Higher Order Symmetries of Underdetermined Systems of Partial Differential Equations and Noether's Second Theorem

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Yanenko 100 Conference, August, 2021

Symmetry Groups of Differential Equations

$\Longrightarrow$ Sophus Lie (1842–1899)

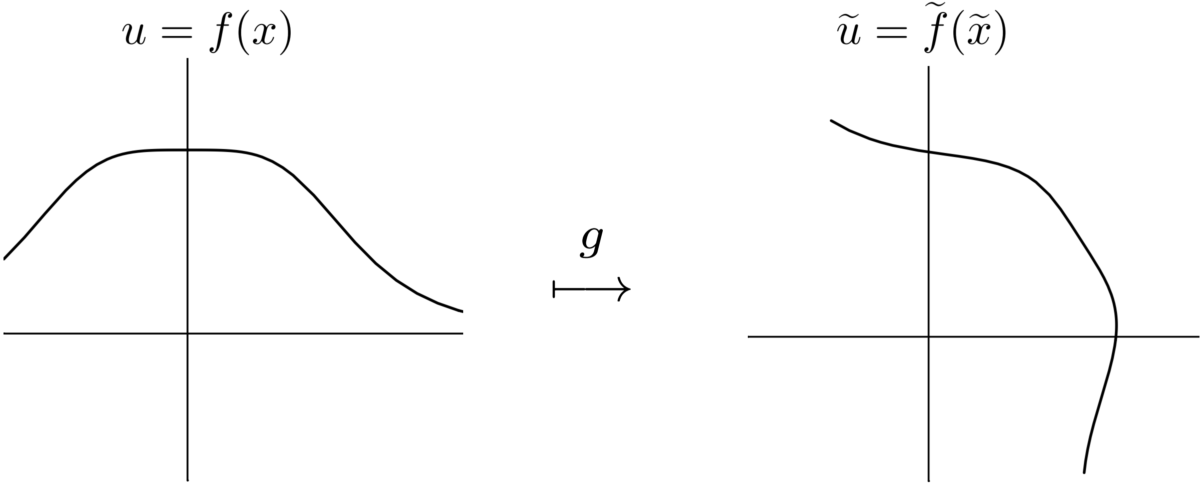
System of differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — Lie group or Lie pseudo-group acting on the space of independent and dependent variables:

$$(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$$

G acts on functions by transforming their graphs:



Definition. G is a symmetry group of the system $\Delta = 0$ if $\widetilde{f} = g \cdot f$ is a solution whenever f is.

Infinitesimal Generators

Every one-parameter group can be viewed as the flow of a vector field $\mathbf{v}$, known as its infinitesimal generator.

In other words, the one-parameter group is realized as the solution to the system of ordinary differential equations governing the vector field's flow:

$$\frac{dz}{d\varepsilon} = \mathbf{v}(z)$$

Equivalently, if one expands the group transformations in powers of the group parameter ε, the infinitesimal generator comes from the linear terms:

$$z(\varepsilon) = z + \varepsilon\, \mathbf{v}(z) + \cdots$$

Infinitesimal Generators = Vector Fields

In differential geometry, it has proven to be very useful to identify a vector field with a first order differential operator

In local coordinates $(\ldots x^i \ldots u^\alpha \ldots)$, the vector field

$$\mathbf{v} = (\;\ldots\; \xi^i(x,u) \;\ldots\; \varphi^\alpha(x,u) \;\ldots\;)$$

that generates the one-parameter group (flow)

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

is identified with the differential operator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha}$$

Prolongation

Since G acts on functions, it acts on their derivatives $u^{(n)}$, leading to the prolonged group action:

$$(\widetilde{x}, \widetilde{u}^{(n)}) = \mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$$

$\Longrightarrow$ formulas provided by implicit differentiation

Prolonged infinitesimal generator:

$$\mathrm{pr}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(n)})\, \frac{\partial}{\partial u_J^\alpha}$$

The Prolongation Formula

The coefficients of the prolonged vector field are given by the explicit prolongation formula (PJO, 1979):

$$\varphi_J^\alpha = D_J\, Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha$$

where $D_i = \sum_{\alpha, J} u_{J,i}^\alpha \dfrac{\partial}{\partial u_J^\alpha}$ $\quad D_J = D_{j_1} \cdots D_{j_k}$ — total derivatives

$$Q = (Q^1, \ldots, Q^q) \quad \text{— characteristic of } \mathbf{v}$$

$$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

★ Invariant functions $u = f(x)$ are solutions to $Q(x, u^{(1)}) = 0$

Lie's Infinitesimal Symmetry Criterion for Differential Equations

Theorem. A connected group of transformations G is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\boxed{\operatorname{pr} \mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0}$$

for every infinitesimal generator $\mathbf{v}$ of G.

Generalized (Higher Order) Symmetries

⋆ Due to Noether (1918)

⋆ *NOT* Lie or Bäcklund, who only got as far as
contact transformations.

Key Idea: Allow the coefficients of the infinitesimal generator to depend on derivatives of u, but drop the requirement that the (prolonged) vector field define a geometrical transformation on any finite order jet space:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic :

$$Q_\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Prolongation formula:

$$\mathrm{pr}\,\mathbf{v} = \mathrm{pr}\,\mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

$$\mathrm{pr}\,\mathbf{v}_Q = \sum_{\alpha, J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}$$

★ $\mathbf{v}$ is a generalized symmetry of a differential equation if and only if its evolutionary form $\mathbf{v}_Q$ is.

Example. Burgers' equation.

$$u_t = u_{xx} + u u_x$$

Characteristics of generalized symmetries:

u_x space translations

$u_{xx} + u\,u_x$ time translations

$$u_{xxx} + \tfrac{3}{2}u\,u_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}u^2 u_x$$

$$u_{xxxx} + 2\,u\,u_{xxx} + 5\,u_x u_{xx} + \tfrac{3}{2}u^2 u_x + 3\,u\,u_x^2 + \tfrac{1}{2}u^3 u_x$$

$$\vdots$$

⟹ See Mikhailov–Shabat–Sokolov and J.P. Wang's thesis for long lists of equations with higher order symmetries.

Question: Which systems of PDE possess higher order generalized symmetries?

- Linear systems of partial differential equations that admit a nontrivial point symmetry group, as well as systems that can be linearized by a point or contact transformation or (in favorable circumstances) a differential substitution

  (C integrable systems)

- Integrable systems solvable by inverse scattering

  (S integrable systems)

⋆ Underdetermined systems that admit a symmetry generator depending on an arbitrary function of the independent variables

$\Longrightarrow$ Almost all equations with one higher order symmetry have infinitely many.

Bakirov's Counterexample:

The "triangular system" of evolution equations

$$u_t = u_{xxxx} + v^2 \qquad v_t = \tfrac{1}{5}\, v_{xxxx}$$

has one sixth order generalized symmetry, but no further higher order symmetries.

Bakirov (1991), Beukers–Sanders–Wang (1998),
van der Kamp–Sanders (2002)

⋆ ⋆ Non-triangular examples?

Recursion operators

$\Longrightarrow$ PJO (1977)

Definition. An operator $\mathcal{R}$ is called a recursion operator for the system $\Delta = 0$ if it maps symmetries to symmetries, i.e., if $\mathbf{v}_Q$ is a generalized symmetry (in evolutionary form), and $\widetilde{Q} = \mathcal{R}Q$, then $\mathbf{v}_{\widetilde{Q}}$ is also a generalized symmetry.

$\Longrightarrow$ A recursion operator generates infinitely many symmetries with characteristics

$$Q, \quad \mathcal{R}Q, \quad \mathcal{R}^2 Q, \quad \mathcal{R}^3 Q, \quad \ldots$$

Theorem. Given the system $\Delta = 0$ with Fréchet derivative (linearization) D_Δ, if

$$[D_\Delta, \mathcal{R}] = 0$$

on solutions, then $\mathcal{R}$ is a recursion operator.

Example. Burgers' equation.

$$u_t = u_{xx} + uu_x$$

$$D_\Delta = D_t - D_x^2 - uD_x - u_x$$

$$\mathcal{R} = D_x + \tfrac{1}{2}u + \tfrac{1}{2}uD_x^{-1}$$

$$D_\Delta \cdot \mathcal{R} = D_t D_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x + \tfrac{1}{2}u_t - \\ - \tfrac{3}{2}u_{xx} - \tfrac{3}{2}uu_x + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1},$$

$$\mathcal{R} \cdot D_\Delta = D_t D_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x - u_{xx} - uu_x$$

hence

$$[D_\Delta, \mathcal{R}] = \tfrac{1}{2}(u_t - u_{xx} - uu_x) + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1}$$

which vanishes on solutions.

Linear Equations

Theorem. Let

$$\Delta[u] = 0$$

be a linear system of partial differential equations. Then any symmetry $\mathbf{v}_Q$ with linear characteristic $Q = \mathcal{D}[u]$ determines a recursion operator $\mathcal{D}$, since

$$[\mathcal{D}, \Delta] = \widetilde{\mathcal{D}} \cdot \Delta$$

If $\mathcal{D}_1, \ldots, \mathcal{D}_m$ determine linear symmetries $\mathbf{v}_{Q_1}, \ldots, \mathbf{v}_{Q_m}$, then any polynomial in the $\mathcal{D}_j$'s also gives a linear symmetry.

Question 1: Given a linear system, when are all symmetries
a) linear? b) generated by first order symmetries?

Question 2: What is the structure of the non-commutative symmetry algebra?

Bi–Hamiltonian systems

$\Longrightarrow$ Magri (1978)

Theorem. Suppose

$$\frac{du}{dt} = F_1 = J_1 \nabla H_1 = J_2 \nabla H_0$$

is a biHamiltonian system, where J_1, J_2 form a compatible pair of Hamiltonian operators. Assume that J_1 is nondegenerate. Then

$$\mathcal{R} = J_2 J_1^{-1}$$

is a recursion operator that generates an infinite hierarchy of biHamiltonian symmetries

$$\frac{du}{dt} = F_n = \mathcal{R}\, F_{n-1} = J_1 \nabla H_n = J_2 \nabla H_{n-1}.$$

The Korteweg–deVries Equation

$$\frac{\partial u}{\partial t} = u_{xxx} + u\,u_x = J_1 \frac{\delta \mathcal{H}_1}{\delta u} = J_2 \frac{\delta \mathcal{H}_0}{\delta u}$$

$$J_1 = D_x \qquad\qquad \mathcal{H}_1[u] = \int \left(\tfrac{1}{6} u^3 - \tfrac{1}{2} u_x^2 \right) dx$$

$$J_2 = D_x^3 + \tfrac{2}{3} u\, D_x + \tfrac{1}{3} u_x \qquad\qquad \mathcal{H}_0[u] = \int \tfrac{1}{2} u^2\, dx$$

★ ★ Bi–Hamiltonian system with recursion operator (Lenard)

$$\mathcal{R} = J_2 \cdot J_1^{-1} = D_x^2 + \tfrac{2}{3} u + \tfrac{1}{3} u_x D_x^{-1}$$

Hierarchy of generalized symmetries and higher order conservation laws:

$$\frac{\partial u}{\partial t} = u_{xxxxx} + \tfrac{5}{3} u\, u_{xxx} + \tfrac{10}{3} u_x u_{xx} + \tfrac{5}{6} u^2 u_x = J_1 \frac{\delta \mathcal{H}_2}{\delta u} = J_2 \frac{\delta \mathcal{H}_1}{\delta u}$$

$$\mathcal{H}_2[u] = \int \left(\tfrac{1}{2} u_{xx}^2 - \tfrac{5}{6} u_x^2 + \tfrac{5}{72} u^4 \right) dx$$

and so on . . . (Gardner, Green, Kruskal, Miura, Lax)

Underdetermined Systems

We assume that the system of differential equations

$$\Delta_\kappa[u] = 0, \quad \kappa = 1, \ldots, q, \tag{$*$}$$

has the same number of equations as unknowns $u = (u^1, \ldots, u^q)$.

Definition. The system of differential equations $(*)$ is *underdetermined* if there exist differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_q$ that do not simultaneously vanish on solutions, such that

$$\mathcal{D}_1 \Delta_1 + \cdots + \mathcal{D}_q \Delta_q \equiv 0.$$

For the general case (which is quite subtle) see

W.M. Seiler, *Involution*, Springer, 2010.

Examples of underdetermined systems arising in basic physics include Maxwell's equations for electromagnetism and Einstein's equations for general relativity.

The Main Theorem

Theorem. Suppose that a system of differential equations admits an infinitesimal symmetry $\mathbf{v}_Q$ whose characteristic

$$Q[u,h] = Q(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ h_K(x)\ \ldots\ )$$

depends on finitely many derivatives $h_K = \partial_K h$ of an arbitrary function $h(x)$ of the independent variables. Let $F[u]$ be an arbitrary differential function. Then the characteristic

$$\widehat{Q}[u] = Q(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ D_K F\ \ldots\ )$$

obtained by replacing the derivatives of h by the corresponding total derivatives of F is also the characteristic of an infinitesimal symmetry $\mathbf{v}_{\widehat{Q}}$ of the system.

Thus, any such underdetermined system of differential equations automatically admits an infinite family of higher order symmetries depending upon an arbitrary function F of the independent variables, the dependent variables, and their derivatives of arbitrarily high order.

★★ Systems that satisfy the hypothesis of the Theorem are necessarily underdetermined, although, as we will see, not every underdetermined system will admit such a symmetry generator.

Proof:

First: note that the partial derivatives of h coincide with its total derivatives: $\partial_K h(x) = D_K h$.

Second: Suppose

$$R(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ \partial_K h(x)\ \ldots\ ) = 0$$

where $h(x)$ is an arbitrary function of all the independent variables. Then, since its partial derivatives $\partial_K h(x)$ can assume any values, we can replace them by any quantities and still have equality. In particular,

$$R(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ D_K F\ \ldots\ ) = 0$$

where $F[u]$ be an arbitrary differential function.

$\Longrightarrow$ Kiselev's Substitution Principle:

`https://preprints.ihes.fr/2012/M/M-12-13.pdf`

Third: according to the prolongation formula for evolutionary vector fields, the coefficients of $\operatorname{pr}\mathbf{v}_Q$ are obtained by total differentiation, so

$$D_I Q_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ \partial_K h(x)\ \ldots\ ),$$

where $R_{\alpha,I}$ are certain functions of the jet coordinates and the partial (total) derivatives of h, then, replacing h by F in Q leads, via the substitution principle, to the same algebraic expressions for its total derivatives

$$D_I \widehat{Q}_\alpha = R_{\alpha,I}(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ D_K F\ \ldots\ ),$$

in terms of the jet coordinates and the total derivatives of F. By the preceding remarks, we can thus replace each partial derivative $h_K(x)$ appearing in the determining equations by the corresponding total derivative $D_K F$ without affecting their validity. We conclude that the evolutionary vector field $\mathbf{v}_{\widehat{Q}}$ with characteristic $\widehat{Q}$ also satisfies the symmetry determining equations for the system of differential equations. *Q.E.D.*

Variational Symmetries

Definition. A (strict) variational symmetry is a transformation $(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$ which leaves the variational problem invariant:

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \widetilde{u}^{(n)})\, d\widetilde{x} = \int_{\Omega} L(x, u^{(n)})\, dx$$

Infinitesimal invariance criterion:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = 0$$

Divergence symmetry (Bessel–Hagen):

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B$$

$\Longrightarrow$ Every divergence symmetry has an equivalent strict variational symmetry

Theorem. Every symmetry of a variational problem is a symmetry of the Euler–Lagrange equations.

★ ★ But not conversely!

★ Almost all examples of non-variational symmetries are scaling symmetries. One known exception is the equations of 3D linear isotropic elasticity which admits the non-variational generalized symmetry whose flow is equivalent to Maxwell's equations! (PJO, 1984)

Noether's Second Theorem

Theorem. A system of Euler-Lagrange equations is underdetermined if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function of the independent variables. The associated conservation laws are trivial.

⋆ Noether's First Theorem gives a one-to-one correspondence between non-trivial symmetries and non-trivial conservation laws. (Martinez Alonso, 1979; PJO, 1986)

Open Question: Are there over-determined systems of Euler–Lagrange equations for which trivial symmetries give non-trivial conservation laws?

Generalized Noether's Second Theorem

Theorem. If $E(L) = 0$ is any underdetermined system of Euler–Lagrange equations, then it admits generalized symmetries of arbitrarily high order depending upon one or more arbitrary differential functions.

$\Longrightarrow$ Fulp-Lada-Stasheff, 2003; Anco, S.C., 2017 (co-symmetries)

Generalized Noether's Second Theorem

Theorem. If $E(L) = 0$ is any underdetermined system of Euler–Lagrange equations, then it admits generalized symmetries of arbitrarily high order depending upon one or more arbitrary differential functions.

This result resolves a mystery concerning Noether's Second Theorem, which relies on infinitesimal symmetries that involve one or more arbitrary functions of the p independent variables. But what about the functions of the dependent variables, and hodograph and reciprocal transformations that interchange independent and dependent variables, etc.? According to the Theorem, once the system admits variational symmetries depending on an arbitrary function of *any* p jet variables, then it automatically admits variational symmetries depending on an arbitrary differential function!

Relativity

Noether's Second Theorem effectively resolved Hilbert's dilemma regarding the law of conservation of energy in Einstein's field equations for general relativity.

Namely, the time translational symmetry that ordinarily leads to conservation of energy in fact belongs to an infinite-dimensional symmetry group, and thus, by Noether's Second Theorem, the corresponding conservation law is trivial, meaning that it vanishes on all solutions.

$\Longrightarrow$ Higher order symmetries of Einstein's equations:

Anderson and Torre, 1993

A Simple Example:

Variational problem:

$$I[u,v] = \iint (u_x + v_y)^2 \, dx \, dy$$

Variational symmetry group:

$$(u,v) \longmapsto (u + \varphi_y, v - \varphi_x)$$

Evolutionary generator:

$$\mathbf{v}_Q = - \frac{\partial h}{\partial y} \frac{\partial}{\partial u} + \frac{\partial h}{\partial x} \frac{\partial}{\partial v}$$

Euler-Lagrange equations:

$$\Delta_1 = E_u(L) = u_{xx} + v_{xy} = 0$$

$$\Delta_2 = E_v(L) = u_{xy} + v_{yy} = 0$$

Differential relation:

$$D_y \Delta_1 - D_x \Delta_2 \equiv 0$$

The Main Theorem implies that, for any differential function $F[u, v]$ depending on x, y and u, v and their derivatives, the evolutionary vector field

$$\widehat{\mathbf{v}} = -D_y F \frac{\partial}{\partial u} + D_x F \frac{\partial}{\partial v}$$

is also a variational symmetry, and thus a higher order symmetry of the underdetermined Euler–Lagrange equations.

For example, the second order variational problem

$$\widetilde{I}[u,v] = \iint \left[\, \tfrac{1}{2}\,(u_{xx} + v_{xy})\,(u_{xy} + v_{yy}) + \tfrac{1}{6}\,(u_x + v_y)^3 \,\right] dx\, dy,$$

with underdetermined nonlinear fourth order Euler–Lagrange equations

$$u_{xxxy} + v_{xxyy} = (u_x + v_y)\,(u_{xx} + v_{xy}),$$
$$u_{xxyy} + v_{xyyy} = (u_x + v_y)\,(u_{xy} + v_{yy}),$$

possesses the aforementioned properties.

While the Theorem implies the existence of higher order symmetries of any underdetermined system of Euler–Lagrange equations, this result does not extend to general underdetermined systems of nonlinear partial differential equations. Indeed, in the present context, if $H[u,v]$ is any differential function, then the underdetermined system

$$\Delta_1 = D_x H = 0, \qquad \Delta_2 = D_y H = 0,$$

satisfies the same linear dependency:

$$D_y \Delta_1 - D_x \Delta_2 = 0.$$

An evolutionary infinitesimal generator $\mathbf{v} = Q[u,v]\partial_u + R[u,v]\partial_v$ will be an infinitesimal symmetry of provided

$$D_x[\mathrm{pr}\,\mathbf{v}(H)] = D_y[\mathrm{pr}\,\mathbf{v}(H)] = 0$$

whenever holds. It is clear that, by making $H[u,v]$ sufficiently complicated, one can ensure that there are no symmetries. Thus, such an underdetermined system does not admit an infinite-dimensional symmetry algebra of the required form, and hence the Theorem does not apply.

Systems of differential equations or variational problems for curves, surfaces, etc., that do not depend on any underlying parametrization thereof are called *parameter-independent*. The symmetry pseudo-group consisting of all local diffeomorphisms of the base space X has infinitesimal generators

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x) \frac{\partial}{\partial x^i},$$

where $\xi^i(x)$ are arbitrary functions.

Theorem. A system of differential equations $\Delta[u] = 0$ is parameter-independent if and only if it admits all generalized infinitesimal symmetry generators of the form

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} \left(\sum_{i=1}^{p} u_i^\alpha F_i[u] \frac{\partial}{\partial u^\alpha} \right),$$

where $F_1[u], \ldots, F_p[u]$ are arbitrary differential functions.

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