Moving Frames

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Moving Frames

Classical contributions:

G. Darboux, É. Cotton, É. Cartan

Modern contributions:

P. Griffiths, M. Green, G. Jensen

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry"
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Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants and Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
- Computer vision
    - object recognition
    - symmetry detection
- Invariant numerical methods
- Poisson geometry & solitons
- Lie pseudogroups

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Equivalence:

Determine when two n-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \;\subset\; M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Self-equivalence or *self-congruence*:

$$N = g \cdot N$$

Classical Geometry

Equivalence Problem: Determine whether or not two given submanifolds N and $\overline{N}$ are congruent under a group transformation: $\overline{N} = g \cdot N$.

Symmetry Problem: Given a submanifold N, find all its symmetries (belonging to the group).

- *Euclidean group* — $G = \mathrm{SE}(n)$ or $\mathrm{E}(n)$
  - $\Rightarrow$ isometries of Euclidean space
  - $\Rightarrow$ translations, rotations (& reflections)

$$z \longmapsto R \cdot z + a \qquad \begin{cases} R \in \mathrm{SO}(n) \text{ or } \mathrm{O}(n) \\ a \in \mathbb{R}^n \\ z \in \mathbb{R}^n \end{cases}$$

- *Equi-affine group*: $G = \mathrm{SA}(n)$
  $R \in \mathrm{SL}(n)$ — area-preserving
- *Affine group*: $G = \mathrm{A}(n)$
  $R \in \mathrm{GL}(n)$
- *Projective group*: $G = \mathrm{PSL}(n)$
  acting on $\mathbb{RP}^{n-1}$

$\Longrightarrow$ Applications in computer vision

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

$\Rightarrow$ multiplier representation of GL(2)
$\Rightarrow$ modular forms

Transformation group:

$$g\colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \, \frac{u}{(\gamma x + \delta)^n} \right)$$

Equivalence of functions $\Longleftrightarrow$ equivalence of graphs

$$N_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Moving Frames

Definition.

A *moving frame* is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Necessity: Let $z \in M$.

Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

Regularity: Suppose

$$z_n = g_n \cdot z \longrightarrow z \qquad \text{as} \qquad n \to \infty$$

By continuity,

$$\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$$

Hence $g_n \longrightarrow e$ in G.

Sufficiency: By construction — “normalization”.

Q.E.D.

Isotropy

Isotropy subgroup for $z \in M$:

$$G_z = \{ \, g \mid \; g \cdot z = z \, \}$$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $G_z = \{e\}$ for all $z \in M$.

- locally free — the orbits all have the same dimension as G:
  G_z is a discrete subgroup of G.

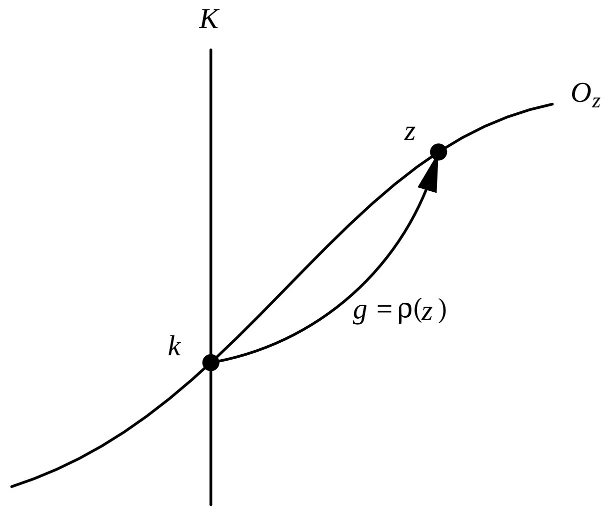
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  ($\not\approx$ irrational flow on the torus)

- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$:

$$G_M = \bigcap_{z \in M} G_z = \{e\}$$

Geometrical Construction

Normalization = choice of cross-section to the group orbits



K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Construction of Moving Frames

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right	
$w(g,z) = g^{-1} \cdot z$	$w(g,z) = g \cdot z$	

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

$\Longrightarrow$ Implicit Function Theorem

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants:

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by $I(z) = F(\rho(z) \cdot z)$.

$$\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, , \ldots c_r, I_1(z), \ldots, I_{m-r}(z))$$

Invariantization amounts to restricting F to the cross-section

$$I \mid K = F \mid K$$

and then requiring that $I = \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota : \quad \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x,u) \;\longmapsto\; g \cdot z = (\, x\cos\theta - u\sin\theta \;,\; x\sin\theta + u\cos\theta \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g,z) = g^{-1} \cdot z = (y,v)$$

$$y = x\cos\theta + u\sin\theta \qquad v = -\,x\sin\theta + u\cos\theta$$

Cross-section

$$K = \{\, u = 0,\; x > 0 \,\}$$

Normalization equation

$$v = -\,x\sin\theta + u\cos\theta = 0$$

Left moving frame:

$$\theta = \tan^{-1}\frac{u}{x} \quad \Longrightarrow \quad \theta = \rho(x,u) \in \mathrm{SO}(2)$$

Fundamental invariant

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization

$$\iota[\, F(x,u) \,] = F(r,0)$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Jet Space

- Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

- Jet space is the proper setting for the geometry of partial differential equations.

M — smooth m-dimensional manifold

$$1 \leq p \leq m - 1$$

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — (extended) jet bundle

$\Longrightarrow$ Defined as the space of equivalence classes of p-dimensional submanifolds under the equivalence relation of n^{th} order contact at a single point.

$\Longrightarrow$ Can be identified as the space of n^{th} order Taylor polynomials for submanifolds given as graphs $u = f(x)$

Local Coordinates on Jet Space

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — n^{th} extended jet bundle for
p-dimensional submanifolds $N \subset M$

Local coordinates:

Assume $N = \{\, u = f(x) \,\}$ is a graph (section).

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$$p + q = m = \dim M$$

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

$$u^\alpha_J = \partial_J u^\alpha \qquad 0 \le \#J \le n$$

— induced jet coordinates

- No bundle structure assumed on M.
- Projective completion of $\mathrm{J}^n E$ when $E \to X$ is a bundle.

Prolongation of Group Actions

G — transformation group acting on M

$\Longrightarrow$ G maps submanifolds to submanifolds
and preserves the order of contact

$G^{(n)}$ — prolonged action of G on the jet space J^n

The prolonged group formulae

$$w^{(n)} = (y, v^{(n)}) = g^{(n)} \cdot z^{(n)}$$

are obtained by implicit differentiation:

$$dy^i = \sum_{j=1}^{p} P^i_j(g, z^{(1)})\, dx^j$$

$$D_{y^j} = \sum_{i=1}^{p} Q^i_j(g, z^{(1)})\, D_{x^i}$$

$$\Longrightarrow \quad Q = P^{-T}$$

$$v^\alpha_J = D_{y^{j_1}} \cdots D_{y^{j_k}}(v^\alpha)$$

Differential invariant $\quad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

$\Longrightarrow$ curvatures

Freeness

Theorem. If G acts (locally) effectively on M, then G acts (locally) freely on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$.

Definition. $N \subset M$ is *regular* at order n if $\mathrm{j}_n N \subset \mathcal{V}^n$.

Corollary. Any regular submanifold admits a (local) moving frame.

Theorem. A submanifold is totally singular, $\mathrm{j}_n N \subset \mathrm{J}^n \setminus \mathcal{V}^n$ for all n, if and only if its symmetry group

$$G_N = \{\, g \mid \, g \cdot N \subset N \,\}$$

does not act freely on N.

Moving Frames on Jet Space

$$w^{(n)} = (y, v^{(n)}) = \begin{cases} g^{(n)} \cdot z^{(n)} & \text{right} \\ (g^{(n)})^{-1} \cdot z^{(n)} & \text{left} \end{cases}$$

Choose $r = \dim G$ jet coordinates

$$z_1, \ldots, z_r \qquad\qquad x^i \text{ or } u^\alpha_J$$

Coordinate cross-section $K \subset \mathrm{J}^n$

$$z_1 = c_1 \quad \ldots \quad z_r = c_r$$

Corresponding lifted differential invariants:

$$w_1, \ldots, w_r \qquad\qquad y^i \text{ or } v^\alpha_J$$

Normalization Equations

$$w_1(g, x, u^{(n)}) = c_1 \quad \ldots \quad w_r(g, x, u^{(n)}) = c_r$$

Solution:

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)}) \qquad \Longrightarrow \quad \text{moving frame}$$

The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = w^{(n)}(\rho^{(n)}(z^{(n)}), z^{(n)})$$

$$\boxed{\begin{aligned} H^i(x, u^{(n)}) &= y^i(\rho^{(n)}(x, u^{(n)}), x, u) \\ I^\alpha_K(x, u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}) \end{aligned}}$$

Phantom differential invariants

$$w_1 = c_1 \; \ldots \; w_r = c_r \qquad \Longrightarrow \quad \text{normalizations}$$

Theorem. Every n^{th} order differential invariant can be locally uniquely written as a function of the non-phantom fundamental differential invariants in $I^{(n)}$.

Invariant Differentiation

Contact-invariant coframe

$$dy^i \quad \longmapsto \quad \omega^i = \sum_{j=1}^{p} P_j^i(\rho^{(n)}(z^{(n)}), z^{(n)})\; dx^i$$

$\Longrightarrow$ arc length element

Invariant differential operators:

$$D_{y^j} \quad \longmapsto \quad \mathcal{D}_j = \sum_{i=1}^{p} Q_j^i(\rho^{(n)}(z^{(n)}), z^{(n)})\, D_{x^i}$$

$\Longrightarrow$ arc length derivative

Duality:

$$dF = \sum_{i=1}^{p} \mathcal{D}_i F \cdot \omega^i$$

Theorem. The higher order differential invariants are obtained by invariant differentiation with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_p$.

Euclidean Curves $G = \mathrm{SE}(2)$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to J^3 via implicit differentiation

$$\left.\begin{aligned} y &= \cos\theta\,(x-a) + \sin\theta\,(u-b) \\ v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-b)$$

$$v_y = \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3}$$

$$v_{yyy} = \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}$$

$$\vdots$$

Normalization $r = \dim G = 3$

$$y = 0, \qquad v = 0, \qquad v_y = 0$$

Left moving frame $\rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x, \qquad b = u, \qquad \theta = \tan^{-1} u_x$$

Differential invariants

$$v_{yy} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Invariant one-form — arc length

$$dy = (\cos\theta + u_x \sin\theta)\, dx \quad \longmapsto \quad ds = \sqrt{1+u_x^2}\ dx$$

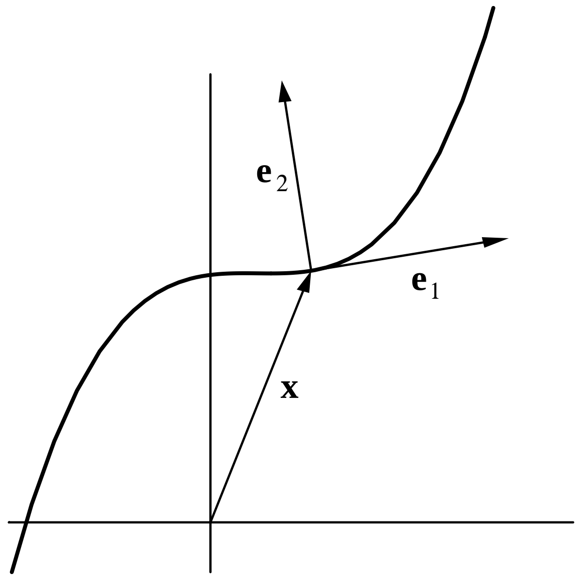
Invariant differential operator

$$\frac{d}{dy} = \frac{1}{\cos\theta + u_x\sin\theta}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

Euclidean Curves



Moving frame $\qquad \rho:\ (x,u,u_x) \longmapsto (R,\mathbf{a}) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1+u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Frenet frame

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix} \qquad \mathbf{e}_2 = \mathbf{e}_1^{\perp} = \begin{pmatrix} -y_s \\ x_s \end{pmatrix}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\, \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = -\,\kappa\, \mathbf{e}_1$$

The Replacement Theorem

Any differential invariant has the form

$$I = F(x, u^{(n)}) = F(y, w^{(n)}) = F(I^{(n)})$$

$\Longrightarrow$ T.Y. Thomas

$$\kappa = \frac{v_{yy}}{(1+v_y^2)^2} = \frac{u_{xx}}{(1+u_x^2)^2}$$

$$\iota(x) = \iota(u) = (u_x) = 0$$

$$\iota(u_{xx}) = \kappa$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\,z + b \qquad A \in \mathrm{SL}(2), \qquad b \in \mathbb{R}^2$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - u_x\beta)\; dx \qquad D_y = \frac{1}{\delta - u_x\beta}\, D_x$$

$$\left.\begin{aligned} y &= \delta(x-a) - \beta(u-b) \\ v &= -\,\gamma(x-a) + \alpha(u-b) \end{aligned}\right\} \qquad w = A^{-1}(z-b)$$

$$v_y = -\,\frac{\gamma - \alpha u_x}{\delta - \beta u_x} \qquad v_{yy} = -\,\frac{u_{xx}}{(\delta - \beta u_x)^3}$$

$$v_{yyy} = -\,\frac{(\delta - \beta u_x)u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5}$$

$$v_{yyyy} = -\,\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 u_{xx}u_{xxx}\beta(\delta - \beta u_x) + 15u_{xx}^3\beta^2}{(\alpha + \beta u_x)^7}$$

$$\vdots$$

Nondegeneracy $\qquad u_{xx} = 0$

$\Longrightarrow$ Straight lines are totally singular

(three-dimensional equi-affine symmetry group)

Normalization $\qquad r = \dim G = 5$

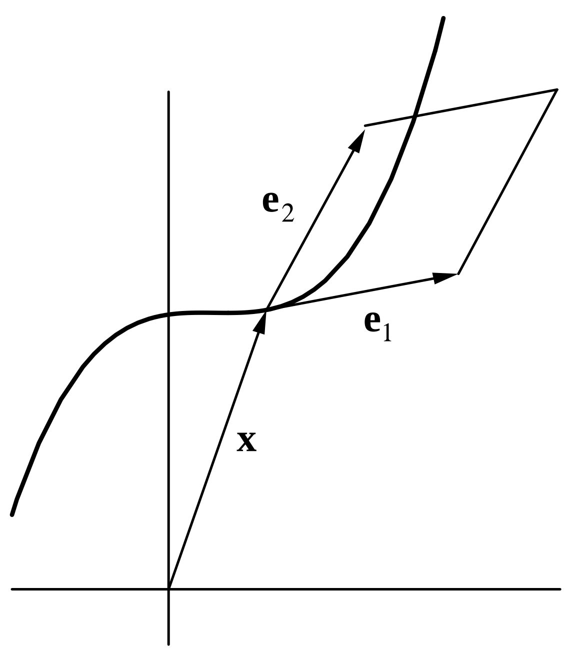
$$y = 0, \quad v = 0, \quad v_y = 0, \quad v_{yy} = 1, \quad v_{yyy} = 0.$$

Left Moving frame $\quad \rho : \mathrm{J}^3 \longrightarrow \mathrm{SA}(2)$

$$A = \begin{pmatrix} \sqrt[3]{u_{xx}} & -\frac{1}{3} u_{xx}^{-5/3} u_{xxx} \\ u_x \sqrt[3]{u_{xx}} & u_{xx}^{-1/3} - \frac{1}{3} u_{xx}^{-5/3} u_{xxx} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{dz}{ds} & \dfrac{d^2 z}{ds^2} \end{pmatrix}$$

$$\mathbf{b} = z = \begin{pmatrix} x \\ u \end{pmatrix}$$



Frenet frame

$$\mathbf{e}_1 = \frac{dz}{ds} \qquad\qquad \mathbf{e}_2 = \frac{d^2 z}{ds^2}$$

Frenet equations = Maurer–Cartan equations:

$$\frac{dz}{ds} = \mathbf{e}_1 \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2 \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\, \mathbf{e}_1$$

Equi-affine arc length

$$dy \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\; dx = \sqrt[3]{\dot z \wedge \ddot z}\; dt$$

Invariant differential operator

$$D_y \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\, D_x = \frac{1}{\sqrt[3]{\dot z \wedge \ddot z}}\, D_t$$

Equi-affine curvature

$$v_{4y} \quad \longmapsto \quad \kappa = \frac{5 u_{xx} u_{xxxx} - 3 u_{xxx}^2}{9 u_{xx}^{8/3}} = z_s \wedge z_{ss}$$

$$v_{5y} \quad \longmapsto \quad \frac{d\kappa}{ds} \qquad\qquad v_{6y} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

Equivalence & Signature

Cartan's main idea: The equivalence and symmetry properties of submanifolds will be found by restricting the differential invariants to the submanifold $J(x) = I(\mathrm{j}_n N|_x)$.

Equivalent submanifolds should have the same invariants.

However, unless an invariant $J(x)$ is constant, it carries little information by itself, since the equivalence map will typically drastically change the dependence of the invariant on the parameter x.

$\Longrightarrow$ Constant curvature submanifolds

However, a functional dependency or *syzygy* among the invariants *is* intrinsic:

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The Signature Map

Equivalence and symmetry properties of submanifolds are governed by the functional dependencies — "syzygies" — among the differential invariants.

$$J_k(x) = \Phi(J_1(x), \ldots, J_{k-1}(x))$$

The syzygies are encoded by the *signature map*

$$\Sigma : N \quad \longrightarrow \quad \mathcal{S}$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\Sigma(x) = (J_1(x), \ldots, J_m(x))$$
$$= (I_1 \mid N, \ \ldots \ , I_m \mid N)$$

The image $\mathcal{S} = \mathrm{Im}\,\Sigma$ is the signature subset (or submanifold) of N.

Geometrically, the signature

$$\mathcal{S} \subset \mathcal{K}$$

is the image of $\mathrm{j}_n N$ in the cross-section $\mathcal{K} \subset \mathrm{J}^n$, where $n \gg 0$ is sufficiently large.

$$\Sigma : N \quad \longrightarrow \quad \mathrm{j}_n N \quad \longrightarrow \quad \mathcal{S} \subset \mathcal{K}$$

Theorem. Two submanifolds are equivalent

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical

$$\mathcal{S} = \overline{\mathcal{S}}$$

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the first two differential invariants κ and κ_s

$$\mathcal{S} = \left\{ \left(\kappa \ , \ \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

Theorem. Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ object recognition

Symmetry

Signature map

$$\Sigma \; : \; N \quad \longrightarrow \quad \mathcal{S}$$

Theorem. Let $\mathcal{S}$ denote the signature of the submanifold N. Then the dimension of its symmetry group $G_N = \{\, g \mid g \cdot N \subset N \,\}$ equals

$$\dim G_N \;=\; \dim N - \dim \mathcal{S}$$

Corollary. For a regular submanifold $N \subset M$,

$$0 \;\leq\; \dim G_N \;\leq\; \dim N$$

$\Longrightarrow$ Only totally singular submanifolds can have larger symmetry groups!

Maximally Symmetric Submanifolds

Theorem. The following are equivalent:

- The submanifold N has a p-dimensional symmetry group
- The signature $\mathcal{S}$ degenerates to a point

$$\dim \mathcal{S} = 0$$

- The submanifold has all constant differential invariants
- $N = H \cdot \{ z_0 \}$ is the orbit of a p-dimensional subgroup $H \subset G$

$\Longrightarrow$ In Euclidean geometry, these are the circles, straight lines, spheres & planes.

$\Longrightarrow$ In equi-affine plane geometry, these are the conic sections.

Discrete Symmetries

Definition. The *index* of a submanifold N equals the number of points in $\mathcal{C}$ which map to a generic point of its signature $\mathcal{S}$:

$$\iota_N = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self–intersections

Theorem. The cardinality of the symmetry group of N equals its index ι_N.

$\Longrightarrow$ Approximate symmetries

Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{u = 0\} \qquad G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \alpha\delta - \beta\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta} \, , \; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

$$\sigma = \gamma \, x + \delta \qquad \Delta = \alpha\delta - \beta\gamma$$

Prolongation:

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}$$

$$v = \sigma^{-n} u$$

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta \, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2 \, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = 0 \qquad v = 1 \qquad v_y = 0 \qquad v_{yy} = \frac{1}{n(n-1)}$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\, u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\, u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\, x u^{(1-n)/n}$$

$$H = n(n-1)uu_{xx} - (n-1)^2 u_x^2 \qquad \text{— Hessian}$$

Nonsingular form: $H \neq 0$

Note: $H \equiv 0$ if and only if $Q(x) = (ax+b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \qquad \deg H = 2n-4 \qquad \deg T = 3n-6 \qquad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left(\frac{T(x)^2}{H(x)^3}\,,\ \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0$.

Signature map

$$\Sigma : N_Q \longrightarrow \mathcal{S}_Q \qquad \Sigma(x) = (J(x)^2, K(x))$$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

$\Longrightarrow$ diagonalizable

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant.

  $\Longrightarrow$ totally singular

- A one-parameter group if and only if $H \not\equiv 0$ and T^2 is a constant multiple of H^3 if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$.

  $\Longrightarrow$ maximally symmetric

- In all other cases, a finite group whose cardinality equals the index

$$\iota_Q = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & \qquad U = cH^2 \\ 4n - 8 & \qquad \text{otherwise} \end{cases}$$

Joint Invariants

Let G act on M.

A k-point *joint invariant* is an invariant of the k-fold Cartesian product action on

$$M \times \cdots \times M$$

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \;=\; I(z_1, \ldots, z_k)}$$

A k-point *joint differential invariant* is an invariant of the prolonged action $G^{(n)}$ on a k-fold Cartesian product of jet space

$$\mathrm{J}^n \times \cdots \times \mathrm{J}^n$$

$$\boxed{I(g^{(n)} \cdot z_1^{(n)}, \ldots, g^{(n)} \cdot z_k^{(n)}) \;=\; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

$\Longrightarrow$ Joint differential invariants are known as "semi-differential invariants" in the computer vision literature, and are proposed as "noise resistant" alternatives for object recognition.

Joint Euclidean Invariants

SE(2) acts on $M = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$:

$$z_i = (x_i, u_i) \qquad w_i = (y_i, v_i) = g^{-1} \cdot z_i \qquad i = 0, 1, 2, \ldots$$

$$y_i = \cos\theta\,(x_i - a) + \sin\theta\,(u_i - b)$$

$$v_i = -\sin\theta\,(x_i - a) + \cos\theta\,(u_i - b)$$

Normalization (cross-section)

$$y_0 = 0 \qquad v_0 = 0 \qquad y_1 > 0 \qquad v_1 = 0$$

Left moving frame $\quad \rho\colon M \to \mathrm{SE}(2)$

$$a = x_0 \qquad b = u_0 \qquad \theta = \tan^{-1}\left(\frac{u_1 - u_0}{x_1 - x_0}\right)$$

Joint invariants:

$$y_i \longmapsto \frac{(z_i - z_0)\cdot(z_1 - z_0)}{\|\, z_1 - z_0 \,\|} \qquad v_i \longmapsto \frac{(z_i - z_0)\wedge(z_1 - z_0)}{\|\, z_1 - z_0 \,\|}$$

Theorem. Every joint Euclidean invariant is a function of the interpoint distances $\| z_i - z_j \|$ and, in the orientation preserving case, a single signed area $A(z_0, z_1, z_2)$

Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy:

$$\Phi(I_1, \ldots, I_\ell) \equiv 0$$

The total number of syzygies is

$$\ell - k\,p$$

Typically, the number of joint invariants is

$$\ell = k\,m - r = (\#\text{points})(\dim M) - \dim G$$

Therefore, to find a joint invariant signature, that involes no differentiation, we need at least

$$k \geq \frac{r}{m-p} + 1$$

points on our submanifold.

Joint Euclidean Signature

For the Euclidean group $G = \mathrm{SE}(2)$ acting on curves $\mathcal{C} \subset \mathbb{R}^2$ (or $\mathbb{R}^3$) we need at least four points

$$z_0, z_1, z_2, z_3 \in \mathcal{C}$$

Joint invariants:

$$a = \| \, z^1 - z^0 \, \|, \qquad b = \| \, z^2 - z^0 \, \|, \qquad c = \| \, z^3 - z^0 \, \|,$$

$$d = \| \, z^2 - z^1 \, \|, \qquad e = \| \, z^3 - z^1 \, \|, \qquad f = \| \, z^3 - z^2 \, \|.$$

$\Longrightarrow$ six functions of four variables

Joint Signature: $\qquad \Sigma \colon \mathcal{C}^{\times 4} \longrightarrow \mathcal{S} \subset \mathbb{R}^6$

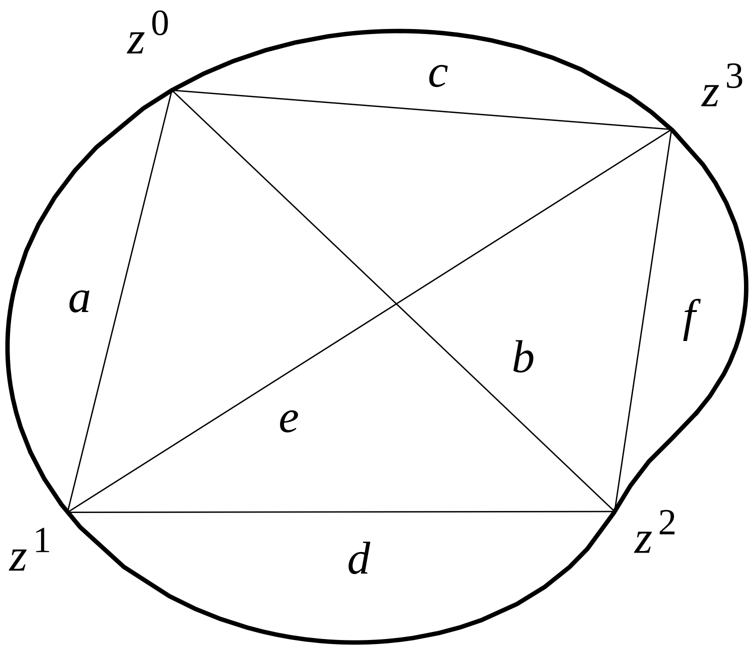
$\dim \mathcal{S} = 4 \quad \Longrightarrow \quad$ two syzygies

$$\Phi_1(a,b,c,d,e,f) = 0 \qquad \Phi_2(a,b,c,d,e,f) = 0$$

Universal Cayley–Menger syzygy:

$$\det \begin{vmatrix} 2a^2 & a^2+b^2-d^2 & a^2+c^2-e^2 \\ a^2+b^2-d^2 & 2b^2 & b^2+c^2-f^2 \\ a^2+c^2-e^2 & b^2+c^2-f^2 & 2c^2 \end{vmatrix} = 0$$

$$\Longleftrightarrow \qquad \mathcal{C} \subset \mathbb{R}^2$$



Four-Point Euclidean Joint Signature

Euclidean Joint Differential Invariants

— Planar Curves

- One–point

  $\Rightarrow$ curvature

$$\kappa = \frac{\dot{z} \wedge \ddot{z}}{\| \dot{z} \|^3}$$

- Two–point

  $\Rightarrow$ distances $\quad \| z^1 - z^0 \|$

  $\Rightarrow$ tangent angles $\quad \phi^k = \measuredangle(z_1 - z_0, \dot{z}_k)$

Equi–Affine Joint Differential Invariants — Planar Curves

- One–point

  $\Rightarrow$ affine curvature

$$\kappa = \frac{(z_t \wedge z_{tttt}) + 4(z_{tt} \wedge z_{ttt})}{3(z_t \wedge z_{tt})^{5/3}} - \frac{5(z_t \wedge z_{ttt})^2}{9(z_t \wedge z_{tt})^{8/3}}$$

$$= z_s \wedge z_{ss}$$

- Two–point

  $\Rightarrow$ tangent triangle area ratio

$$\frac{\dot{z}_0 \wedge \ddot{z}_0}{[\,(z_1 - z_0) \wedge \dot{z}_0\,]^3} = \frac{[\,\dot{0}\;\ddot{0}\,]}{[\,0\;1\;\dot{0}\,]^3}$$

- Three–point

  $\Rightarrow$ triangle area

$$\tfrac{1}{2}\,(z_1 - z_0) \wedge (z_2 - z_0) = \tfrac{1}{2}\,[\,0\;1\;2\,]$$

Projective Joint Differential Invariants — Planar Curves

- One–point

  ⇒ projective curvature

$$\kappa = \ldots$$

- Two–point

  ⇒ tangent triangle area ratio

$$\frac{[\,0\ 1\ \dot{0}\,]^3\,[\,\dot{1}\ \ddot{1}\,]}{[\,0\ 1\ \dot{1}\,]^3\,[\,\dot{0}\ \ddot{0}\,]}$$

- Three–point

  ⇒ tangent triangle ratio

$$\frac{[\,0\ 2\ \dot{0}\,]\,[\,0\ 1\ \dot{1}\,]\,[\,1\ 2\ \dot{2}\,]}{[\,0\ 1\ \dot{0}\,]\,[\,1\ 2\ \dot{1}\,]\,[\,0\ 2\ \dot{2}\,]}\,.$$

- Four–point

  ⇒ area cross–ratio

$$\frac{[\,0\ 1\ 2\,]\,[\,0\ 3\ 4\,]}{[\,0\ 1\ 3\,]\,[\,0\ 2\ 4\,]}$$

Transformation Groups and Jets

$(x^1, \ldots, x^p)$ — independent variables

$(u^1, \ldots, u^q)$ — dependent variables

$z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$ — n^{th} order jet space

u^α_J — derivative coordinates on J^n

G — transformation group

$G^{(n)}$ — prolonged action on J^n

$\mathbf{v} \in \mathfrak{g}$ — Lie algebra

$\mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$ — Prolonged inf. gens.

The Prolongation Formula

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha, J}^{n} \varphi^\alpha_J(x, u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}$$

$$\varphi^\alpha_J = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i\, u^\alpha_{J,i}$$

Characteristic

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i\,\frac{\partial u^\alpha}{\partial x^i}$$

Rotation group — SO(2)

$$(x,u) \longmapsto (x\cos\theta - u\sin\theta,\ x\sin\theta + u\cos\theta)$$

Transformed function $\quad v = \bar{f}(y)$:

$$y = x\cos\theta - f(x)\sin\theta,$$

$$v = x\sin\theta + f(x)\cos\theta,$$

Second prolongation

$$(x,u,u_x,u_{xx}) \longmapsto \left(x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta},\ \frac{u_{xx}}{(\cos\theta - u_x\sin\theta)^3}\right)$$

Infinitesimal generator

$$\mathbf{v} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u}$$

Second prolongation

$$\mathbf{v}^{(2)} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u} + (1+u_x^2)\,\frac{\partial}{\partial u_x} + 3u_x u_{xx}\,\frac{\partial}{\partial u_{xx}}$$

$$Q = x + uu_x$$

$$\varphi^x = D_x Q + \xi u_{xx} = D_x(x + uu_x) - uu_{xx} = 1 + u_x^2$$

$$\varphi^{xx} = D_x^2 Q + \xi u_{xxx} = D_x^2(x + uu_x) - uu_{xxx} = 3u_x u_{xx}$$

Differential invariant:

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

Infinitesimal criterion:

$$\mathbf{v}^{(n)}(I) = 0 \qquad \text{for all} \qquad \mathbf{v}^{(n)} \in \mathfrak{g}^{(n)}$$

$\Longrightarrow$ Solve the first order linear partial differential equation by the method of characteristics.

$\Longrightarrow$ Moving frames avoids integration!

Note: If $I_1, \ldots, I_k$ are differential invariants, so is $\Phi(I_1, \ldots, I_k)$.

$\Longrightarrow$ Classify differential invariants up to functional independence.

$\Longrightarrow$ Local results on open subsets of jet space.

Theorem. Any transformation group admits a finite system of fundamental differential invariants

$$J_1, \ldots, J_\ell$$

and p invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

such that every differential invariant is a function of the differentiated invariants:

$$I = \Phi(\; \ldots \; \mathcal{D}_K J_\nu \; \ldots \;)$$

Classification Problem.

How many fundamental differential invariants $J_1, \ldots, J_\ell$ are required?

$\Longrightarrow$ For curves ($p = 1$), we have $\ell = q$.

Syzygy Problem.

Determine the algebraic relations

$$\Phi(\ \ldots \ \mathcal{D}_K J_\nu \ \ldots \ ) = 0$$

among the differentiated invariants.

Commutation Formulae.

The order of invariant differentiation matters

$$[\, \mathcal{D}_i \, , \, \mathcal{D}_j \,] \; = \; ???$$

$\Longrightarrow$ Only an issue when $p > 1$.

The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1} \cdot z^{(n)}$$

$$\boxed{\begin{aligned} H^i(x,u^{(n)}) &= y^i(\rho^{(n)}(x,u^{(n)}),x,u) \\ I^\alpha_K(x,u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x,u^{(n)}),x,u^{(k)}) \end{aligned}}$$

Recurrence Formulae:

$$\boxed{\begin{aligned} \mathcal{D}_j H^i &= \delta^i_j + M^i_j \\ \mathcal{D}_j I^\alpha_K &= I^\alpha_{K,j} + M^\alpha_{K,j} \end{aligned}}$$

$M^i_j, M^\alpha_{K,j}$ — correction terms

Commutation Formulae:

$$\boxed{[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k}$$

- The correction terms can be computed directly from the infinitesimal generators!

Generating Invariants

Theorem. A generating system of differential invariants consists of

- all non-phantom differential invariants H^i and I^α coming from the un-normalized zero$^{\text{th}}$ order lifted invariants y^i, v^α, and
- all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant.

$$\text{order} \ \leq \ \text{order}\, \rho + 1$$

In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $\mathcal{D}_K H^i$, $\mathcal{D}_K I^\alpha_{J,i}$.

$\Longrightarrow$ Not necessarily a minimal set!

Syzygies

A syzygy is a functional relation among differentiated invariants:

$$H(\ \ldots\ \mathcal{D}_J I_\nu\ \ldots\ )\ \equiv\ 0$$

Derivatives of syzygies are syzygies
$\Longrightarrow$ find a minimal basis

Remark: There are no syzygies among the normalized differential invariants $I^{(n)}$ except for the "phantom syzygies"

$$I_\nu = c_\nu$$

corresponding to the normalizations.

Classification of Syzygies

Theorem. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j H^i = \delta^i_j + M^i_j}$$

— H^i non-phantom

$$\boxed{\mathcal{D}_J I^\alpha_K = c_\nu + M^\alpha_{K,J}}$$

— I^α_K generating

— $I^\alpha_{J,K} = w_\nu = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}}$$

— I^α_{LK}, I^α_{LJ} generating, $K \cap J = \varnothing$

$\Longrightarrow$ Not necessarily a minimal system!

Right Regularization

If G acts on M, then the *lifted action*

$$(h, z) \quad \longmapsto \quad (h \cdot g^{-1}, g \cdot z)$$

on the trivial right principal bundle

$$\mathcal{B} \;=\; G \times M$$

is always regular and free!

The functions $w \colon \mathcal{B} \longrightarrow M$ given by

$$w(g, z) = g \cdot z$$

provide a complete system of global invariants for the lifted action.

Example. $G = \mathrm{SO}(2)$ $\qquad M = \mathbb{R}^2$

$$\mathcal{B} = \mathrm{SO}(2) \times \mathbb{R}^2 \qquad \text{solid torus}$$

$$(x, u, \phi) \longmapsto$$

$$(x\cos\theta - u\sin\theta,\; x\sin\theta + u\cos\theta, \phi + \theta \bmod 2\,\pi)$$

Jet Regularization

$$\mathcal{B}^n = \mathrm{J}^n \times G$$

J^n J^n

$$w = w^{(n)} = g^{(n)} \cdot z^{(n)}$$

$$\sigma^{(n)}(z^{(n)}) = \big(z^{(n)}, \rho^{(n)}(z^{(n)})\big)$$

$$I^{(n)}(z^{(n)}) = w^{(n)} \circ \sigma^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}$$

Invariantization

$$\iota(F) = (\sigma^{(n)})^* \circ (w^{(n)})^* \, F = F \circ I^{(n)}$$

General Philosophy of Lifting

All invariant objects on $\mathcal{B}^n = \mathrm{J}^n \times G$

are well-behaved and easily understood.

$\Longrightarrow$ lifted invariants

We use the G-equivariant *moving frame section*

$$\sigma^{(n)} : \mathrm{J}^n \longrightarrow \mathcal{B} \qquad\qquad \sigma(z^{(n)}) = (\rho(z^{(n)}), z^{(n)})$$

to pull back lifted invariants to construct ordinary invariants on J^n.

For example,

$$\sigma^* w^{(n)} = w^{(n)} \circ \sigma = I^{(n)}$$

gives the fundamental differential invariants.

Similarly for lifted invariant differential forms, differential operators, tensors, etc.

$\Longrightarrow$ The key complication is that the pull-back process does not commute with differentiation!

The Variational Bicomplex

Infinite jet space

$$M = \mathrm{J}^0 \longleftarrow \mathrm{J}^1 \longleftarrow \mathrm{J}^2 \longleftarrow \cdots$$

Inverse limit

$$\mathrm{J}^\infty = \lim_{n \to \infty} \mathrm{J}^n$$

Local coordinates

$$z^{(\infty)} = (x, u^{(\infty)}) = (\; \ldots \; x^i \; \ldots \; u_J^\alpha \; \ldots \;)$$

Coframe — basis for the cotangent space $T^*\mathrm{J}^\infty$:

Horizontal one-forms

$$dx^1, \ldots, dx^p$$

Contact (vertical) one-forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha \, dx^i$$

Intrinsic definition of contact form

$$\theta \mid \mathrm{j}_\infty N = 0 \qquad \Longleftrightarrow \qquad \theta = \sum A_J^\alpha \, \theta_J^\alpha$$

Vertical and Horizontal Differentials

Bigrading of the differential forms on J^∞

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

Differential

$$d = d_H + d_V$$

$$d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s}$$

$$d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1}$$

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \qquad \text{— total derivatives}$$

$$d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u_J^\alpha}\, \theta_J^\alpha \qquad \text{— variation}$$

$\Longrightarrow$ *Vinogradov, Tsujishita, I. Anderson*

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Horizontal form

$$dx$$

Contact (vertical) forms

$$\theta = du - u_x \, dx$$

$$\theta_x = du_x - u_{xx} \, dx$$

$$\theta_{xx} = du_{xx} - u_{xxx} \, dx$$

$$\vdots$$

Differential

$$\begin{aligned} dF &= \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial u} \, du + \frac{\partial F}{\partial u_x} \, du_x + \frac{\partial F}{\partial u_{xx}} \, du_{xx} + \cdots \\ &= (D_x F) \, dx + \frac{\partial F}{\partial u} \theta + \frac{\partial F}{\partial u_x} \theta_x + \frac{\partial F}{\partial u_{xx}} \theta_{xx} + \cdots \\ &= \; d_H F + \, d_V F \end{aligned}$$

Total derivative

$$D_x F = \frac{\partial F}{\partial u} \, u_x + \frac{\partial F}{\partial u_x} \, u_{xx} + \frac{\partial F}{\partial u_{xx}} \, u_{xxx} + \cdots$$

Lifted Variational Tricomplex

$$\mathcal{B}^\infty = \mathrm{J}^\infty \times G$$

- Lifted horizontal forms

$$d_J\, y^i \qquad\qquad i = 1, \ldots, p$$

- Lifted invariant contact forms

$$\Theta^\alpha_J = d_J\, v^\alpha_J - \sum_{i=1}^{p} v^\alpha_{J,i}\, d_J\, y^i$$

- Right-invariant Maurer–Cartan forms

$$\boldsymbol{\mu} = dg \cdot g^{-1} \quad \Longrightarrow \quad \mu^1, \ldots, \mu^r \qquad r = \dim G$$

Differential forms on $\mathcal{B}^\infty$

$$\Omega^* = \bigoplus_{r,s,t} \widehat{\Omega}^{r,s,t}$$

Differential

$$d = d_H + d_V + d_G$$

$$d_H : \quad \widehat{\Omega}^{r,s,t} \longrightarrow \widehat{\Omega}^{r+1,s,t}$$

$$d_V : \quad \widehat{\Omega}^{r,s,t} \longrightarrow \widehat{\Omega}^{r,s+1,t}$$

$$d_G : \quad \widehat{\Omega}^{r,s,t} \longrightarrow \widehat{\Omega}^{r,s,t+1}$$

Invariantization

$$\iota : \begin{array}{lcl} \text{Functions} & \longrightarrow & \text{Invariants} \\ \text{Forms} & \longrightarrow & \text{Invariant Forms} \end{array}$$

Functions:

$$\iota(F) \;=\; \sigma^* \circ w^* \,(F) \;=\; F \circ I^{(\infty)}$$

Differential Forms:

$$\iota(\Omega) = \sigma^*(\pi_J(w^*\,\Omega)).$$

π_J — Jet projection

$$T^*\mathcal{B}^\infty \;=\; T^*(\mathrm{J}^\infty \times G) \;\simeq\; T^*\mathrm{J}^\infty \oplus T^*G$$

Invariant Variational Complex

Fundamental differential invariants

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)$$

Invariant horizontal one-forms

$$\varpi^i = \iota(dx^i) = \omega^i + \eta^i$$

ω^i — contact-invariant forms

η^i — contact "corrections"

Invariant contact forms

$$\vartheta^\alpha_K = \iota(\theta^\alpha_J)$$

Differential forms $\quad \Omega^* = \bigoplus_{r,s} \widehat{\Omega}^{r,s}$

Differential $\quad d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$

$$d_{\mathcal{H}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r+1,s}$$

$$d_{\mathcal{V}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r,s+1}$$

$$d_{\mathcal{W}} : \widehat{\Omega}^{r,s} \longrightarrow \widehat{\Omega}^{r-1,s+2}$$

The Key Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{k=1}^{p} \nu^\kappa \wedge \, \iota[\mathbf{v}_\kappa(\Omega)]$$

$\mathbf{v}_1, \ldots, \mathbf{v}_r$ — basis for $\mathfrak{g}$

$$\nu^\kappa = \sigma^* \mu^\kappa = \gamma^\kappa + \varepsilon^\kappa \qquad \kappa = 1, \ldots, r$$

$$\gamma^\kappa \in \widehat{\Omega}^{1,0} \qquad \varepsilon^\kappa \in \widehat{\Omega}^{0,1}$$

— pull back of the dual basis Maurer–Cartan forms via the moving frame section

$$\sigma^* : \mathrm{J}^\infty \to \mathcal{B}^\infty$$

⋆⋆⋆ All recurrence formulae, syzygies, commutation formulae, etc. are found by applying the key formula for various forms and functions Ω

Euclidean Curves

Lifted invariants

$$y = w^*(x) = x\cos\phi - u\sin\phi + a$$

$$v = w^*(u) = x\cos\phi + u\sin\phi + b$$

$$v_y = w^*(u_x) = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = w^*(u_{xx}) = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = w^*(u_{xx}) = \frac{(\cos\phi\ - u_x\sin\phi\ )u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi\ - u_x\sin\phi\ )^5}$$

$$dy = (\cos\phi - u_x\sin\phi)\,dx - (\sin\phi)\,\theta + da - v\,d\phi$$

$$d_J y = \pi_J(dy) = (\cos\phi - u_x\sin\phi)\,dx - (\sin\phi)\,\theta$$

$$D_y = \frac{1}{\cos\phi - u_x\sin\phi}\,D_x \qquad\qquad \theta = du - u_x\,dx$$

Normalization

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

Right moving frame $\qquad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\ \tan^{-1} u_x \qquad a = -\frac{x + uu_x}{\sqrt{1+u_x^2}} \qquad b = \frac{xu_x - u}{\sqrt{1+u_x^2}}$$

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa$$

$$\iota(u_{xxx}) = I_3 = \kappa_s$$

$$\iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

Invariant horizontal one-form

$$\begin{aligned} \iota(dx) = \sigma^*(\,d_J\,y) = \varpi \quad &= \quad \omega \quad + \quad \eta \\ &= \sqrt{1+u_x^2}\,dx \quad + \quad \frac{u_x}{\sqrt{1+u_x^2}}\,\theta \end{aligned}$$

Invariant contact forms

$$\iota(\theta) = \vartheta = \frac{\theta}{\sqrt{1+u_x^2}}$$

$$\iota(\theta_x) = \vartheta_1 = \frac{(1+u_x^2)\,\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad\qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

$$d_{\mathcal{H}}\, I = D_s I \cdot \varpi$$

Horizontal recurrence formula

$$d_{\mathcal{H}}\,\iota(F) = \iota(d_H F) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

Use phantom invariants

$$0 = d_{\mathcal{H}}\, H = \iota(d_H x) + \sum \iota(\mathbf{v}_\kappa(x))\,\gamma^\kappa = \varpi + \gamma^1,$$

$$0 = d_{\mathcal{H}}\, I_0 = \iota(d_H u) + \sum \iota(\mathbf{v}_\kappa(u))\,\gamma^\kappa = \gamma^2,$$

$$0 = d_{\mathcal{H}}\, I_1 = \iota(d_H u_x) + \sum \iota(\mathbf{v}_\kappa(u_x))\,\gamma^\kappa = \kappa\,\varpi + \gamma^3,$$

to solve for

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

$$\gamma^1 = -\varpi \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,\varpi$$

Recurrence formulae

$$\begin{aligned}\kappa_s \varpi = d_{\mathcal{H}}\,\kappa = d_{\mathcal{H}}\,(I_2) &= \iota(d_H u_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3 \\ &= \iota(u_{xxx}\,dx) - \iota(3u_x u_{xx}))\,\kappa\,\varpi = I_3\,\varpi \\ \kappa_{ss}\varpi = d_{\mathcal{H}}\,(I_3) &= \iota(d_H u_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3 \\ &= \iota(u_{xxxx}\,dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\,\varpi = I_4 - 3I_2^3\,\varpi\end{aligned}$$

$$\begin{aligned}\kappa &= I_2 & I_2 &= \kappa \\ \kappa_s &= I_3 & I_3 &= \kappa_s \\ \kappa_{ss} &= I_4 - 3I_2^3 & I_4 &= \kappa_{ss} + 3\kappa^3 \\ \kappa_{sss} &= I_5 - 19I_2^2 I_3 & I_4 &= \kappa_{sss} + 19\kappa^2\kappa_s\end{aligned}$$

Vertical recurrence formula

$$d_{\mathcal{V}}\,\iota(F) = \iota(\,d_V\,F) + \iota(\mathbf{v}_1(F))\,\varepsilon^1 + \iota(\mathbf{v}_2(F))\,\varepsilon^2 + \iota(\mathbf{v}_3(F))\,\varepsilon^3$$

Use phantom invariants

$$0 = \,d_{\mathcal{V}}\,H = \varepsilon^1$$

$$0 = \,d_{\mathcal{V}}\,I_0 = \vartheta + \varepsilon^2$$

$$0 = \,d_{\mathcal{V}}\,I_1 = \vartheta_1 + \varepsilon^3$$

to solve for

$$\varepsilon^1 = 0 \qquad\qquad \varepsilon^2 = -\,\vartheta = -\,\iota(\theta) \qquad\qquad \varepsilon^3 = -\,\vartheta_1 = -\,\iota(\theta_1)$$

Recurrence formulae

$$d_{\mathcal{V}}\,I_2 = \,d_{\mathcal{V}}\,\kappa = \iota(\theta_2) + \iota(\mathbf{v}_3(u_{xx}))\,\varepsilon^3 = \vartheta_2 = (\mathcal{D}^2 + \kappa^2)\,\vartheta,$$

$d_{\mathcal{H}}\,\vartheta$:

$$\mathcal{D}\vartheta = \vartheta_1 \qquad\qquad \mathcal{D}\vartheta_1 = \vartheta_2 - \kappa^2\,\vartheta$$

$$d_{\mathcal{V}}\,\varpi = -\,\kappa\,\vartheta \wedge \varpi$$

Example

$(x^1, x^2, u) \in M = \mathbb{R}^3$ $\qquad G = \mathrm{GL}(2)$

$$(x^1, x^2, u) \longmapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$$

$$\lambda = \alpha\delta - \beta\gamma$$

$\Longrightarrow$ Classical invariant theory

Prolongation (lifted differential invariants):

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2) \qquad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2)$$

$$v = \lambda^{-1} u$$

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda} \qquad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}$$

$$v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda}$$

$$v_{12} = \frac{\alpha\beta u_{11} + (\alpha\delta + \beta\gamma) u_{12} + \gamma\delta u_{22}}{\lambda}$$

$$v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}$$

Normalization

$$y^1 = 1 \qquad y^2 = 0 \qquad v_1 = 1 \qquad v_2 = 0$$

Nondegeneracy

$$x^1 \frac{\partial u}{\partial x^1} + x^2 \frac{\partial u}{\partial x^2} \neq 0$$

First order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix}$$

Normalized differential invariants

$$J^1 = 1 \qquad J^2 = 0$$

$$I = \frac{u}{x^1 u_1 + x^2 u_2}$$

$$I_1 = 1 \qquad I_2 = 0$$

$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

Phantom differential invariants

$$I_1 \qquad I_2$$

Generating differential invariants

$$I \qquad I_{11} \qquad I_{12} \qquad I_{22}$$

Invariant differential operators

$\mathcal{D}_1 = x^1 D_1 + x^2 D_2$ — scaling process
$\mathcal{D}_2 = -u_2 D_1 + u_1 D_2$ — Jacobian process

Recurrence formulae

$$\mathcal{D}_1 J^1 = \delta^1_1 - 1 = 0 \qquad \mathcal{D}_2 J^1 = \delta^1_2 - 0 = 0$$

$$\mathcal{D}_1 J^2 = \delta^2_1 - 0 = 0 \qquad \mathcal{D}_2 J^2 = \delta^2_2 - 1 = 0$$

$$\mathcal{D}_1 I = I_1 - I(1 + I_{11}) = -I(1 + I_{11}) \qquad \mathcal{D}_2 I = I_2 - I\, I_{12} = -I\, I_{12}$$

$$\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0 \qquad \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0 \qquad \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0$$

$$\mathcal{D}_1 I_{11} = I_{111} + (1 - I_{11}) I_{11} \qquad \mathcal{D}_2 I_{11} = I_{112} + (2 - I_{11}) I_{12}$$

$$\mathcal{D}_1 I_{12} = I_{112} - I_{11} I_{12} \qquad \mathcal{D}_2 I_{12} = I_{122} + (1 - I_{11}) I_{22}$$

$$\mathcal{D}_1 I_{22} = I_{122} + (I_{11} - 1) I_{22} - 2 I_{12}^2 \qquad \mathcal{D}_2 I_{22} = I_{222} - I_{12} I_{22}$$

$\Longrightarrow$ Use I to generate I_{11} and I_{12}

Syzygies

$$\begin{aligned}
\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2 I_{12} \\
\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11} - 1) I_{22} - 2 I_{12}^2 \\
(\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} &= \\
&= 2 I_{22} \mathcal{D}_1 I_{11} + (5 I_{12} - 2) \mathcal{D}_1 I_{12} + (3 I_{11} - 5) \mathcal{D}_1 I_{22} - \\
&\quad - (2 I_{11} - 5)(I_{11} - 1) I_{12} + 4 (I_{11} - 1) I_{12}^2
\end{aligned}$$

Commutation formulae

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12} \mathcal{D}_1 + (I_{11} - 1) \mathcal{D}_2$$

Invariant Variational Problems

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

$I_1, \ldots, I_\ell$ — fundamental differential invariants

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — contact-invariant volume form

Invariant Euler-Lagrange equations

$$\mathbf{E}(L) = F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Problem.

Construct F directly from P.

$\Longrightarrow$ P. Griffiths, I. Anderson

Example. Planar Euclidean group $G = \mathrm{SE}(2)$

Invariant variational problem

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) = F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

Euler-Lagrange equation

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

$$\int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

Invariantized Euler operator

$$\mathcal{E} = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial}{\partial \kappa_n} \qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian operator

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\, (-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Invariant Euler-Lagrange formula

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\, \mathcal{E}(P) + \kappa\, \mathcal{H}(P).$$

Elastica

$$P = \tfrac{1}{2}\,\kappa^2 \qquad \mathcal{E}(P) = \kappa \qquad \mathcal{H}(P) = -P = -\tfrac{1}{2}\,\kappa^2$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

Euler-Lagrange Equations

Integration by Parts:

$$\pi : \Omega^{p,1} \longrightarrow \mathcal{F}^1 = \Omega^{p,1} \,/\, d_H\, \Omega^{p-1,1}$$

$$\Longrightarrow \quad \text{Source forms}$$

Variational derivative or Euler operator:

$$\delta = \pi \circ d_V : \Omega^{p,0} \quad \longrightarrow \quad \mathcal{F}^1$$

Variational Problems $\longrightarrow$ Source Forms

$$\delta : \lambda = L\, d\mathbf{x} \quad \longrightarrow \quad \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\ \theta^\alpha \wedge\ d\mathbf{x}$$

Hamiltonian

$$\mathbf{H}(L) = \sum_{\alpha=1}^{m} \sum_{i>j\geq 0} u^\alpha_{i-j} (-D_x)^j \frac{\partial L}{\partial u^\alpha_i} - L$$

The Simplest Example. $M = \mathbb{R}^2 \quad x, u \in \mathbb{R}$

Lagrangian form

$$\lambda = L(x, u^{(n)})\, dx$$

Vertical derivative

$$\begin{aligned} d\lambda &= d_V \lambda \\ &= \left(\frac{\partial L}{\partial u}\,\theta + \frac{\partial L}{\partial u_x}\,\theta_x + \frac{\partial L}{\partial u_{xx}}\,\theta_{xx} + \cdots \right) \wedge\, dx \in \Omega^{1,1} \end{aligned}$$

Integration by parts

$$\begin{aligned} d_H\,(A\,\theta) &= (D_x A)\, dx \wedge \theta - A\,\theta_x \wedge dx \\ &= -[\,(D_x A)\,\theta + A\,\theta_x\,] \wedge dx \end{aligned}$$

Variational derivative

$$\begin{aligned} \delta\lambda &= \left(\frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge\, dx \\ &= \mathbf{E}(L)\,\theta \wedge\, dx \in \mathcal{F}^1 \end{aligned}$$

Plane Curves

Invariant Lagrangian

$$\int P(\kappa, \kappa_s, \ldots)\, \varpi$$

κ — fundamental differential invariant (curvature)

$\varpi = \omega + \eta$ — fully invariant horizontal form

$\omega = ds$ — contact-invariant arc length

Invariant integration by parts

$$d_{\mathcal{V}}\,(P\,\varpi) = \mathcal{E}(P)\ d_{\mathcal{V}}\,\kappa \wedge \varpi - \mathcal{H}(P)\ d_{\mathcal{V}}\,\varpi$$

Vertical differentiation formulae

$d_{\mathcal{V}}\,\kappa = \mathcal{A}(\vartheta)$ $\qquad$ $\mathcal{A}$ — Eulerian operator

$d_{\mathcal{V}}\,\varpi = \mathcal{B}(\vartheta) \wedge \varpi$ $\qquad$ $\mathcal{B}$ — Hamiltonian operator

$\Longrightarrow$ The explicit formulae follow from our fundamental recurrence formula, based on the infinitesimal generators of the action.

Invariant Euler-Lagrange equation

$$\boxed{\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = 0}$$

General Framework

Fundamental differential invariants

$$I^1, \ldots, I^\ell$$

Invariant horizontal coframe

$$\varpi^1, \ldots, \varpi^p$$

Dual invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

Invariant volume form

$$\boldsymbol{\varpi} = \varpi^1 \wedge \cdots \wedge \varpi^p$$

Differentiated invariants

$$I^\alpha_{,K} = \mathcal{D}^K J^\alpha = \mathcal{D}_{k_1} \cdots \mathcal{D}_{k_n} J^\alpha$$

$\Longrightarrow$ order is important!

Eulerian operator

$$d_{\mathcal{V}}\, I^\alpha = \sum_{\beta=1}^{q} \mathcal{A}^\alpha_\beta(\vartheta^\beta) \qquad \mathcal{A} = (\, \mathcal{A}^\alpha_\beta \,)$$

$\Longrightarrow$ $m \times q$ matrix of invariant differential operators

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Hamiltonian operator complex

$$d_{\mathcal{V}}\, \varpi^j = \sum_{\beta=1}^{q} \mathcal{B}^j_{i,\beta}(\vartheta^\beta) \wedge \varpi^i \qquad \mathcal{B}^j_i = (\, \mathcal{B}^j_{i,\beta} \,)$$

$\Longrightarrow$ p^2 row vectors of invariant differential operators

---

$$\boldsymbol{\varpi}_{(i)} = (-1)^{i-1}\, \varpi^1 \wedge \cdots \wedge \varpi^{i-1} \wedge \varpi^{i+1} \wedge \cdots \wedge \varpi^p$$

Twist invariants

$$d_{\mathcal{H}}\, \boldsymbol{\varpi}_{(i)} = Z_i\, \boldsymbol{\varpi}$$

Twisted adjoint

$$\mathcal{D}_i^\dagger = -(\mathcal{D}_i + Z_i)$$

Invariant variational problem

$$\int P(I^{(n)})\, \boldsymbol{\varpi}$$

Invariant Eulerian

$$\mathcal{E}_\alpha(P) = \sum_K \mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,K}}$$

Invariant Hamiltonian tensor

$$\mathcal{H}^i_j(P) = -\,P\,\delta^i_j + \sum_{\alpha=1}^{q} \sum_{J,K} I^\alpha_{,J,j}\, \mathcal{D}_K^\dagger \frac{\partial P}{\partial I^\alpha_{,J,i,K}}\,,$$

Invariant Euler-Lagrange equations

$$\mathcal{A}^\dagger\, \mathcal{E}(P) - \sum_{i,j=1}^{p} (\mathcal{B}^j_i)^\dagger\, \mathcal{H}^i_j(P) = 0.$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates $z = (x, y, u)$

Group: $G = \mathrm{E}(3)$

$$z \longmapsto R\,z + a, \qquad R \in \mathrm{O}(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Left moving frame

$$a = z \qquad R = (\ \mathbf{t}_1\ \mathbf{t}_2\ \mathbf{n}\ )$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- $\mathbf{n}$ — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \qquad \kappa^2 = \iota(u_{yy})$$

$\Longrightarrow$ principal curvatures

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \qquad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$\Longrightarrow$ Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \qquad \text{with} \qquad \begin{aligned} \kappa^1_{,22} &= \mathcal{D}_2^2 \iota(u_{xx}) \\ \kappa^2_{,11} &= \mathcal{D}_1^2 \iota(u_{yy}) \end{aligned}$$

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0$$

$\Longrightarrow$ Codazzi equations

Twisted adjoints

$$\mathcal{D}_1^\dagger = -\,(\mathcal{D}_1 + Z_1) \qquad Z_1 = \frac{\kappa^2_{,1}}{\kappa^1 - \kappa^2}$$

$$\mathcal{D}_2^\dagger = -(\mathcal{D}_2 + Z_2) \qquad Z_2 = \frac{\kappa^1_{,2}}{\kappa^2 - \kappa^1}$$

Gauss curvature — Codazzi equations:

$$\begin{aligned} K = \kappa^1\kappa^2 &= \mathcal{D}_1^\dagger(Z_1) + \mathcal{D}_2^\dagger(Z_2) \\ &= -\,(\mathcal{D}_1 + Z_1)Z_1 - (\mathcal{D}_2 + Z_2)Z_2 \end{aligned}$$

K is an invariant divergence

$\Longrightarrow$ Gauss–Bonnet Theorem!

Invariant contact form

$$\vartheta = \iota(\theta) = \iota(du - u_x\, dx - u_y\, dy)$$

Invariant vertical derivatives

$$d_{\mathcal{V}}\, \kappa^1 = \iota(\theta_{xx}) = (\, \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2\,)\, \vartheta$$

$$d_{\mathcal{V}}\, \kappa^2 = \iota(\theta_{yy}) = (\, \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2\,)\, \vartheta$$

Eulerian operator

$$\mathcal{A} = \begin{pmatrix} \mathcal{D}_1^2 + Z_2\, \mathcal{D}_2 + (\kappa^1)^2 \\ \mathcal{D}_2^2 + Z_1\, \mathcal{D}_1 + (\kappa^2)^2 \end{pmatrix}$$

$$d_{\mathcal{V}}\, \varpi^1 = \kappa^1\, \vartheta \wedge \varpi^1 - \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,)\vartheta \wedge \varpi^2$$

$$d_{\mathcal{V}}\, \varpi^2 = \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_2 \mathcal{D}_1 - Z_1 \mathcal{D}_2\,)\vartheta \wedge \varpi^1 + \kappa^2\, \vartheta \wedge \varpi^2$$

Hamiltonian operator complex

$$\begin{aligned} \mathcal{B}_1^1 &= \kappa^1, \\ \mathcal{B}_2^2 &= \kappa^2, \end{aligned} \qquad \mathcal{B}_2^1 = \frac{1}{\kappa^1 - \kappa^2}(\, \mathcal{D}_1 \mathcal{D}_2 - Z_2 \mathcal{D}_1\,) = -\,\mathcal{B}_1^2$$

Euclidean-invariant variational problem

$$\int P(\kappa^{(n)})\,\omega^1 \wedge \omega^2 = \int P(\kappa^{(n)})\;dA$$

Euler-Lagrange equations

$$\mathbf{E}(L) = \mathcal{A}^{\dagger}\,\mathcal{E}(P) - \mathcal{B}^{\dagger}\,\mathcal{H}(P) = 0,$$

Special case: $P(\kappa^1, \kappa^2)$

$$\mathbf{E}(L) = \left[\,(\mathcal{D}_1^{\dagger})^2 - \mathcal{D}_2^{\dagger}\cdot Z_2 + (\kappa^1)^2\,\right]\frac{\partial P}{\partial \kappa^1} + \\ + \left[\,(\mathcal{D}_2^{\dagger})^2 - \mathcal{D}_1^{\dagger}\cdot Z_1 + (\kappa^2)^2\,\right]\frac{\partial P}{\partial \kappa^2} + (\kappa^1 + \kappa^2)\,P$$

Minimal surfaces: $P = 1$

$$\kappa^1 + \kappa^2 = 2H = 0$$

Minimizing mean curvature: $P = H = \frac{1}{2}(\kappa^1 + \kappa^2)$

$$\tfrac{1}{2}\left[\,(\kappa^1)^2 + (\kappa^2)^2 + \kappa^1 + \kappa^2\,\right] = 2H^2 + H - K = 0.$$

Willmore surfaces: $P = \frac{1}{2}(\kappa^1)^2 + \frac{1}{2}(\kappa^2)^2$

$$\Delta(\kappa^1 + \kappa^2) + \tfrac{1}{2}(\kappa^1 + \kappa^2)(\kappa^1 - \kappa^2)^2 = 2\,\Delta H + 4(H^2 - K)H = 0$$

Laplace–Beltrami operator

$$\Delta = (\mathcal{D}_1 + Z_1)\mathcal{D}_1 + (\mathcal{D}_2 + Z_2)\mathcal{D}_2 = -\,\mathcal{D}_1^{\dagger}\cdot\mathcal{D}_1 - \mathcal{D}_2^{\dagger}\cdot\mathcal{D}_2$$

Multi–Space

Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

Jet space is the proper setting for the geometry of partial differential equations.

In this talk, I will propose a setting, named *multi-space*, for the geometry of numerical approximations to derivatives and differential equations.

$\Longrightarrow$ *Multi-space is the context for geometric integration.*

Invariant Numerical Approximations

Key remark: Every (finite difference) numerical approximation to the derivatives of a function require evaluating the function at several points $z_i = (x_i, u_i) = (x_i, f(x_i))$.

In other words, we seek to approximate the n^{th} order jet of a submanifold $N \subset M$ by a function $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)} = M \times \cdots \times M$, or, more correctly, on the "off-diagonal" part

$$M^{\diamond(n+1)} = \{\, z_i \neq z_j \text{ for all } i \neq j \,\}$$

$$\Longrightarrow \quad \textit{distinct } (n+1)\text{-tuples of points.}$$

Thus, multi-space should contain both the jet space and the off-diagonal Cartesian product space as submanifolds:

$$\left.\begin{array}{c} M^{\diamond(n+1)} \\ \downarrow \\ \mathrm{J}^n(M,p) \end{array}\right\} \quad \subset \quad M^{(n)}$$

Functions $F : M^{(n)} \longrightarrow \mathbb{R}$ are given by

$$F(z_0, \ldots, z_n) \qquad \text{on} \qquad M^{\diamond(n+1)}$$

and extend smoothly to J^n as the points coalesce. In this manner, $F \mid M^{\diamond(n+1)}$
provides a finite difference approximation to the differential function $F \mid \mathrm{J}^n$.

Construction of $M^{(n)}$

Definition. An $(n+1)$-*pointed manifold*

$$\mathbf{M} = (z_0, \ldots, z_n; M)$$

M — smooth manifold

$z_0, \ldots, z_n \in M$ — not necessarily distinct

Given $\mathbf{M}$, let

$$\#i \;=\; \#\left\{\, j \;\middle|\; z_j = z_i \,\right\}$$

denote the number of points which coincide with the i^{th} one.

Multi-contact for Curves

Definition. Two $(n+1)$-pointed curves

$$\mathbf{C} = (z_0, \ldots, z_n; C), \qquad \widetilde{\mathbf{C}} = (\tilde{z}_0, \ldots, \tilde{z}_n; \widetilde{C}),$$

have n^{th} *order multi-contact* if and only if

$$z_i = \tilde{z}_i, \qquad \text{and} \qquad \mathrm{j}_{\#i-1} C|_{z_i} = \mathrm{j}_{\#i-1} \widetilde{C}|_{z_i},$$

for each $i = 0, \ldots, n$.

$$\#i \;=\; \#\left\{\, j \;\middle|\; z_j = z_i \,\right\}$$

Definition. The n^{th} order *multi-space* $M^{(n)}$ is the set of equivalence classes of $(n+1)$-pointed curves in M under the equivalence relation of n^{th} order multi-contact.

The Fundamental Theorem

Theorem. If M is a smooth m-dimensional manifold, then its n^{th} order multi-space $M^{(n)}$ is a smooth manifold of dimension $(n+1)m$, which contains the off-diagonal part $M^{\diamond(n+1)}$ of the Cartesian product space as an open, dense submanifold, and the n^{th} order jet space J^n as a smooth submanifold.

$$\left.\begin{matrix} M^{\diamond(n+1)} \\[2ex] \mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_\nu} \\[2ex] \mathrm{J}^n(M,p) \end{matrix}\right\} \quad \subset \quad M^{(n)}$$

Example. Let $M = \mathbb{R}^m$

(i) $M^{(1)}$ is the space of two-pointed lines

$$M^{(1)} \simeq \{ (z_0, z_1; L) \mid z_0, z_1 \in L \quad \text{—} \quad \text{line} \} .$$

$\Longrightarrow$ Blow-up construction in algebraic geometry

(ii) $M^{(2)}$ is the space of three-pointed circles, i.e.,

$$M^{(2)} \simeq \{ (z_0, z_1, z_2, C) \mid z_0, z_1, z_2 \in C \quad \text{—} \quad \text{circle} \} .$$

Straight lines are included as circles of infinite radius, but points are not included (even though they could be viewed as circles of zero radius).

$\Longrightarrow$ Grassmann bundles.

(iii) $M^{(3)}$????

Topology — local and global.

Finite Differences

Local coordinates on J^n are provided by the coefficients of Taylor polynomials

$\Longrightarrow$ derivatives

Local coordinates on $M^{(n)}$ are provided by the coefficients of interpolating polynomials.

$\Longrightarrow$ finite differences

Given $(z_0, \ldots, z_n) \in M^{\diamond(n+1)}$, define the classical *divided differences* by the standard recursive rule

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,] = \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z_k \,] - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]}{x_k - x_{k-1}}$$

$$[\, z_j \,] = u_j$$

$\Longrightarrow$ Well-defined provided no two points lie on the same vertical line.

$\Longrightarrow$ Symmetric functions of z_i.

Definition. Given an $(n+1)$-pointed graph $\mathbf{C} = (z_0, \ldots, z_n; C)$, its divided differences are defined by

$$[\, z_j \,]_C = f(x_j)$$

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,]_C = \lim_{z \to z_k} \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z \,]_C - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]_C}{x - x_{k-1}}$$

$\Longrightarrow$ When taking the limit, the point $z = (x, f(x))$ must lie on the graph C, and take limiting values $x \to x_k$ and $f(x) \to f(x_k)$.

Theorem. Two $(n+1)$-pointed graphs $\mathbf{C}, \widetilde{\mathbf{C}}$ have n^{th} order multi-contact if and only if they have the same divided differences:

$$[\, z_0 z_1 \ldots z_k \,]_C = [\, z_0 z_1 \ldots z_k \,]_{\widetilde{C}}, \qquad k = 0, \ldots, n.$$

Local coordinates on $M^{(n)}$

They consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n \qquad \begin{array}{ll} u^{(0)} = u_0 = [\, z_0 \,]_C & u^{(1)} = [\, z_0 z_1 \,]_C \\ u^{(2)} = 2\, [\, z_0 z_1 z_2 \,]_C \quad \ldots & u^{(n)} = n!\, [\, z_0 z_1 \ldots z_n \,]_C \end{array}$$

prescribed by $(n+1)$-pointed graphs

$$\mathbf{C} = (z_0, \ldots, z_n; C)$$

The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n.

Numerical Approximations

$\Delta(x, u^{(n)})$ — differential function

$$\Delta : \mathrm{J}^n \to \mathbb{R}$$

System of differential equations:

$$\Delta_1(x, u^{(n)}) = \cdots = \Delta_k(x, u^{(n)}) = 0.$$

Definition. An $(n+1)$-*point numerical approximation of order* k to a differential function $\Delta : \mathrm{J}^n \to \mathbb{R}$ is a k^{th} order extension $F : M^{(n)} \to \mathbb{R}$ of Δ to multi-space, based on the inclusion $\mathrm{J}^n \subset M^{(n)}$.

$$F(x_0, \ldots, x_n, u^{(0)}, \ldots, u^{(n)})$$
$$\longrightarrow \quad F(x, \ldots, x, u^{(0)}, \ldots, u^{(n)}) = \Delta(x, u^{(n)})$$

Invariant Numerical Approximations

G — Lie group acting on M

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Differential Invariant

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

Joint Invariant

$$J(g \cdot z_0, \ldots, g \cdot z_k) = J(z_0, \ldots, z_k)$$

Semi-differential invariant =

Joint differential invariant

$\Longrightarrow$ *Approximate differential invariants by joint invariants*

Euclidean Invariants

Joint Euclidean invariant:

$$\mathbf{d}(z, w) = \| \, z - w \, \|$$

Euclidean curvature:

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

Euclidean arc length:

$$ds = \sqrt{1 + u_x^2} \, dx$$

Higher order differential invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Euclidean–invariant differential equation:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0$$

Three point approximation

Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \qquad \text{— semi-perimeter}$$

Expansion:

$$\begin{aligned}\widetilde{\kappa} = \kappa \quad & + \frac{1}{3}(b-a)\,\frac{d\kappa}{ds} + \frac{1}{12}(b^2 - ab + a^2)\,\frac{d^2\kappa}{ds^2} + \\ & + \frac{1}{60}(b^3 - ab^2 + a^2 b - a^3)\,\frac{d^3\kappa}{ds^3} + \\ & + \frac{1}{120}(b-a)\,(3b^2 + 5ab + 3a^2)\,\kappa^2\,\frac{d\kappa}{ds} + \cdots .\end{aligned}$$

Higher order invariants

$$\kappa_s = \frac{d\kappa}{ds}$$

Invariant finite difference approximation:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_i, P_{i-1})}$$

Unbiased centered difference:

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}, P_{i+2}) = \frac{\widetilde{\kappa}(P_i, P_{i+1}, P_{i+2}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}(P_{i+1}, P_{i-1})}$$

Better approximation (M. Boutin):

$$\widetilde{\kappa}_s(P_{i-2}, P_{i-1}, P_i, P_{i+1}) = 3\ \frac{\widetilde{\kappa}(P_{i-1}, P_i, P_{i+1}) - \widetilde{\kappa}(P_{i-2}, P_{i-1}, P_i)}{\mathbf{d}_{i-2} + 2\mathbf{d}_{i-1} + 2\mathbf{d}_i + \mathbf{d}_{i+1}}$$

$$\mathbf{d}_j = \mathbf{d}(P_j, P_{j+1})$$

Affine Joint Invariants

$$\mathbf{x} \to A\mathbf{x} + b \qquad \det A = 1$$

Area is the fundamental joint affine invariant

$$\begin{aligned}
[\,ijk\,] &= (P_i - P_j) \wedge (P_i - P_k) \\
&= \det \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} \\
&= \text{Area of parallelogram} \\
&= 2 \times \text{Area of triangle } \Delta(P_i, P_j, P_k)
\end{aligned}$$

Syzygies:

$$[\,ijl\,] + [\,jkl\,] = [\,ijk\,] + [\,ikl\,]$$

$$[\,ijk\,]\,[\,ilm\,] - [\,ijl\,]\,[\,ikm\,] + [\,ijm\,]\,[\,ikl\,] = 0$$

Affine Differential Invariants

Affine curvature

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}}$$

Affine arc length

$$ds = \sqrt[3]{u_{xx}}\; dx$$

Higher order affine invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Conic Sections

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Affine curvature:

$$\kappa = \frac{S}{T^{2/3}}$$

$$S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

$$T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

Ellipse:

$$\kappa = (\pi/\mathbf{A})^{2/3}$$

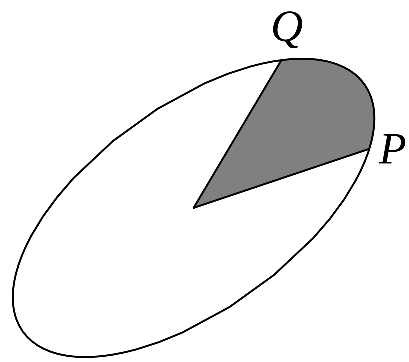
$$\mathbf{A} = \pi \frac{T}{S^{3/2}} = \text{Area}$$

Affine arc length of ellipse:

$$\int_P^Q ds = \frac{T^{1/3}}{S^{1/2}} \arcsin \sqrt{\frac{-CT}{S^2}} \left(x + \frac{CD - BE}{S} \right) \Bigg|_P^Q$$

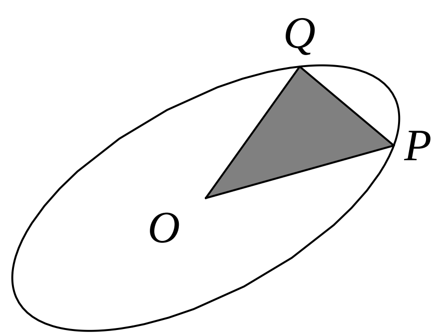
$$= 2ST^{-2/3}\mathbf{A}(P,Q)$$

$\mathbf{A}(P, Q)$:



Triangular approximation:

$\Delta(O, P, Q)$:



Total affine arc length:

$$\mathbf{L} = 2\sqrt[3]{\mathbf{A}} = -2\pi \frac{\sqrt[3]{T}}{\sqrt{S}}$$

Conic through five points $P_0, \ldots, P_4$:

$$[013][024][\mathbf{x}12][\mathbf{x}34] = [012][034][\mathbf{x}13][\mathbf{x}24]$$

$$\mathbf{x} = (x, y)$$

Affine curvature and arc length:

$$\kappa = \frac{S}{T^{2/3}}$$

$$ds = \text{Area } \Delta(O, P_1, P_3) = \tfrac{1}{2}[O, P_1, P_3] = \frac{N}{2S}$$

$$4T = \prod_{0 \leq i < j < k \leq 4} [ijk]$$

$$\begin{aligned} 4S = \; & [013]^2[024]^2([124] - [123])^2 + \\ & + [012]^2[034]^2([134] + [123])^2 - \\ & - 2[012][034][013][024]([123][234] + [124][134]) \end{aligned}$$

$$\begin{aligned} 4N = \; & -[123][134]\,\{[023]^2[014]^2([124] - [123]) + \\ & + [012]^2[034]^2([134] + [123]) + \\ & + [012][023][014][034]([134] - [123])\} \end{aligned}$$

Theorem. P_0, P_1, P_2, P_3, P_4 — points on the convex curve $\mathcal{C}$.

κ — affine curvature of $\mathcal{C}$ at P_2

$\widetilde{\kappa} = \widetilde{\kappa}(P_0, P_1, P_2, P_3, P_4)$

— affine curvature of conic

$L_i = \int_{P_2}^{P_i} ds$

— affine arc length of conic

Expansion:

$$\widetilde{\kappa} = \kappa + \frac{1}{5}\left(\sum_{i=0}^{4} L_i\right)\frac{d\kappa}{ds} + \frac{1}{30}\left(\sum_{0\le i\le j\le 4} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots$$

Multi-Invariants

G — Lie group which acts smoothly on M

$\Longrightarrow$ G preserves the multi-contact equivalence relation

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$G^{(n)}$ — n^{th} *multi-prolongation* to $M^{(n)}$

$\Longrightarrow$ On $\mathrm{J}^n \subset M^{(n)}$ it coincides with the usual jet space prolongation

$\Longrightarrow$ On $M^{\diamond(n+1)} \subset M^{(n)}$ it coincides with the $(n+1)$-fold Cartesian product action.

$K : M^{(n)} \to \mathbb{R}$ — *multi-invariant*

$$K(g^{(n)} \cdot z^{(n)}) = K(z^{(n)})$$

$\Longrightarrow$ $K \mid \mathrm{J}^n$ — differential invariant

$\Longrightarrow$ $K \mid M^{\diamond(n+1)}$ — joint invariant

$\Longrightarrow$ $K \mid \mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_\nu}$ — joint diff. invariant

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The theory of multi-invariants *is* the theory of invariant numerical approximations!

Moving frames provide a

systematic algorithm for

constructing multi-invariants!

A moving frame on multi-space

$$\rho : M^{(n)} \quad \longrightarrow \quad G$$

is called a *multi-frame*.

Example. $G = \mathbb{R}^2 \ltimes \mathbb{R}$

$$(x, u) \quad \longmapsto \quad (\lambda^{-1} x + a, \lambda u + b)$$

Multi-prolonged action: compute the divided differences of the basic lifted invariants

$$y_k = \lambda^{-1} x_k + a, \qquad v_k = \lambda\, u_k + b.$$

We find

$$v^{(1)} = [\, w_0 w_1 \,] = \frac{v_1 - v_0}{y_1 - y_0}$$

$$= \lambda^2 \, \frac{u_1 - u_0}{x_1 - x_0} = \lambda^2 \, [\, z_0 z_1 \,] = \lambda^2 \, u^{(1)},$$

$$v^{(n)} = \lambda^{n+1} \, u^{(n)}.$$

Moving frame cross-section

$$y_0 = 0 \qquad v_0 = 0 \qquad v^{(1)} = 1$$

Solve for the group parameters

$$a = -\sqrt{u^{(1)}}\; x_0 \qquad b = -\frac{u_0}{\sqrt{u^{(1)}}} \qquad \lambda = \frac{1}{\sqrt{u^{(1)}}}$$

$$\Longrightarrow \quad \text{multi-frame } \rho \colon M^{(n)} \to G.$$

Multi-invariants:

$$y_k: \qquad H_k = (x_k - x_0)\sqrt{u^{(1)}} = (x_k - x_0)\sqrt{\frac{u_1 - u_0}{x_1 - x_0}}$$

$$u_k: \qquad K_k = \frac{u_k - u_0}{\sqrt{u^{(1)}}} = (u_k - u_0)\sqrt{\frac{x_1 - x_0}{u_1 - u_0}}$$

$$u^{(n)}: \qquad K^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}} = \frac{n!\,[\,z_0 z_1 \ldots z_n\,]}{[\,z_0 z_1\,]^{(n+1)/2}}$$

$$K^{(0)} = K_0 = 0 \qquad K^{(1)} = 1$$

Coalescent limit

$$K^{(n)} \quad \longrightarrow \quad I^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}}$$

$\Longrightarrow$ $K^{(n)}$ is a first order invariant numerical approximation to the differential invariant $I^{(n)}$.

$\Longrightarrow$ Higher order invariant numerical approximations are obtained by invariantization of higher order divided difference approximations.

$$F(\ \ldots\ x_k\ \ldots\ u^{(n)}\ \ldots\ )$$

$$\longrightarrow \quad F(\ \ldots\ H_k\ \ldots\ K^{(n)}\ \ldots\ )$$

Invariantization

To construct an invariant numerical scheme for any similarity-invariant ordinary differential equation

$$F(x, u, u^{(1)}, u^{(2)}, \ldots u^{(n)}) = 0,$$

we merely *invariantize* the defining differential function, leading to the general similarity–invariant numerical approximation

$$F(0, 0, 1, K^{(2)}, \ldots, K^{(n)}) = 0.$$

$\Longrightarrow$ Nonsingular!

Example. Euclidean group SE(2)

$$y = x\cos\theta - u\sin\theta + a \qquad v = x\sin\theta + u\cos\theta + b$$

Multi-prolonged action on $M^{(1)}$:

$$y_0 = x_0\cos\theta - u_0\sin\theta + a \qquad v_0 = x_0\sin\theta + u_0\cos\theta + b$$

$$y_1 = x_1\cos\theta - u_1\sin\theta + a \qquad v^{(1)} = \frac{\sin\theta + u^{(1)}\cos\theta}{\cos\theta - u^{(1)}\sin\theta}$$

Cross-section

$$y_0 = v_0 = v^{(1)} = 0$$

Right moving frame

$$a = -\,x_0\cos\theta + u_0\sin\theta = -\,\frac{x_0 + u^{(1)}\,u_0}{\sqrt{1+(u^{(1)})^2}}$$

$$b = -\,x_0\sin\theta - u_0\cos\theta = \frac{x_0\,u^{(1)} - u_0}{\sqrt{1+(u^{(1)})^2}}$$

$$\tan\theta = -\,u^{(1)}\,.$$

Euclidean multi-invariants

$$(y_k, v_k) \quad \longrightarrow \quad I_k = (H_k, K_k)$$

$$H_k = \frac{(x_k - x_0) + u^{(1)}\,(u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{1 + [\,z_0 z_1\,]\,[\,z_0 z_k\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

$$K_k = \frac{(u_k - u_0) - u^{(1)}\,(x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{[\,z_0 z_k\,] - [\,z_0 z_1\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

Difference quotients

$$[\,I_0 I_k\,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\,z_0 z_1 z_k\,]}{1 + [\,z_0 z_k\,]\,[\,z_0 z_1\,]}$$

$$I^{(1)} = [\,I_0 I_1\,] = 0$$

$$I^{(2)} = 2\,[\,I_0 I_1 I_2\,] = 2\,\frac{[\,I_0 I_2\,] - [\,I_0 I_1\,]}{H_2 - H_1}$$

$$= \frac{2\,[\,z_0 z_1 z_2\,]\sqrt{1 + [\,z_0 z_1\,]^2}}{(\,1 + [\,z_0 z_1\,]\,[\,z_1 z_2\,]\,)(\,1 + [\,z_0 z_1\,]\,[\,z_0 z_2\,]\,)}$$

$$= \frac{u^{(2)}\sqrt{1 + (u^{(1)})^2}}{\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_0)\,\right]\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_1)\,\right]}$$

Invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \to z_0} I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}}$$

Euclidean–invariant approximation for $\kappa_s = \iota(u_{xxx})$:

$$I^{(3)} = 6\,[\,I_0 I_1 I_2 I_3\,] = 6\,\frac{[\,I_0 I_1 I_3\,] - [\,I_0 I_1 I_2\,]}{H_3 - H_2}$$

Higher Dimensional Submanifolds

$T^{(n)}M|_z$ — n^{th} order tangent space

Proposition.

Two p-dimensional submanifolds $N, \widetilde{N}$ have n^{th} order *contact* at a common point $z \in N \cap \widetilde{N}$ if and only if

$$T^{(n)}N|_z = T^{(n)}\widetilde{N}|_z$$

$\Longrightarrow$ Requires $\binom{p+n}{n}$ coalescing points to approximate n^{th} order derivatives

Surfaces $p = 2$

n	$\binom{p+n}{n}$
0	1
1	3
2	6
3	10
$\vdots$	$\vdots$

Definition. A subspace $V \subset T^{(n)}M|_z$ is called *admissible* if for every vector

$\mathbf{v} \in V \cap T^{(k)}M|_z, \quad 1 \leq k \leq n,$

there exists a submanifold $N \subset M$ such that

$\mathbf{v} \in T^{(k)}N|_z \subset V.$

Definition. Two submanifolds $N, \widetilde{N}$ have r^{th} *order subcontact* at a common point if and only if for some n, there exists an admissible common r-dimensional subspace

$$S \subset T^{(n)}N|_z \cap T^{(n)}\widetilde{N}|_z \subset T^{(n)}M|_z$$

Example. Surfaces: $S, \widetilde{S} \subset M$

order	Conditions
0	$z \in S \cap \widetilde{S}$ — common point
1	tangent curves: $TC\|_z = T\widetilde{C}\|_z$
2	tangent surfaces: $TS\|_z = T\widetilde{S}\|_z$ osculating curves: $T^{(2)}C\|_z = T^{(2)}\widetilde{C}\|_z$
3	$TS\|_z = T\widetilde{S}\|_z$ and $T^{(2)}C\|_z = T^{(2)}\widetilde{C}\|_z$ $T^{(3)}C\|_z = T^{(3)}\widetilde{C}\|_z$
⋮	⋮
5	$T^{(2)}S\|_z = T^{(2)}\widetilde{S}\|_z$ $TS\|_z = T\widetilde{S}\|_z,\ T^{(3)}C\|_z = T^{(3)}\widetilde{C}\|_z,\ T^{(2)}C'\|_z = T^{(2)}\widetilde{C}'\|_z$ $TS\|_z = T\widetilde{S}\|_z,\ T^{(4)}C\|_z = T^{(4)}\widetilde{C}\|_z$ $T^{(5)}C\|_z = T^{(5)}\widetilde{C}\|_z$

Multi-space and Multi-variate Interpolation

Definition. Let M be a smooth manifold.

The n^{th} order *multi-space* $M^{(n)}$ is the set of all *n-point interpolant data*

$$\mathbf{Z} = (z_0, \ldots, z_{n-1}; V_0, \ldots, V_{n-1}),$$

consisting of

(a) an ordered set of n points $z_0, \ldots, z_{n-1} \in M$.

$$\#i = \#\left\{ j \;\middle|\; z_j = z_i \right\}$$

(b) an ordered collection of admissible subspaces $V_i \subset T^{(n)}M|_{z_i}$ such that

$$\begin{cases} V_i = V_j \quad \text{if} \quad z_i = z_j \\ \dim V_i = \#i - 1 \end{cases}$$

In particular, if $\#i = 1$, and so z_i only appears once in $\mathbf{Z}$, then $V_i = \{0\}$ is trivial.

Multivariate Hermite Interpolation

Definition. An *interpolant* to $\mathbf{Z}$ is a submanifold $N \subset M$ such that $z_i \in N$ and $V_i \subset T^{(n)}N|_{z_i}$.

Conjecture. The multispace $M^{(n)}$ is a manifold of dimension $(n+1)m$. It contains

- $M^{\diamond n}$ as an open, dense submanifold
- *all* $\mathrm{J}^k(M,p)$ that have dimension $\le (n+1)m$ as submanifolds
- various off-diagonal copies of multi-jet spaces $\mathrm{J}^{i_1}(M,p) \diamond \cdots \diamond \mathrm{J}^{i_k}(M,p)$ for $i_1 + \cdots + i_k = n - k$ as submanifolds.

$\Longrightarrow$ smooth or analytic

Difficulties

♠ Multi-variate interpolation theory.

♠ Multi-variate divided differences.

♠ Coordinates at coalescent points.

♠ Topological structure — local and global