Lie Pseudo–Groups

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui se trouve dans les mémoires de Cartan, premiere exploration à travers une jungle presque impénétrable; mais cell-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

What's the Difficulty with Infinite–Dimensional Groups?

- Lie invented Lie groups to study symmetry and solution of differential equations.

♢ In Lie's time, there were no abstract Lie groups. All groups were realized by their action on a space.

♠ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

♡ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!

Ehresmann's Trinity

1953:

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- Lie pseudo-groups

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- Jets

Ehresmann's Trinity

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- Lie pseudo-groups
- Jets
- Groupoids

Lie Pseudo-groups — History

- Lie, Medolaghi, Vessiot
- É. Cartan
- Ehresmann, Libermann
- Kuranishi, Spencer, Singer, Sternberg, Guillemin, Kumpera, ...

Lie Pseudo-groups — Applications

- Relativity
- Noether's (Second) Theorem
- Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, ...
- Fluid mechanics, metereology:
  Navier–Stokes, Euler, boundary layer, quasi-geostropic, ...
- Linear and linearizable PDEs
- Solitons (in $2 + 1$ dimensions): K–P, Davey-Stewartson, ...
- Kac–Moody
- Morphology and shape recognition
- Control theory
- Geometric numerical integration
- *Lie groups*!

Lie Pseudo-groups — Moving Frames

♢ Motivation: To develop an algorithmic invariant calculus for Lie group and pseudo-group actions. Classify and construct differential invariants — including their generators and syzygies — invariant differential forms, invariant differential operators, invariant differential equations, invariant variational problems, etc.

♠ Tools: The equivariant approach to moving frames — which can be implemented for arbitrary Lie group and most Lie pseudo-group actions — along with the induced invariant variational bicomplex.

♡ Additional benefits: A new, elementary approach to the structure theory for Lie pseudo-groups, including explicit construction of Maurer–Cartan forms and direct, elementary determination of structure equations from the infinitesimal generators.

$\Longrightarrow$ PJO, Fels, Pohjanpelto, Cheh, Itskov, Valiquette

Lie Pseudo-groups — Further applications

- Symmetry groups of differential equations
- Vessiot group splitting; explicit solutions
- Gauge theories
- Calculus of variations
- Invariant geometric flows
- Computer vision and mathematical morphology
- Geometric numerical integration

Pseudo-groups

M — analytic (smooth) manifold

Definition. A pseudo-group is a collection of local analytic diffeomorphisms $\phi\colon \operatorname{dom}\phi \subset M \to M$ such that

- *Identity:* $\mathbf{1}_M \in \mathcal{G}$
- *Inverses:* $\phi^{-1} \in \mathcal{G}$
- *Restriction:* $U \subset \operatorname{dom}\phi \implies \phi \mid U \in \mathcal{G}$
- *Continuation:* $\operatorname{dom}\phi = \bigcup U_\kappa$ and $\phi \mid U_\kappa \in \mathcal{G} \implies \phi \in \mathcal{G}$
- *Composition:* $\operatorname{im}\phi \subset \operatorname{dom}\psi \implies \psi \circ \phi \in \mathcal{G}$

The Diffeomorphism Pseudo-group

M — m-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$ — pseudo-group of all local analytic diffeomorphisms

$$Z = \phi(z)$$

$$\begin{cases} z = (z^1, \ldots, z^m) & \text{— source coordinates} \\ Z = (Z^1, \ldots, Z^m) & \text{— target coordinates} \end{cases}$$

$$\begin{cases} L_\psi(\phi) = \psi \circ \phi & \text{— left action} \\ R_\psi(\phi) = \phi \circ \psi^{-1} & \text{— right action} \end{cases}$$

Jets

For $0 \leq n \leq \infty$:

Given a smooth map $\phi\colon M \to M$, written in local coordinates as $Z = \phi(z)$, let $\mathrm{j}_n\phi|_z$ denote its n–jet at $z \in M$, i.e., its n^{th} order Taylor polynomial or series based at z.

$\mathrm{J}^n(M, M)$ is the n^{th} order jet bundle, whose points are the jets.

Local coordinates on $\mathrm{J}^n(M, M)$:

$$(z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b\ \ldots\ Z^b_A\ \ldots\ ), \qquad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

Diffeomorphism Jets

The n^{th} order diffeomorphism jet bundle is the subbundle

$$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$$

consisting of n^{th} order jets of local diffeomorphisms $\phi \colon M \to M$.

The Inverse Function Theorem tells us that $\mathcal{D}^{(n)}$ is defined by the non-vanishing of the Jacobian determinant:

$$\det(\, Z^a_b \,) = \det(\, \partial Z^a / \partial z^b \,) \neq 0$$

⋆ $\mathcal{D}^{(n)}$ forms a groupoid under composition of Taylor polynomials/series.

Groupoid Structure

Double fibration:

$$\begin{array}{ccc} & \mathcal{D}^{(n)} & \\ \boldsymbol{\sigma}^{(n)} \swarrow & & \searrow \boldsymbol{\tau}^{(n)} \\ M & & M \end{array}$$

$$\boldsymbol{\sigma}^{(n)}(z, Z^{(n)}) = z \qquad \text{— source map}$$

$$\boldsymbol{\tau}^{(n)}(z, Z^{(n)}) = Z \qquad \text{— target map}$$

You are only allowed to multiply $h^{(n)} \cdot g^{(n)}$ if

$$\boldsymbol{\sigma}^{(n)}(h^{(n)}) = \boldsymbol{\tau}^{(n)}(g^{(n)})$$

♢ Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.

One-dimensional case: $M = \mathbb{R}$

Source coordinate: x Target coordinate: X

Local coordinates on $\mathcal{D}^{(n)}(\mathbb{R})$

$$g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n)$$

Diffeomorphism jet:

$$X[\![\, h \,]\!] = X + X_x\, h + \tfrac{1}{2}\, X_{xx}\, h^2 + \tfrac{1}{6}\, X_{xxx}\, h^3 + \ \cdots$$

$\Longrightarrow$ Taylor polynomial/series at a source point x

Groupoid multiplication of diffeomorphism jets:

$$(X, \mathbf{X}, \mathbf{X}_X, \mathbf{X}_{XX}, \ \ldots\ ) \cdot (x, X, X_x, X_{xx}, \ \ldots\ )$$
$$= (x, \mathbf{X}, \mathbf{X}_X\, X_x, \mathbf{X}_X\, X_{xx} + \mathbf{X}_{XX}\, X_x^2, \ \ldots\ )$$

$\Longrightarrow$ Composition of Taylor polynomials/series

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate X of the first multiplicand matches the target coordinate X of the second.
- The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

Pseudo-group Jets

Any pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines

a Lie sub-groupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$.

Definition. $\mathcal{G}$ is regular if, for $n \gg 0$, its jets $\boldsymbol{\sigma} : \mathcal{G}^{(n)} \to M$ form an embedded subbundle of $\boldsymbol{\sigma} : \mathcal{D}^{(n)} \to M$ and the projection $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a fibration.

Definition. A regular, analytic pseudo-group $\mathcal{G}$ is called a Lie pseudo-group of order $n^\star \geq 1$ if *every* local diffeomorphism $\phi \in \mathcal{D}$ satisfying $\mathrm{j}_{n^\star}\phi \subset \mathcal{G}^{(n^\star)}$ belongs it: $\phi \in \mathcal{G}$.

In local coordinates, $\mathcal{G}^{(n^\star)} \subset \mathcal{D}^{(n^\star)}$ forms a system of differential equations

$$F^{(n^\star)}(z, Z^{(n^\star)}) = 0$$

called the determining system of the pseudo-group. The Lie condition requires that *every* local solution to the determining system belongs to the pseudo-group.

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What about integrability/involutivity?

Lemma. In the analytic category, for sufficiently large $n \gg 0$ the determining system $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ of a regular pseudo-group is an involutive system of partial differential equations.

Proof: regularity + Cartan–Kuranishi + local solvability.

Lie Completion of a Pseudo-group

Definition. The Lie completion $\overline{\mathcal{G}} \supset \mathcal{G}$ of a regular pseudo-group is defined as the space of all analytic diffeomorphisms ϕ that solve the determining system $\mathcal{G}^{(n^\star)}$.

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Theorem. $\mathcal{G}$ and $\overline{\mathcal{G}}$ have the same differential invariants, the same invariant differential forms, etc.

★ Thus, for local geometry, there is no loss in generality assuming all (regular) pseudo-groups are Lie pseudo-groups!

A Non-Lie Pseudo-group

$$X = \phi(x) \qquad Y = \phi(y) \qquad \text{where} \qquad \phi \in \mathcal{D}(\mathbb{R})$$

On the off-diagonal set $M = \{ (x, y) \mid x \neq y \}$, the pseudo-group $\mathcal{G}$ is regular of order 1, and $\mathcal{G}^{(1)} \subset \mathcal{D}^{(1)}$ is defined by the first order determining system

$$X_y = Y_x = 0 \qquad X_x, Y_y \neq 0$$

The general solution to the determining system $\mathcal{G}^{(1)}$ forms the Lie completion $\overline{\mathcal{G}}$:

$$X = \phi(x) \qquad Y = \psi(y) \qquad \text{where} \qquad \phi, \psi \in \mathcal{D}(\mathbb{R})$$

Structure of Lie Pseudo-groups

Recall:

The structure of a finite-dimensional Lie group G is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C_{ij}^k \, \mu^i \wedge \mu^j$$

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I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a Lie group and hence Lie pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $\mathcal{G}^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.

The Variational Bicomplex

The differential one-forms on an infinite jet bundle split into two types:

- horizontal forms
- contact forms

Consequently, the exterior derivative on $\mathcal{D}^{(\infty)}$ splits

$$d = d_M + d_G$$

into horizontal (manifold) and contact (group) components, leading to the variational bicomplex structure on the algebra of differential forms on $\mathcal{D}^{(\infty)}$.

For the diffeomorphism jet bundle

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$$

Local coordinates:

$$\underbrace{z^1, \ \ldots \ z^m,}_{\text{source}} \quad \underbrace{Z^1, \ \ldots \ Z^m,}_{\text{target}} \quad \underbrace{\ldots \ Z^b_A, \ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Basis contact forms:

$$\Theta^b_A = d_G \, Z^b_A = dZ^b_A - \sum_{a=1}^m Z^a_{A,a} \, dz^a$$

One-dimensional case: $M = \mathbb{R}$

Local coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R})$

$$(x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n, \ \ldots \ )$$

Horizontal form:

$$dx$$

Contact forms:

$$\Theta = dX - X_x \, dx$$

$$\Theta_x = dX_x - X_{xx} \, dx$$

$$\Theta_{xx} = dX_{xx} - X_{xxx} \, dx$$

$$\vdots$$

Maurer–Cartan Forms

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Key observation: Since the right action only affects source coordinates, the target coordinate functions Z^a are right-invariant.

Thus, when we decompose

$$dZ^a = \underbrace{\sigma^a}_{\text{horizontal}} + \underbrace{\mu^a}_{\text{contact}}$$

both components σ^a, μ^a are right-invariant one forms.

Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b$$

Dual invariant total differentiation operators:

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} \left(Z^a_b \right)^{-1} \mathbb{D}_{z^b}$$

Thus, the invariant contact forms μ^b_A are obtained by invariant differentiation of the order zero contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \mu^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \mu^b \qquad b = 1, \ldots, m, \ \#A \geq 0$$

One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = dX - X_x\, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx}\, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx}\, dx$$

Right-invariant horizontal form:

$$\sigma = d_M X = X_x\, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x}\, \mathbb{D}_x$$

Invariant contact forms:

$$\mu = \Theta = dX - X_x\, dx$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x} = \frac{dX_x - X_{xx}\, dx}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x\, \Theta_{xx} - X_{xx}\, \Theta_x}{X_x^3}$$

$$= \frac{X_x\, dX_{xx} - X_{xx}\, dX_x + (X_{xx}^2 - X_x X_{xxx})\, dx}{X_x^3}$$

$$\vdots$$

$$\mu_n = \mathbb{D}_X^n \mu$$

The Structure Equations for the Diffeomorphism Pseudo–group

$$d\mu_A^b = \sum C_{A,c,d}^{b,B,C}\ \mu_B^c \wedge \mu_C^d$$

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Formal Maurer–Cartan series:

$$\mu^b[\![\,H\,]\!] = \sum_A \frac{1}{A!}\,\mu_A^b\,H^A$$

$H = (H^1, \ldots, H^m)$ — formal parameters

$$d\mu[\![\,H\,]\!] = \nabla\mu[\![\,H\,]\!] \wedge (\,\mu[\![\,H\,]\!] - dZ\,)$$

$$d\sigma = -\,d\mu[\![\,0\,]\!] = \nabla\mu[\![\,0\,]\!] \wedge \sigma$$

One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![H]\!] = \frac{d\mu}{dH}[\![H]\!] \wedge (\mu[\![H]\!] - dZ)$$

where

$$\sigma = X_x\, dx = dX - \mu$$

$$\mu[\![H]\!] = \mu + \mu_X H + \tfrac{1}{2}\mu_{XX} H^2 + \cdots$$

$$\mu[\![H]\!] - dZ = -\sigma + \mu_X H + \tfrac{1}{2}\mu_{XX} H^2 + \cdots$$

$$\frac{d\mu}{dH}[\![H]\!] = \mu_X + \mu_{XX} H + \tfrac{1}{2}\mu_{XXX} H^2 + \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$d\mu_n = -\,\mu_{n+1} \wedge \sigma \;+\; \sum_{i=0}^{n-1} \binom{n}{i}\, \mu_{i+1} \wedge \mu_{n-i}$$

$$= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1}\binom{n+1}{j}\, \mu_j \wedge \mu_{n+1-j}.$$

$\Longrightarrow$ Cartan

The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for a pseudo-group $\mathcal{G} \subset \mathcal{D}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms σ^a, μ_A^b to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

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$\star\star$ The resulting one-forms are no longer linearly independent, but the dependencies can be determined directly from the infinitesimal generators of $\mathcal{G}$.

Infinitesimal Generators

$\mathfrak{g}$ — Lie algebra of infinitesimal generators of the pseudo-group $\mathcal{G}$

$z = (x, u)$ — local coordinates on M

Vector field:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha}$$

Vector field jet:

$$\mathrm{j}_n \mathbf{v} \longmapsto \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ )$$

$$\zeta^b_A = \frac{\partial^{\# A} \zeta^b}{\partial z^A} = \frac{\partial^k \zeta^b}{\partial z^{a_1} \cdots \partial z^{a_k}}$$

The infinitesimal generators of $\mathcal{G}$ are the solutions to the

infinitesimal determining equations

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- If $\mathcal{G}$ is the symmetry group of a system of differential equations, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

Theorem. The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariantized infinitesimal determining equations

$$\mathcal{L}(\ \ldots\ Z^a\ \ldots\ \mu^b_A\ \ldots\ ) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations

$$\mathcal{L}(\ \ldots\ z^a\ \ldots\ \zeta^b_A\ \ldots\ ) = 0 \qquad (\star)$$

by replacing

- source variables z^a by target variables Z^a
- derivatives of vector field coefficients ζ^b_A by right-invariant Maurer–Cartan forms μ^b_A

The Structure Equations for a Lie Pseudo-group

Theorem. The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla\mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the solution space of the linear algebraic system

$$\mathcal{L}(\ \ldots \ Z^a, \ \ldots \ \mu^b_A, \ \ldots \ ) = 0.$$

Comparison of Structure Equations

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- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).

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- For finite-dimensional intransitive Lie group actions, Cartan's pseudo-group structure equations do not coincide with the standard Maurer–Cartan equations. Ours do (upon restriction to a source fiber).
- Cartan's structure equations for isomorphic pseudo-groups can be non-isomorphic. Ours are always isomorphic.

Lie–Kumpera Example

$$X = f(x) \qquad\qquad U = \frac{u}{f'(x)}$$

Infinitesimal generators:

$$\mathbf{v} = \xi \, \frac{\partial}{\partial x} + \varphi \, \frac{\partial}{\partial u} = \xi(x) \, \frac{\partial}{\partial x} - \xi'(x) \, u \, \frac{\partial}{\partial u}$$

Linearized determining system

$$\xi_x = -\,\frac{\varphi}{u} \qquad\qquad \xi_u = 0 \qquad\qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\, dx = f_x\, dx, \qquad \tau = U_x\, dx + \frac{U}{u}\, du = \frac{-\,u\, f_{xx}\, dx + f_x\, du}{{f_x}^2}$$

$$\mu = dX - \frac{U}{u}\, dx = df - f_x\, dx, \qquad \nu = dU - U_x\, dx - \frac{U}{u}\, du = -\,\frac{u}{{f_x}^2}\,(\, d f_x - f_{xx}\, dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\, dx}{U} = \frac{d f_x - f_{xx}\, dx}{f_x}, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}\,(dU_x - U_{xx}\, dx) - \frac{U_x}{u}\,(dU - U_x\, dx)$$

$$= -\,\frac{u}{{f_x}^3}(\, d f_{xx} - f_{xxx}\, dx\,) + \frac{u\, f_{xx}}{{f_x}^4}(\, d f_x - f_{xx}\, dx\,)$$

$$\nu_U = -\,\frac{du}{u} + \frac{dU - U_x\, dx}{U} = -\,\frac{d f_x - f_{xx}\, dx}{f_x}$$

First order linearized determining system:

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

First order Maurer–Cartan determining system:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

Substituting into the full diffeomorphism structure equations yields the (first order) structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}, \qquad d\nu = -\,\nu_X \wedge \sigma \;-\; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \;-\; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

By a symmetry, we mean a transformation that maps solutions to solutions.

Lie: To find the symmetry group of the differential equations, work infinitesimally.

The vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry if its flow $\exp(t\,\mathbf{v})$ is a one-parameter symmetry group of the differential equation.

We prolong $\mathbf{v}$ to the jet space whose coordinates are the derivatives appearing in the differential equation:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\varphi_\alpha^J = D_J \left(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \, \xi^i \right) + \sum_{i=1}^{p} u_{J,i}^\alpha \, \xi^i$$

D_J — total derivatives

Infinitesimal invariance criterion:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

★ ★ We can determine the structure of the symmetry group without solving the determining equations!

The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \ \cdots\ + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\varphi^t = \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u$$

$$\varphi^x = \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u$$

$$\varphi^{xxx} = \varphi_{xxx} + 3\,u_x\varphi_u + \ \cdots$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\,u\tau_t \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \;\cdots\; = \varphi_{uu} = 0$$

General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2c_4 u.$$

Basis for symmetry algebra $\mathfrak{g}_{KdV}$:

$$\begin{aligned}
\mathbf{v}_1 &= \partial_t, \\
\mathbf{v}_2 &= \partial_x, \\
\mathbf{v}_3 &= t\,\partial_x + \partial_u, \\
\mathbf{v}_4 &= 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.
\end{aligned}$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \longmapsto (\lambda^3\, t + a, \lambda\, x + c\, t + b, \lambda^{-2}\, u + c)$$

$$\mathbf{v}_1 = \partial_t, \qquad \mathbf{v}_2 = \partial_x,$$

$$\mathbf{v}_3 = t\,\partial_x + \partial_u, \qquad \mathbf{v}_4 = 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

Commutator table:

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$\mathbf{v}_1$	$3\,\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\,\mathbf{v}_1$	0	$-2\,\mathbf{v}_3$
$\mathbf{v}_4$	$-\,\mathbf{v}_1$	$-3\,\mathbf{v}_2$	$2\,\mathbf{v}_3$	0

Entries: $[\mathbf{v}_i, \mathbf{v}_j] = \sum_k C_{ij}^k \mathbf{v}_k.$ C_{ij}^k — structure constants of $\mathfrak{g}$

Diffeomorphism Maurer–Cartan forms:

$$\mu^t,\ \mu^x,\ \mu^u,\ \mu^t_T,\ \mu^t_X,\ \mu^t_U,\ \mu^x_T,\ \ldots,\ \mu^u_U,\ \mu^t_{TT},\ \mu^T_{TX},\ \ldots$$

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\, u\, \tau_t \qquad \varphi_u = -\tfrac{2}{3}\, \tau_t = -\, 2\, \xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \ \cdots\ = \varphi_{uu} = 0$$

Maurer–Cartan determining equations:

$$\mu^t_X = \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0,$$

$$\mu^u = \mu^x_T - \tfrac{2}{3}\, U \mu^t_T, \qquad \mu^u_U = -\tfrac{2}{3}\, \mu^t_T = -\, 2\, \mu^x_X,$$

$$\mu^t_{TT} = \mu^t_{TX} = \mu^t_{XX} = \ \cdots\ = \mu^u_{UU} = \ \ldots\ = 0.$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Substituting into the full diffeomorphism structure equations yields the structure equations for $\mathfrak{g}_{KdV}$:

$$\begin{aligned}
d\mu^1 &= -\,\mu^1 \wedge \mu^4, \\
d\mu^2 &= -\mu^1 \wedge \mu^3 - \tfrac{2}{3} U\, \mu^1 \wedge \mu^4 - \tfrac{1}{3}\, \mu^2 \wedge \mu^4, \\
d\mu^3 &= \tfrac{2}{3}\, \mu^3 \wedge \mu^4, \\
d\mu^4 &= 0.
\end{aligned}$$

$$\boxed{d\mu^i = C^i_{jk}\, \mu^j \wedge \mu^k}$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Substituting into the full diffeomorphism structure equations yields the structure equations for $\mathfrak{g}_{KdV}$:

$$d\mu^1 = -\,\mu^1 \wedge \mu^4,$$

$$d\mu^2 = -\mu^1 \wedge \mu^3 - \tfrac{2}{3}\,U\,\mu^1 \wedge \mu^4 - \tfrac{1}{3}\,\mu^2 \wedge \mu^4,$$

$$d\mu^3 = \tfrac{2}{3}\,\mu^3 \wedge \mu^4,$$

$$d\mu^4 = 0.$$

$$\boxed{d\mu^i = C^i_{jk}(Z)\,\mu^j \wedge \mu^k}$$

Essential Invariants

- The pseudo-group structure equations live on the bundle $\boldsymbol{\tau} : \mathcal{G}^{(\infty)} \to M$, and the structure coefficients C^i_{jk} constructed above may vary from point to point.

♡ In the case of a finite-dimensional Lie group action, $\mathcal{G}^{(\infty)} \simeq G \times M$, and this means the basis of Maurer–Cartan forms on each fiber of $\mathcal{G}^{(\infty)}$ is varying with the target point $Z \in M$. However, we can always make a Z–dependent change of basis to make the structure coefficients constant.

⋆ However, for infinite-dimensional pseudo-groups, it may not be possible to find such a change of Maurer–Cartan basis, leading to the concept of essential invariants.

Kadomtsev–Petviashvili (KP) Equation

$$(u_t + \tfrac{3}{2} u\, u_x + \tfrac{1}{4} u_{xxx})_x \pm \tfrac{3}{4} u_{yy} = 0$$

Symmetry generators:

$$\mathbf{v}_f = f(t)\,\partial_t + \tfrac{2}{3} y\, f'(t)\,\partial_y + \Big(\tfrac{1}{3} x\, f'(t) \mp \tfrac{2}{9} y^2 f''(t) \Big)\,\partial_x$$

$$+ \Big(-\tfrac{2}{3} u\, f'(t) + \tfrac{2}{9} x\, f''(t) \mp \tfrac{4}{27} y^2 f'''(t) \Big)\,\partial_u,$$

$$\mathbf{w}_g = g(t)\,\partial_y \mp \tfrac{2}{3} y\, g'(t)\,\partial_x \mp \tfrac{4}{9} y\, g''(t)\,\partial_u,$$

$$\mathbf{z}_h = h(t)\,\partial_x + \tfrac{2}{3} h'(t)\,\partial_u.$$

$\Longrightarrow$ Kac–Moody loop algebra $A_4^{(1)}$

Navier–Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \, \Delta \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0.$$

Symmetry generators:

$$\mathbf{v}_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t) \cdot \partial_{\mathbf{x}} + \boldsymbol{\alpha}'(t) \cdot \partial_{\mathbf{u}} - \boldsymbol{\alpha}''(t) \cdot \mathbf{x} \, \partial_p$$

$$\mathbf{v}_0 = \partial_t$$

$$\mathbf{s} = \mathbf{x} \cdot \partial_{\mathbf{x}} + 2\, t \, \partial_t - \mathbf{u} \cdot \partial_{\mathbf{u}} - 2\, p \, \partial_p$$

$$\mathbf{r} = \mathbf{x} \wedge \partial_{\mathbf{x}} + \mathbf{u} \wedge \partial_{\mathbf{u}}$$

$$\mathbf{w}_h = h(t) \, \partial_p$$

Action of Pseudo-groups on Submanifolds a.k.a. Solutions of Differential Equations

$\mathcal{G}$ — Lie pseudo-group acting on p-dimensional submanifolds:

$$N = \{ u = f(x) \} \subset M$$

For example, $\mathcal{G}$ may be the symmetry group of a system of differential equations

$$\Delta(x, u^{(n)}) = 0$$

and the submanifolds are the graphs of solutions $u = f(x)$.

Goal: Understand $\mathcal{G}$–invariant objects (moduli spaces)

Prolongation

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} order submanifold jet bundle

Local coordinates :

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

Differential Invariants

A differential invariant is an invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(\mathcal{G})$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(\mathcal{G})$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Computing Differential Invariants

♠ The infinitesimal method:

$$\mathbf{v}(I) = 0 \qquad \text{for every infinitesimal generator} \qquad \mathbf{v} \in \mathfrak{g}$$

$\Longrightarrow$ Requires solving differential equations.

♡ Moving frames.

- Completely algebraic.
- Can be adapted to arbitrary group and pseudo-group actions.
- Describes the complete structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ — using only linear algebra & differentiation!
- Prescribes differential invariant signatures for equivalence and symmetry detection.

Moving Frames

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

Moving Frames for Pseudo–Groups

Definition. A (right) *moving frame* of *order* n is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

$\Longrightarrow$ Groupoid action.

Moving Frames for Pseudo–Groups

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$\Longrightarrow$ Groupoid action.

Proposition. A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

Freeness

For Lie group actions, freeness means trivial isotropy:

$$G_z = \{ g \in G \mid g \cdot z = z \} = \{e\}.$$

For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order n, using the n^{th} order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

Definition. At a jet $z^{(n)} \in \mathrm{J}^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \mathbf{1}^{(n)}_z \}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have dimension $r_n = \dim \mathcal{G}^{(n)}_z$

$\Longrightarrow$ Kumpera's growth bounds on Spencer cohomology.

Persistence of Freeness

Theorem. If $n \geq 1$ and $\mathcal{G}^{(n)}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$, then it acts (locally) freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$.

The Normalization Algorithm

To construct a moving frame :

I. Compute the prolonged pseudo-group action

$$u_K^\alpha \quad \longmapsto \quad U_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$$

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

$$u_{J_\kappa}^{\alpha_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

III. Solve the normalization equations

$$U_{J_\kappa}^{\alpha_\kappa} = F_{J_\kappa}^{\alpha_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the n^{th} order pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

IV. Substitute the moving frame formulas into the unnormalized jet coordinates $u_K^\alpha = F_K^\alpha(x, u^{(n)}, g^{(n)})$. The resulting functions form a complete system of n^{th} order differential invariants

$$I_K^\alpha(x, u^{(n)}) = F_K^\alpha(x, u^{(n)}, \rho^{(n)}(x, u^{(n)}))$$

Invariantization

A moving frame induces an invariantization process, denoted ι, that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.

Invariantization

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants: $\iota(x^i) = H^i \qquad \iota(u_J^\alpha) = I_J^\alpha$

- Phantom differential invariants: $I_{J_\kappa}^{\alpha_\kappa} = c_\kappa$
- The non-constant invariants form a complete system of functionally independent differential invariants
- Replacement Theorem

$$I(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ ) = \iota(\,I(\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ )\,)$$

$$= I(\ \ldots\ H^i\ \ldots\ I_J^\alpha\ \ldots\ )$$

The Invariant Variational Bicomplex

♢ Differential functions $\implies$ differential invariants

$$\iota(x^i) = H^i \qquad\qquad \iota(u^\alpha_J) = I^\alpha_J$$

♢ Differential forms $\implies$ invariant differential forms

$$\iota(\,dx^i) = \varpi^i \qquad\qquad \iota(\theta^\alpha_K) = \vartheta^\alpha_K$$

♢ Differential operators $\implies$ invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i$$

Recurrence Formulae

⋆⋆ Invariantization and differentiation
do not commute ⋆⋆

The *recurrence formulae* connect the differentiated invariants with their invariantized counterparts:

$$\boxed{\boxed{\mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha}}$$

$\Longrightarrow$ $M_{J,i}^\alpha$ — correction terms

♡ Once established, the recurrence formulae completely prescribe the structure of the differential invariant algebra $\mathcal{I}(\mathcal{G})$ — thanks to the functional independence of the non-phantom normalized differential invariants.

⋆⋆ The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!

Lie–Tresse–Kumpera Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$:

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^{\,3}} - \frac{3u_x\, f_{xx}}{f_x^{\,4}} - \frac{u\, f_{xxx}}{f_x^{\,4}} + \frac{3u\, f_{xx}^{\,2}}{f_x^{\,5}}$$

$$U_{XY} = \frac{u_{xy}}{f_x^{\,2}} - \frac{u_y\, f_{xx}}{f_x^{\,3}} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$f, f_x, f_{xx}, f_{xxx}, \ldots$ — pseudo-group parameters

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^{\,2}} - \frac{u\, f_{xx}}{f_x^{\,3}} = 0, \quad U_{XX} = \cdots = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y = \frac{u_y}{f_x} \quad \longmapsto \quad J = \iota(u_y) = \frac{u_y}{u}$$

$$U_{XY} = \cdots \quad \longmapsto \quad J_1 = \iota(u_{xy}) = \frac{u u_{xy} - u_x u_y}{u^3}$$

$$U_{YY} = \cdots \quad \longmapsto \quad J_2 = \iota(u_{xy}) = \frac{u_{yy}}{u}$$

$$U_{XXY} \longmapsto J_3 = \iota(u_{xxy}) \qquad U_{XYY} \longmapsto J_4 = \iota(u_{xyy}) \qquad U_{YYY} \longmapsto J_5 = \iota(u_{yyy})$$

Invariant horizontal forms

$$d_H X = f_x \, dx \longmapsto u \, dx, \qquad d_H Y = dy \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u} \, \mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m \, \mathcal{D}_2^n \, J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

Recurrence formulae:

$$\mathcal{D}_1 J = J_1, \qquad \mathcal{D}_2 J = J_2 - J^2,$$

$$\mathcal{D}_1 J_1 = J_3, \qquad \mathcal{D}_2 J_1 = J_4 - 3 \, J \, J_1,$$

$$\mathcal{D}_1 J_2 = J_4, \qquad \mathcal{D}_2 J_2 = J_5 - J \, J_2,$$

Korteweg–deVries Equation

Prolonged Symmetry Group Action:

$$T = e^{3\lambda_4}(t + \lambda_1)$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1\lambda_3 + \lambda_2)$$

$$U = e^{-2\lambda_4}(u + \lambda_3)$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x)$$

$$U_X = e^{-3\lambda_4}u_x$$

$$U_{TT} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + {\lambda_3}^2 u_{xx})$$

$$U_{TX} = D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx})$$

$$U_{XX} = e^{-4\lambda_4}u_{xx}$$

$$\vdots$$

Cross Section:

$$T = e^{3\lambda_4}(t + \lambda_1) = 0$$

$$X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) = 0$$

$$U = e^{-2\lambda_4}(u + \lambda_3) = 0$$

$$U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1$$

Moving Frame:

$$\lambda_1 = -t, \qquad \lambda_2 = -x, \qquad \lambda_3 = -u, \qquad \lambda_4 = \tfrac{1}{5}\log(u_t + u u_x)$$

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}}$$

$$\vdots$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + uu_x)^{-1/5} D_x.$$

Commutation formula:

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = I_{01}\,\mathcal{D}_1$$

Recurrence formulae:

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5}\,I_{01}^2 - \tfrac{3}{5}\,I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}\,I_{01}^3 - \tfrac{3}{5}\,I_{01}I_{11},$$

$$\mathcal{D}_1 I_{20} = I_{30} + 2\,I_{11} - \tfrac{8}{5}\,I_{01}I_{20} - \tfrac{8}{5}\,I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2\,I_{01}I_{11} - \tfrac{8}{5}\,I_{01}^2 I_{20} - \tfrac{8}{5}\,I_{11}I_{20},$$

$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5}\,I_{01}I_{11} - \tfrac{6}{5}\,I_{11}I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01}I_{02} - \tfrac{6}{5}\,I_{01}^2 I_{11} - \tfrac{6}{5}\,I_{11}^2,$$

$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5}\,I_{01}I_{02} - \tfrac{4}{5}\,I_{02}I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5}\,I_{01}^2 I_{02} - \tfrac{4}{5}\,I_{02}I_{11},$$

$$\vdots \qquad\qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}, \qquad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}.$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01}\right) \mathcal{D}_1 I_{01}$$
$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

The Master Recurrence Formula

$$d_H I^\alpha_J = \sum_{i=1}^{p} (\mathcal{D}_i I^\alpha_J)\,\omega^i = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J$$

where

$$\widehat{\psi}\,^\alpha_J = \iota(\,\widehat{\varphi}\,^\alpha_J\,) = \Phi^\alpha_J(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ ;\ \ldots\ \gamma^b_A\ \ldots\ )$$

are the invariantized prolonged vector field coefficients, which are particular linear combinations of

$\gamma^b_A \ = \ \iota(\zeta^b_A)$ — invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.

• The invariantized Maurer–Cartan forms are subject to the *invariantized determining equations*:

$$\mathcal{L}(H^1,\ldots,H^p,I^1,\ldots,I^q,\ \ldots\ ,\gamma^b_A,\ \ldots\ ) = 0$$

$$d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)$$

Step 1: Solve the phantom recurrence formulas

$$0 = d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i}\,\omega^i + \widehat{\psi}\,^\alpha_J(\;\ldots\;\gamma^b_A\;\ldots\;)$$

for the invariantized Maurer–Cartan forms:

$$\gamma^b_A = \sum_{i=1}^{p} J^b_{A,i}\,\omega^i \qquad (*)$$

Step 2: Substitute $(*)$ into the non-phantom recurrence formulae to obtain the explicit correction terms.

♢ Only uses linear differential algebra based on the specification of cross-section.

♡ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

Lie–Tresse–Kumpera Example (continued)

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Phantom recurrence formulae:

$$0 = dH = \varpi^1 + \gamma, \qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \gamma_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \gamma_1, \qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \gamma_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\gamma = -\,\varpi^1, \qquad \gamma_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\gamma_1 = J\,\varpi^2 + \vartheta, \qquad \gamma_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\qquad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$

The Korteweg–deVries Equation (continued)

Recurrence formula:

$$dI_{jk} = I_{j+1,k}\omega^1 + I_{j,k+1}\omega^2 + \iota(\varphi^{jk})$$

Invariantized Maurer–Cartan forms:

$$\iota(\tau) = \lambda, \qquad \iota(\xi) = \mu, \qquad \iota(\varphi) = \psi = \nu, \qquad \iota(\tau_t) = \psi^t = \lambda_t, \qquad \ldots$$

Invariantized determining equations:

$$\lambda_x = \lambda_u = \mu_u = \nu_t = \nu_x = 0$$

$$\nu = \mu_t \qquad \nu_u = -\,2\,\mu_x = -\,\tfrac{2}{3}\,\lambda_t$$

$$\lambda_{tt} = \lambda_{tx} = \lambda_{xx} = \;\cdots\; = \nu_{uu} = \;\cdots\; = 0$$

Invariantizations of prolonged vector field coefficients:

$$\iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \nu, \quad \iota(\varphi^t) = -I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\iota(\varphi^x) = -I_{01}\lambda_t, \quad \iota(\varphi^{tt}) = -2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t, \quad \ldots$$

Phantom recurrence formulae:

$$0 = d_H\, H^1 = \omega^1 + \lambda,$$

$$0 = d_H\, H^2 = \omega^2 + \mu,$$

$$0 = d_H\, I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu,$$

$$0 = d_H\, I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \tfrac{5}{3}\,\lambda_t,$$

$$\Longrightarrow \quad \text{Solve for} \qquad \lambda = -\omega^1, \qquad \mu = -\omega^2, \qquad \nu = -\omega^1 - I_{01}\omega^2,$$

$$\lambda_t = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2.$$

Non-phantom recurrence formulae:

$$d_H\, I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t,$$

$$d_H\, I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \tfrac{8}{3}I_{20}\lambda_t,$$

$$d_H\, I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t,$$

$$d_H\, I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\lambda_t,$$

$$\vdots$$

$$
\begin{aligned}
\mathcal{D}_1 I_{01} &= I_{11} - \tfrac{3}{5} I_{01}^2 - \tfrac{3}{5} I_{01} I_{20}, & \mathcal{D}_2 I_{01} &= I_{02} - \tfrac{3}{5}\, I_{01}^3 - \tfrac{3}{5}\, I_{01} I_{11},\\
\mathcal{D}_1 I_{20} &= I_{30} + 2 I_{11} - \tfrac{8}{5} I_{01} I_{20} - \tfrac{8}{5} I_{20}^2, & \mathcal{D}_2 I_{20} &= I_{21} + 2 I_{01} I_{11} - \tfrac{8}{5} I_{01}^2 I_{20} - \tfrac{8}{5} I_{11} I_{20},\\
\mathcal{D}_1 I_{11} &= I_{21} + I_{02} - \tfrac{6}{5} I_{01} I_{11} - \tfrac{6}{5} I_{11} I_{20}, & \mathcal{D}_2 I_{11} &= I_{12} + I_{01} I_{02} - \tfrac{6}{5} I_{01}^2 I_{11} - \tfrac{6}{5} I_{11}^2,\\
\mathcal{D}_1 I_{02} &= I_{12} - \tfrac{4}{5} I_{01} I_{02} - \tfrac{4}{5} I_{02} I_{20}, & \mathcal{D}_2 I_{02} &= I_{03} - \tfrac{4}{5} I_{01}^2 I_{02} - \tfrac{4}{5} I_{02} I_{11},\\
&\vdots & &\vdots
\end{aligned}
$$

Gröbner Basis Approach

Suppose $\mathcal{G}$ acts freely at order $n^\star$.

The differential invariants of order $> n^\star$ are naturally identified with polynomials belonging to a certain algebraic module $\mathcal{J}$, called the invariantized prolonged symbol module which is defined as the invariantized pull-back of the symbol module for the infinitesimal determining equations under a certain explicit linear map.

Constructive Basis Theorem

Theorem. A system of generating differential invariants in one-to-one correspondence with the Gröbner basis elements of the invariantized prolonged symbol module $\mathcal{J}^{>n^\star}$ plus, possibly, a finite number of differential invariants of order $\leq n^\star$.

Syzygy Theorem

Theorem. Every differential syzygy among the generating differential invariants is either a syzygy among those of order $\leq n^\star$, or arises from an algebraic syzygy among the Gröbner basis polynomials in $\mathcal{J}^{>n^\star}$, or comes from a commutator syzygy among the invariant differential operators.

The Symbol Module

Linearized determining equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0$$

$$t = (t_1, \ldots, t_m), \qquad T = (T_1, \ldots, T_m)$$

$$\mathcal{T} = \left\{ P(t,T) = \sum_{a=1}^{m} P_a(t)\, T_a \right\} \simeq \mathbb{R}[t] \otimes \mathbb{R}^m \subset \mathbb{R}[t,T]$$

Symbol module:

$$\boxed{\mathcal{I} \subset \mathcal{T}}$$

The Prolonged Symbol Module

$$s = (s_1, \ldots, s_p), \qquad S = (S_1, \ldots, S_q),$$

$$\widehat{\mathcal{S}} = \left\{ T(s,S) = \sum_{\alpha=1}^{q} T_\alpha(s)\, S_\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s,S]$$

Define the linear map

$$s_i = \beta_i(t) = t_i + \sum_{\alpha=1}^{q} u_i^\alpha\, t_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S_\alpha = B_\alpha(T) = T_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha\, T_i, \qquad \alpha = 1, \ldots, q.$$

$\Longrightarrow$ "symbol"" of the vector field prolongation operation

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{I})}$$