Dispersive Quantization

Peter J. Olver
University of Minnesota
http://www.math.umn.edu/ ∼ olver

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Peter J. Olver

Introduction to Partial Differential Equations

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Dispersion

Definition. A linear partial differential equation is called dispersive if the different Fourier modes travel unaltered but at different speeds.

Substituting

$$u(t,x) = e^{\,\mathrm{i}\,(k\,x - \omega\,t)}$$

produces the dispersion relation

$$\omega = \omega(k)$$

relating frequency ω and wave number k.

Phase velocity: $\quad c_p = \dfrac{\omega(k)}{k}$

Group velocity: $\quad c_g = \dfrac{d\omega}{dk} \qquad$ (stationary phase)

The simplest linear dispersive wave equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

Dispersion relation: $\omega = -\,k^3$

Phase velocity: $c_p = \dfrac{\omega}{k} = -\,k^2$

Group velocity: $c_g = \dfrac{d\omega}{dk} = -3\,k^2$

Thus, wave packets (and energy) move *faster* (to the left) than the individual waves.

Linear Dispersion on the Line

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \qquad u(0,x) = f(x)$$

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Fourier transform solution:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k)\, e^{\,\mathrm{i}\,(k\,x + k^3\,t)}\, dk$$

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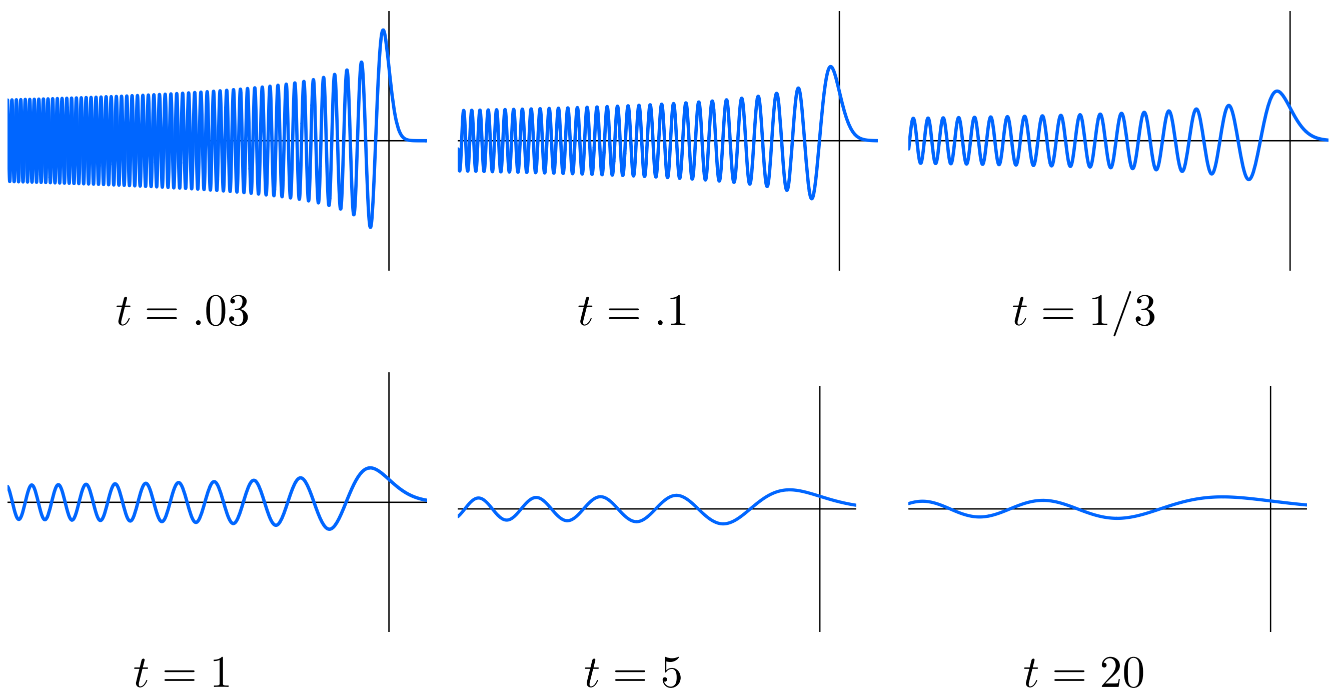
Fundamental solution $\qquad u(0,x) = \delta(x)$

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\,\mathrm{i}\,(k\,x + k^3\,t)}\, dk = \frac{1}{\sqrt[3]{3\,t}}\ \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3\,t}}\right)$$



$t = .03$
$t = .1$
$t = 1/3$
$t = 1$
$t = 5$
$t = 20$

Linear Dispersion on the Line

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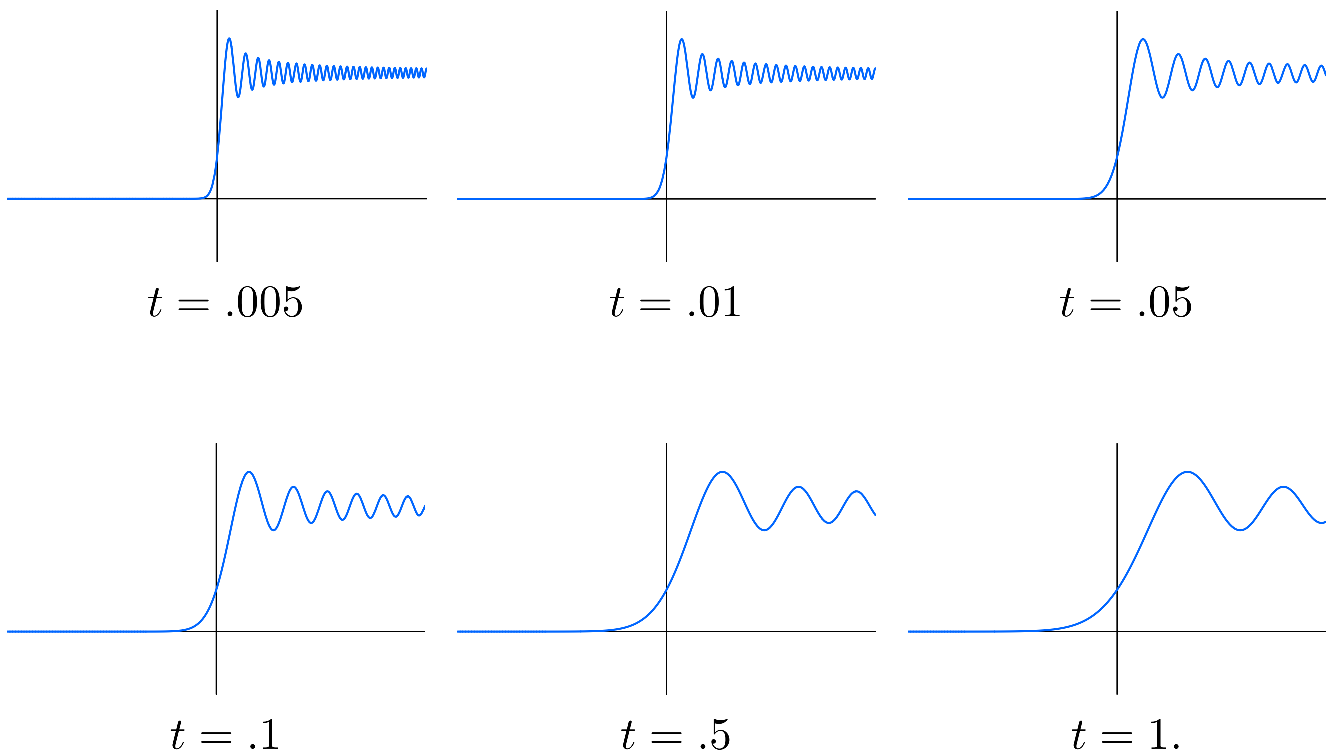
$$u(t,x) = \frac{1}{3} \ - \ H\left(-\frac{x}{\sqrt[3]{3\,t}}\right)$$

$$H(z) = \frac{z\,\Gamma\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{1}{3};\,\frac{2}{3},\,\frac{4}{3};\,\frac{1}{9}\,z^3\right)}{3^{5/3}\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)} - \frac{z^2\,\Gamma\left(\frac{2}{3}\right)\,{}_1F_2\left(\frac{2}{3};\,\frac{4}{3},\,\frac{5}{3};\,\frac{1}{9}\,z^3\right)}{3^{7/3}\,\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)}$$


$t = .005$
$t = .01$
$t = .05$
$t = .1$
$t = .5$
$t = 1.$

Periodic Linear Dispersion

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$$

$$u(t,0) = u(t,2\pi) \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,2\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,2\pi)$$

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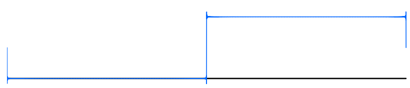
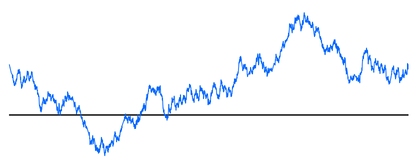
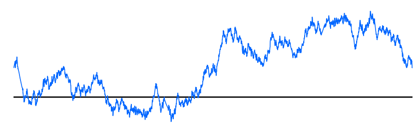
$$u^\star(t,x) \ \sim \ \frac{1}{2} \ + \ \frac{2}{\pi} \sum_{j=0}^{\infty} \frac{\sin(\,(2j+1)\,x - (2j+1)^3\,t\,)}{2j+1}\,.$$



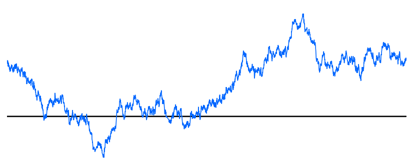
$t = 0.$



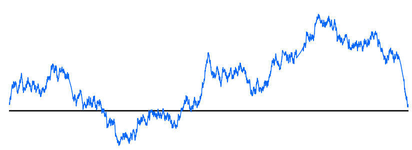
$t = .1$



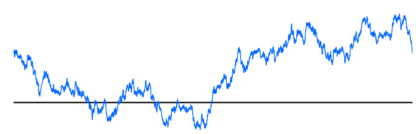
$t = .2$



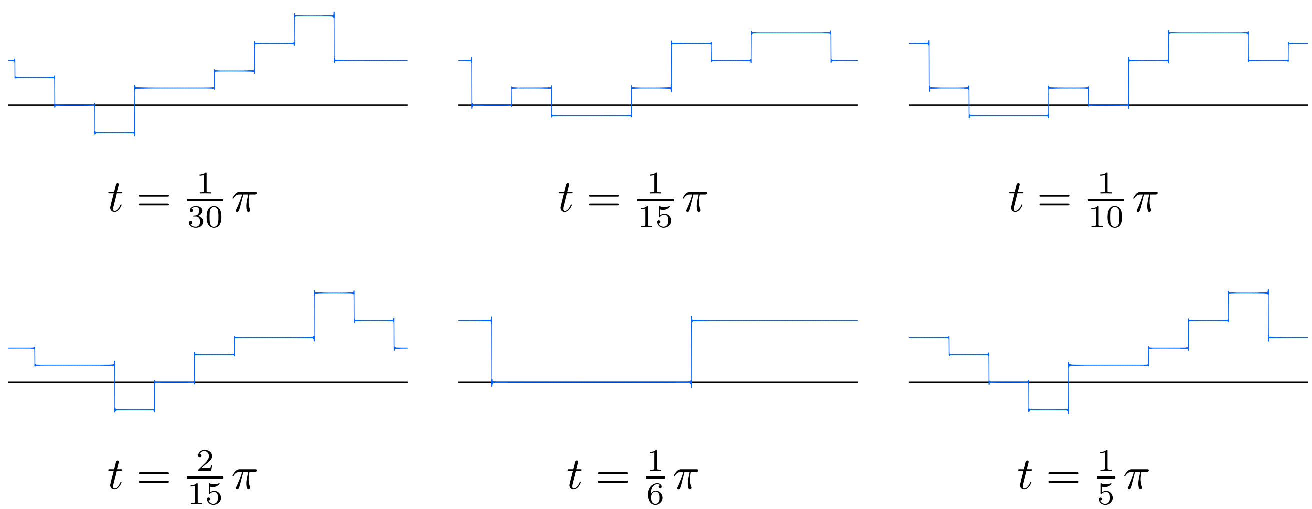
$t = .3$

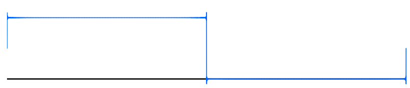


$t = .4$



$t = .5$


$t = \frac{1}{30}\pi$
$t = \frac{1}{15}\pi$
$t = \frac{1}{10}\pi$
$t = \frac{2}{15}\pi$
$t = \frac{1}{6}\pi$
$t = \frac{1}{5}\pi$

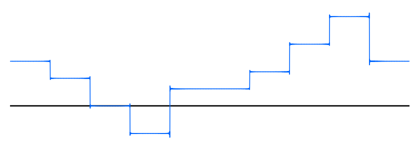


$t = \pi$

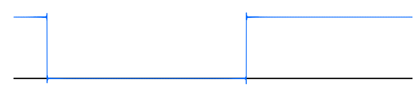
$t = \frac{1}{2}\pi$

$t = \frac{1}{3}\pi$

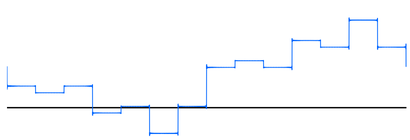
$t = \frac{1}{4}\pi$



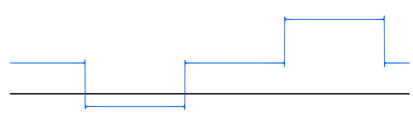
$t = \frac{1}{5}\pi$



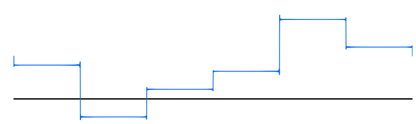
$t = \frac{1}{6}\pi$



$t = \frac{1}{7}\pi$



$t = \frac{1}{8}\pi$



$t = \frac{1}{9}\pi$

Theorem. At rational time $t = \pi\, p/q$, the solution $u^\star(t, x)$ is constant on every subinterval $\pi\, j/q < x < \pi(j + 1)/q$. At irrational time $u^\star(t, x)$ is a non-differentiable continuous function.

Lemma.

$$f(x) \; \sim \sum_{k=-\infty}^{\infty} c_k \, e^{\,\mathrm{i}\,k\,x}$$

is piecewise constant on intervals $p\,\pi/q < x < (p+1)\,\pi/q$ if and only if

$$\widehat{c}_k = \widehat{c}_l, \quad k \equiv l \not\equiv 0 \mod 2\,q, \qquad \widehat{c}_k = 0, \quad 0 \neq k \equiv 0 \mod 2\,q.$$

where

$$\widehat{c}_k = \frac{\pi\,k\,c_k}{\mathrm{i}\,q\,(e^{-\,\mathrm{i}\,\pi\,k/q} - 1)} \qquad\qquad k \not\equiv 0 \mod 2\,q.$$

$\Longrightarrow$ DFT

The Fourier coefficients of the solution $u^\star(t,x)$ at rational time $t = \pi\, p/q$ are

$$c_k = b_k\left(\pi\,\frac{p}{q}\right) = b_k(0)\, e^{\,\mathrm{i}\,(k\,x - k^3 \pi\, p/q)},$$

where

$$b_k(0) = \begin{cases} -\,\mathrm{i}\,/(\pi\, k), & k \text{ odd}, \\ 1/2, & k = 0, \\ 0, & 0 \neq k \;\; \text{even}. \end{cases}$$

Crucial observation:

$$\text{if} \quad k \equiv l \mod 2\,q, \quad \text{then} \quad k^3 \equiv l^3 \mod 2\,q$$

and so

$$e^{\,\mathrm{i}\,(k\,x - k^3 \pi\, p/q)} = e^{\,\mathrm{i}\,(l\,x - l^3 \pi\, p/q)}$$

Theorem. At rational time $t = \pi\, p/q$, the fundamental solution to the initial-boundary value problem is a linear combination of finitely many delta functions.

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Corollary. At rational time, any solution profile $u(\pi p/q, x)$ to the periodic initial-boundary value problem depends on only finitely many values of the initial data, namely $u(0, x_j) = f(x_j)$ where $x_j = \pi j/q$ for $j = 0, \ldots, 2q - 1$ when p is odd, or $x_j = 2\pi j/q$ for $j = 0, \ldots, q - 1$ when p is even.

★★ The same phenomenon appears in any linearly dispersive equation with "integral" dispersion relation:

$$\omega(k) = \sum_{m=0}^{n} c_m k^m$$

where

$$c_m / c_n \in \mathbb{Q}$$

Linear Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Dispersion relation: $\omega = k^2$

Phase velocity: $c_p = \dfrac{\omega}{k} = -\,k$

Group velocity: $c_g = \dfrac{d\omega}{dk} = -2\,k$

Periodic Linear Schrödinger Equation

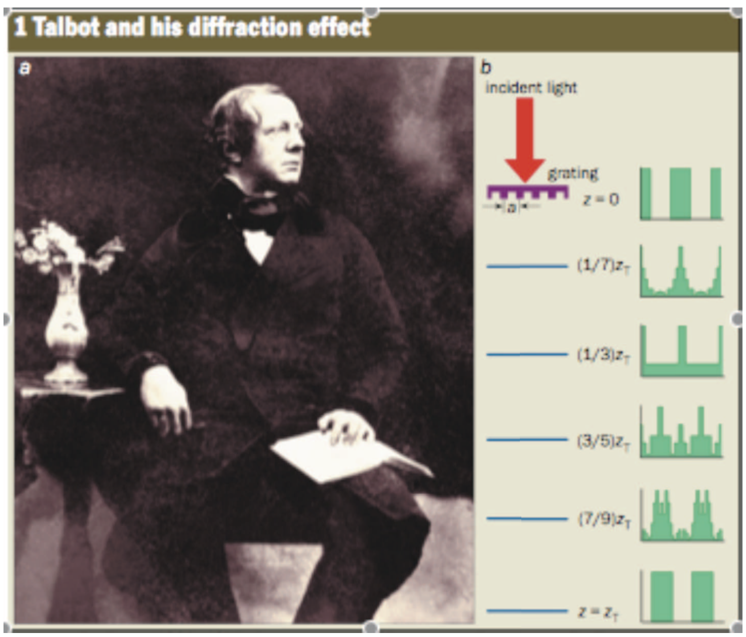
$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

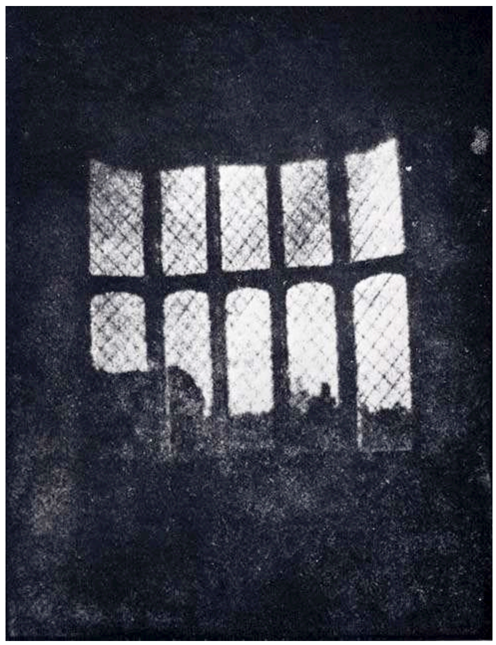
$$u(t,0) = u(t,2\pi) \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,2\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,2\pi)$$

- Michael Berry, et. al.
- Oskolkov
- Michael Taylor
- Fulling, Güntürk
- Kapitanski, Rodnianski

"Does a quantum particle know the time?"

William Henry Fox Talbot (1800–1877)





★ Talbot's 1835 image of a latticed window in Lacock Abbey

⟹ oldest photographic negative in existence.

The Talbot Effect

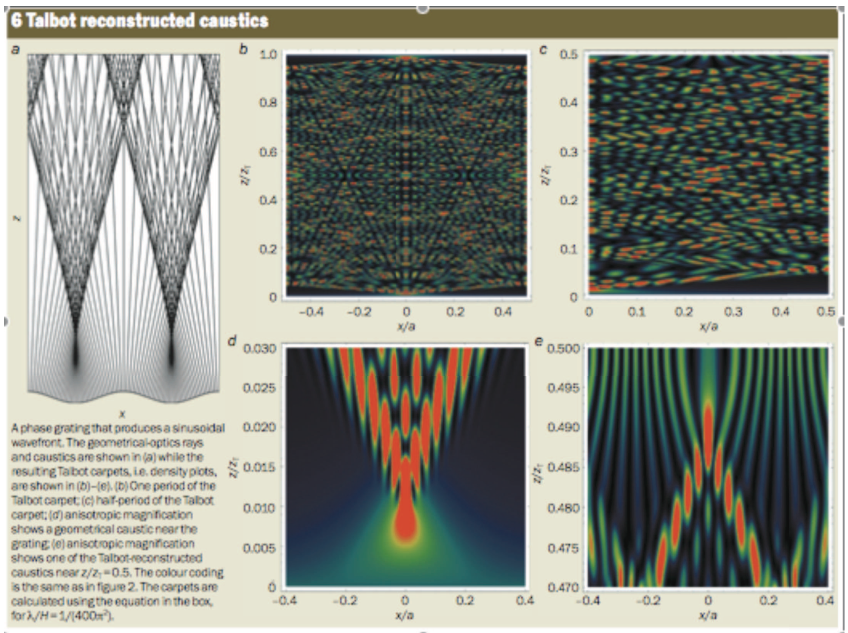
Fresnel diffraction by periodic gratings (1836)

"It was very curious to observe that though the grating was greatly out of the focus of the lens . . . the appearance of the bands was perfectly distinct and well defined . . . the experiments are communicated in the hope that they may prove interesting to the cultivators of optical science."

— *Fox Talbot*

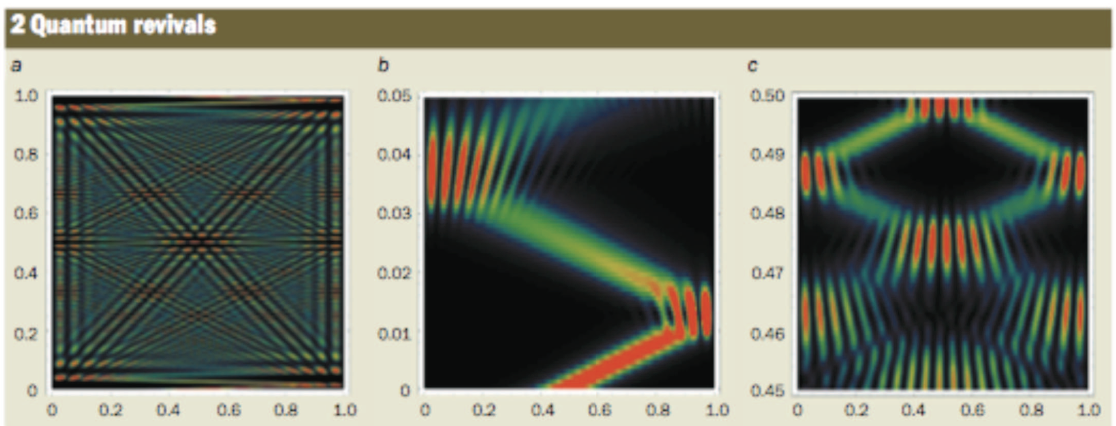
$\Longrightarrow$ Lord Rayleigh calculates the Talbot distance (1881)

The Quantized/Fractal Talbot Effect



- Optical experiments — Berry & Klein
- Diffraction of matter waves (helium attoms) — Nowak et. al.

Quantum Revival



- Electrons in potassium ions — Yeazell & Stroud
- Vibrations of bromine molecules — Vrakking, Villeneuve, Stolow

Periodic Schrödinger Equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(t,0) = u(t,2\pi) \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,2\pi) \qquad \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,2\pi)$$

Integrated fundamental solution:

$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x + k^2 t)}}{k}\,.$$

★ For $x/t \in \mathbb{Q}$, this is known as a Gauss (or, more generally, Weyl) sum, of importance in number theory

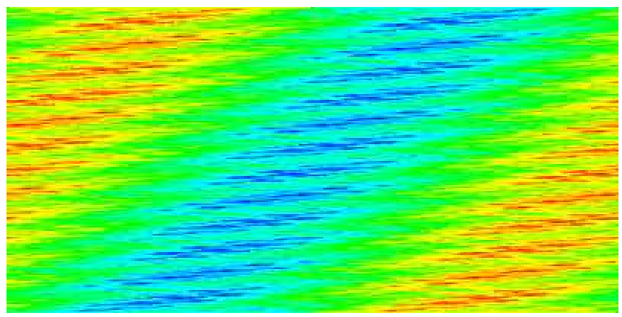
$\Longrightarrow$ Hardy, Littlewood, Weil, I. Vinogradov, etc.

Integrated fundamental solution:

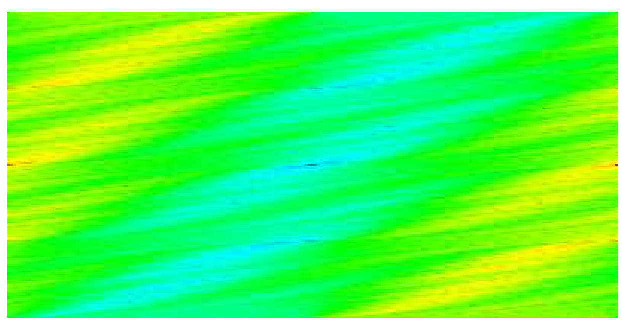
$$u(t,x) = \frac{1}{2\pi} \sum_{0 \neq k = -\infty}^{\infty} \frac{e^{\,\mathrm{i}\,(k\,x + k^2 t)}}{k}\,.$$

Theorem.

- The fundamental solution $\partial u/\partial x$ is a Jacobi theta function.
  At rational times $t = p\pi/q$, it linear combination of delta functions concentrated at rational nodes $x_j = \pi j/q$.
- At irrational times t, the integrated fundamental solution is a continuous but nowhere differentiable function.
  (The fractal dimension of its graph is $\frac{3}{2}$.)



Dispersive Carpet



Schrödinger Carpet

Future Directions

- Other boundary conditions (Fokas/Bona)
- Higher space dimensions and other domains (e.g., tori, spheres)
- Numerical solution techniques?
- Dispersive nonlinear partial differential equations:
  periodic Korteweg–deVries — Zabusky & Kruskal
- Experimental verification in dispersive media?