# Repères Mobiles
# Moving Frames

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Classical contributions:

G. Darboux, É. Cotton, É. Cartan

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Modern contributions:

P. Griffiths, M. Green, G. Jensen

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Fels, M., Olver, P.J.,

Part I, *Acta Appl. Math.* **51** (1998) 161–213

Part II, *Acta Appl. Math.*, to appear

Olver, P.J., *Classical Invariant Theory*,

Cambridge Univ. Press, 1999

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http://www.math.umn.edu/~olver

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear."

"Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

— Hermann Weyl

"Cartan on groups and differential geometry", *Bull. Amer. Math. Soc.* **44** (1938) 598–601

## Applications of Moving Frames

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- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Invariant differential forms and tensors
- Classical invariant theory
- Identities and syzygies
- Computer vision
- Invariant numerical methods
- Lie pseudogroups

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## The Basic Equivalence Problem

$M$ — smooth $m$-dimensional manifold.

$G$ — transformation group acting on $M$

- finite-dimensional Lie groups
- infinite-dimensional Lie pseudo-groups

**Equivalence:**

Determine when two $n$-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \subset M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

**Symmetry:**

Self-equivalence or *self-congruence*:

$$N = g \cdot N$$

# Euclidean & Affine Geometry

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- *Euclidean group* — $G = \mathrm{SE}(n)$ or $\mathrm{E}(n)$
  isometries of Euclidean space
  translations, rotations (& reflections)

$$z \longmapsto R \cdot z + a \qquad \begin{cases} R \in \mathrm{SO}(n) \text{ or } \mathrm{O}(n) \\ a \in \mathbb{R}^n \\ z \in \mathbb{R}^n \end{cases}$$

**Equivalence Problem**: Can given submanifolds $N$ and $\overline{N}$ be transformed into each other by a Euclidean transformation, i.e., a combination of translations, rotations, and, possibly, reflections?

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- *Equi-affine group*: $G = \mathrm{SA}(n)$
  $R \in \mathrm{SL}(n)$ — area-preserving
- *Affine group*: $G = \mathrm{A}(n)$
  $R \in \mathrm{GL}(n)$
- *Projective group*: $G = \mathrm{PSL}(n)$
  acting on $\mathbb{RP}^{n-1}$

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$\Longrightarrow$ Applications in computer vision

**Classical Invariant Theory**

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

$\Longrightarrow$ multiplier representation of GL(2)

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Transformation group:

$$g : \; (x, u) \longmapsto \left( \frac{\alpha x + \beta}{\gamma x + \delta} \, , \; \frac{u}{(\gamma x + \delta)^n} \right)$$

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We identify a polynomial with its graph

$$N_Q = \{ \, (x, u) = (x, Q(x)) \, \} \subset \mathbb{C}^2$$

Then

$$\overline{Q} = g \cdot Q \quad \Longleftrightarrow \quad N_{\overline{Q}} = g \cdot N_Q$$

# Moving Frames

**Definition.**

A *moving frame* is a $G$-equivariant map

$$\rho : M \longrightarrow G$$

$\Longrightarrow$ *Cartan, Griffiths*

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Equivariance:

$$\rho(g\cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

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Note

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

**Theorem.** A moving frame exists in a neighborhood of a point $z \in M$ if and only if $G$ acts freely and regularly near $z$.

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- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity: $g \cdot z = z$ iff $g = e$.
- locally free — the orbits have the same dimension as $G$.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once ( $\not\sim$ irrational flow on the torus)
- effective — the only group element $g \in G$ which fixes *every* point $z \in M$ is the identity: $g \cdot z = z$ for all $z \in M$ iff $g = e$.
- locally effective — ...

## Normalization

Set

$$w(g, z) = g^{-1} \cdot z$$

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \quad \ldots \quad w_r(g, z) = c_r$$

Assuming $\det(\, \partial w_i / \partial g_j \,) \neq 0$, the solution

$$g = \rho(z)$$

*is* a (local) left moving frame.

$\Longrightarrow$ Implicit Function Theorem

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## Invariants

Substituting the moving frame normalizations into the remaining $m - r$ components of $w(g, z)$ produces a comple system of (functionally independent) *fundamental invariants*:

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

**Theorem.** Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$

# Geometrical Interpretation

| Normalization = choice of cross-section to the group orbits |
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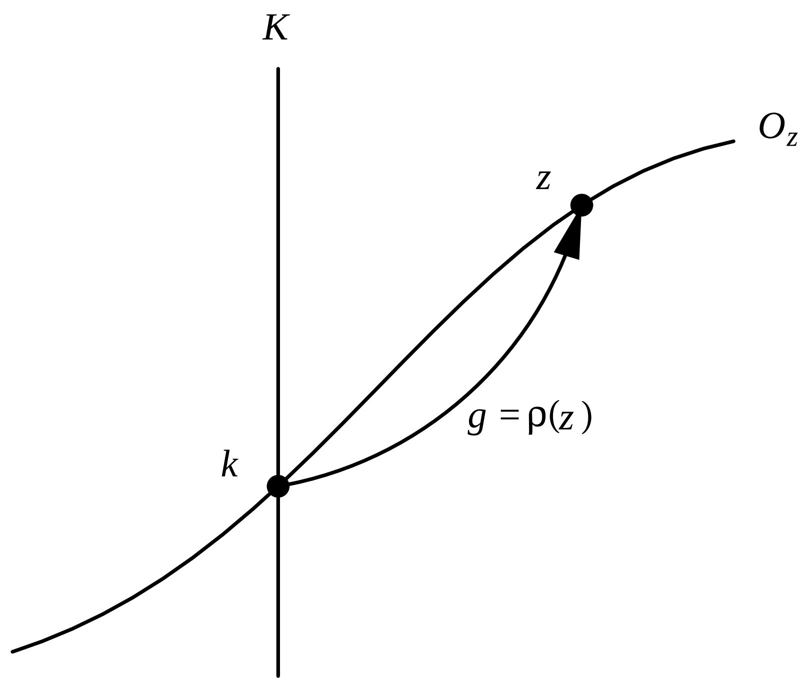


$K$ — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the cross-section and in the orbit through $z$

- $k$ is the *canonical form* of $z$
- The (nonconstant) coordinates of $k$ are the fundamental invariants

$g \in G$ — unique group element mapping $k$ to $z$

$$z = g \cdot k \qquad \Longrightarrow \quad \text{freeness}$$

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## Normalization Equations

The map

$$g = \rho(z)$$

satisfying the *normalization equations*

$$w(g, z) = g^{-1} \cdot z = k \in K$$

defines a (local) left moving frame

$$\rho : \, M \longrightarrow G$$

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Coordinate cross-section

$$K = \{ \; z_1 = c_1 \quad \ldots \quad z_r = c_r \; \}$$

Normalization equations:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

# Prolongation

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Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

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Any non-free, effective action can be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{\times n} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint differential invariants
$\Longrightarrow$ invariant numerical approximations

## Joint Euclidean Invariants

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E(2) acts on $M = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$: $\qquad z_i = (x_i, u_i)$

$$y_i = \cos\theta\,(x_i - a) + \sin\theta\,(u_i - b)$$
$$v_i = -\sin\theta\,(x_i - a) + \cos\theta\,(u_i - b)$$
$$i = 0, 1, 2, \ldots$$

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Normalization

$$y_0 = 0 \qquad v_0 = 0 \qquad v_1 = 0$$

Moving frame $\quad \rho : M \to \mathrm{E}(2)$

$$a = x_0 \qquad b = u_0 \qquad \theta = \tan^{-1}\left(\frac{u_1 - u_0}{x_1 - x_0}\right)$$

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Joint invariants:

$$y_i \longmapsto \frac{(z_i - z_0)\cdot(z_1 - z_0)}{\|\, z_1 - z_0 \,\|}$$
$$v_i \longmapsto \frac{(z_i - z_0)\wedge(z_1 - z_0)}{\|\, z_1 - z_0 \,\|}$$

**Theorem.** Every joint Euclidean invariant is a function of the interpoint distances:

$$\left\|\, z_i - z_j \,\right\|$$

## Jets and Prolongation

$M$ — $m$-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — $n^{\text{th}}$ extended jet bundle for $p$-dimensional submanifolds $N \subset M$

$\Longrightarrow$ equivalence classes of submanifolds under $n^{\text{th}}$ order contact

$G$ — transformation group acting on $M$

$G^{(n)}$ — $n^{\text{th}}$ prolonged action of $G$ on $\mathrm{J}^n$

Local coordinates:

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables; $q = m - p$

$(x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots)$ — jet coordinates

$$u^\alpha_J = \partial_J u^\alpha$$

# Euclidean Curves

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Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

Prolong to $\mathrm{J}^3$ — implicit differentiation

$$\begin{aligned}
y &= \cos\theta\,(x-a) + \sin\theta\,(u-b) \\
v &= -\sin\theta\,(x-a) + \cos\theta\,(u-b) \\
v_y &= \frac{-\sin\theta + u_x\cos\theta}{\cos\theta + u_x\sin\theta} \\
v_{yy} &= \frac{u_{xx}}{(\cos\theta + u_x\sin\theta)^3} \\
v_{yyy} &= \frac{(\cos\theta + u_x\sin\theta)u_{xxx} - 3u_{xx}^2\sin\theta}{(\cos\theta + u_x\sin\theta)^5}
\end{aligned}$$

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Normalization

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

Moving frame

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

Differential invariants (curvatures)

$$v_{yy} \longmapsto \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

$$v_{yyy} \longmapsto \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}$$

$$v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

Invariant one-form — arc length

$$dy = (\cos t + u_x \sin t)\, dx \longmapsto ds = \sqrt{1+u_x^2}\, dx$$

Invariant differential operator

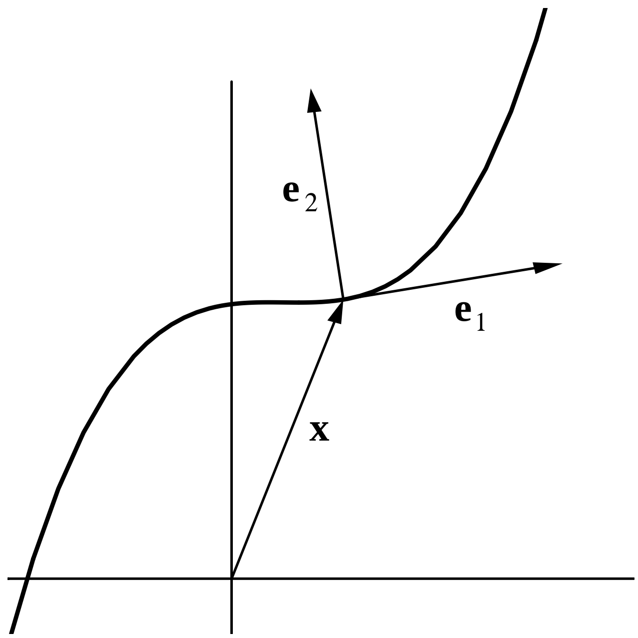
$$\frac{d}{dy} = \frac{1}{\cos t + u_x \sin t}\ \frac{d}{dx}$$

$$\longmapsto \quad \frac{d}{ds} = (1+u_x^2)^{-1/2}\ \frac{d}{dx}$$

**Theorem.** All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \cdots$$

## Euclidean Curves




Moving frame:

$$a = x \qquad b = u \qquad \theta = \tan^{-1} u_x$$

$$\rho : \ (x, u, u_x) \ \longmapsto \ (R, \mathbf{a}) \in \mathrm{SE}(2)$$

$$R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2) \qquad \mathbf{a} = \begin{pmatrix} x \\ u \end{pmatrix}$$

$$\mathbf{e}_1 = \frac{d\mathbf{x}}{ds} = (x_s, y_s) \qquad \mathbf{e}_2 = \mathbf{e}_1^{\perp} = (-y_s, x_s)$$

$\Longrightarrow$ Frenet frame

## Equivalence & Signature

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**Definition.** The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the first two differential invariants $\kappa$ and $\kappa_s$

$$\mathcal{S} = \left\{ \left( \kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \quad \subset \quad \mathbb{R}^2$$

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**Theorem.** Two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical

$$\overline{\mathcal{S}} = \mathcal{S}$$

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$\Longrightarrow$ object recognition

## Maximally Symmetric Curves

**Theorem.** The following are equivalent:

- The signature curve $\mathcal{S}$ degenerates to a point
- The curve $\mathcal{C}$ has constant curvature
- $\mathcal{C} = \{\exp(t\mathbf{v})x_0\}$ is the orbit of a one-parameter subgroup

$\Longrightarrow$ In Euclidean geometry, these are the circles and straight lines.

$\Longrightarrow$ In equi-affine geometry, these are the conic sections.

# Discrete Symmetries

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Let $\Sigma : \mathcal{C} \longrightarrow \mathcal{S}$ denote the signature map, so

$$\Sigma(z) = (\kappa(z), \kappa_s(z)), \qquad z \in \mathcal{C}$$

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**Definition.** The *index* of a curve $\mathcal{C}$ equals the number of points in $\mathcal{C}$ which map to a generic point of $\mathcal{S}$:

$$\iota_{\mathcal{C}} = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

$\Longrightarrow$ Self intersections

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**Theorem.** The cardinality of the symmetry group of $\mathcal{C}$ equals its index $\iota_{\mathcal{C}}$.

$\Longrightarrow$ Approximate symmetries

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## Rigidity — Euclidean Curves

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**Theorem.** If

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

are equivalent curves and have third order contact at a point, then

$$\overline{\mathcal{C}} = \mathcal{C}$$

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**Theorem.** For each $z \in \mathcal{C}$, there exists $g_z \in G$ such that $\overline{\mathcal{C}}$ and $g_z \cdot \mathcal{C}$ have third order contact at $\bar{z} = g_z \cdot z$, if and only if

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

for fixed $g \in G$.

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- Rigidity order
  = order of signature curve
  = order $\kappa_s$
- $\mathcal{C}$ maximally symmetric $\Longrightarrow$
  rigidity order = order $\kappa$

## Moving Frames in Classical Invariant Theory

$M = \mathbb{R}^2, \quad G = \mathrm{GL}(2)$

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$$(x, u) \longmapsto \left( \frac{\alpha x + \beta}{\gamma x + \delta} \,, \; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

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$$\sigma = \gamma p + \delta \qquad\qquad \Delta = \alpha\delta - \beta\gamma$$

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Prolongation:

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}$$

$$v = \sigma^{-n} u$$

$$v_y = \frac{\sigma u_x - n\gamma u}{\Delta \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2 u_{xx} - 2(n-1)\gamma\sigma u_x + n(n-1)\gamma^2 u}{\Delta^2 \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

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Normalization:

$$y = 0 \qquad v = 1 \qquad v_y = 0 \qquad v_{yy} = \frac{1}{n(n-1)}$$

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Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\,u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\,xu^{(1-n)/n}$$

$$H = n(n-1)uu_{xx} - (n-1)^2 u_x \qquad \text{— Hessian}$$

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Nondegeneracy:

$$H \neq 0$$

*Note*: $H \equiv 0$ if and only if $Q(x) = (ax+b)^n$

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Differential invariants:

$$v_{yyy} = \frac{J}{n^2(n-1)} \qquad v_{yyyy} = \frac{K + 3(n-2)}{n^3(n-1)}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

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$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \qquad \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \qquad \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \quad \deg H = 2n-4 \quad \deg T = 3n-6 \quad \deg U = 4n-8$$

## Signatures of Binary Forms

**Definition.** The *signature curve* of a nondegenerate complex-valued binary form $Q(x)$ is the rational curve parametrized by the two fundamental absolute rational covariants,

$$\mathcal{S}_Q = \left\{ (J(x)^2, K(x)) = \left( \frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \;\middle|\; H(x) \neq 0 \right\}$$

*Nonsingular*: $H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0$.

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**Theorem.** Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

## Maximally Symmetric Binary Forms

**Theorem.** The following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- $T^2$ is a constant multiple of $H^3$
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- the graph of $Q$ coincides with the orbit of a one-parameter subgroup

## Symmetries of Binary Forms

Signature map:

$$\Sigma(x) = (J(x)^2, K(x))$$

Index:

$$\iota_Q = \min\left\{\#\,\Sigma^{-1}\{w\}\;\middle|\; w \in \mathcal{S}\right\}$$

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**Theorem.** The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree $n$ is:

- A two-parameter group if and only if $H \equiv 0$ if and only if $Q$ is equivalent to a constant.
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2$ is a constant multiple of $H^3$ if and only if $Q$ is complex-equivalent to a monomial $x^k$, with $k \neq 0, n$.
- A finite group in all other cases. The cardinality of the group equals the index $\iota_Q$ of the signature curve.

$\Longrightarrow$ More general analytic functions can admit infinite, discrete symmetry groups, e.g., periodic functions.

Let $Q(x)$ be a binary form of degree $n$ which is not complex equivalent to a monomial.

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} \quad T = (Q,\, H)^{(1)} \quad U = (Q,\, T)^{(1)}$$

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Define the bivariate polynomials

$$A(x, y) = H(x)^3\, T(y)^2 - H(y)^3\, T(x)^2 \qquad \deg A = 6n - 12$$

$$B(x, y) = H(x)^2\, U(y) - H(y)^2\, U(x) \qquad \deg B = 4n - 8$$

Their $k^{\text{th}}$ *subresultant*, taken with respect to $y$:

$$R_k(x) = R_k[A, B]$$

$\Longrightarrow$ Subresultants detect multiple common roots.

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**Theorem.** Assuming simple roots, the index of $Q(x)$ equals the first integer $k$ for which $R_k(x) \not\equiv 0$.

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**Theorem.** Let $\iota_Q$ denote the index of a binary form $Q$ of degree $n$ which is not complex-equivalent to a monomial. Then

- $\iota_Q \le 6n - 12$ if $U = cH^2$ for some constant $c$, or
- $\iota_Q \le 4n - 8$ in all other cases.

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$\Longrightarrow$ The equation $U = 0$ can be transformed into a Schwarz-type hypergeometric equation and into the Chazy equation from Painlevé analysis

# Regularization

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If $G$ acts on $M$, then the lifted action

$$(h, z) \longmapsto (g \cdot h, g \cdot z)$$

on the trivial left principal bundle

$$\mathcal{B} = G \times M$$

is always regular and free!

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The functions

$$w(g, z) = g^{-1} \cdot z$$

provide a complete system of invariants for the lifted action.

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A moving frame $\rho : M \longrightarrow G$ defines a $G$-equivariant section

$$\sigma : M \longrightarrow \mathcal{B} \qquad \sigma(z) = (\rho(z), z)$$

# General Philosophy of Lifting

Invariant objects on $\mathcal{B}$ — *lifted invariants* — are well-behaved and easily understood.

The moving frame section

$$\sigma : M \longrightarrow \mathcal{B}$$

allows us to "pull-back" lifted invariants to construct ordinary invariants on $M$.

For example,

$$\sigma^* w = w \circ \sigma = I$$

defines the fundamental invariant functions

$$I(z) = w(\rho(z), z) = \rho(z)^{-1} \cdot z$$

Similarly for lifted invariant differential forms, differential operators, tensors, etc.

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$\Longrightarrow$ The key complication is that the pull-back process does not commute with differentiation!

## Invariantization

Given a moving frame

$$\rho : M \longrightarrow G$$

If

$$F : M \longrightarrow \mathbb{R} \qquad\qquad F(z)$$

is any function then

$$L = F \circ w : \mathcal{B} \longrightarrow \mathbb{R} \qquad\qquad L(g, z) = F(g^{-1} \cdot z)$$

defines a lifted invariant. Further

$$I = F \circ w \circ \sigma : M \longrightarrow \mathbb{R} \qquad\qquad I(z) = F(\rho(z)^{-1} \cdot z)$$

defines an ordinary invariant function

$$\Longrightarrow \qquad \text{the } \textit{invariantization} \text{ of } F.$$

If $F$ is already an invariant, then $I = F$.

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The invariantization process

$$\mathcal{I}_\rho : F(z) \longmapsto I(z) = F(\rho(z)^{-1} \cdot z)$$

defines a *projection* from the space of functions to the space of invariants.

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# Computing Lifted Differential Invariants

Lifted ordinary invariants:

$$w(g,z) = g^{-1} \cdot z$$

Lifted independent and dependent variables:

$$z = (x,u) \qquad w = (y,v)$$

Explicitly:

$$y^1 = w^1(g,z) \quad \ldots \quad y^p = w^p(g,x)$$
$$v^1 = w^{p+1}(g,z) \quad \ldots \quad v^q = w^m(g,z)$$

Differentiate the $v$'s with respect to the $y$'s:

$$v^\alpha_K = \mathcal{E}_K v^\alpha$$

Lifted invariant differential operators:

$$\mathcal{E}_j F = \mathrm{D}_{y^j} F = \frac{\mathbf{D}(y^1,\ldots,y^{j-1},F,y^{j+1},\ldots y^p)}{\mathbf{D}(y^1,\ldots,y^p)}$$

$$\mathcal{E} = \mathbf{D}y(g,x,u^{(n)})^{-T} \cdot \mathbf{D}$$

## Jet Normalization

Choose $r = \dim G$ jet coordinates

$$z_1, \ldots, z_r \qquad\qquad x^i \text{ or } u^\alpha_J$$

Coordinate cross-section:

$$z_1 = c_1 \quad \ldots \quad z_r = c_r$$

Lifted differential invariants

$$w_1, \ldots, w_r \qquad\qquad y^i \text{ or } v^\alpha_J$$

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**Normalization Equations**

$$w_1(g, x, u^{(n)}) = c_1 \quad \ldots \quad w_r(g, x, u^{(n)}) = c_r$$

Solution:

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)}) \qquad \Longrightarrow \quad \text{moving frame}$$

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## The Fundamental Differential Invariants

$$I^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1} \cdot z^{(n)}$$

$$\boxed{\begin{aligned} J^i(x,u^{(n)}) &= y^i(\rho^{(n)}(x,u^{(n)}),x,u) \\ I^\alpha_K(x,u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x,u^{(n)}),x,u^{(k)}) \end{aligned}}$$

Phantom differential invariants

$$w_1 = c_1 \ \ldots \ w_r = c_r \qquad \Longrightarrow \quad \text{normalizations}$$

**Theorem.** Every $n^{\text{th}}$ order differential invariant can be locally written as a unique function of the non-phantom fundamental differential invariants in $I^{(n)}$.

Invariant differential operators:

$$\mathcal{D} = \mathbf{D}y(\rho^{(n)}(x,u^{(n)}),x,u^{(n)})^{-T} \cdot \mathbf{D}$$

**Theorem.** The higher order differential invariants are obtained by invariant differentiation with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_p$.

**Important:**

$$\mathcal{E}_i v^\alpha_K = v^\alpha_{K,i} \qquad [\mathcal{E}_i, \mathcal{E}_j] = 0$$

*but*

$$\mathcal{D}_i I^\alpha_K \neq I^\alpha_{K,i} \qquad [\mathcal{D}_i, \mathcal{D}_j] \neq 0$$

$\Longrightarrow$ *Pull-back does not commute with differentiation*!

**Recurrence Formulae**:

$$\boxed{\begin{aligned} \mathcal{D}_j J^i &= \delta^i_j + M^i_j \\ \mathcal{D}_j I^\alpha_K &= I^\alpha_{K,j} + M^\alpha_{K,j} \end{aligned}}$$

$M^i_j, M^\alpha_{K,j}$ — correction terms

**Commutation Formulae**:

$$\boxed{[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k}$$

- The correction terms can be computed directly from the infinitesimal generators!

## Generating Invariants

**Theorem.** A generating system of differential invariants consists of

- all non-phantom differential invariants $J^i$ and $I^\alpha$ coming from the un-normalized zero$^{\text{th}}$ order lifted invariants $y^i$, $v^\alpha$, and
- all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where $I^\alpha_J$ is a phantom differential invariant.

In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $\mathcal{D}_K J^i$, $\mathcal{D}_K I^\alpha_{J,i}$.

$\Longrightarrow$ Not necessarily a minimal set!

## Syzygies

A syzygy is a functional relation among differentiated invariants:

$$H(\ldots \mathcal{D}_J I_\nu \ldots) \equiv 0$$

Derivatives of syzygies are syzygies
$\Longrightarrow$ find a minimal basis

*Remark*: There are no syzygies among the normalized differential invariants $I^{(n)}$ except for the "phantom syzygies"

$$I_\nu = c_\nu$$

corresponding to the normalizations.

## Syzygies

**Theorem.** All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j J^i = \delta^i_j + M^i_j}$$

— $J^i$ non-phantom

$$\boxed{\mathcal{D}_J I^\alpha_K = c_\nu + M^\alpha_{K,J}}$$

— $I^\alpha_K$ generating

— $I^\alpha_{J,K} = w_\nu = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}}$$

— $I^\alpha_{LK}$, $I^\alpha_{LJ}$ generating, $K \cap J = \varnothing$

$\Longrightarrow$ Not necessarily a minimal system!

## Invariantization

$F(z^{(n)})$

— arbitrary diff. function

$L = F(w^{(n)})$

— lifted differential invariant

$J = F(I^{(n)}(z^{(n)}))$

— differential invariant

*Invariantization*

$$F(z^{(n)}) \longmapsto F(I^{(n)}(z^{(n)}))$$

$\Longrightarrow$ projection

## Recurrence Formulae

$$\boxed{\begin{aligned} \mathcal{D}_j J^i &= \delta^i_j + M^i_j \\ \mathcal{D}_j I^\alpha_K &= I^\alpha_{K,j} + M^\alpha_{K,j} \end{aligned}}$$

$V = V^{(n)}$

— coefficient matrix of infinitesimal generators

$W = V \circ I^{(n)}$

— invariantized version

$P$

— Gauss–Jordan row reduction of $W$ w.r.t. normalization variables

$S = \mathbf{D} z$

— total Jacobian matrix

$R = S \circ I^{(n)}$

— invariantized version

$M = -R \cdot P$

— correction term matrix

## Commutation Formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A_{ij}^k \, \mathcal{D}_k$$

$X^k = \mathbf{D}\xi^k$

— total Jacobian matrix

$Y^k = X^k \circ I^{(1)}$

— invariantization

$B^k$

— Gauss–Jordan reduction of $Y^k$

$A^k = R \cdot B^k - (R \cdot B^k)^T$

— commutation coefficients

## Example

Space: $M = \mathbb{R}^3$ coordinates $x^1, x^2, u$

Group: $G = \mathrm{GL}(2)$

$$(x^1, x^2, u) \longmapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$$

$$\lambda = \alpha\delta - \beta\gamma$$

$\Longrightarrow$ Classical invariant theory

Prolongation (lifted differential invariants):

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2) \qquad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2)$$

$$v = \lambda^{-1} u$$

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda} \qquad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}$$

$$v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda}$$

$$v_{12} = \frac{\alpha\beta\delta u_{11} + (\alpha\delta + \beta\gamma) u_{12} + \gamma\delta u_{22}}{\lambda}$$

$$v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}$$

Normalization

$$y^1 = 1 \quad y^2 = 0 \quad v_1 = 1 \quad v_2 = 0$$

Nondegeneracy

$$x^1 \frac{\partial u}{\partial x^1} + x^2 \frac{\partial u}{\partial x^2} \neq 0$$

First order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix}$$

Normalized differential invariants

$$J^1 = 1 \qquad J^2 = 0$$

$$I = \frac{u}{x^1 u_1 + x^2 u_2}$$

$$I_1 = 1 \qquad I_2 = 0$$

$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

Phantom differential invariants

$$I_1 \qquad I_2$$

Generating differential invariants

$$I \qquad I_{11} \qquad I_{12} \qquad I_{22}$$

Invariant differential operators

$\mathcal{D}_1 = x^1 D_1 + x^2 D_2$ — scaling process

$\mathcal{D}_2 = -u_2 D_1 + u_1 D_2$ — Jacobian process

Prolonged infinitesimal generator coefficients

$$V = \begin{pmatrix} x^1 & 0 & u & 0 & u_2 & u_{11} & 0 & u_{22} \\ x^2 & 0 & 0 & 0 & -u_1 & 0 & -u_{11} & -2u_{12} \\ 0 & x^1 & 0 & -u_2 & 0 & -2u_{12} & -u_{22} & 0 \\ 0 & x^2 & u & u_1 & 0 & u_{11} & 0 & -u_{22} \end{pmatrix}$$

Invariantization

$$W = \begin{pmatrix} 1 & 0 & I & 0 & 0 & -I_{11} & 0 & I_{22} \\ 0 & 0 & 0 & 0 & -1 & 0 & -I_{11} & -2I_{12} \\ 0 & 1 & 0 & 0 & 0 & -2I_{12} & -I_{22} & 0 \\ 0 & 0 & I & 1 & 0 & I_{11} & 0 & -I_{22} \end{pmatrix}$$

Gauss–Jordan reduction

$$P = \begin{pmatrix} 1 & 0 & I & 0 & 0 & -I_{11} & 0 & I_{22} \\ 0 & 1 & 0 & 0 & 0 & -2I_{12} & -I_{22} & 0 \\ 0 & 0 & I & 1 & 0 & I_{11} & 0 & -I_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 & I_{11} & 2I_{12} \end{pmatrix}$$

"Differentiated" phantom invariants $\quad J^1, J^2, I_1, I_2$

$$T = \begin{pmatrix} 1 & 0 & I_{11} & I_{12} \\ 0 & 1 & I_{12} & I_{22} \end{pmatrix}$$

Correction matrix

$$\begin{aligned} M &= -T \cdot P \\ &= -\begin{pmatrix} 1 & 0 & I(1+I_{11}) & I_{11} & I_{12} & (I_{11}-1)I_{11} & \ldots \\ 0 & 1 & II_{12} & I_{12} & I_{22} & (I_{11}-2)I_{12} & \ldots \end{pmatrix} \end{aligned}$$

Recurrence formulae

$$\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0 \qquad \mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0$$

$$\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0 \qquad \mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0$$

$$\mathcal{D}_1 I = I_1 - I(1 + I_{11}) = -I(1 + I_{11}) \qquad \mathcal{D}_2 I = I_2 - I\, I_{12} = -I\, I_{12}$$

$$\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0 \qquad \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0 \qquad \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0$$

$$\mathcal{D}_1 I_{11} = I_{111} + (1 - I_{11}) I_{11} \qquad \mathcal{D}_2 I_{11} = I_{112} + (2 - I_{11}) I_{12}$$

$$\mathcal{D}_1 I_{12} = I_{112} - I_{11} I_{12} \qquad \mathcal{D}_2 I_{12} = I_{122} + (1 - I_{11}) I_{22}$$

$$\mathcal{D}_1 I_{22} = I_{122} + (I_{11} - 1) I_{22} - 2 I_{12}^2 \qquad \mathcal{D}_2 I_{22} = I_{222} - I_{12} I_{22}$$

$\Longrightarrow$ Use $I$ to generate $I_{11}$ and $I_{12}$

Syzygies

$$\begin{aligned}
\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2 I_{12} \\
\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11} - 1) I_{22} - 2 I_{12}^2 \\
(\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} &= \\
&= 2 I_{22} \mathcal{D}_1 I_{11} + (5 I_{12} - 2) \mathcal{D}_1 I_{12} + (3 I_{11} - 5) \mathcal{D}_1 I_{22} - \\
&\quad - (2 I_{11} - 5)(I_{11} - 1) I_{12} + 4 (I_{11} - 1) I_{12}^2
\end{aligned}$$

Commutation formulae

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12} \mathcal{D}_1 + (I_{11} - 1) \mathcal{D}_2$$