Classification of Lie Groups Acting on the Plane

Table 1

Transitive, Imprimitive Lie Algebras of Vector Fields in $\mathbb{C}^2$

	Generators	Dim	Structure
1.1.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - 2xu\partial_u$	3	$\mathfrak{sl}(2)$
1.2.	$\partial_x, x\partial_x - u\partial_u, x^2\partial_x - (2xu+1)\partial_u$	3	$\mathfrak{sl}(2)$
1.3.	$\partial_x, x\partial_x, u\partial_u, x^2\partial_x - xu\partial_u$	4	$\mathfrak{gl}(2)$
1.4.	$\partial_x, x\partial_x, x^2\partial_x, \partial_u, u\partial_u, u^2\partial_u$	6	$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$
1.5.	$\partial_x, \eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
1.6.	$\partial_x, u\partial_u, \eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	$k+2$	$\mathbb{C}^2 \ltimes \mathbb{C}^k$
1.7.	$\partial_x, x\partial_x + \alpha u\partial_u, \partial_u, x\partial_u, \ldots, x^{k-1}\partial_u$	$k+2$	$\mathfrak{a}(1) \ltimes \mathbb{C}^k$
1.8.	$\partial_x, x\partial_x + (ku + x^k)\partial_u, \partial_u, x\partial_u, \ldots, x^{k-1}\partial_u$	$k+2$	$\mathbb{C} \ltimes (\mathbb{C} \ltimes \mathbb{C}^k)$
1.9.	$\partial_x, x\partial_x, u\partial_u, \partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+3$	$(\mathfrak{a}(1) \oplus \mathbb{C}) \ltimes \mathbb{C}^k$
1.10.	$\partial_x, 2x\partial_x + (k-1)u\partial_u, x^2\partial_x + (k-1)xu\partial_u,$ $\partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+3$	$\mathfrak{sl}(2) \ltimes \mathbb{C}^k$
1.11.	$\partial_x, x\partial_x, x^2\partial_x + (k-1)xu\partial_u, u\partial_u,$ $\partial_u, x\partial_u, x^2\partial_u, \ldots, x^{k-1}\partial_u$	$k+4$	$\mathfrak{gl}(2) \ltimes \mathbb{C}^k$

In Cases 1.5 and 1.6, the functions $\eta_1(x), \ldots, \eta_k(x)$ satisfy a k^{th} order constant coefficient homogeneous linear ordinary differential equation $\mathcal{D}[u] = 0$.

In Cases 1.5 – 1.11 we require $k \geq 1$. Note, though, that if we set $k = 0$ in Case 1.10, and replace u by u^2, we obtain Case 1.1. Similarly, if we set $k = 0$ in Case 1.11, we obtain Case 1.3. Cases 1.7 and 1.8 for $k = 0$ are equivalent to the Lie algebra $\{\partial_x, e^x\partial_u\}$ of type 1.5. Case 1.9 for $k = 0$ is equivalent to the Lie algebra $\{\partial_x, \partial_u, u\partial_u\}$ of type 1.6.

Table 2

Primitive Lie Algebras of Vector Fields in $\mathbb{C}^2$

	Generators	Dim	Structure
2.1.	$\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u$	5	$\mathfrak{sa}(2)$
2.2.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u$	6	$\mathfrak{a}(2)$
2.3.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u,$ $x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$

Table 3

Intransitive Lie Algebras of Vector Fields in $\mathbb{C}^2$

	Generators	Dim	Structure
3.1.	$\eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u$	k	$\mathbb{C}^k$
3.2.	$\eta_1(x)\partial_u, \ldots, \eta_k(x)\partial_u, u\partial_u$	$k+1$	$\mathbb{C} \ltimes \mathbb{C}^k$
3.3.	$\partial_u, u\partial_u, u^2\partial_u$	3	$\mathfrak{sl}(2)$

Table 4

Primitive Lie Algebras of Vector Fields in $\mathbb{R}^2$

	Generators	Dim	Structure	$\mathbb{C}$ Type
6.1.	$\partial_x, \partial_u, \alpha(x\partial_x + u\partial_u) + u\partial_x - x\partial_u$	3	$\mathbb{R} \ltimes \mathbb{R}^2$	1.7
6.2.	$\partial_x, x\partial_x + u\partial_u, (x^2 - u^2)\partial_x + 2xu\partial_u$	3	$\mathfrak{sl}(2)$	1.2
6.3.	$u\partial_x - x\partial_u, (1 + x^2 - u^2)\partial_x + 2xu\partial_u,$ $2xu\partial_x + (1 - x^2 + u^2)\partial_u$	3	$\mathfrak{so}(3)$	1.1
6.4.	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u$	4	$\mathbb{R}^2 \ltimes \mathbb{R}^2$	1.9
6.5.	$\partial_x, \partial_u, x\partial_x - u\partial_u, u\partial_x, x\partial_u$	5	$\mathfrak{sa}(2)$	2.1
6.6.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u$	6	$\mathfrak{a}(2)$	2.2
6.7.	$\partial_x, \partial_u, x\partial_x + u\partial_u, u\partial_x - x\partial_u,$ $(x^2 - u^2)\partial_x + 2xu\partial_u,$ $2xu\partial_x + (u^2 - x^2)\partial_u$	6	$\mathfrak{so}(3,1)$	1.4
6.8.	$\partial_x, \partial_u, x\partial_x, u\partial_x, x\partial_u, u\partial_u,$ $x^2\partial_x + xu\partial_u, xu\partial_x + u^2\partial_u$	8	$\mathfrak{sl}(3)$	2.3

Table 5

Differential Invariants of Transformation Groups in $\mathbb{C}^2$

	Fundamental differential invariant(s)	Invariant one-form	Lie determinant
1.1.	$u^{-4}(2uu_2 - 3u_1^2)$	$u\,dx$	u^2
1.2.	$(u_1 - u^2)^{-3/2}(u_2 - 6uu_1 + 4u^3)$	$\sqrt{u_1 - u^2}\,dx$	$u_1 - u^2$
1.3.	$Q_2^{-3/2} S_3$	$u^{-1}\sqrt{Q_2}\,dx$	$u\,Q_2$
1.4.	$Q_3^{-3} U_5$	$u_1^{-1}\sqrt{Q_3}\,dx$	$u_1 Q_3^2$
1.5.	$W(x)^{-1}\mathcal{D}[u]$	dx	$W(x)$
1.6.	$D_x \log \mathcal{D}[u]$	dx	$W(x)\mathcal{D}[u]$
1.7a.	$u_k^{(\alpha-k)^{-1}-1} u_{k+1}$	$u_k^{-(\alpha-k)^{-1}}\,dx$	u_k
1.7b.	$u_k, \quad u_{k+1}^{-2} u_{k+2}$	$u_{k+1}\,dx$	u_{k+1}
1.8.	$u_{k+1} e^{u_k/k!}$	$e^{-u_k/k!}\,dx$	1
1.9.	$u_{k+1}^{-2} u_k u_{k+2}$	$u_k^{-1} u_{k+1}\,dx$	$u_k u_{k+1}$
1.10.	$u_k^{-2(k+3)/(k+1)} Q_{k+2}$	$u_k^{2/(k+1)}\,dx$	u_k^2
1.11.	$Q_{k+2}^{-3/2} S_{k+3}$	$u_k^{-1}\sqrt{Q_{k+2}}\,dx$	$u_k Q_{k+2}$
2.1.	$u_2^{-8/3} R_4$	$u_2^{1/3}\,dx$	u_2^3
2.2.	$R_4^{-3/2} S_5$	$u_2^{-1}\sqrt{R_4}\,dx$	$u_2^2 R_4$
2.3.	$S_5^{-8/3} V_7$	$u_2^{-1} S_5^{1/3}\,dx$	$u_2 S_5^2$
3.1.	$x, \quad \mathcal{D}[u]$	dx	$W(x)$
3.2.	$x, \quad D_x \log \mathcal{D}[u]$	dx	$W(x)\mathcal{D}[u]$
3.3.	$x, \quad u_1^{-2} Q_3$	dx	u_1^3

In Table 5, for given functions $\eta_1(x), \ldots, \eta_k(x)$, we let $\mathcal{D}$ be a k^{th} order linear ordinary differential operator whose kernel is spanned by $\eta_1(x), \ldots, \eta_k(x)$, and let $W(x)$ denote their Wronskian determinant. Also,

$$Q_{k+2} = (k+1)u_k u_{k+2} - (k+2)u_{k+1}^2, \qquad R_4 = 3u_2 u_4 - 5u_3^2,$$
$$S_{k+3} = (k+1)^2 u_k^2 u_{k+3} - 3(k+1)(k+3)u_k u_{k+1} u_{k+2} + {} + 2(k+2)(k+3)u_{k+1}^3,$$
$$T_7 = 10u_3^3 u_7 - 70u_3^2 u_4 u_6 - 49u_3^2 u_5^2 + 280u_3 u_4^2 u_5 - 175u_4^4,$$
$$U_5 = u_1^2\big[Q_3 D_x^2 Q_3 - \tfrac{5}{4}(D_x Q_3)^2\big] + u_1 u_2 Q_3 D_x Q_3 - (2u_1 u_3 - u_2^2)Q_3^2,$$
$$V_7 = u_2^2\big[S_5 D_x^2 S_5 - \tfrac{7}{6}(D_x S_5)^2\big] + u_2 u_3 S_5 D_x S_5 - \tfrac{1}{2}(9u_2 u_4 - 7u_3^2)S_5^2.$$

