Computational Aspects of Moving Frames

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- Reid, Lisle
- Mansfield, Hubert, Kogan, . . .
- Morosov
- McLenaghan, Smirnov, The, . . .
- Hickman, Hann
- Shakiban, Lloyd, . . .

Applications of Moving Frames

- Differential geometry
- Equivalence
- Symmetry
- Differential invariants
- Rigidity
- Joint Invariants & Semi-Differential Invariants
- Invariant differential forms and tensors
- Identities and syzygies
- Classical invariant theory
- Computer vision
    - object recognition
    - symmetry detection
- Invariant numerical methods
- Relativity & separation of variables: Killing tensors, etc.
- Poisson geometry & solitons
- Lie pseudogroups
- Explict solutions of PDEs (group splitting)

Equivariant Moving Frames

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group

Definition.

A *moving frame* is a G-equivariant map

$$\rho : M \longrightarrow G$$

Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Theorem

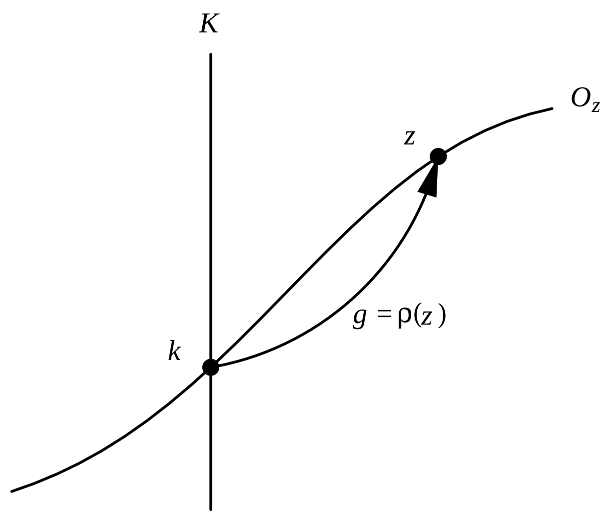
Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

$G_z = \{ g \mid g \cdot z = z \}$ $\quad\Longrightarrow\quad$ Isotropy subgroup

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity:
  $\Longrightarrow \quad G_z = \{e\}$ for all $z \in M$.
- locally free — the orbits all have the same dimension as G:
  $\Longrightarrow \quad G_z$ is a discrete subgroup of G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once
  $\not\approx$ irrational flow on the torus

Geometric Construction of Moving Frames

Normalization = choice of cross-section to the group orbits



K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction of Moving Frames

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\;, z_r = c_r \,\}$$

left	right	
$w(g,z) = g^{-1} \cdot z$	$w(g,z) = g \cdot z$	

Choose $r = \dim G$ components to *normalize*:

$$w_1(g,z) = c_1 \qquad \ldots \qquad w_r(g,z) = c_r$$

The solution

$$g = \rho(z)$$

is a (local) moving frame.

$\Longrightarrow$ Implicit Function Theorem

$\Longrightarrow$ Rational algebraic functions

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g,z)$ produces the fundamental invariants:

$$I_1(z) = w_{r+1}(\rho(z),z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z),z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ldots, I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F(z)$ with respect to a right moving frame ρ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

$$\iota(z_j) = \begin{cases} c_j, & 1 \leq j \leq r, \\ \\ I_{j-r}(z), & r+1 \leq j \leq m. \end{cases}$$

$$\iota\,[\,F(z_1, \ldots, z_m)\,] \;=\; F(c_1, , \ldots c_r, I_1(z), \ldots, I_{m-r}(z))$$

Invariantization amounts to restricting F to the cross-section $I \,|\, K \;=\; F \,|\, K$ and then requiring that $I \;=\; \iota(F)$ be constant along the orbits.

In particular, if $I(z)$ is an invariant, then $\iota(I) = I$.

Invariantization defines a canonical projection

$$\iota: \quad \begin{array}{ccc} \text{functions} & \longmapsto & \text{invariants} \\ \text{differential forms} & \longmapsto & \text{invariant forms} \end{array}$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

An effective action can usually be made free by:

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to “multi-space”

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

$\Longrightarrow$ joint or semi-differential invariants
$\Longrightarrow$ invariant numerical approximations

Jet Space

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} extended jet bundle for
p-dimensional submanifolds $N \subset M$

Assume $N = \{ u = f(x) \}$ is a graph (section).

$x = (x^1, \ldots, x^p)$ — independent variables

$u = (u^1, \ldots, u^q)$ — dependent variables

$$p + q = m = \dim M$$

Local coordinates on J^n:

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots \ x^i \ \ldots \ u^\alpha_J \ \ldots \ )$$

$$u^\alpha_J = \partial_J u^\alpha \qquad 0 \le \# J \le n$$

— derivatives of dependent variables

= Taylor coefficients

Prolongation of Group Actions

G — transformation group acting on M

$G^{(n)}$ — prolonged action of G on the jet space J^n

Differential invariant: $I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$$

$\Longrightarrow$ curvatures

Theorem. If G acts (locally) effectively on M, then G acts (locally) freely on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$ for $n \gg 0$.

The prolonged group formulae

$$w^{(n)} = (y, v^{(n)}) = g^{(n)} \cdot z^{(n)}$$

are obtained by implicit differentiation.

Moving Frames on Jet Space

Write out prolonged group action:

$$w^{(n)} = (y, v^{(n)}) = \begin{cases} g^{(n)} \cdot z^{(n)} & \text{right} \\ (g^{(n)})^{-1} \cdot z^{(n)} & \text{left} \end{cases}$$

Choose a coordinate cross-section $\quad K \subset \mathrm{J}^n$

$$z_1 = c_1 \quad \ldots \quad z_r = c_r \qquad\qquad r = \dim G$$

$$\Longrightarrow \quad z_\nu \text{ are jet coordinates } (x^i \text{ or } u^\alpha_J)$$

Solve the *Normalization Equations*

$$w_1(g, x, u^{(n)}) = c_1 \quad \ldots \quad w_r(g, x, u^{(n)}) = c_r$$

for the group parameters

$$g = \rho^{(n)}(z^{(n)}) = \rho^{(n)}(x, u^{(n)})$$

The result is a moving frame of order n.

The Fundamental Differential Invariants

Invariantization of jet coordinates:

$$\boxed{H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(k)}) = \iota(u^\alpha_K)}$$

- phantom (constant) differential invariants = normalization variables
- fundamental (nonconstant) differential invariants

Theorem. Every differential invariant can be (locally) uniquely written as a function of the fundamental differential invariants:

$$I = \Phi(\;\ldots\; H^i \;\ldots\; I^\alpha_K \;\ldots\;)$$

Invariant Differentiation

Invariant differential operators:

$$\boxed{\mathcal{D}_1 = \iota(D_{x^1}) \quad \ldots \quad \mathcal{D}_p = \iota(D_{x^p})}$$

Theorem. The higher order differential invariants are obtained by invariant differentiation of a finite number of fundamental differential invariants $I_1, \ldots, I_N$ with respect to $\mathcal{D}_1, \ldots, \mathcal{D}_p$.

The Algebra of Differential Invariants

- Fundamental differential invariants

$$I_1, \ldots, I_N \qquad\qquad N = ???$$

- Invariant differential operators

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

- Recurrence Formulae:

$$\mathcal{D}_i I^\alpha_J = I^\alpha_{Ji} + R^\alpha_{J,i}$$

- Commutation formulae

$$[\,\mathcal{D}_i,\, \mathcal{D}_j\,] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k$$

- Classification of syzygies

$$F(\ \ldots\ \mathcal{D}^J I^\alpha\ \ \ldots\ ) = 0$$

- Noncommutative differential
  Gröbner bases

The Key Formula

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma_\kappa \wedge \iota[\mathbf{v}_\kappa(\Omega)]$$

Ω — any function or differential form

$\mathbf{v}_1, \ldots, \mathbf{v}_r$
— prolonged infinitesimal generators

$\gamma_1, \ldots, \gamma_r$
— invariantized dual Maurer–Cartan forms

$\star\star\star$ All recurrence formulae, syzygies, commutation formulae, etc. are found by applying the key formula with Ω replaced by the basic forms and functions!

$\star\star\star$ To determine $\gamma_1, \ldots, \gamma_r$, let Ω range over the phantom differential invariants

Infinitesimal Generators

G — transformation group

$G^{(n)}$ — prolonged action on J^n

$\mathfrak{g}$ — Lie algebra of infinitesimal generators $\mathbf{v}$

$\mathbf{v}^{(n)}$ — Prolonged inf. gens.

The Prolongation Formula

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha,J} \varphi_J^\alpha(x,u^{(j)})\,\frac{\partial}{\partial u_J^\alpha}$$

$$\varphi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha$$

Characteristic

$$Q^\alpha(x,u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i\,\frac{\partial u^\alpha}{\partial x^i}$$

Euclidean group — SE(2)

$$(x,u) \longmapsto (x\cos\phi - u\sin\phi + a,\ x\sin\phi + u\cos\phi + b)$$

Transformed function $\quad v = \overline{f}(y)$:

$$y = x\cos\phi - f(x)\sin\phi,$$

$$v = x\sin\phi + f(x)\cos\phi,$$

Second prolongation

$$(x,u,u_x,u_{xx}) \longmapsto \Big(x\cos\phi - u\sin\phi, x\sin\phi + u\cos\phi, \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi},\ \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}\Big)$$

Infinitesimal generators

$$\mathbf{v}_1 = \frac{\partial}{\partial x} \qquad \mathbf{v}_2 = \frac{\partial}{\partial u} \qquad \mathbf{v}_3 = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u}$$

Second prolongation

$$\mathbf{v}_3^{(2)} = -u\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial u} + (1+u_x^2)\,\frac{\partial}{\partial u_x} + 3u_xu_{xx}\,\frac{\partial}{\partial u_{xx}}$$

$$Q = x + uu_x$$

$$\varphi^x = D_xQ + \xi u_{xx} = D_x(x+uu_x) - uu_{xx} = 1 + u_x^2$$

$$\varphi^{xx} = D_x^2Q + \xi u_{xxx} = D_x^2(x+uu_x) - uu_{xxx} = 3u_xu_{xx}$$

Generating Differential Invariants

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

$\Longrightarrow$ moving frame of order n

Recurrence Formulae:

$$\mathcal{D}_i I^\alpha_J = I^\alpha_{Ji} + R^\alpha_{J,i}$$

$$\operatorname{order}(R^\alpha_{J,i}) \ \leq \ \max\{\operatorname{order} J + 1, n + 1\}$$

Theorem. The non-phantom differential invariants of order $\leq n+1$ generate all differential invariants through invariant differentiation.

$$\mathcal{D}^K H^i, \qquad \mathcal{D}^K I^\alpha_J,$$

for $\#J \leq n+1, \ \alpha = 1, \ldots, q, \ i = 1, \ldots, p, \ \#K \geq 0.$

$\Longrightarrow$ Not necessarily a minimal system!

Commutation formulae

$$\boxed{[\,\mathcal{D}_i,\,\mathcal{D}_j\,] = \sum_{i=1}^{p} A_{ij}^{k}\,\mathcal{D}_k}$$

The coefficients A_{ij}^k can be obtained from the key formula by choosing $\Omega = dx^1, \ \ldots \ dx^p$.

Syzygies

Theorem. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:

$$\boxed{\mathcal{D}_j H^i = \delta^i_j + M^i_j}$$

— H^i non-phantom

$$\boxed{\mathcal{D}_J I^\alpha_K = c_\nu + M^\alpha_{K,J}}$$

— I^α_K generating

— $I^\alpha_{J,K} = c_\nu$ phantom

$$\boxed{\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}}$$

— I^α_{LK}, I^α_{LJ} generating, $K \cap J = \varnothing$

$\Longrightarrow$ Not necessarily a minimal system!

Euclidean Plane Curves

Lifted invariants

$$y = x\cos\phi - u\sin\phi + a$$

$$v = x\cos\phi + u\sin\phi + b$$

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi\ - u_x\sin\phi\,)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi\ - u_x\sin\phi\,)^5}$$

$$dy = (\cos\phi - u_x\sin\phi)\,dx$$

$$D_y = \frac{1}{\cos\phi - u_x\sin\phi}\,D_x$$

Normalization

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

Right moving frame $\qquad \rho\colon \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$\phi = -\,\tan^{-1} u_x \qquad a = -\,\frac{x + uu_x}{\sqrt{1+u_x^2}} \qquad b = \frac{xu_x - u}{\sqrt{1+u_x^2}}$$

Fundamental normalized differential invariants

$$\left.\begin{aligned} \iota(x) &= H = 0 \\ \iota(u) &= I_0 = 0 \\ \iota(u_x) &= I_1 = 0 \end{aligned}\right\} \qquad \text{phantom diff. invs.}$$

$$\iota(u_{xx}) = I_2 = \kappa$$

$$\iota(u_{xxx}) = I_3 = \kappa_s$$

$$\iota(u_{xxxx}) = I_4 = \kappa_{ss} + 3\kappa^3$$

Invariant horizontal one-form

$$\iota(dx) = \sqrt{1 + u_x^2}\; dx = ds$$

Invariant differentiation

$$\iota(D_x) = \frac{1}{\sqrt{1 + u_x^2}}\, D_x = D_s$$

Prolonged infinitesimal generators

$$\mathbf{v}_1 = \partial_x \qquad\qquad \mathbf{v}_2 = \partial_u$$

$$\mathbf{v}_3 = -u\,\partial_x + x\,\partial_u + (1+u_x^2)\,\partial_{u_x} + 3u_x u_{xx}\,\partial_{u_{xx}} + \cdots$$

$$dI = D_s I \cdot ds$$

Horizontal recurrence formula

$$d\iota(F) = \iota(dF) + \iota(\mathbf{v}_1(F))\,\gamma^1 + \iota(\mathbf{v}_2(F))\,\gamma^2 + \iota(\mathbf{v}_3(F))\,\gamma^3$$

Use phantom invariants

$$0 = dH = \iota(dx) + \sum \iota(\mathbf{v}_\kappa(x))\,\gamma^\kappa = ds + \gamma^1,$$

$$0 = dI_0 = \iota(du) + \sum \iota(\mathbf{v}_\kappa(u))\,\gamma^\kappa = \gamma^2,$$

$$0 = dI_1 = \iota(du_x) + \sum \iota(\mathbf{v}_\kappa(u_x))\,\gamma^\kappa = \kappa\,ds + \gamma^3,$$

to solve for

$$\gamma^1 = -ds \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\,ds$$

$$\gamma^1 = -ds \qquad \gamma^2 = 0 \qquad \gamma^3 = -\kappa\, ds$$

Recurrence formulae

$$\begin{aligned} \kappa_s\, ds = d\kappa = d(I_2) &= \iota(du_{xx}) + \iota(\mathbf{v}_3(u_{xx}))\,\gamma^3 \\ &= \iota(u_{xxx}\, dx) - \iota(3u_x u_{xx})\,\kappa\, ds = I_3\, ds \end{aligned}$$

$$\begin{aligned} \kappa_{ss}\, ds = d(I_3) &= \iota(du_{xxx}) + \iota(\mathbf{v}_3(u_{xxx})\,\gamma^3 \\ &= \iota(u_{xxxx}\, dx) - \iota(4u_x u_{xxx} + 3u_{xx}^2)\,\kappa\, ds = (I_4 - 3\, I_2^3)\, ds \end{aligned}$$

$$\begin{aligned} \kappa &= I_2 & I_2 &= \kappa \\ \kappa_s &= I_3 & I_3 &= \kappa_s \\ \kappa_{ss} &= I_4 - 3I_2^3 & I_4 &= \kappa_{ss} + 3\kappa^3 \\ \kappa_{sss} &= I_5 - 19I_2^2 I_3 & I_4 &= \kappa_{sss} + 19\kappa^2\kappa_s \end{aligned}$$

Example

$(x^1, x^2, u) \in M = \mathbb{R}^3$ $\qquad G = \mathrm{GL}(2)$

$$(x^1, x^2, u) \longmapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$$

$$\lambda = \alpha\delta - \beta\gamma$$

$\Longrightarrow$ Classical invariant theory

Prolongation (lifted differential invariants):

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2) \qquad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2)$$

$$v = \lambda^{-1} u$$

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda} \qquad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}$$

$$v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda}$$

$$v_{12} = \frac{\alpha\beta u_{11} + (\alpha\delta + \beta\gamma) u_{12} + \gamma\delta u_{22}}{\lambda}$$

$$v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}$$

Normalization

$$y^1 = 1 \qquad y^2 = 0 \qquad v_1 = 1 \qquad v_2 = 0$$

Nondegeneracy

$$x^1 \frac{\partial u}{\partial x^1} + x^2 \frac{\partial u}{\partial x^2} \neq 0$$

First order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix}$$

Normalized differential invariants

$$J^1 = 1 \qquad\qquad J^2 = 0$$

$$I = \frac{u}{x^1 u_1 + x^2 u_2}$$

$$I_1 = 1 \qquad\qquad I_2 = 0$$

$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2}$$

$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}$$

Phantom differential invariants

$$I_1 \qquad\qquad I_2$$

Generating differential invariants

$$I \qquad I_{11} \qquad I_{12} \qquad I_{22}$$

Invariant differential operators

$\mathcal{D}_1 = x^1 D_1 + x^2 D_2$ — scaling process

$\mathcal{D}_2 = -u_2 D_1 + u_1 D_2$ — Jacobian process

Recurrence formulae

$$\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0 \qquad \mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0$$

$$\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0 \qquad \mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0$$

$$\mathcal{D}_1 I = I_1 - I(1 + I_{11}) = -I(1 + I_{11}) \qquad \mathcal{D}_2 I = I_2 - I\, I_{12} = -I\, I_{12}$$

$$\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0 \qquad \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0$$

$$\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0 \qquad \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0$$

$$\mathcal{D}_1 I_{11} = I_{111} + (1 - I_{11}) I_{11} \qquad \mathcal{D}_2 I_{11} = I_{112} + (2 - I_{11}) I_{12}$$

$$\mathcal{D}_1 I_{12} = I_{112} - I_{11} I_{12} \qquad \mathcal{D}_2 I_{12} = I_{122} + (1 - I_{11}) I_{22}$$

$$\mathcal{D}_1 I_{22} = I_{122} + (I_{11} - 1) I_{22} - 2 I_{12}^2 \qquad \mathcal{D}_2 I_{22} = I_{222} - I_{12} I_{22}$$

$\Longrightarrow$ Use I to generate I_{11} and I_{12}

Syzygies

$$\begin{aligned} \mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2 I_{12} \\ \mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11} - 1) I_{22} - 2 I_{12}^2 \\ (\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} &= \\ = \ & 2 I_{22} \mathcal{D}_1 I_{11} + (5 I_{12} - 2) \mathcal{D}_1 I_{12} + (3 I_{11} - 5) \mathcal{D}_1 I_{22} - \\ & - (2 I_{11} - 5)(I_{11} - 1) I_{12} + 4 (I_{11} - 1) I_{12}^2 \end{aligned}$$

Commutation formulae

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12} \mathcal{D}_1 + (I_{11} - 1) \mathcal{D}_2$$

Euclidean Surfaces

$S \subset M = \mathbb{R}^3$ coordinates: $z = (x, y, u)$

Group: $G = \mathrm{E}(3)$

$$z \longmapsto R\,z + a, \qquad R \in \mathrm{O}(3)$$

Normalization — coordinate cross-section

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Left moving frame

$$a = z \qquad R = (\ \mathbf{t}_1\ \mathbf{t}_2\ \mathbf{n}\ )$$

- $\mathbf{t}_1, \mathbf{t}_2 \in TS$ — Frenet frame
- $\mathbf{n}$ — unit normal

Fundamental differential invariants

$$\kappa^1 = \iota(u_{xx}) \qquad \kappa^2 = \iota(u_{yy})$$

$\Longrightarrow$ principal curvatures

$H = \frac{1}{2}(\kappa^1 + \kappa^2)$ — mean curvature

$K = \kappa^1 \kappa^2$ — Gaussian curvature

Frenet coframe

$$\varpi^1 = \iota(dx^1) = \omega^1 + \eta^1 \qquad \varpi^2 = \iota(dx^2) = \omega^2 + \eta^2$$

Invariant differential operators

$$\mathcal{D}_1 \qquad \mathcal{D}_2$$

$\Longrightarrow$ Frenet differentiation

Fundamental Syzygy:

Use the recurrence formula to compare

$$\iota(u_{xxyy}) \qquad \text{with} \qquad \begin{aligned} \kappa^1_{,22} &= \mathcal{D}_2^2 \iota(u_{xx}) \\ \kappa^2_{,11} &= \mathcal{D}_1^2 \iota(u_{yy}) \end{aligned}$$

$$\kappa^1_{,22} - \kappa^2_{,11} + \frac{\kappa^1_{,1}\kappa^2_{,1} + \kappa^1_{,2}\kappa^2_{,2} - 2(\kappa^2_{,1})^2 - 2(\kappa^1_{,2})^2}{\kappa^1 - \kappa^2} - \kappa^1\kappa^2(\kappa^1 - \kappa^2) = 0$$

$\Longrightarrow$ Codazzi equations

Lie Pseudogroups

Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, premiere exploration à travers une jungle presque impénétrable; mais cell-ci menace de refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, *Oeuvres Scientifiques*

Pseudogroups in Action

- Lie — Medolaghi — Vessiot
- Cartan ... Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, ...
- Relativity
- Gauge theory and field theories
  Maxwell, Yang–Mills, conformal, string, ...
- Noether's Second Theorem
- Fluid Mechanics, Metereology
  Euler, Navier–Stokes,
  boundary layer, quasi-geostropic , ...
- Solitons (in $2 + 1$)
  K–P, Davey-Stewartson, ...
- Kac–Moody
- *Lie groups*!

$\Longrightarrow$ with J. Pohjanpelto

Pseudo-groups

Let M be a manifold. A *pseudo-group* is a collection of local diffeomrophisms $\varphi\colon M \to M$ such that, for any $\varphi, \psi \in \mathcal{G}$,

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses*: $\varphi^{-1} \in \mathcal{G}$,
- *Restriction*: $U \subset \operatorname{dom}\varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Composition*: $\operatorname{im}\varphi \subset \operatorname{dom}\psi \implies \psi \circ \varphi \in \mathcal{G}$.

Definition. A *Lie pseudo-group* $\mathcal{G}$ is defined by an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

$\implies$ *Analytic* (*Cartan–Kähler*)

$\implies$ Abstract object $\mathcal{G}$???

Infinitesimal Generators

$\mathfrak{g}$ — space of infinitesimal generators of $\mathcal{G}$

Locally defined vector fields:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a}$$

Infinitesimal determining equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \qquad (*)$$

$\Longrightarrow$ obtained by linearization

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

Groupoids

Define $G^{(n)}$ to be the space of Taylor polynomials (jets) of pseudo-group diffeomorphisms.

$G^{(n)}$ forms a *groupoid* since you can only compose two Taylor series (algebraically) if the target of the first matches the source of the second!

One-dimensional case:

$$\varphi: \quad y = a_0 + a_1(x - x_0) + \frac{a_2}{2}(x - x_0)^2 + \frac{a_3}{3!}(x - x_0)^3 + \cdots ,$$

$$\psi: \quad z = b_0 + b_1(y - y_0) + \frac{b_2}{2}(y - y_0)^2 + \frac{b_3}{3!}(y - y_0)^3 + \cdots ,$$

Composition:

$$\varphi \circ \psi: \quad z = c_0 + c_1(x - x_0) + \frac{c_2}{2}(x - x_0)^2 + \frac{c_3}{3!}(x - x_0)^3 + \cdots ,$$

where, **provided** $y_0 = a_0$ (i.e., source = target)

$$c_0 = b_0, \qquad c_1 = b_1 a_1, \qquad c_2 = b_2 a_1^2 + b_1 a_2,$$
$$c_3 = b_3 a_1^3 + 3 b_2 a_1 a_2 + b_1 a_3, \qquad \ldots$$

$\Longrightarrow$ Faà–di–Bruno formula.

We will view the pseudo-group $\mathcal{G}$ as the projective limit of its Taylor groupoids:

$$G^{(0)} \longleftarrow G^{(1)} \longleftarrow G^{(2)} \longleftarrow \cdots$$

Moving frame of order n:

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G^{(n)}$$

$\Longrightarrow$ requires freeness!

$\Longrightarrow$ compatibility

The moving frame $\rho^{(n)}$ is fixed by the same basic normalization algorithm, using coordinates on $G^{(n)}$ provided by the independent Taylor coefficients.

Theorem. (O–P) For $n \gg 0$, if $\mathcal{G}$ acts freely on J^n then it acts freely on J^k for all $k \geq n$.

$\Longrightarrow$ Bezout's theorem.

Differential Invariants for Pseudo-groups

Invariantization of jet coordinates:

$$H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(k)}) = \iota(u^\alpha_K)$$

- phantom (constant) differential invariants
- fundamental (nonconstant) differential invariants

$\Longrightarrow$ For pseudo-groups, phantom invariants appear at arbitrary order.

Theorem. Every differential invariant can be (locally) uniquely written as a function of the (non-phantom) fundamental differential invariants.

Invariant Differentiation

Invariant differential operators:

$$\boxed{\mathcal{D}_j = \iota(D_{x^j})}$$

Theorem. (Tresse–Kumpera)

The higher order differential invariants are obtained by invariant differentiation of a finite number of fundamental differential invariants $I_1, \ldots, I_N$.

Gröbner Basis Approach

Identify the cross-section variables with the complementary monomials to a certain algebraic module $\mathcal{J}$, which is the pull-back of the symbol module of the pseudo-group under a certain explicit linear map.

$\Longrightarrow$ Compatible term ordering.

$\Longrightarrow$ Algebraic specification of compatible moving frames of all orders $n > n^\star$.

Theorem. (O–P) For $n \gg 0$, the differential invariants are in one-to-one correspondence with the leading monomials in $\mathcal{J}$. The (higher order) generating differential invariants correspond to the Gröbner basis monomials.

Structure Equations

The full diffeomorphism pseudo-group:

$$d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ)$$

$$\mu^a[\![H]\!] = \sum_J \frac{1}{J!}\, \mu^a_J \, H^J$$

μ^a_J — Maurer–Cartan forms
= right-invariant contact forms on $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$.

$\mathcal{G}$ — Lie pseudo-group acting on M

The Maurer–Cartan forms are obtained by restricting the μ^a_J to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

Theorem. (O–P) The right-invariant Maurer–Cartan forms for $\mathcal{G}$ satisfy the invariantized linear determining equations

$$L^{(n)}(Z, \mu^{(n)}) = 0, \qquad (**)$$

Thus, the structure equations for the pseudo-group are obtained by restricting the diffeomorphism structure equations to the solution space of $(**)$.

The Key Formula

$$d\,\iota(\Omega) \;=\; \iota\left[\; d\Omega + \widehat{\mathbf{v}}^{(\infty)}(\Omega) \right]$$

where $$\widehat{\mathbf{v}}^{(\infty)} = \sum_{i=1}^{p} \eta^i \frac{\partial}{\partial x^i} + \sum_{\alpha,K} \psi_\alpha^K \frac{\partial}{\partial u_J^\alpha}$$

The coefficients of $\widehat{\mathbf{v}}^{(\infty)}$ are obtained from the coefficients φ_α^K of a prolonged vector field by replacing:

- source jet variables u_J^α by their invariantizations $I_J^\alpha = \iota(u_J^\alpha)$
- derivatives of vector field coefficients ζ_J^a by the pull-backs ν_J^α of the Maurer–Cartan forms

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- All recurrence formulae for differential invariants and invariant differential forms follow from specialization.
- In particular, the formulae for the pull-backs of the Maurer–Cartan forms ν_J^α are found by solving the recurrence formula for the phantom differential invariants.

First Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \, \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3u\, f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = 0, \qquad U = 1, \qquad U_X = 0, \qquad U_{XX} = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \quad \longmapsto \quad J = \frac{u_y}{u}$$

$$U_{XY} \quad \longmapsto \quad J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad\qquad U_{YY} \quad \longmapsto \quad J_2 = \frac{u_{yy}}{u}$$

Invariant coframe

$$dX \quad \longmapsto \quad u\,dx, \qquad\qquad dY \quad \longmapsto \quad dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x \qquad\qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m\,\mathcal{D}_2^n\,J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Prolonged infinitesimal generator $a = a(x)$:

$$\mathbf{v}^{(\infty)} = a\,\partial_x - u\,a_x\,\partial_u - (\,u\,a_{xx} + 2\,u_x\,a_x\,)\,\partial_{u_x} - u_y\,a_x\,\partial_{u_y} -$$
$$- (\,u\,a_{xxx} + 3\,u_x\,a_{xx} + 3\,u_{xx}\,a_x\,)\,\partial_{u_{xx}} -$$
$$- \Big(\,u_y\,a_{xx} + 2\,u_{xy}\,a_x\Big)\,\partial_{u_{xy}} - u_{yy}\,a_x\,\partial_{u_{yy}} - \cdots$$

Normalization

$$X = 0, \quad U = 1, \quad U_X = 0, \quad U_{XX} = 0, \quad \ldots .$$

Fundamental differential invariants:

$$\iota(x) = F = 0, \quad \iota(y) = y, \quad \iota(u) = I_{00} = 1,$$
$$\iota(u_x) = I_{10} = 0, \quad \iota(u_y) = I_{10} = J,$$
$$\iota(u_{xx}) = I_{20} = 0, \quad \iota(u_{xy}) = I_{11} = J_1, \quad \iota(u_{yy}) = I_{02} = J_2,$$

Invariant horizontal forms:

$$\varpi^1 = \iota(dx) = u\,dx, \qquad \varpi^2 = \iota(dy) = dy,$$

Pulled-back Maurer–Cartan forms:

$$\nu, \qquad \nu_1, \qquad \nu_2, \qquad \ldots$$

$$\widehat{\mathbf{v}}^{(\infty)} = \nu\,\partial_x - u\,\nu_1\,\partial_u - (\,u\,\nu_2 + 2\,u_x\,\nu_1\,)\,\partial_{u_x} - u_y\,\nu_1\,\partial_{u_y} - \;\cdots$$

Phantom invariants:

$$0 = dH = \varpi^1 + \nu, \qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \nu_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \nu_1, \qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \nu_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\nu = -\,\varpi^1, \qquad \nu_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\nu_1 = J\,\varpi^2 + \vartheta, \qquad \nu_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\qquad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$

$$\mathcal{D}_1 J = J_1, \quad \mathcal{D}_2 J = J_2 - J^2, \quad d_{\mathcal{V}}\,J = \vartheta_2 - J\,\vartheta,$$

$$\mathcal{D}_1 J_1 = J_3, \quad \mathcal{D}_2 J_1 = J_4 - 3\,J\,J_1, \quad d_{\mathcal{V}}\,J_1 = \vartheta_4 - J\,\vartheta_{10} - J_1\,\vartheta,$$

$$\mathcal{D}_1 J_2 = J_4, \quad \mathcal{D}_2 J_2 = J_5 - J\,J_2, \quad d_{\mathcal{V}}\,J_2 = \vartheta_5 - J_2\,\vartheta,$$

$\Longrightarrow$ All higher order differential invariants are obtained from J by invariant differentiation

Invariant horizontal forms

$$d\varpi^1 = -\,J\,\varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1, \qquad d\varpi^2 = 0.$$

Commmutation formula

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = J\,\mathcal{D}_1.$$