$$

which, when solved for the group parameters $g^{(n)} = \gamma^{(n)}(x, u^{(n)})$, produces the moving frame section (16.3).

With the moving frame in place, the general invariantization procedure introduced in [**87**] in the finite-dimensional case adapts straightforwardly. To compute the invariantization of a function, differential form, differential operator, etc., one writes out how it explicitly transforms under the pseudo-group, and then replaces the pseudo-group parameters by their moving frame expressions (16.3). Invariantization defines a morphism that projects the exterior algebra differential functions and forms onto the algebra of invariant differential functions and forms. In particular, invariantizing the coordinate functions on J^∞ leads to the *normalized differential invariants*

$$H^i = \iota(x^i), \qquad i = 1, \ldots, p, \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0, \quad (16.6)$$

collectively denoted by $(H, I^{(n)}) = \iota(x, u^{(n)})$. The normalized differential invariants naturally split into two subspecies: those appearing in the normalization equations (16.5) will be constant, and are known as the *phantom differential invariants*. The remaining $s_n = \dim J^n - r_n$ components, called the *basic differential invariants*, form a complete system of functionally independent differential invariants of order $\leq n$ for the prolonged pseudo-group action on submanifolds.

Secondly, invariantization of the basis horizontal one-forms leads to the invariant one-forms

$$\varpi^i = \iota(dx^i) = \omega^i + \kappa^i, \qquad i = 1, \ldots, p, \qquad (16.7)$$

where ω^i, κ^i denote, respectively, the horizontal and vertical (contact) components. If the pseudo-group acts projectably, then the contact components vanish: $\kappa^i = 0$. The horizontal forms $\omega^1, \ldots, \omega^p$ provide, in the language of [**117**], a contact-invariant coframe on J^∞. The dual invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are uniquely defined by the formula

$$dF = \sum_{i=1}^{p} \mathcal{D}_i F \, \varpi^i + \, \cdots \, , \qquad (16.8)$$

valid for any differential function F, where the dots indicate contact components which are not needed here, but do play an important role in the study of invariant variational problems, cf. [**87**]. The invariant differential operators $\mathcal{D}_i$ map differential invariants to differential invariants. In general, they do not commute, but are subject to linear commutation relations of the form

$$\left[\, \mathcal{D}_i, \mathcal{D}_j \,\right] = \sum_{k=1}^{p} Y_{ij}^k \, \mathcal{D}_k, \qquad i, j = 1, \ldots, p, \qquad (16.9)$$

where the coefficients Y_{ij}^k are certain differential invariants. Finally, invariantizing the basis contact one-forms

$$\vartheta_K^\alpha = \iota(\theta_K^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#K \geq 0, \qquad (16.10)$$

provide a complete system of invariant contact one-forms. The invariant coframe serves to define the invariant variational complex for the pseudo-group, [**87**].

The *Basis Theorem* for differential invariants states that, assuming freeness of the sufficiently high order prolonged pseudo-group action, then locally, there exist a finite number of *generating differential invariants* $I_1, \ldots, I_\ell$, with the property that every differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_k} I_\kappa.$$

The differentiated invariants are not necessarily independent, but may be subject to certain functional relations or *differential syzygies* of the form

$$H(\, \ldots \, \mathcal{D}_J I_\kappa \, \ldots \,) \equiv 0. \qquad (16.11)$$

A consequence of our moving frame methods is a constructive algorithm for producing a (not necessarily minimal) system of generating differential invariants, as well as a proof that there are finitely many generating syzygies, meaning that any other syzygy is a differential consequence thereof.

Example 16.5. Consider the action of the pseudo-group (15.12) on surfaces $u = h(x, y)$. Under the pseudo-group transformations, the basis horizontal forms dx, dy are mapped to the one-forms

$$d_H X = f_x \, dx, \qquad\qquad d_H Y = e_x \, dx + f_x \, dy. \qquad (16.12)$$

The prolonged pseudo-group transformations are found by applying the dual implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \, \mathrm{D}_x - \frac{e_x}{f_x^2} \, \mathrm{D}_y, \qquad\qquad \mathrm{D}_Y = \frac{1}{f_x} \, \mathrm{D}_y,$$

successively to $U = u + e_x / f_x$, so that

$$U_X = \frac{u_x}{f_x} + \frac{e_{xx} - e_x \, u_y}{f_x^2} - 2 \, \frac{f_{xx} \, e_x}{f_x^3}, \qquad\qquad U_Y = \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2},$$

$$U_{XX} = \frac{u_{xx}}{f_x^2} + \frac{e_{xxx} - e_{xx} \, u_y - 2 \, e_x \, u_{xy} - f_{xx} \, u_x}{f_x^3} +$$

$$+ \frac{e_x^2 \, u_{yy} + 3 \, e_x f_{xx} \, u_y - 4 \, e_{xx} \, f_{xx} - 3 \, e_x \, f_{xxx}}{f_x^4} + 8 \, \frac{e_x \, f_{xx}^2}{f_x^5}, \qquad (16.13)$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} + \frac{f_{xxx} - f_{xx} \, u_y - e_x \, u_{yy}}{f_x^3} - 2 \, \frac{f_{xx}^2}{f_x^4}, \qquad\qquad U_{YY} = \frac{u_{yy}}{f_x^2},$$

and so on. In these formulae, the jet coordinates $f, f_x, f_{xx}, \ldots, e, e_x, e_{xx}, \ldots$ are to be regarded as the independent pseudo-group parameters. The pseudo-group cannot act freely on J^1 since $r_1 = \dim \mathcal{G}^{(1)}|_z = 6 > \dim \mathrm{J}^1 = 5$. On the other hand, $r_2 = \dim \mathcal{G}^{(2)}|_z = 8 = \dim \mathrm{J}^2$, and the action on J^2 is, in fact, locally free and transitive on the sets $\mathcal{V}^2_+ = \mathrm{J}^2 \cap \{u_{yy} > 0\}$ and $\mathcal{V}^2_- = \mathrm{J}^2 \cap \{u_{yy} < 0\}$. Moreover, as predicted by Theorem 16.3, $\mathcal{G}^{(n)}$ acts locally freely on the corresponding open subsets of J^n for any $n \geq 2$.

To construct the moving frame, we successively solve the following coordinate cross-section equations for the pseudo-group parameters:

$$
\begin{aligned}
X &= 0, & f &= 0, \\
Y &= 0, & e &= 0, \\
U &= 0, & e_x &= -u \, f_x, \\
U_Y &= 0, & f_{xx} &= -u_y \, f_x, \\
U_X &= 0, & e_{xx} &= (u \, u_y - u_x) \, f_x, \\
U_{YY} &= 1, & f_x &= \sqrt{u_{yy}}, \\
U_{XY} &= 0, & f_{xxx} &= -\sqrt{u_{yy}} \, \left(u_{xy} + u u_{yy} - u_y^2 \right), \\
U_{XX} &= 0, & e_{xxx} &= -\sqrt{u_{yy}} \, \left(u_{xx} - u u_{xy} - 2 \, u^2 u_{yy} - 2 u_x u_y + u u_y^2 \right).
\end{aligned}
$$

At this stage, we can construct the first two fundamental differential invariants:

$$J_1 = \iota(u_{xyy}) = \frac{u_{xyy} + u u_{yyy} + 2 \, u_y u_{yy}}{u_{yy}^{3/2}}, \qquad\qquad J_2 = \iota(u_{yyy}) = \frac{u_{yyy}}{u_{yy}^{3/2}}. \qquad (16.14)$$

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Higher order differential invariants are found by continuing this procedure, or by employing the more powerful Taylor series method developed in [**129**]. Further, substituting the pseudo-group normalizations into (16.12) fixes the invariant horizontal coframe

$$\omega^1 = \iota(dx) = \sqrt{u_{yy}}\, dx, \qquad \omega^2 = \iota(dy) = \sqrt{u_{yy}}\,(dy - u\, dx). \tag{16.15}$$

The dual invariant total derivative operators are

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,(\mathrm{D}_x + u\,\mathrm{D}_y), \qquad \mathcal{D}_2 = \frac{1}{\sqrt{u_{yy}}}\,\mathrm{D}_y. \tag{16.16}$$

The higher-order differential invariants can be generated by successively applying these differential operators to the pair of basic differential invariants (16.14). The commutation relation is

$$[\mathcal{D}_1, \mathcal{D}_2] = -\tfrac{1}{2}\,J_2 \mathcal{D}_1 + \tfrac{1}{2}\,J_1 \mathcal{D}_2. \tag{16.17}$$

Finally, there is a single generating syzygy

$$\mathcal{D}_1 J_2 - \mathcal{D}_2 J_1 = 2 \tag{16.18}$$

among the differentiated invariants from which all others can be deduced by invariant differentiation.

Example 16.6. We determine the differential invariants of the Korteweg–deVries equation symmetry group, as determined in (4.14). to obtain the explicit formulas, we begin by using invariant differentiation to prolong the action:

$$
\begin{aligned}
T &= e^{3\lambda_4}(t + \lambda_1),\\
X &= e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2),\\
\widehat{U} &= U = e^{-2\lambda_4}(u + \lambda_3),\\
\widehat{U}_T &= D_T\widehat{U} = e^{-5\lambda_4}(u_t - \lambda_3 u_x),\\
\widehat{U}_X &= D_X\widehat{U} = e^{-3\lambda_4}u_x,\\
\widehat{U}_{TT} &= D_T^2\widehat{U} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}),\\
\widehat{U}_{TX} &= D_X D_T\widehat{U} = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}),\\
\widehat{U}_{XX} &= D_X^2\widehat{U} = e^{-4\lambda_4}u_{xx},\\
\widehat{U}_{TTT} &= D_T^3\widehat{U} = e^{-11\lambda_4}(u_{ttt} - 3\lambda_3 u_{ttx} + 3\lambda_3^2 u_{txx} - \lambda_3^3 u_{xxx}),\\
\widehat{U}_{TTX} &= D_X D_T^2\widehat{U} = e^{-9\lambda_4}(u_{ttx} - 2\lambda_3 u_{txx} + \lambda_3^2 u_{xxx}),\\
\widehat{U}_{TXX} &= D_X^2 D_T\widehat{U} = e^{-7\lambda_4}(u_{txx} - \lambda_3 u_{xxx}),\\
\widehat{U}_{XXX} &= D_X^3\widehat{U} = e^{-5\lambda_4}u_{xxx}.
\end{aligned}
\tag{16.19}
$$

Let us choose the coordinate cross-section defined by the four normalization equations

$$
\begin{aligned}
T &= e^{3\lambda_4}(t + \lambda_1) = 0, & \widehat{U} &= e^{-2\lambda_4}(u + \lambda_3) = 0,\\
X &= e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2) = 0, & \widehat{U}_T &= e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1.
\end{aligned}
\tag{16.20}
$$

On the subset $V = \{u_t + uu_x > 0\}$, the normalization equations can be solved for the group parameters:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + uu_x), \qquad (16.21)$$

thereby prescribing the moving frame. The existence of a moving frame implies that the action of G is locally free on the subset $\mathcal{V}^n = \{u_t + uu_x > 0\} \subset \mathrm{J}^n$ for all $n \geq 1$.

The differential invariants are obtained by invariantizing the jet coordinates

$$t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ \ldots \ ,$$

which is equivalent to substituting the moving frame expressions (16.21) into the prolonged action formulas (16.19). The constant phantom differential invariants

$$H^1 = \iota(t) = 0, \qquad H^2 = \iota(x) = 0, \qquad I_0 = \iota(u) = 0, \qquad I_{10} = \iota(u_t) = 1, \qquad (16.22)$$

result from our particular choice of normalization (16.20). Invariantizing the remaining coordinate functions yields a complete system of functionally independent normalized differential invariants:

$$
\begin{aligned}
I_{01} &= \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}, \\
I_{20} &= \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}, \\
I_{11} &= \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}, \\
I_{02} &= \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}, \\
I_{30} &= \iota(u_{ttt}) = \frac{u_{ttt} + 3uu_{ttx} + 3u^2 u_{txx} + u^3 u_{xxx}}{(u_t + uu_x)^{11/5}}, \\
I_{21} &= \iota(u_{ttx}) = \frac{u_{ttx} + 2uu_{txx} + u^2 u_{xxx}}{(u_t + uu_x)^{9/5}}, \\
I_{12} &= \iota(u_{txx}) = \frac{u_{txx} + uu_{xxx}}{(u_t + uu_x)^{7/5}}, \\
I_{03} &= \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x}, \qquad \ldots \ .
\end{aligned}
\qquad (16.23)
$$

The Replacement Rule (5.9) allows us to immediately rewrite the KdV equation in terms of the differential invariants by applying the invariantization process to it:

$$0 = \iota(u_t + uu_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + uu_x + u_{xxx}}{u_t + uu_x}.$$

Note the appearance of a nonzero multiplier indicating that the KdV equation is initially defined by a relative differential invariant. The invariant horizontal coframe

$$\omega^1 = (u_t + uu_x)^{3/5}\, dt, \qquad \omega^2 = -u(u_t + uu_x)^{1/5}\, dt + (u_t + uu_x)^{1/5}\, dx, \qquad (16.24)$$

is obtained by substituting (16.21) into the lifted horizontal coframe

$$d_H T = (T_t + u_t T_u)\, dt + (T_x + u_x T_u)\, dx = e^{3\lambda_4}\, dt,$$
$$d_H X = (X_t + u_t X_u)\, dt + (X_x + u_x X_u)\, dx = \lambda_3 e^{\lambda_4}\, dt + e^{\lambda_4}\, dx,$$

while the corresponding invariant differential operators

$$\mathcal{D}_1 = (u_t + u u_x)^{-3/5} D_t + u(u_t + u u_x)^{-3/5} D_x, \qquad \mathcal{D}_2 = (u_t + u u_x)^{-1/5} D_x,$$

can be found either by invoking duality (16.8). The invariant horizontal one-forms ω^1, ω^2 satisfy the structure equations

$$d_H \omega^1 = -\tfrac{3}{5}(I_{11} + I_{01}^2)\omega^1 \wedge \omega^2, \qquad d_H \omega^2 = \tfrac{1}{5}(I_{20} + 6I_{01})\omega^1 \wedge \omega^2. \qquad (16.25)$$

These imply the commutation formula

$$[\mathcal{D}_1, \mathcal{D}_2] = \tfrac{3}{5}(I_{11} + I_{01}^2)\mathcal{D}_1 - \tfrac{1}{5}(I_{20} + 6I_{01})\mathcal{D}_2. \qquad (16.26)$$

Higher order differential invariants can now be constructed by repeatedly applying the invariant differential operators to the lower order differential invariants, and hence can be expressed in terms of the normalized differential invariants. For example,

$$\mathcal{D}_1 I_{01} = -\tfrac{3}{5}I_{01}^2 + I_{11} - \tfrac{3}{5}I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = -\tfrac{3}{5}I_{01}^3 + I_{02} - \tfrac{3}{5}I_{01}I_{11},$$

as can be checked by a somewhat tedious explicit calculation. Similarly, the commutation formula (16.26) can be used to derive syzygies among the differentiated invariants. In the next section, we will develop an algorithm for constructing the recurrence formulas and syzygies in a much simpler, direct fashion.

Since the basic differential invariants arising from invariantization of the jet coordinates form a complete system, any other differential invariant, e.g., those constructed by application of the invariant differential operators, can be locally written as a function thereof. The recurrence formulae, cf. [52, 87], connect the differentiated invariants and forms with their normalized counterparts. These formulae are fundamental, since they prescribe the structure of the algebra of (local) differential invariants, underly a full classification of generating differential invariants and their differential syzygies, as well as the structure of invariant variational problems and, indeed, the entire invariant variational bicomplex. As in the finite-dimensional version, the recurrence formulae are established, through just linear algebra and differentiation, using only the formulas for the prolonged infinitesimal generators and the cross-section. In particular, they do *not* require the explicit formulae for either the moving frame, or the Maurer–Cartan forms, or the normalized differential invariants and invariant forms, or even the invariant differential operators!

Let $\nu^{(\infty)} = (\rho^{(\infty)})^* \mu^{(\infty)}$ denote the pulled-back Maurer–Cartan forms via the complete moving frame section $\rho^{(\infty)}$, with individual components

$$\nu_A^b = (\rho^{(\infty)})^* (\mu_A^b) = \sum_{i=1}^p S_{A,i}^b\, \omega^i + \sum_{\alpha,K} T_{A,\alpha}^{b,K}\, \vartheta_K^\alpha, \qquad b = 1, \ldots, m, \qquad \#A \geq 0, \quad (16.27)$$

where the coefficients $S_{A,i}^b, T_{A,\alpha}^{b,K}$ will be called the *Maurer–Cartan invariants*. Their precise formulas will follow directly from the recurrence relations for the phantom differential invariants. In view of Theorem 15.3, the pulled-back Maurer–Cartan forms are subject to the linear relations

$$L^{(n)}(H, I, \nu^{(n)}) = \iota\big[\, L^{(n)}(z, \zeta^{(n)}) \,\big] = 0, \qquad n \geq 0, \qquad (16.28)$$

obtained by invariantizing the original linear determining equations (15.10), where we set $\iota(\zeta_A^b) = \nu_A^b$, and where $(H, I) = \iota(x, u) = \iota(z)$ are the zero$^{\text{th}}$ order differential invariants in (16.6). In particular, if $\mathcal{G}$ acts transitively on M, then, since we are using a minimal order moving frame, (H, I) are constant phantom invariants.

Given a locally defined vector field

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \in \mathcal{X}(M), \qquad (16.29)$$

let

$$\mathbf{v}^{(\infty)} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J \geq 0} \widehat{\varphi}_J^\alpha(x, u^{(k)}) \frac{\partial}{\partial u_J^\alpha} \in \mathcal{X}(\mathrm{J}^\infty(M, p)) \qquad (16.30)$$

denote its infinite prolongation. The coefficients are computed via the usual prolongation formula,

$$\widehat{\varphi}_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} u_{J,i}^\alpha \xi^i, \qquad \text{where} \qquad Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi^i, \qquad \alpha = 1, \ldots, q, \qquad (16.31)$$

are the components of the *characteristic* of $\mathbf{v}$; cf. [**116**, **117**]. Consequently, each prolonged vector field coefficient

$$\widehat{\varphi}_J^\alpha = \Phi_J^\alpha(u^{(n)}, \zeta^{(n)}) \qquad (16.32)$$

is a certain universal linear combination of the vector field jet coordinates, whose coefficients are polynomials in the submanifold jet coordinates u_K^β for $1 \leq \#K \leq n$. Let

$$\eta^i = \iota(\xi^i) = \nu^i, \qquad \widehat{\psi}_J^\alpha = \iota(\widehat{\varphi}_J^\alpha) = \Phi_J^\alpha(I^{(n)}, \nu^{(n)}), \qquad (16.33)$$

denote their invariantizations, which are certain linear combinations of the pulled-back Maurer–Cartan forms ν_A^b, whose coefficients are polynomials in the normalized differential invariants I_K^β for $1 \leq \#K \leq \#J$.

With all these in hand, the desired *universal recurrence formula* is as follows.

Theorem 16.7. *If Ω is any differential form on J^∞, then*

$$d \iota(\Omega) = \iota\big[\, d\Omega + \mathbf{v}^{(\infty)}(\Omega) \,\big], \qquad (16.34)$$

where $\mathbf{v}^{(\infty)}(\Omega)$ denotes the Lie derivative of Ω with respect to the prolonged vector field (16.30), and we use (16.33) and its analogs for the partial derivatives of the prolonged vector field coefficients when invariantizing the result.

Specializing Ω in (16.34) to be one of the coordinate functions x^i, u_J^α yields recurrence formulae for the normalized differential invariants (16.6),

$$dH^i = \iota\left(dx^i + \xi^i \right) = \varpi^i + \eta^i,$$

$$dI_J^\alpha = \iota\left(du_J^\alpha + \widehat{\varphi}_J^\alpha \right) = \iota\left(\sum_{i=1}^p u_{J,i}^\alpha \, dx^i + \theta_J^\alpha + \widehat{\varphi}_J^\alpha \right) = \sum_{i=1}^p I_{J,i}^\alpha \, \varpi^i + \vartheta_J^\alpha + \widehat{\psi}_J^\alpha, \qquad (16.35)$$

where, as in (16.33), each $\widehat{\psi}_J^\alpha$ is written in terms of the pulled-back Maurer–Cartan forms ν_A^b, which are subject to the linear constraints (16.28). Each phantom differential invariant is, by definition, normalized to a constant value, and hence has zero differential. Consequently, the phantom recurrence formulae in (16.35) form a system of linear algebraic equations which can, as a result of the transversality of the cross-section, be uniquely solved for the pulled-back Maurer–Cartan forms.

Theorem 16.8. *If the pseudo-group acts locally freely on $\mathcal{V}^n \subset \mathrm{J}^n$, then the nth order phantom recurrence formulae can be uniquely solved to express the pulled-back Maurer–Cartan forms ν_A^b of order $\#A \leq n$ as invariant linear combinations of the invariant horizontal and contact one-forms $\varpi^i, \vartheta_J^\alpha$.*

Substituting the resulting expressions (16.27) into the remaining, non-phantom recurrence formulae in (16.35) leads to a complete system of recurrence relations, for both the vertical and horizontal differentials of all the normalized differential invariants. In particular, equating the coefficients of the forms ω^i leads to individual recurrence formulae for the normalized differential invariants:

$$\mathcal{D}_i H^j = \delta_i^j + M_i^j, \qquad\qquad \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha, \qquad (16.36)$$

where δ_i^j is the Kronecker delta, and the *correction terms* $M_i^j, M_{J,i}^\alpha$ are certain invariant linear combinations of the Maurer–Cartan invariants $S_{A,i}^b$. One complication, which will be dealt with in the following section, is that the correction term $M_{J,i}^\alpha$ can have the same order as the initial differential invariant $I_{J,i}^\alpha$.

It is worth pointing out that, since the prolonged vector field coefficients $\widehat{\varphi}_J^\alpha$ are polynomials in the jet coordinates u_K^β of order $\#K \geq 1$, their invariantizations are polynomial functions of the differential invariants I_K^β for $\#K \geq 1$. Since the correction terms are constructed by solving a linear system for the invariantized Maurer–Cartan forms (16.27), the Maurer–Cartan invariants depend rationally on these differential invariants. Thus, in most cases (including the majority of applications), the resulting differential invariant algebra is endowed with an entirely rational algebraic recurrence structure.

Theorem 16.9. *If $\mathcal{G}$ acts transitively on M, or, more generally, its infinitesimal generators depend polynomially on the coordinates $z = (x, u) \in M$, then the correction terms $M_i^j, M_{J,i}^\alpha$ in the recurrence formulas (16.35) are rational functions of the basic differential invariants.*

Let (15.10) be the formally integrable completion of the linearized determining equations of a pseudo-group $\mathcal{G}$. At each $z \in M$, we let $\mathcal{I}|_z$ denote the *symbol module* of the

determining equations, which, by formal integrability, forms a submodule of $\mathcal{T} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m$ consisting of real polynomials

$$\eta(t, T) = \sum_{a=1}^{m} \eta_a(t) \, T^a$$

in $t = (t_1, \ldots, t_m)$ and $T = (T^1, \ldots, T^m)$ that are linear in the T's.

Analogously, let $\widehat{\mathcal{S}} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q$ denote the module consisting of polynomials

$$\widehat{\sigma}(s, S) = \sum_{\alpha=1}^{q} \widehat{\sigma}_\alpha(s) \, S^\alpha$$

in $s = (s_1, \ldots, s_p)$, $S = (S^1, \ldots, S^q)$, which are linear in the S's. At each submanifold 1-jet $z^{(1)} = (x, u^{(1)}) \in \mathrm{J}^1(M, p)$, we define a linear map $\boldsymbol{\beta}|_{z^{(1)}} \colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ by the formulas

$$s_i = \beta_i(z^{(1)}; t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha \, t_{p+\alpha}, \qquad\qquad i = 1, \ldots, p,$$

$$S^\alpha = B^\alpha(z^{(1)}; T) = T^{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha \, T^i, \qquad\qquad \alpha = 1, \ldots, q. \tag{16.37}$$

Definition 16.10. The *prolonged symbol submodule* at $z^{(1)} \in \mathrm{J}^1|_z$ is the inverse image of the symbol module under the pull-back map $(\boldsymbol{\beta}|_{z^{(1)}})^*$:

$$\mathcal{J}|_{z^{(1)}} = ((\boldsymbol{\beta}|_{z^{(1)}})^*)^{-1}(\mathcal{I}|_z) = \left\{ \, \sigma(s, S) \mid (\boldsymbol{\beta}|_{z^{(1)}})^*(\sigma) \in \mathcal{I}|_z \, \right\} \subset \widehat{\mathcal{S}}. \tag{16.38}$$

It can be proved that, when the pseudo-group admits a moving frame, the module $\mathcal{J}|_{z^{(1)}}$ coincides with the symbol module associated with the prolonged infinitesimal generators.

To relate this construction to the differential invariant algebra, we invariantize the modules using a moving frame. In general, the invariantization of a prolonged symbol polynomial

$$\sigma(x, u^{(1)}; s, S) = \sum_{\alpha, J} h_\alpha^J(x, u^{(1)}) \, s_J S^\alpha \in \mathcal{J}|_{z^{(1)}},$$

is given by

$$\widetilde{\sigma}(H, I^{(1)}; s, S) = \iota\left[\sigma(x, u^{(1)}; s, S)\right] = \sum_{\alpha, J} h_\alpha^J(H, I^{(1)}) \, s_J S^\alpha, \tag{16.39}$$

which we identify with the differential invariant

$$I_{\widetilde{\sigma}} = \sum_{\alpha, J} h_\alpha^J(H, I^{(1)}) \, I_J^\alpha.$$

Let $\widetilde{\mathcal{J}}|_{(H, I^{(1)})} = \iota(\mathcal{J}|_{z^{(1)}})$ denote the *invariantized prolonged symbol submodule*.

The recurrence formulae for the differential invariants $I_{\widetilde{\sigma}}$ take the form

$$\mathcal{D}_i I_{\widetilde{\sigma}} = I_{s_i\,\widetilde{\sigma}} + M_{\widetilde{\sigma},i}, \tag{16.40}$$

in which, unlike in (16.36), when $\deg \widetilde{\sigma} \gg 0$, the *leading term* $I_{s_i\,\widetilde{\sigma}}$ is strictly of higher order that the *correction term*. Now iteration of (16.40) leads to the Constructive Basis Theorem for differential invariants.

Theorem 16.11. *Let $\mathcal{G}$ be a Lie pseudo-group admitting a moving frame on an open subset of the submanifold jet bundle at order $n^\star$. Then a finite generating system for its algebra of local differential invariants is given by:*

(a) *the differential invariants $I_\nu = I_{\sigma_\nu}$, where $\sigma_1, \ldots, \sigma_l$ form a Gröbner basis for the invariantized prolonged symbol submodule $\widetilde{\mathcal{J}}$, and, possibly,*

(b) *a finite number of additional differential invariants of order $\leq n^\star$.*

We are also able to exhibit a finite generating system of differential invariant syzygies. First, owing to the non-commutative nature of the the invariant differential operators $\mathcal{D}_i$, we have the *commutator syzygies*

$$\mathcal{D}_J I_{\widetilde{\sigma}} - \mathcal{D}_{\widetilde{J}} I_{\widetilde{\sigma}} = M_{\widetilde{\sigma},J} - M_{\widetilde{\sigma},\widetilde{J}} \equiv N_{J,\widetilde{J},\widetilde{\sigma}}, \quad \text{whenever} \quad \widetilde{J} = \pi(J) \tag{16.41}$$

for some permutation π. Provided $\deg \widetilde{\sigma} > n^\star$, the right hand side $N_{J,\widetilde{J},\widetilde{\sigma}}$ is of lower order than the terms on the left hand side.

In addition, any algebraic syzygy satisfied by polynomials in $\widetilde{\mathcal{J}}|_{(H,I^{(1)})}$ provides an additional syzygy amongst the differentiated invariants. In detail, to each invariantly parametrized polynomial

$$q(H, I^{(1)}; s) = \sum_J q_J(H, I^{(1)}) s_J \in \mathbb{R}[s]$$

we associate an invariant differential operator

$$q(H, I^{(1)}; \mathcal{D}) = \sum_J q_J(H, I^{(1)}) \mathcal{D}_J, \tag{16.42}$$

where the sum ranges over non-decreasing multi-indices. In view of (16.40), whenever $\widetilde{\sigma}(H, I^{(1)}; s, S) \in \widetilde{\mathcal{J}}|_{(H,I^{(1)})}$, we can write

$$q(H, I^{(1)}; \mathcal{D})\, I_{\widetilde{\sigma}(H,I^{(1)};s,S)} = I_{q(H,I^{(1)};s)\,\widetilde{\sigma}(H,I^{(1)};s,S)} + R_{q,\widetilde{\sigma}}, \tag{16.43}$$

where $R_{q,\widetilde{\sigma}}$ has order $< \deg q + \deg \widetilde{\sigma}$. Thus, any algebraic syzygy

$$\sum_{\nu=1}^{l} q_\nu(H, I^{(1)}, s)\, \sigma_\nu(H, I^{(1)}; s, S) = 0$$

of the Gröbner basis polynomials of $\widetilde{\mathcal{J}}|_{(H,I^{(1)})}$ induces a syzygy among the generating differential invariants,

$$\sum_{\nu=1}^{l} q_\nu(H, I^{(1)}, \mathcal{D})\, I_{\widetilde{\sigma}_\nu} = R, \qquad \text{where} \qquad \text{order}\, R < \max\{\deg q_\nu + \deg \widetilde{\sigma}_\nu\}.$$

Theorem 16.12. *Every differential syzygy among the generating differential invariants is a combination of the following:*

(a) *the syzygies among the differential invariants of order $\leq n^\star$,*

(b) *the commutator syzygies,*

(c) *syzygies coming from an algebraic syzygy among the Gröbner basis polynomials.*

In this manner, we deduce a finite system of generating differential syzygies for the differential invariant algebra of our pseudo-group.

Further details and applications of these results can be found in our papers listed in the references.

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