Object Recognition, Symmetry Detection, Jigsaw Puzzles, and Melanomas

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Symmetry $\Longrightarrow$ Group Theory!

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

— P.S. Alexandroff

Groups

Definition. A group G is a set with a binary operation

$$(g,h) \longmapsto g\cdot h \in G \quad \text{satisfying}$$

- Associativity: $\quad g\cdot(h\cdot k) = (g\cdot h)\cdot k$
- Identity: $\quad g\cdot e = g = e\cdot g$
- Inverse: $\quad g\cdot g^{-1} = e = g^{-1}\cdot g$

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Examples:

- $G = \mathbb{R}$ — addition
- $G = \mathbb{R}^+$ — multiplication
- $G = \mathrm{GL}(n) = \{ \det A \neq 0 \}$ — matrix multiplication
- $G = \mathrm{SO}(n) = \{ A^T = A^{-1}, \ \det A = +1 \}$ — rotation group

Symmetry

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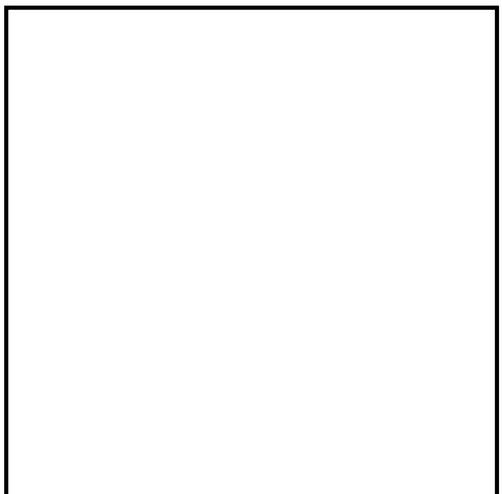
⋆⋆ The set of symmetries forms a group G_S, called the symmetry group of the set S.

Discrete Symmetry Group



Rotations by 90°:

$$G_S = \mathbb{Z}_4$$

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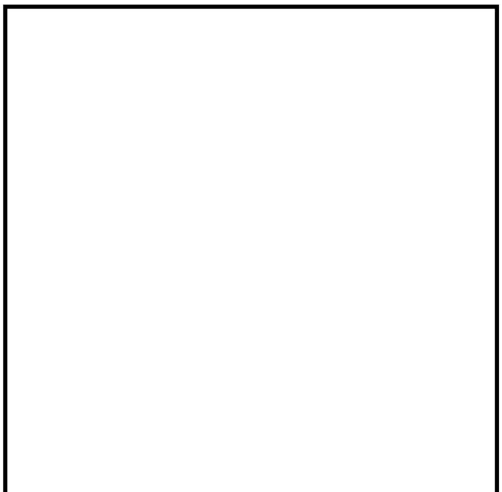


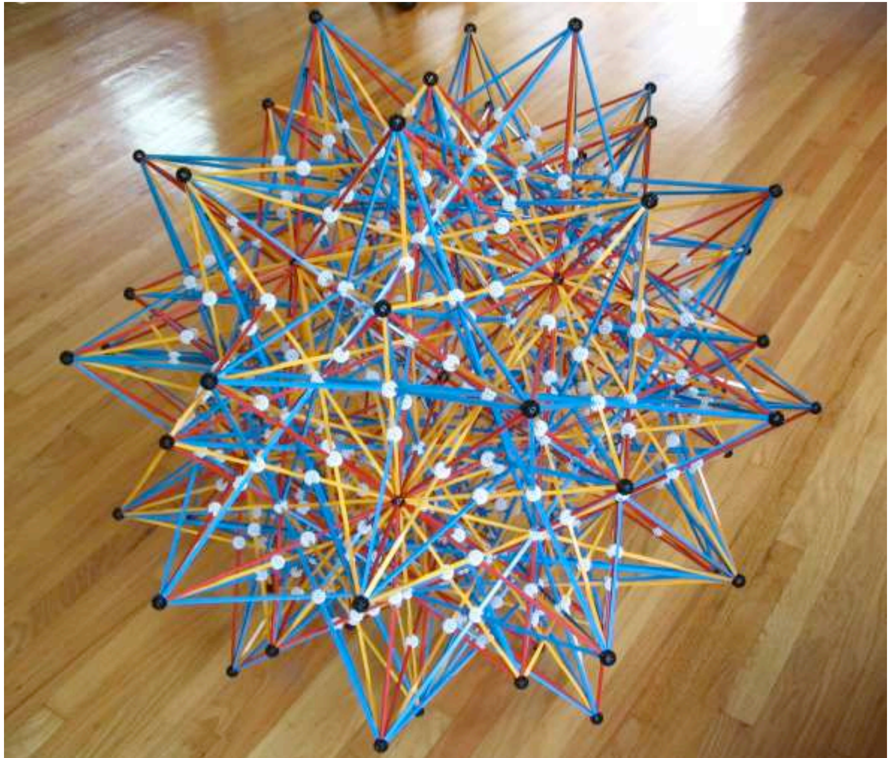
Rotations by 90°:

$$G_S = \mathbb{Z}_4$$

Rotations + reflections:

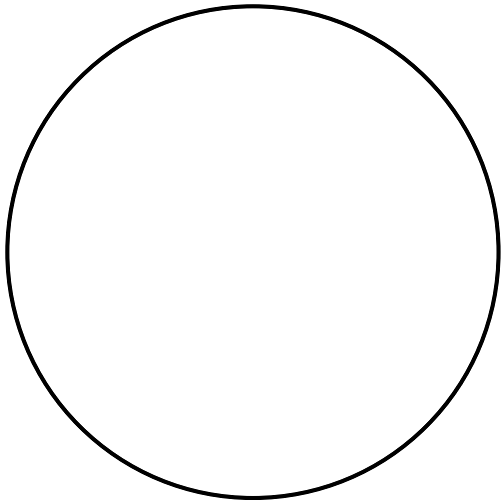
$$G_S = \mathbb{Z}_4 \times \mathbb{Z}_4$$

More Complicated Discrete Symmetry



⟹ David Richter

Continuous Symmetry Group



Rotations:

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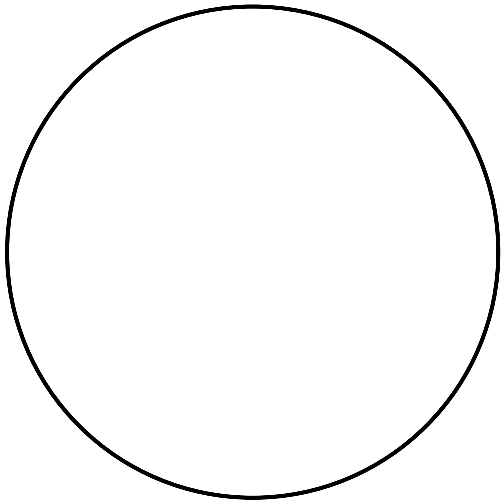

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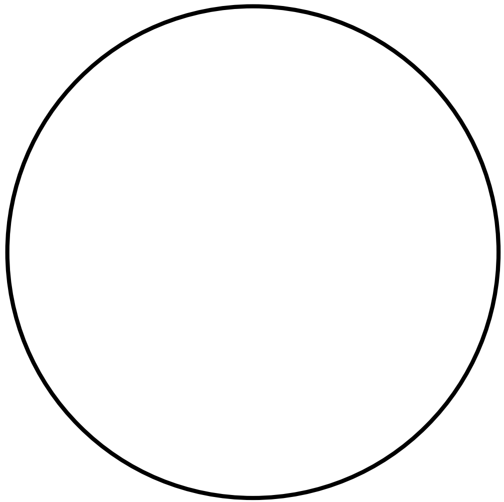

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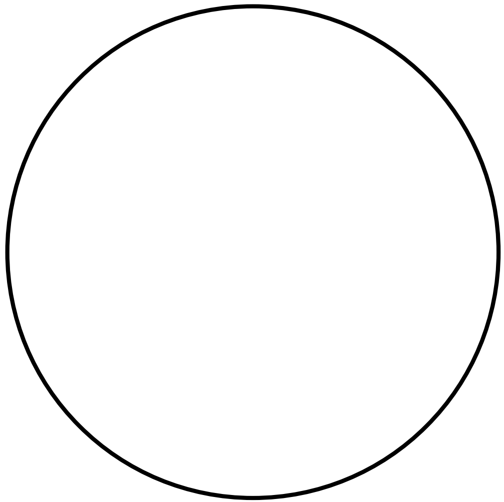

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Inversions:

$$\bar{x} = \frac{x}{x^2 + y^2} \qquad \bar{y} = \frac{y}{x^2 + y^2}$$

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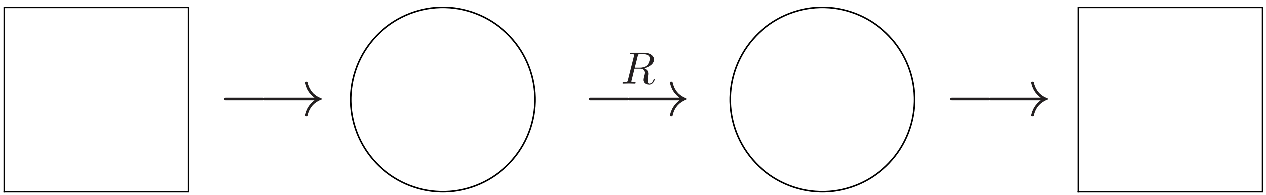
Continuous Symmetries of a Square



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★★ The symmetry group G_S depends on the underlying group of allowable transformations or, equivalently, the geometry of the space!

Geometry = Group Theory

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on
a transformation group.

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A group G acts on a space M via $z \longmapsto g \cdot z$, with

- $g \cdot (h \cdot z) = (g \cdot h) \cdot z$
- $e \cdot z = z$

for all $g, h \in G$ and $z \in M$.

Plane Geometries/Groups

Euclidean geometry:

SE(2) — rigid motions (rotations and translations)

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

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Equi-affine geometry:

SA(2) — area-preserving affine transformations:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \qquad \alpha\,\delta - \beta\,\gamma = 1$$

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Projective geometry:

PSL(3) — projective transformations:

$$\bar{x} = \frac{\alpha\,x + \beta\,y + \gamma}{\rho\,x + \sigma\,y + \tau} \qquad \bar{y} = \frac{\lambda\,x + \mu\,y + \nu}{\rho\,x + \sigma\,y + \tau}$$

Tennis, Anyone?

The Basic Equivalence Problem

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Equivalence:

Determine when two subsets

$$N \quad \text{and} \quad \overline{N} \subset M$$

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Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Invariants

The solution to an equivalence problem rests on understanding its invariants.

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Definition. If G is a group acting on M, then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of G:

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⋆ If G acts transitively, there are no (non-constant) invariants.

Joint Invariants

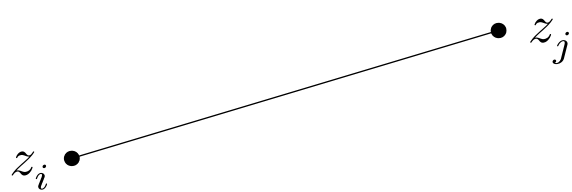
A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) = I(z_1, \ldots, z_k)}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

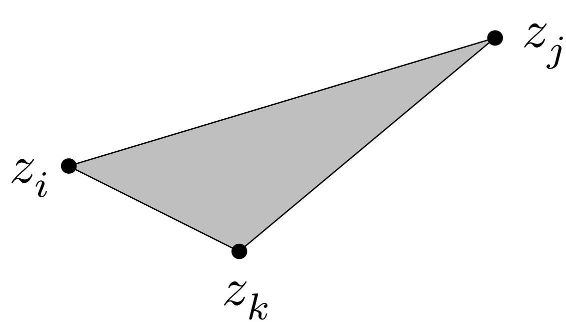
$$d(z_i, z_j) = \| z_i - z_j \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

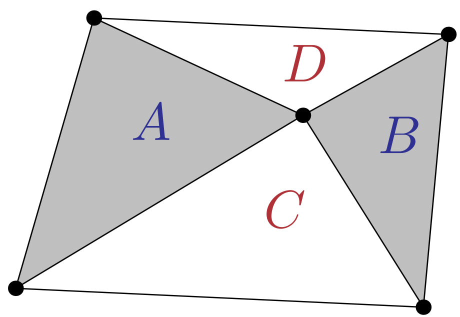
$$A(i,j,k) = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

Theorem. Every joint projective invariant is a function of the planar cross-ratios

$$[\, z_i, z_j, z_k, z_l, z_m \,] \; = \; \frac{A\,B}{C\,D}$$



Differential Invariants

Given a submanifold (curve, surface, ...) $N \subset M$, a differential invariant is an invariant of the action of G on N and its derivatives (jets).

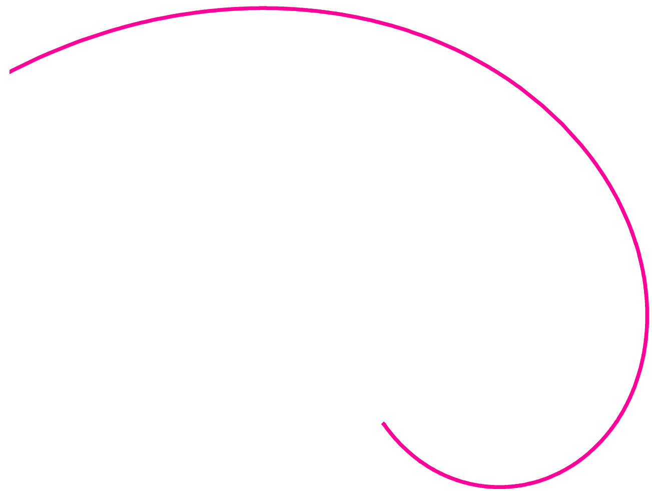
$$\boxed{I(g \cdot z^{(k)}) = I(z^{(k)})}$$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

The simplest differential invariant is the curvature, defined as the reciprocal of the radius of the osculating circle:

$$\kappa = \frac{1}{r}$$

Curvature



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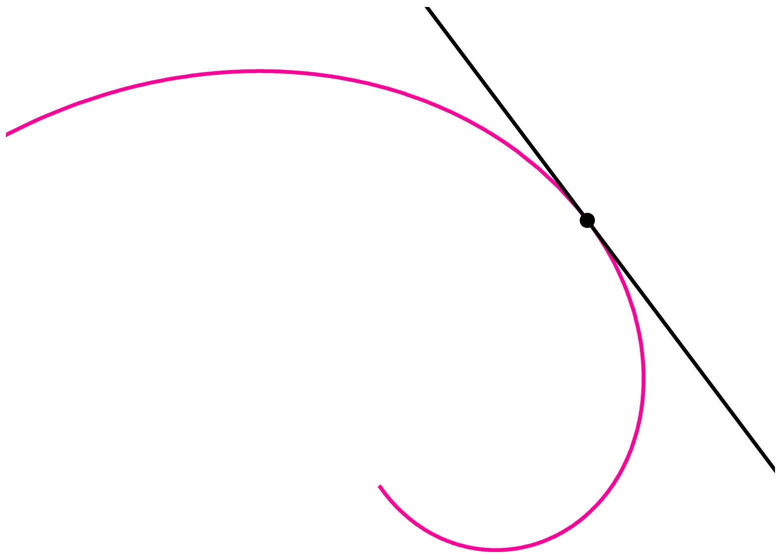

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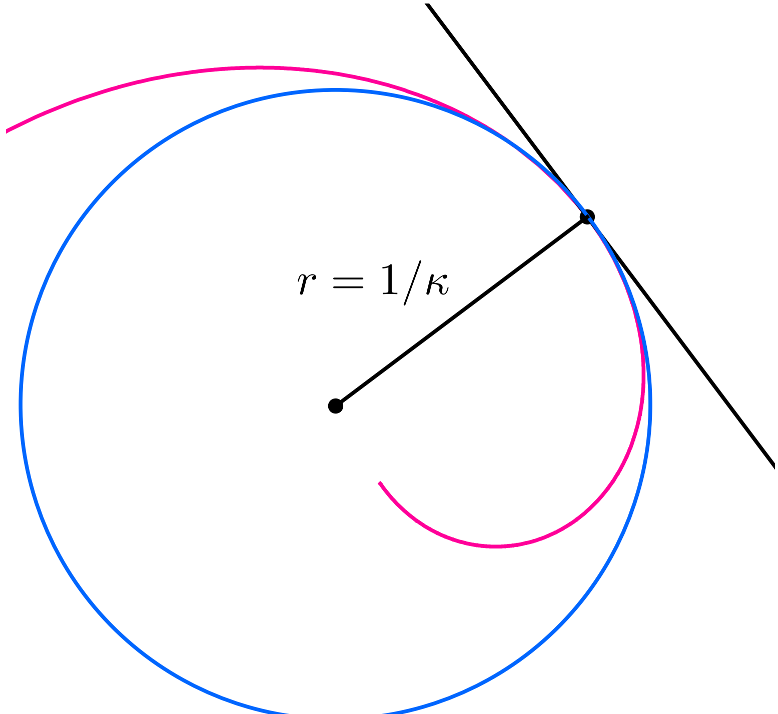

Euclidean Plane Curves: $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

Assume the curve is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\, dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}} \frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length:

$$\kappa, \qquad \kappa_s, \qquad \kappa_{ss}, \qquad \cdots$$

Moving Frames

The equivariant method of moving frames provides a systematic calculus for determining complete systems of invariants (joint invariants, differential invariants, joint differential invariants, etc.) and invariant objects (invariant differential forms, invariant differential operators, invariant tensors, invariant variational problems, invariant numerical approximations).

Equivalence & Invariants

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Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

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- Distinguishing syzygies.

Theorem. (Cartan)
Two regular submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

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♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \tag{$*$}$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or . . .) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Signature Curves

Definition. The *signature curve* $\mathcal{S} \subset \mathbb{R}^2$ of a plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\Sigma \; : \; C \longrightarrow \mathcal{S} = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

Theorem. Two regular curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent:

$$\overline{\mathcal{C}} = g \cdot \mathcal{C}$$

if and only if their signature curves are identical:

$$\overline{\mathcal{S}} = \mathcal{S}$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0.$

Symmetry and Signature

Continuous Symmetries

Theorem. The following are equivalent:

- The curve C has a 1-dimensional symmetry group $H \subset G$
- C is the orbit of a 1-dimensional subgroup $H \subset G$
- The signature $\mathcal{S}$ degenerates to a point: $\dim \mathcal{S} = 0$
- All the differential invariants are constant:

$$\kappa = c, \qquad \kappa_s = 0, \qquad \ldots$$

$\Longrightarrow$ Euclidean plane geometry: circles, lines

$\Longrightarrow$ Equi-affine plane geometry: conic sections.

$\Longrightarrow$ Projective plane geometry: W curves (*Lie & Klein*)

Symmetry and Signature

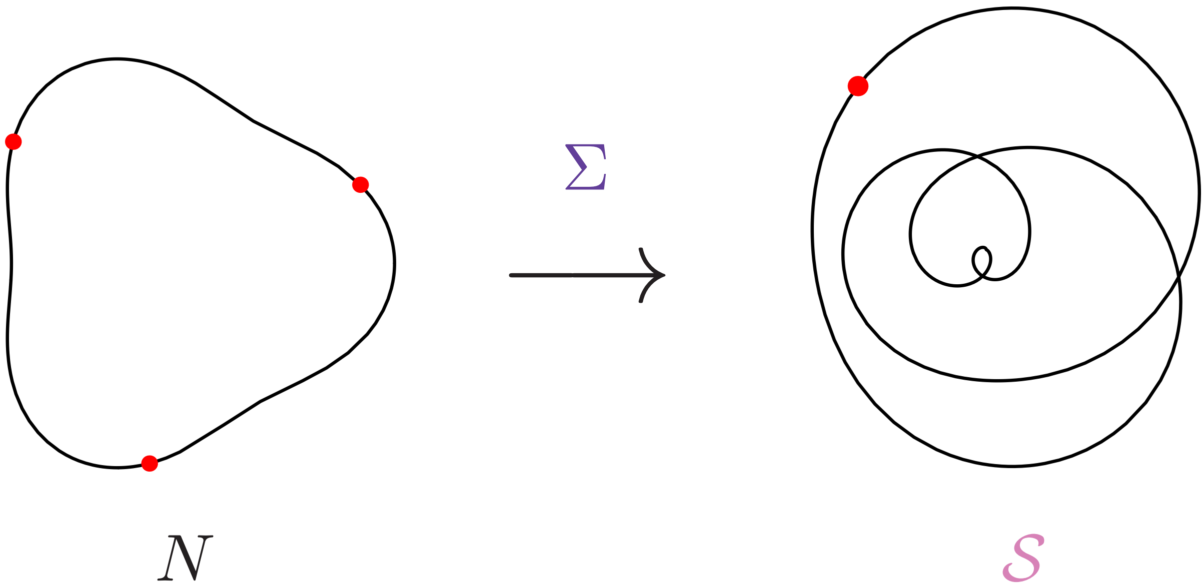
Discrete Symmetries

Definition. The index of a curve C equals the number of points in C which map to a single generic point of its signature:

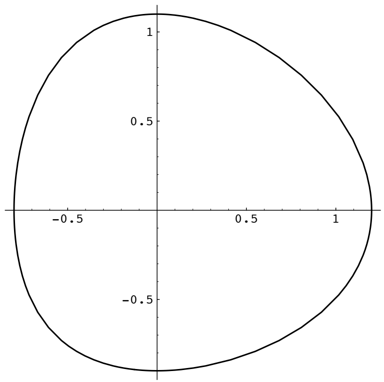
$$\iota_C = \min \left\{ \# \, \Sigma^{-1}\{w\} \;\middle|\; w \in \mathcal{S} \right\}$$

Theorem. The number of discrete symmetries of C equals its index ι_C.

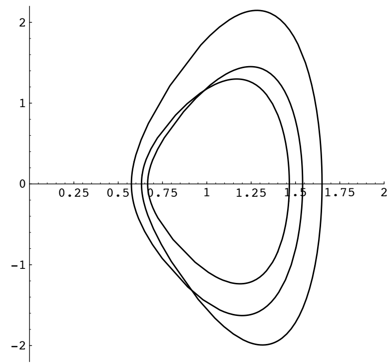
The Index



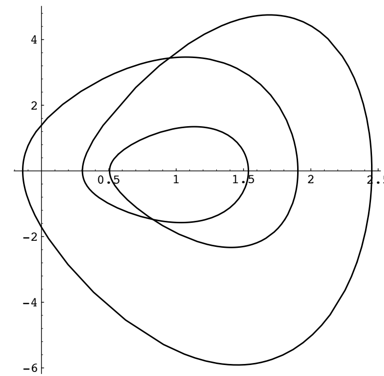
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

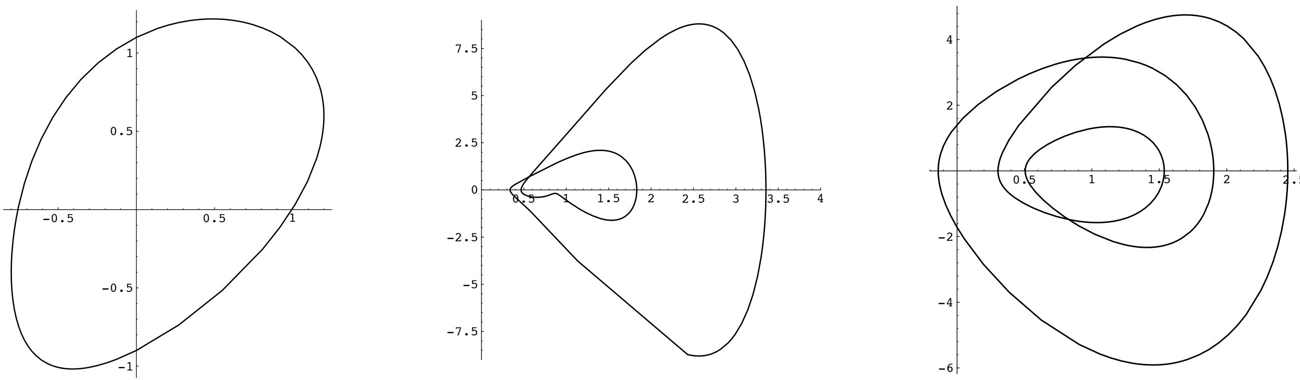


Euclidean Signature



Equi-affine Signature

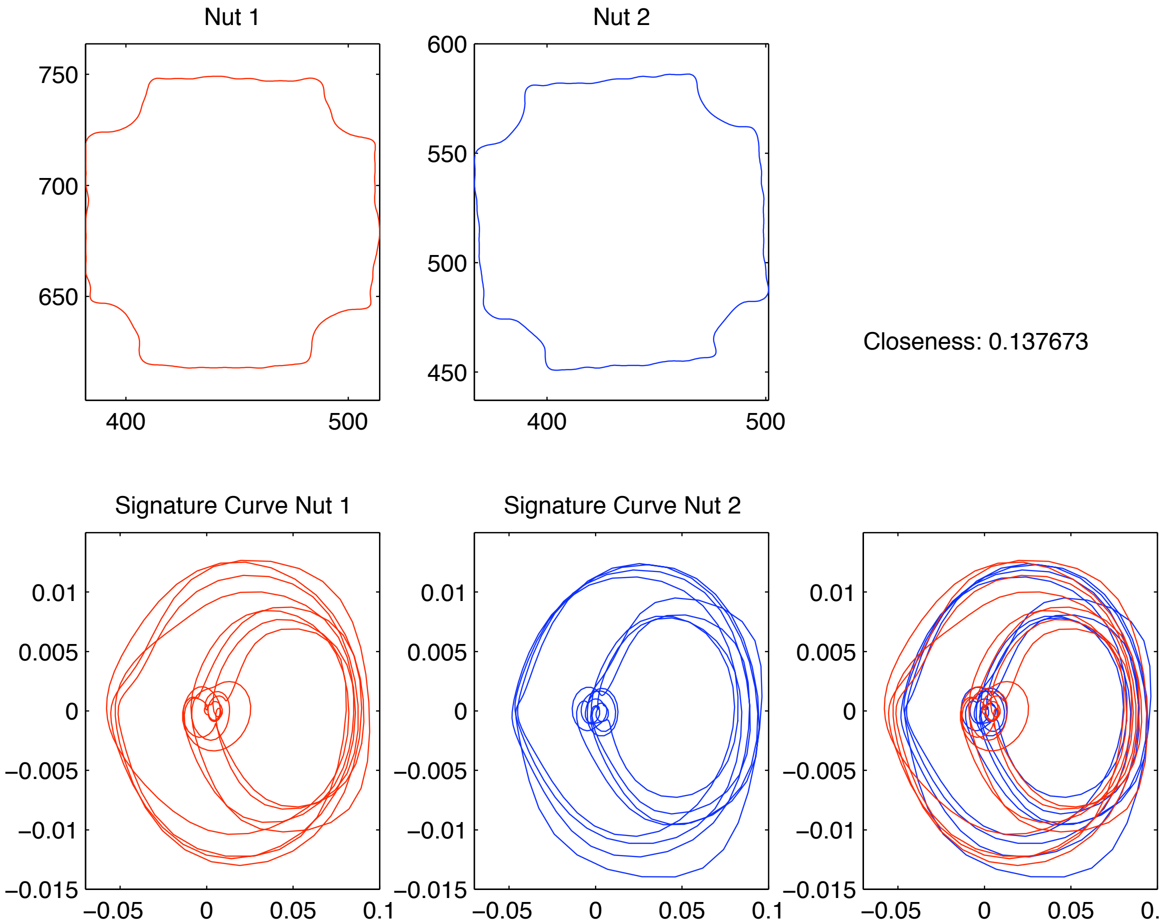
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



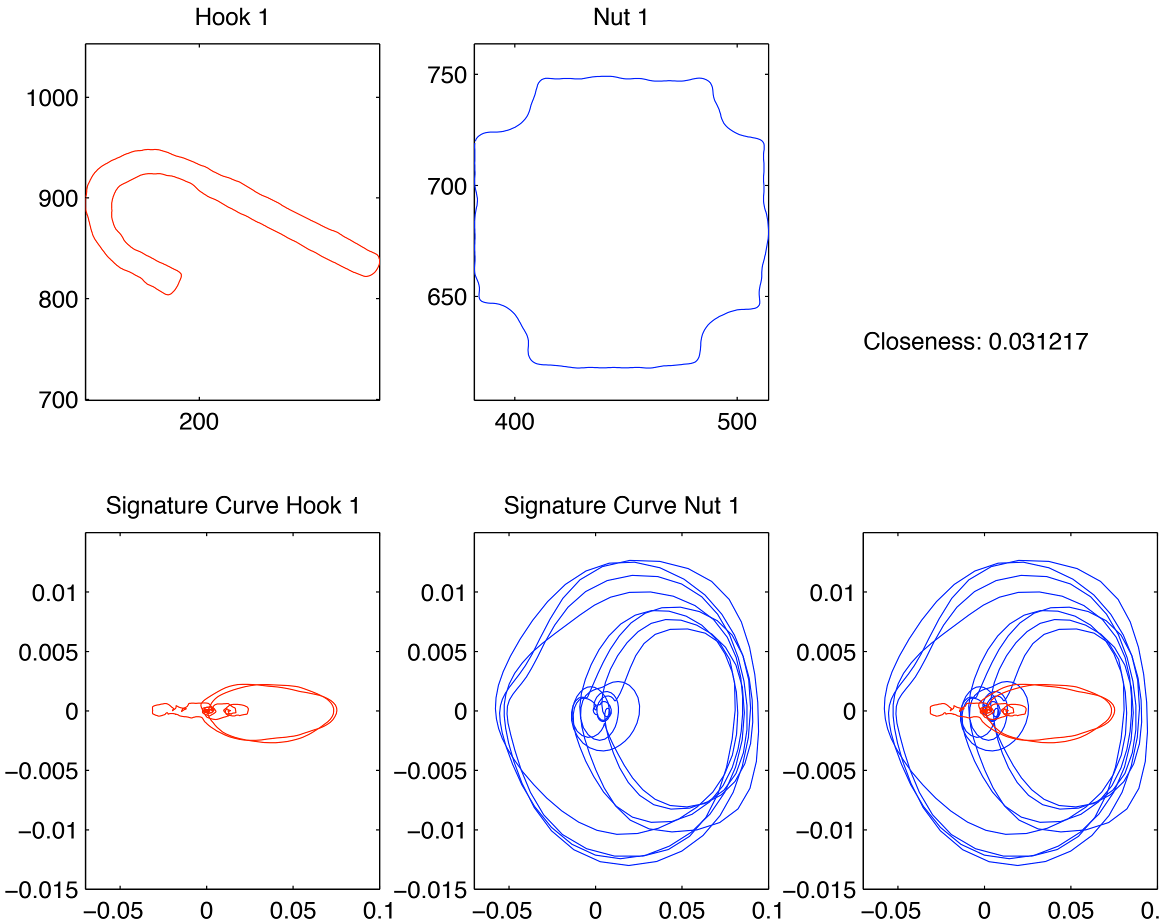
The Original Curve Euclidean Signature Equi-affine Signature

Object Recognition

⟹ Steve Haker



Nut 1
Nut 2
750
700
650
400
500
600
550
500
450
400
500
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1

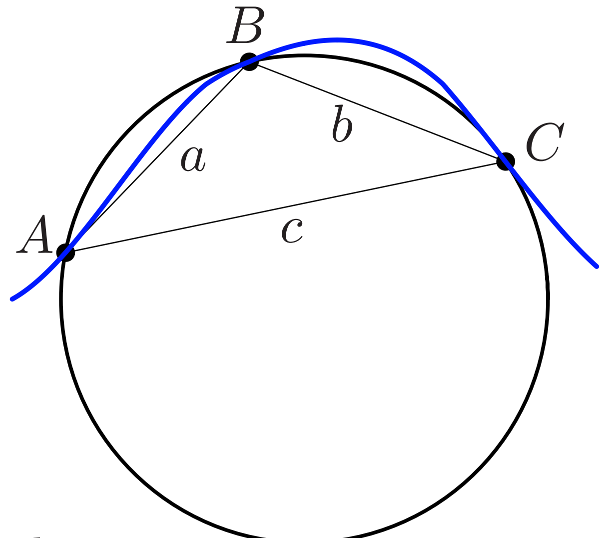
Invariant Numerical Approximations

When dealing with digital images, the calculation of the differential invariant signatures relies on invariant finite difference numerical approximations based on several sample points on the curve.

In other words, one approximates differential invariants by suitable joint invariants.

⋆⋆ Moving frames & multi-space.

Invariant numerical approximation to curvature

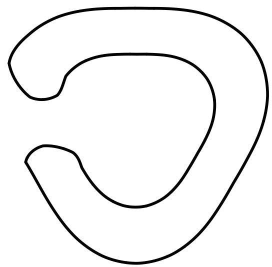


Heron's formula

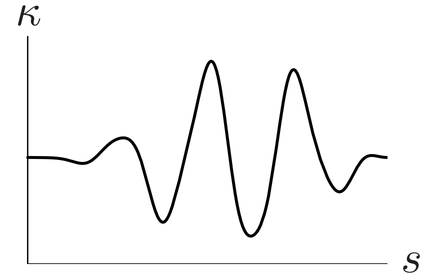
$$\kappa(B) \approx \widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \qquad \text{---} \qquad \text{semi-perimeter}$$

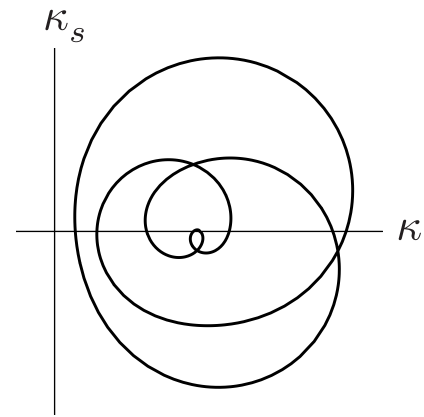
Signatures



Original curve

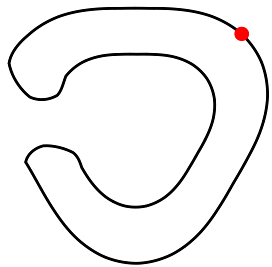


Classical Signature

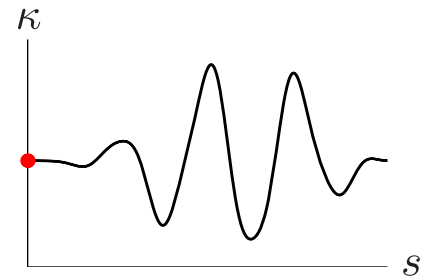


Differential invariant signature

Signatures



Original curve



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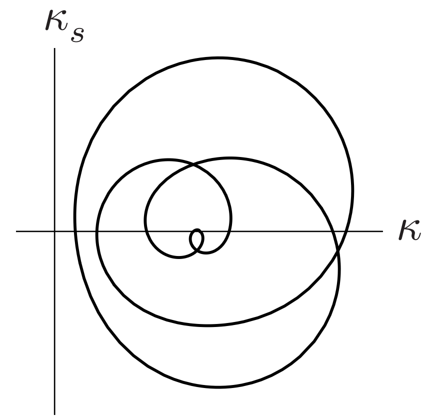
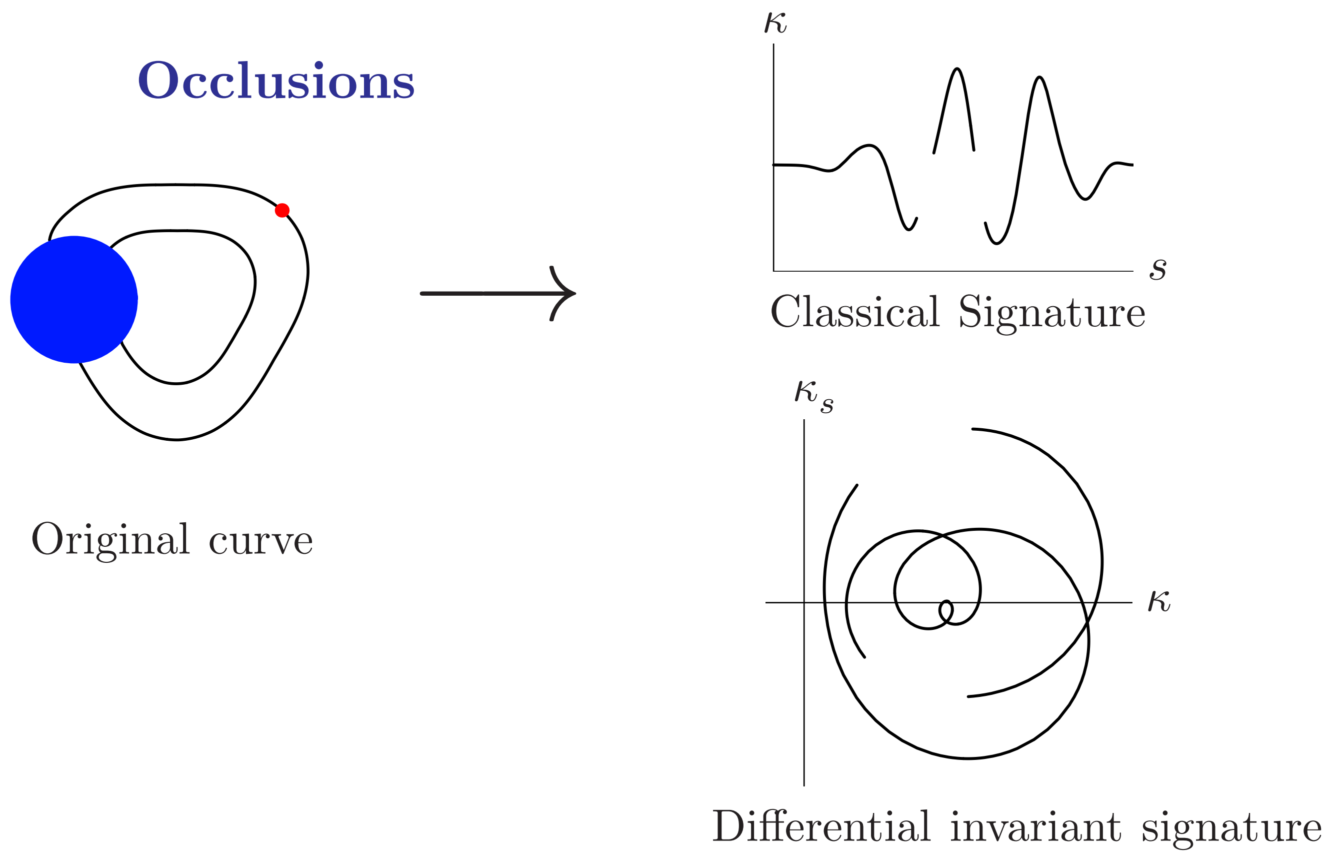


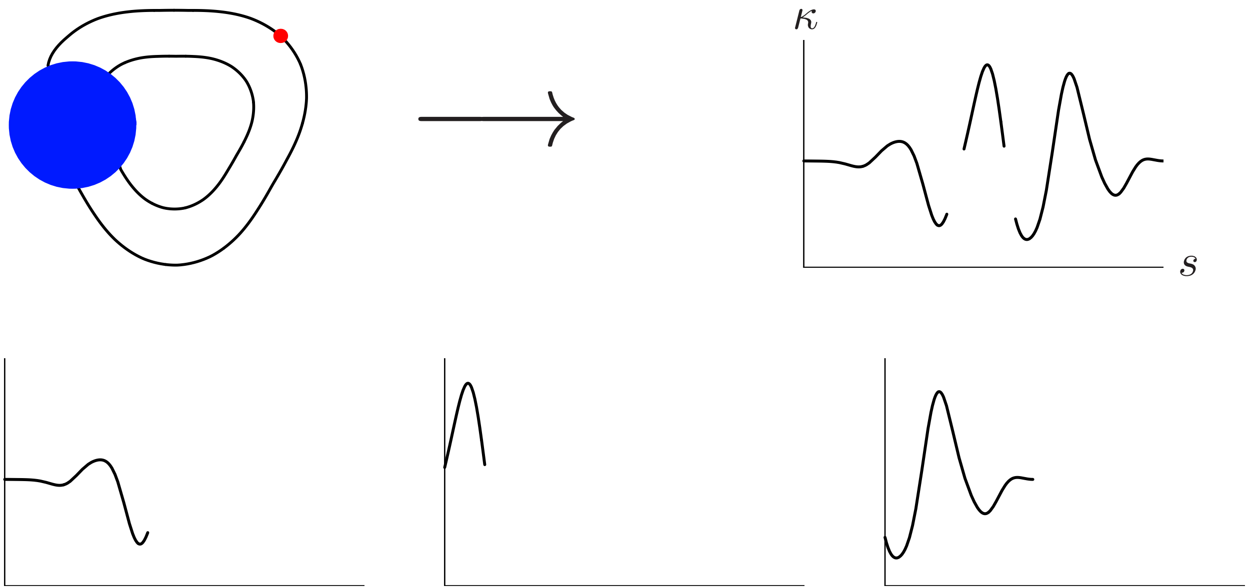
Differential invariant signature

Occlusions



Original curve
κ
s
Classical Signature
κ_s
κ
Differential invariant signature

Classical Occlusions



3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\mathcal{S} = \{\, (\kappa\,,\, \kappa_s\,,\, \tau)\,\} \;\subset\; \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{\, \left(\, H\,,\, K\,,\, H_{,1}\,,\, H_{,2}\,,\, K_{,1}\,,\, K_{,2} \right)\,\right\} \;\subset\; \mathbb{R}^6$$

or $$\widehat{\mathcal{S}} = \left\{\, \left(\, H\,,\, H_{,1}\,,\, H_{,2}\,,\, H_{,11} \right)\,\right\} \;\subset\; \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\mathcal{S} = \left\{\, \left(\, P\,,\, P_{,1}\,,\, P_{,2},\, P_{,11} \right)\,\right\} \;\subset\; \mathbb{R}^4$$

- P — Pick invariant

Generalized Vertices

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

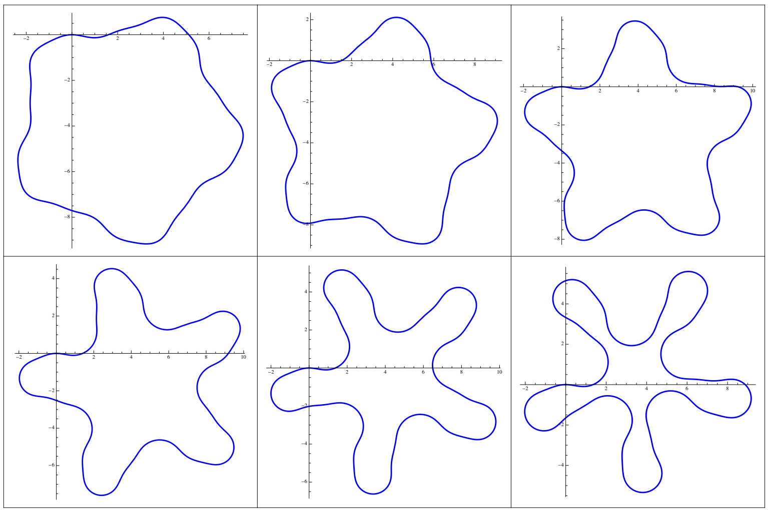
- critical point
- circular arc
- straight line segment

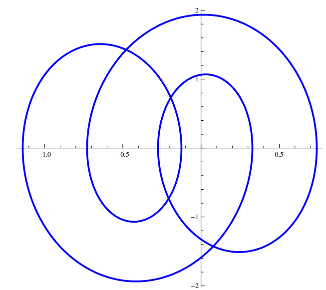
Mukhopadhya's Four Vertex Theorem:

A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.

“Counterexamples”

These degenerate curves all have the same signature:



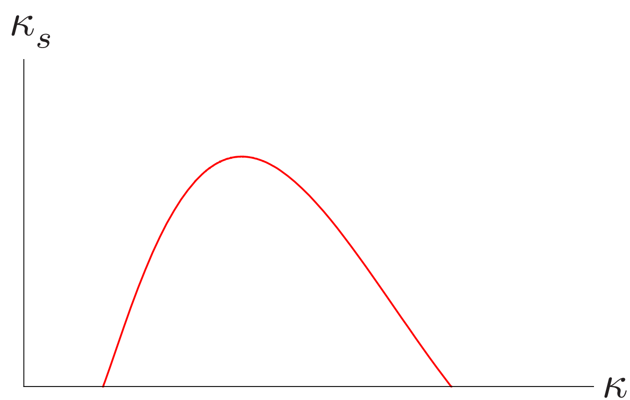


★ Replace vertices with circular arcs: *Musso–Nicoldi*

Bivertex Arcs

Bivertex arc: $\kappa_s \neq 0$ everywhere
except $\kappa_s = 0$ at the two endpoints

The signature $\mathcal{S}$ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition.

v-regular curve — finitely many generalized vertices

$$\boxed{C = \bigcup_{j=1}^{m} B_j \cup \bigcup_{k=1}^{n} V_k}$$

$B_1, \ldots, B_m$ — bivertex arcs

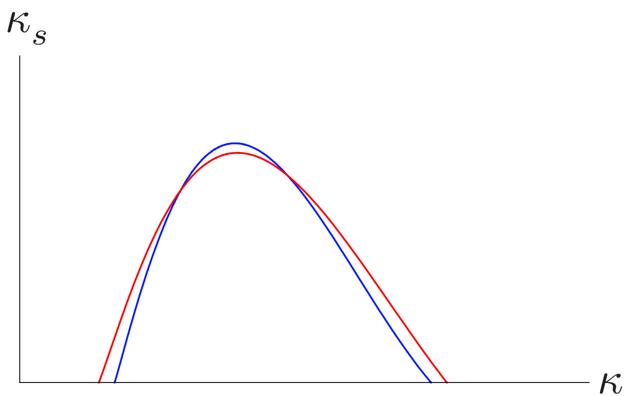
$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then determine whether the rigid equivalences are (approximately) the same. (semi-local)

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013) 176–185.

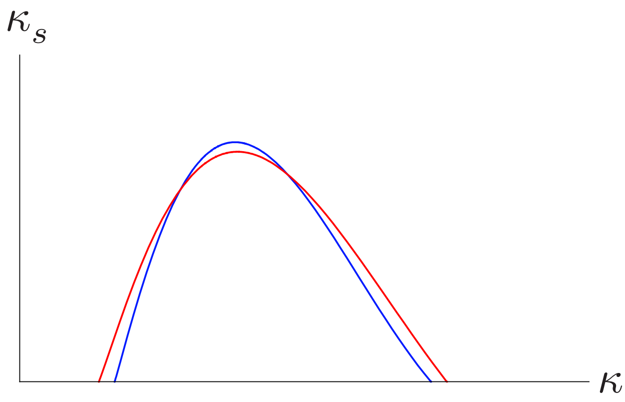
Gravitational/Electrostatic Attraction

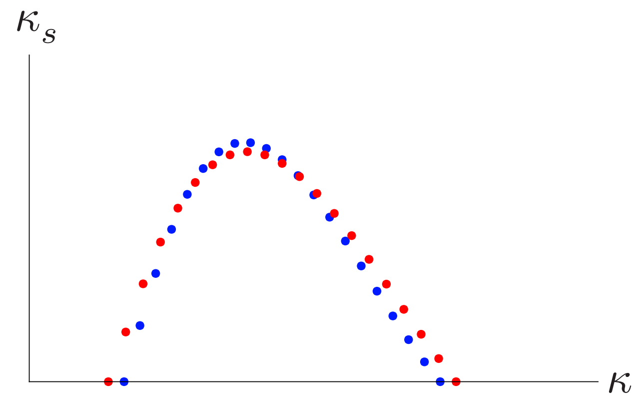
- Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.



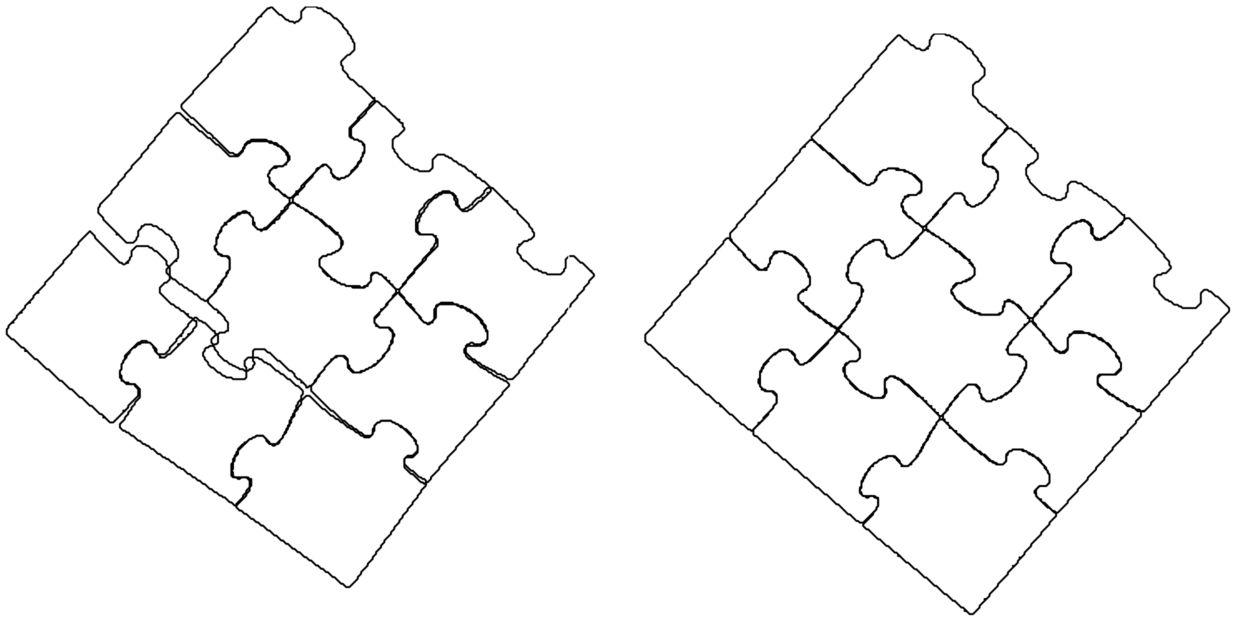
Gravitational/Electrostatic Attraction

- ★ Treat the two (signature) curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ★ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.



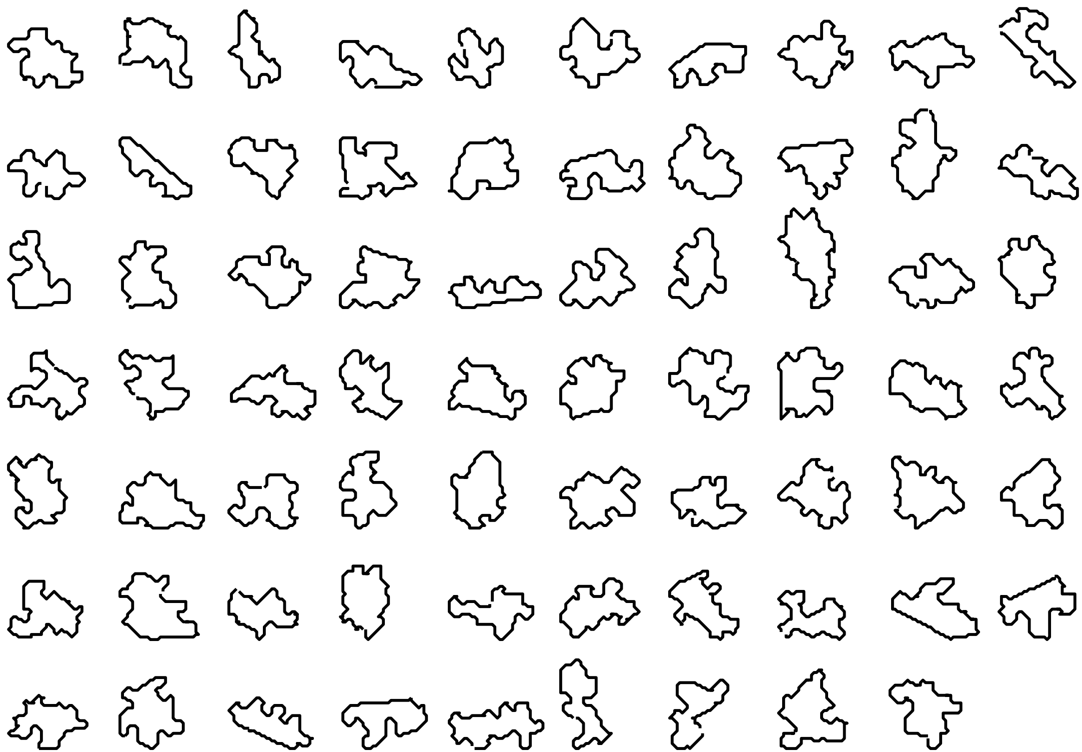


Piece Locking

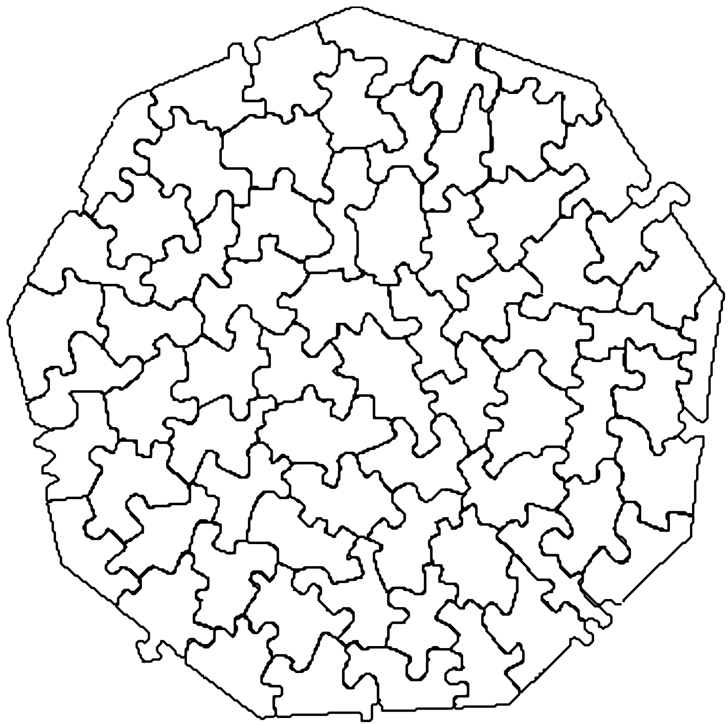


★★ Minimize force and torque based on gravitational attraction of the two matching edges.

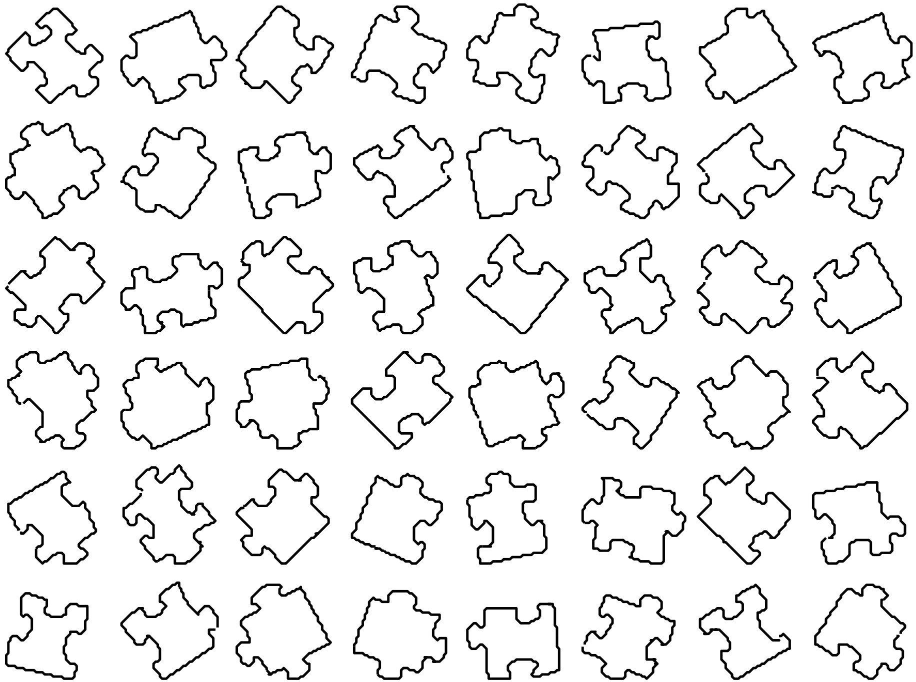
The Baffler Jigsaw Puzzle



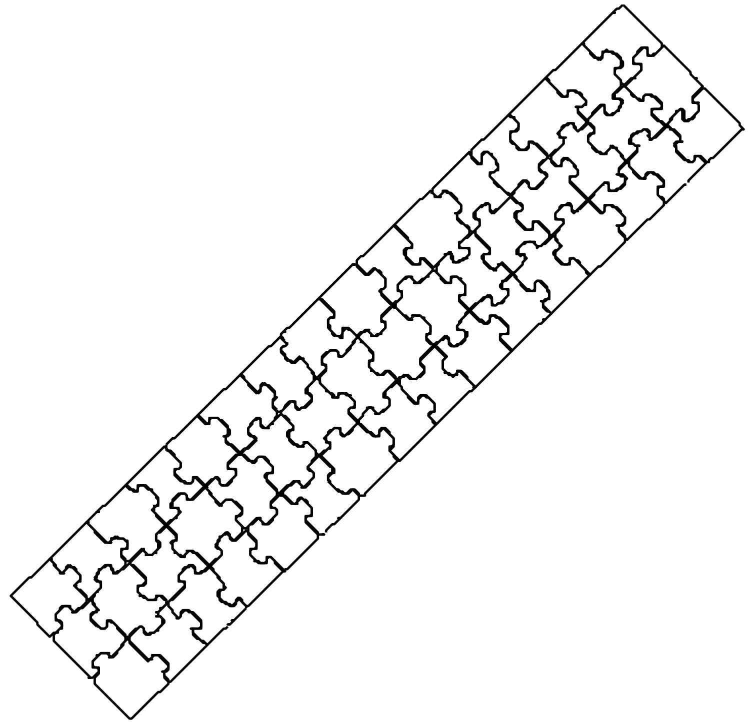
The Baffler Solved



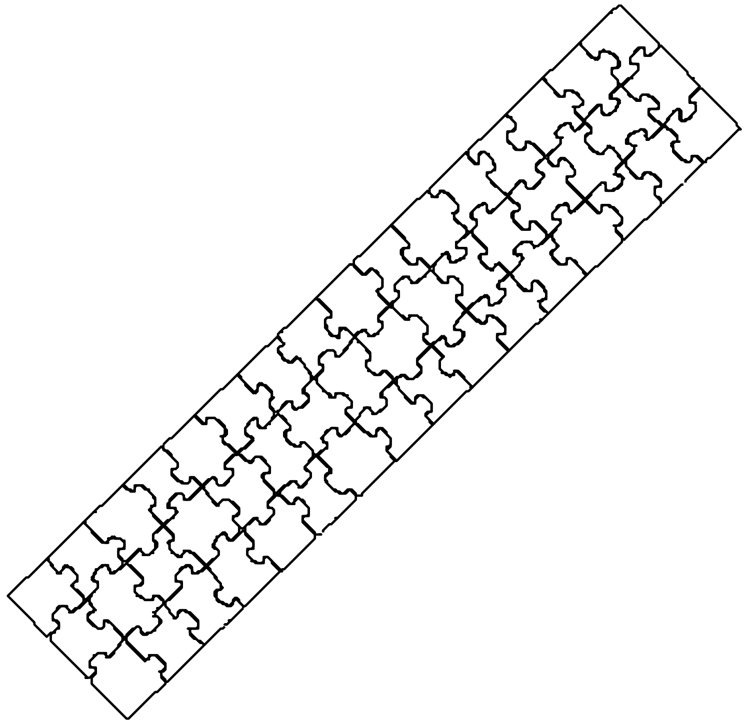
The Rain Forest Giant Floor Puzzle



The Rain Forest Puzzle Solved



The Rain Forest Puzzle Solved



⟹ D. Hoff & PJO, Automatic solution of jigsaw puzzles,
J. Math. Imaging Vision, to appear.

The Distance Histogram

Definition. The distance histogram of a finite set of points $P = \{z_1, \ldots, z_n\} \subset V$ is the function

$$\eta_P(r) = \#\left\{ (i,j) \;\middle|\; 1 \le i < j \le n,\ d(z_i, z_j) = r \right\}.$$

Characterization of Point Sets

Note: If $\widetilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\widetilde{P}}$.

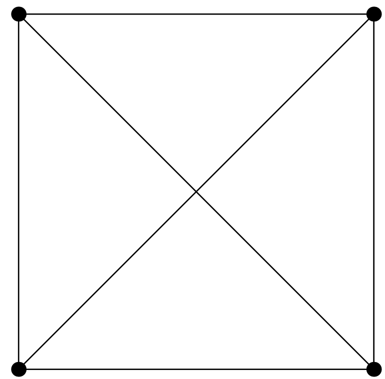
Characterization of Point Sets

Note: If $\widetilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\widetilde{P}}$.

Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\Longrightarrow$ Tinkertoy problem.

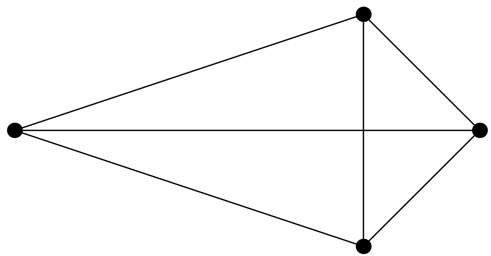
Yes:



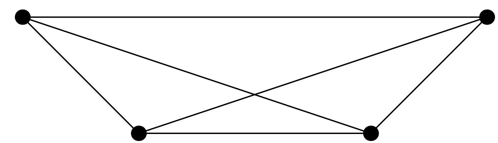
$$\eta \;=\; 1,\;\; 1,\;\; 1,\;\; 1,\;\; \sqrt{2},\;\; \sqrt{2}.$$

No:

Kite



Trapezoid



$$\eta = \sqrt{2}, \quad \sqrt{2}, \quad 2, \quad \sqrt{10}, \quad \sqrt{10}, \quad 4.$$

No:

$$\left.\begin{aligned} P &= \{0, 1, 4, 10, 12, 17\} \\ Q &= \{0, 1, 8, 11, 13, 17\} \end{aligned}\right\} \subset \mathbb{R}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

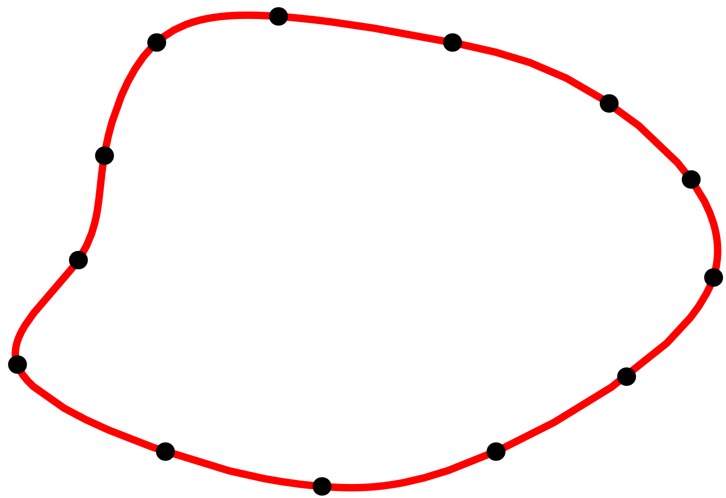
Characterizing Point Sets by their Distance Histograms

Theorem. Suppose $n \leq 3$ or $n \geq m + 2$.
Then there is a Zariski dense open subset in the space of n point configurations in $\mathbb{R}^m$ that are uniquely characterized, up to rigid motion, by their distance histograms.

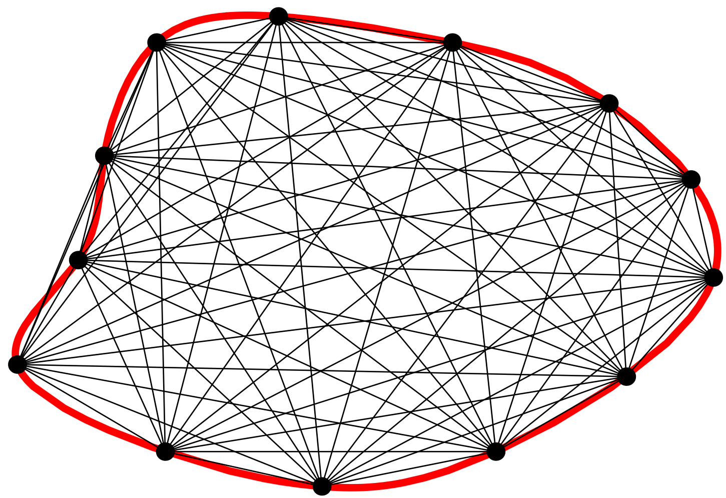
$\Longrightarrow$ M. Boutin & G. Kemper, *Adv. Appl. Math.* **32** (2004) 709–735

Limiting Curve Histogram

Limiting Curve Histogram



Limiting Curve Histogram



Sample Point Histograms

Cumulative distance histogram: $\quad n = \#P$:

$$\Lambda_P(r) = \frac{1}{n} + \frac{2}{n^2} \sum_{s \le r} \eta_P(s) = \frac{1}{n^2} \# \left\{ (i,j) \;\middle|\; d(z_i, z_j) \le r \right\},$$

Note:

$$\eta_P(r) = \tfrac{1}{2} n^2 [\, \Lambda_P(r) - \Lambda_P(r - \delta) \,] \qquad \delta \ll 1.$$

Local cumulative distance histogram:

$$\lambda_P(r, z) = \frac{1}{n} \# \left\{ j \;\middle|\; d(z, z_j) \le r \right\} = \frac{1}{n} \#(P \cap B_r(z))$$

$$\Lambda_P(r) = \frac{1}{n} \sum_{z \in P} \lambda_P(r, z) = \frac{1}{n^2} \sum_{z \in P} \#(P \cap B_r(z)).$$

Ball of radius r centered at z:

$$B_r(z) = \left\{ v \in V \;\middle|\; d(v, z) \le r \right\}$$

Limiting Curve Histogram Functions

Length of a curve

$$l(C) = \int_C ds < \infty$$

Local curve distance histogram function

$$h_C(r,z) = \frac{l(C \cap B_r(z))}{l(C)}$$

$\Longrightarrow$ The fraction of the curve contained in the ball of radius r centered at z.

Global curve distance histogram function:

$$H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s))\, ds.$$

Convergence of Histograms

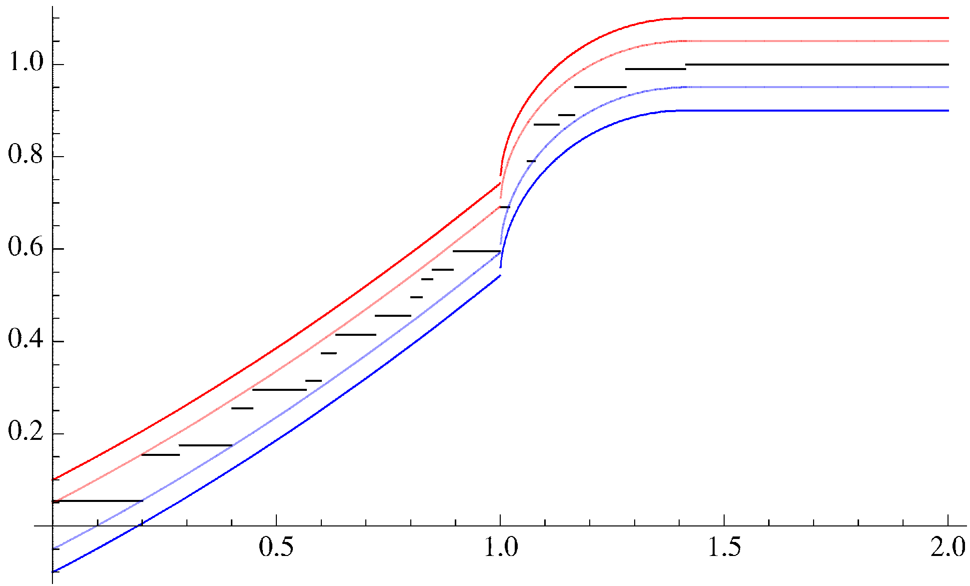
Theorem. Let C be a regular plane curve. Then, for both uniformly spaced and randomly chosen sample points $P \subset C$, the cumulative local and global histograms converge to their continuous counterparts:

$$\lambda_P(r,z) \longrightarrow h_C(r,z), \qquad \Lambda_P(r) \longrightarrow H_C(r),$$

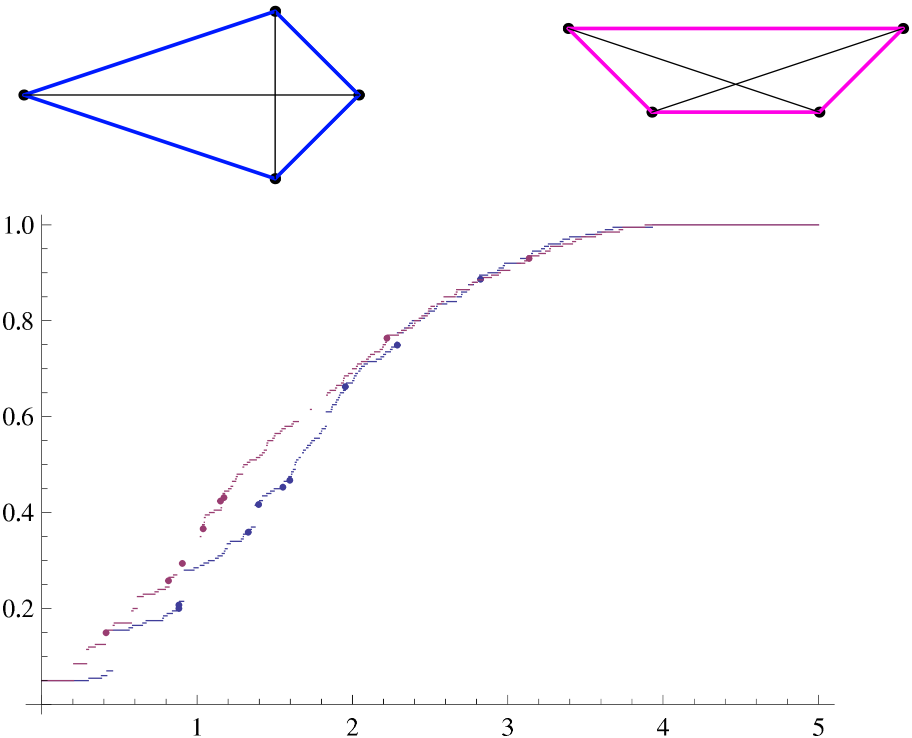
as the number of sample points goes to infinity.

D. Brinkman & PJO, Invariant histograms,
Amer. Math. Monthly **118** (2011) 2–24.

Square Curve Histogram with Bounds



Kite and Trapezoid Curve Histograms

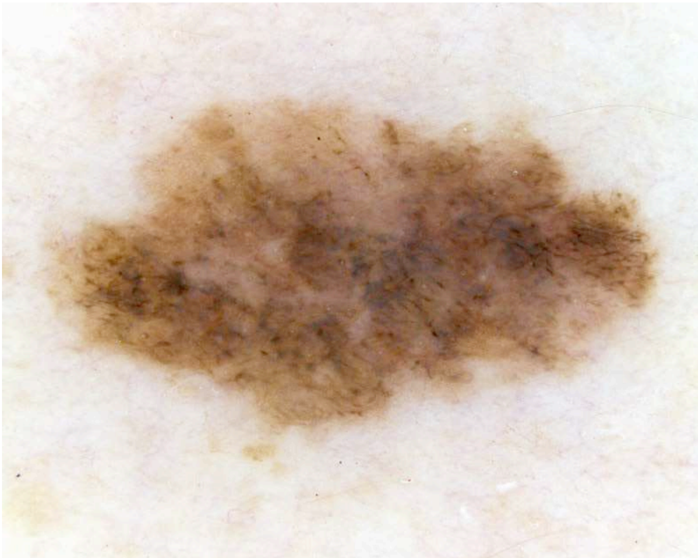


Histogram–Based Shape Recognition

500 sample points

Shape	(a)	(b)	(c)	(d)	(e)	(f)
(a) triangle	2.3	20.4	66.9	81.0	28.5	76.8
(b) square	28.2	.5	81.2	73.6	34.8	72.1
(c) circle	66.9	79.6	.5	137.0	89.2	138.0
(d) 2×3 rectangle	85.8	75.9	141.0	2.2	53.4	9.9
(e) 1×3 rectangle	31.8	36.7	83.7	55.7	4.0	46.5
(f) star	81.0	74.3	139.0	9.3	60.5	.9

Distinguishing Melanomas from Moles

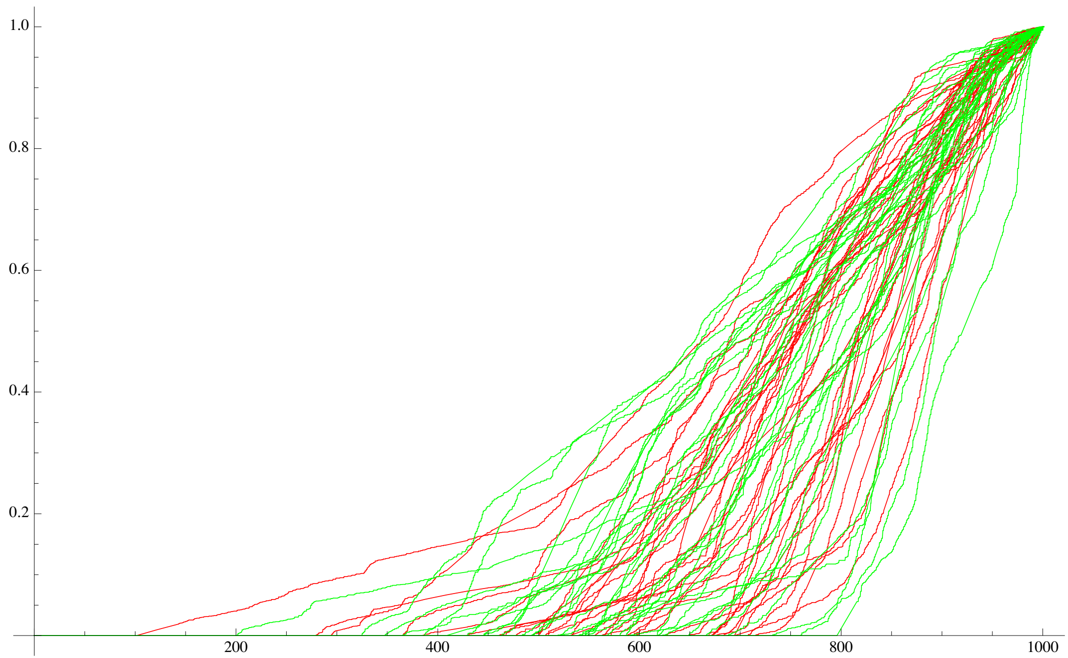


Melanoma

Mole

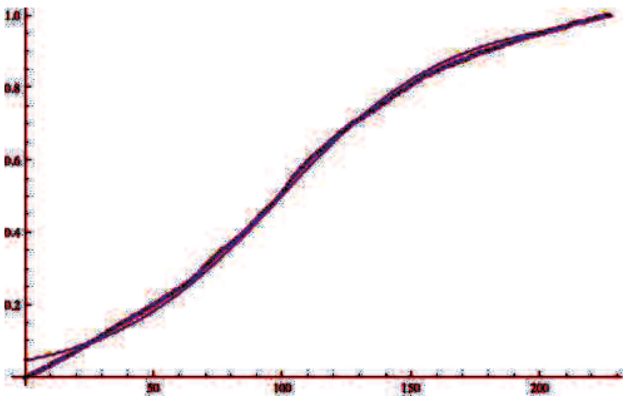
$\Longrightarrow$ A. Rodriguez, J. Stangl, C. Shakiban

Cumulative Global Histograms

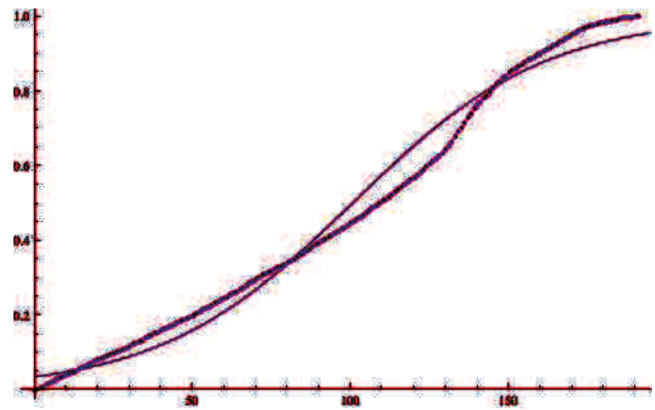


Red: melanoma Green: mole

Logistic Function Fitting

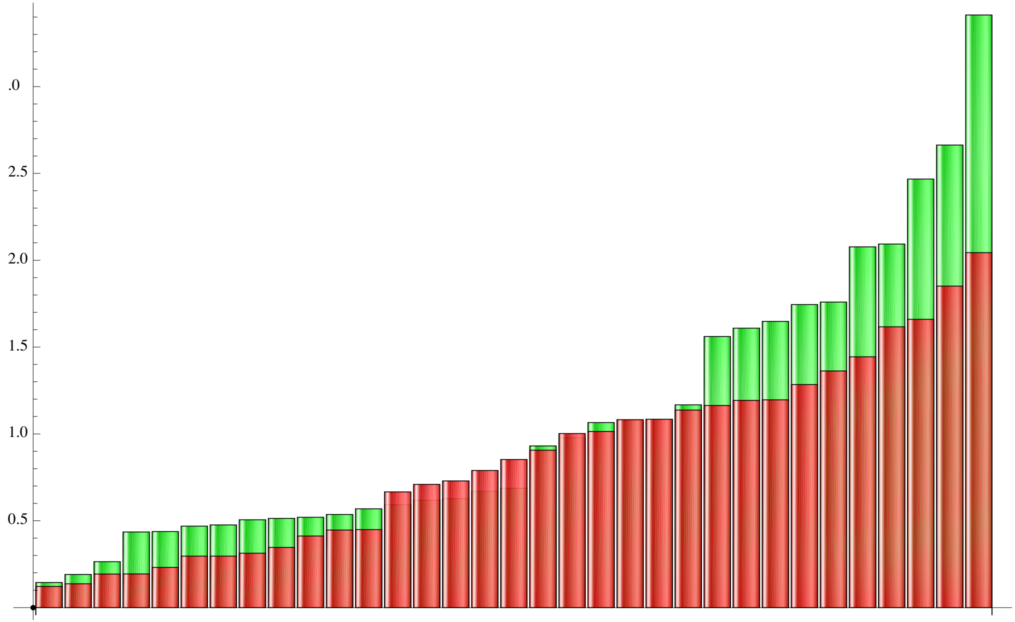


Melanoma



Mole

Logistic Function Fitting — Residuals



Melanoma $= 17.1336 \pm 1.02253$

Mole $= 19.5819 \pm 1.42892$

58.7% Confidence

Curve Histogram Conjecture

Two sufficiently regular plane curves C and $\widetilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\widetilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \widetilde{C}$.

Possible Proof Strategies

- Show that any polygon obtained from (densely) discretizing a curve does not lie in the Boutin–Kemper exceptional set.
- Polygons with obtuse angles: taking r small, one can recover (i) the set of angles and (ii) the shortest side length from $H_C(r)$. Further increasing r leads to further geometric information about the polygon ...
- Expand $H_C(r)$ in a Taylor series at $r = 0$ and show that the corresponding integral invariants characterize the curve.

Taylor Expansions

Local distance histogram function:

$$L\,h_C(r,z) = 2r + \tfrac{1}{12}\kappa^2 r^3 + \left(\tfrac{1}{40}\kappa\,\kappa_{ss} + \tfrac{1}{45}\kappa_s^2 + \tfrac{3}{320}\kappa^4\right) r^5 + \;\cdots\;.$$

Global distance histogram function:

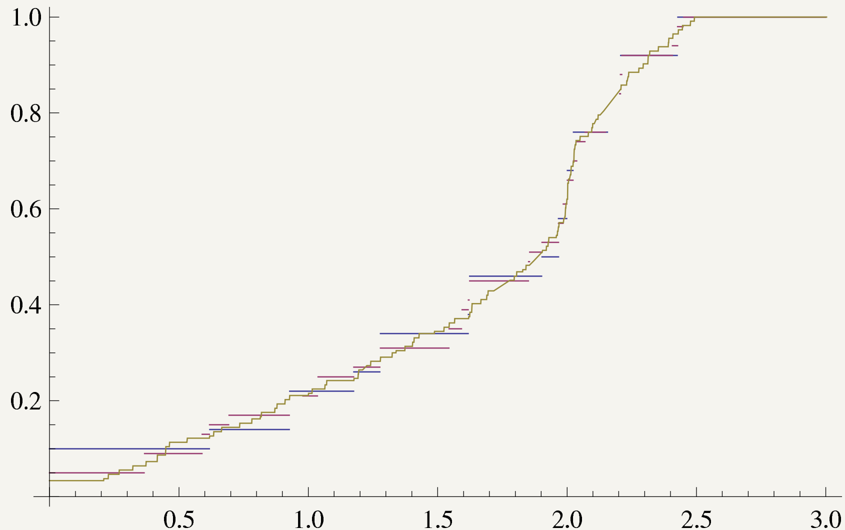
$$H_C(r) = \frac{2r}{L} + \frac{r^3}{12\,L^2}\oint_C \kappa^2\,ds + \frac{r^5}{40\,L^2}\oint_C \left(\tfrac{3}{8}\kappa^4 - \tfrac{1}{9}\kappa_s^2\right) ds + \;\cdots\;.$$

Space Curves

Saddle curve:

$$z(t) = (\cos t, \sin t, \cos 2t), \qquad 0 \leq t \leq 2\pi.$$

Convergence of global curve distance histogram function:

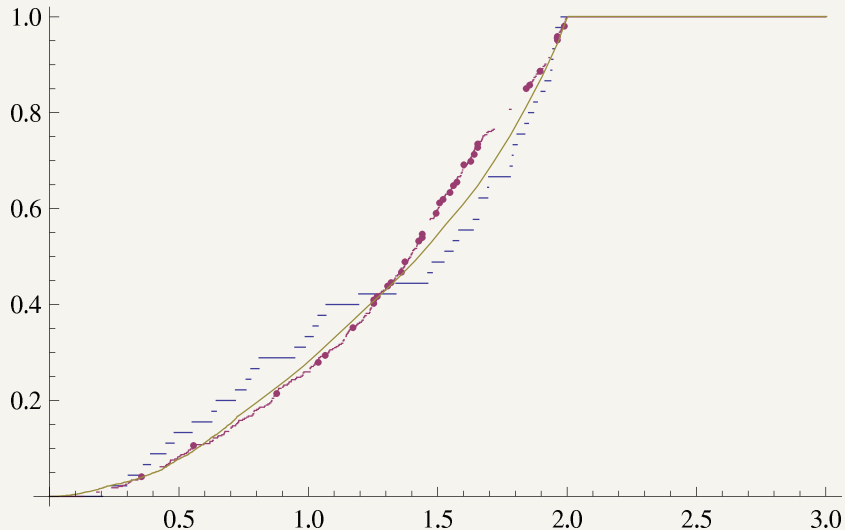


Surfaces

Local and global surface distance histogram functions:

$$h_S(r,z) = \frac{\text{area}\,(S \cap B_r(z))}{\text{area}\,(S)}, \qquad H_S(r) = \frac{1}{\text{area}\,(S)} \iint_S h_S(r,z)\, dS.$$

Convergence for sphere:



Area Histograms

Rewrite global curve distance histogram function:

$$H_C(r) = \frac{1}{L}\oint_C h_C(r, z(s))\, ds = \frac{1}{L^2}\oint_C \oint_C \chi_r(d(z(s), z(s')))\, ds\, ds'$$

$$\text{where} \qquad \chi_r(t) = \begin{cases} 1, & t \le r, \\ 0, & t > r, \end{cases}$$

Global curve area histogram function:

$$A_C(r) = \frac{1}{L^3}\oint_C \oint_C \oint_C \chi_r(\text{area}\,(z(\hat s), z(\hat s'), z(\hat s'')))\, d\hat s\, d\hat s'\, d\hat s'',$$

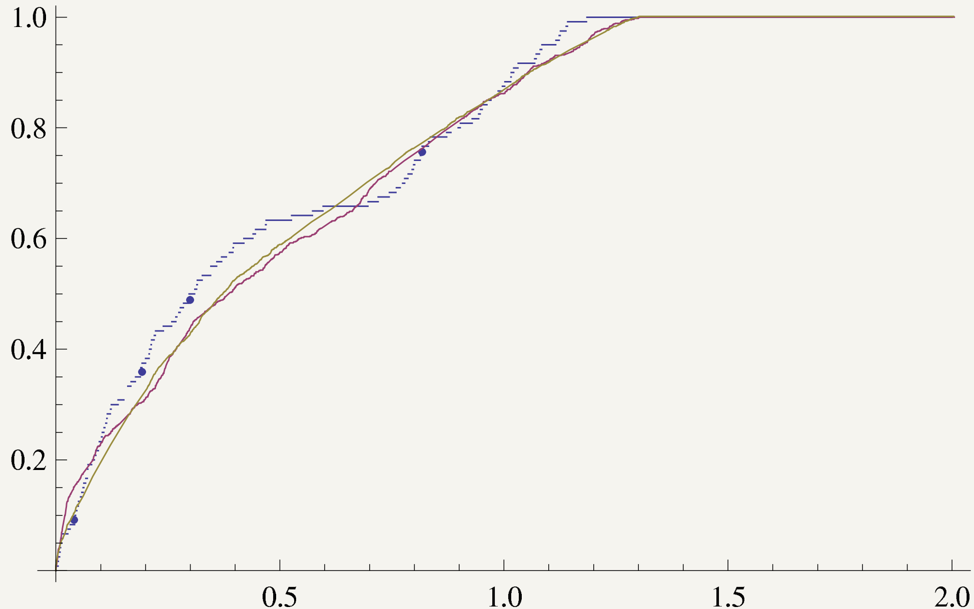
$d\hat s$ — equi-affine arc length element $\qquad L = \int_C d\hat s$

Discrete cumulative area histogram

$$A_P(r) = \frac{1}{n(n-1)(n-2)} \sum_{z \ne z' \ne z'' \in P} \chi_r(\text{area}\,(z, z', z'')),$$

Boutin & Kemper: The area histogram uniquely determines generic point sets $P \subset \mathbb{R}^2$ up to equi-affine motion.

Area Histogram for Circle



★ ★ Joint invariant histograms — convergence???

Triangle Distance Histograms

$Z = (\ldots z_i \ldots) \subset M$ —
sample points on a subset $M \subset \mathbb{R}^n$ (curve, surface, etc.)

$T_{i,j,k}$ — triangle with vertices z_i, z_j, z_k.

Side lengths:

$$\sigma(T_{i,j,k}) = (\, d(z_i, z_j), d(z_i, z_k), d(z_j, z_k) \,)$$

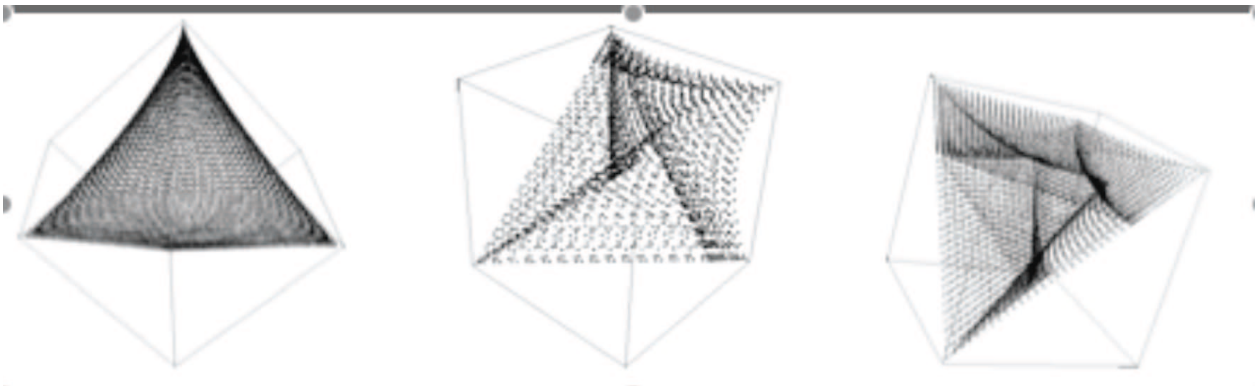
Discrete triangle histogram:

$$\mathcal{S} = \sigma(\mathcal{T}) \subset K$$

Triangle inequality cone:

$$K = \{ \, (x, y, z) \mid \; x, y, z \geq 0, \; x + y \geq z, \; x + z \geq y, \; y + z \geq x \, \} \subset \mathbb{R}^3.$$

Triangle Histogram Distributions



Circle Triangle Square

Convergence to measures . . .

$\Longrightarrow$ Madeleine Kotzagiannidis

Practical Object Recognition

- Scale-invariant feature transform (SIFT) (Lowe)
- Shape contexts (Belongie–Malik–Puzicha)
- Integral invariants (Krim, Kogan, Yezzi, Pottman, . . .)
- Shape distributions (Osada–Funkhouser–Chazelle–Dobkin)
  Surfaces: distances, angles, areas, volumes, etc.
- Gromov–Hausdorff and Gromov-Wasserstein distances (Mémoli)
  $\Longrightarrow$ lower bounds & stability

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