Symmetry, Invariants, Puzzles, and Cancer

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Why Math?

Math is alive

Math is fun

Math is important

Math is everywhere!

No matter what your career path will be,

the more math you learn,

the better you will do, and

the more opportunities you will have!

What math do you need to learn?

- ✸✸ Calculus (derivative = rate of change)
- ✸✸ Linear algebra (matrices)

Symmetry → *Group Theory!*

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

— P.S. Alexandroff

Groups

Definition. A group G is a set with a binary operation $g \cdot h$ satisfying

- Associativity: $g \cdot (h \cdot k) = (g \cdot h) \cdot k$
- Identity: $g \cdot e = g = e \cdot g$
- Inverse: $g \cdot g^{-1} = e = g^{-1} \cdot g$

$\Longrightarrow$ not necessarily commutative: $g \cdot h \neq h \cdot g$

Examples of groups

The integers

…, -4, -3, -2, -1, 0, 1, 2, 3, 4, …

Group operation: addition $3 + 5 = 8$

Identity: zero $3 + 0 = 3 = 0 + 3$

Inverse: negative $7 + (-7) = 0 = (-7) + 7$

Examples of groups

The rational numbers (fractions)

Group operation: addition 1/4 + 5/3 = 23/12

Identity: zero 5/3 + 0 = 5/3 = 0 + 5/3

Inverse: negative 7/2 + (-7/2) = 0 = (-7/2) + 7/2

Examples of groups

The positive rational numbers

Group operation: multiplication 1/4 x 5/3 = 5/12

Identity: one 5/3 x 1 = 5/3 = 1 x 5/3

Inverse: reciprocal 7/2 x 2/7 = 1 = 2/7 x 7/2

Examples of groups

The positive real numbers

Group operation: multiplication

$\sqrt{2}$ x π $= \sqrt{2}\,\pi$ = 4.44288293815836624701588099006....

Identity: one π x 1 = π = 1 x π

Inverse: reciprocal π x 1/ π = 1 = 1/ π x π

Examples of groups

Non-singular matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad h = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \qquad ad - bc \neq 0 \neq xw - yz$$

Group operation:

$$g{\cdot}h = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} \neq \begin{pmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{pmatrix} = h{\cdot}g$$

Identity: $\qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e \cdot g = g = g \cdot e$

Inverse: $\qquad g^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad g \cdot g^{-1} = e^{-1} = g \cdot g$

Symmetry Groups

A symmetry g of a geometric object S is a transformation that preserves it: $g \cdot S = S$

The set of symmetries of a geometric object forms a group

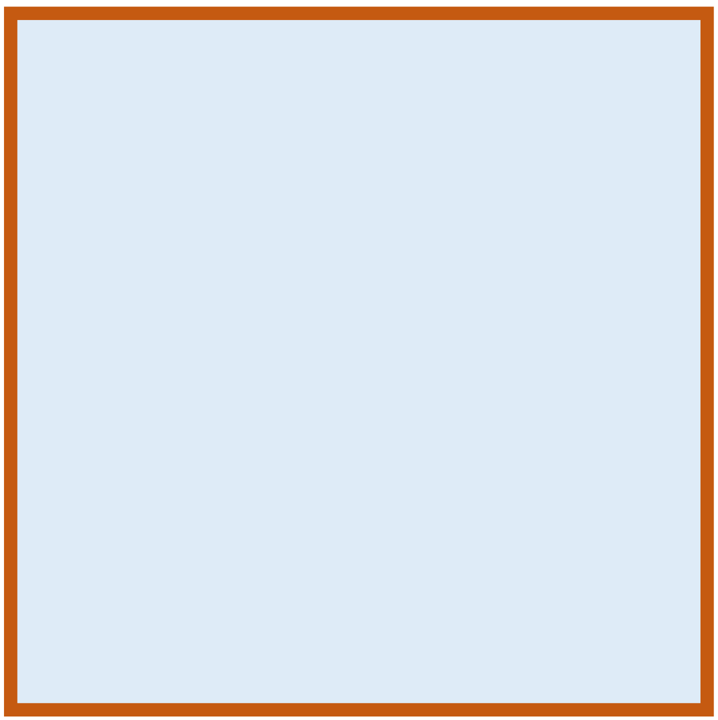
The group operation is composition: $g \cdot h$ = first do h, then do g

The composition of two symmetries is a symmetry

The identity (do nothing) is always a symmetry

The inverse of a symmetry (undo it) is a symmetry

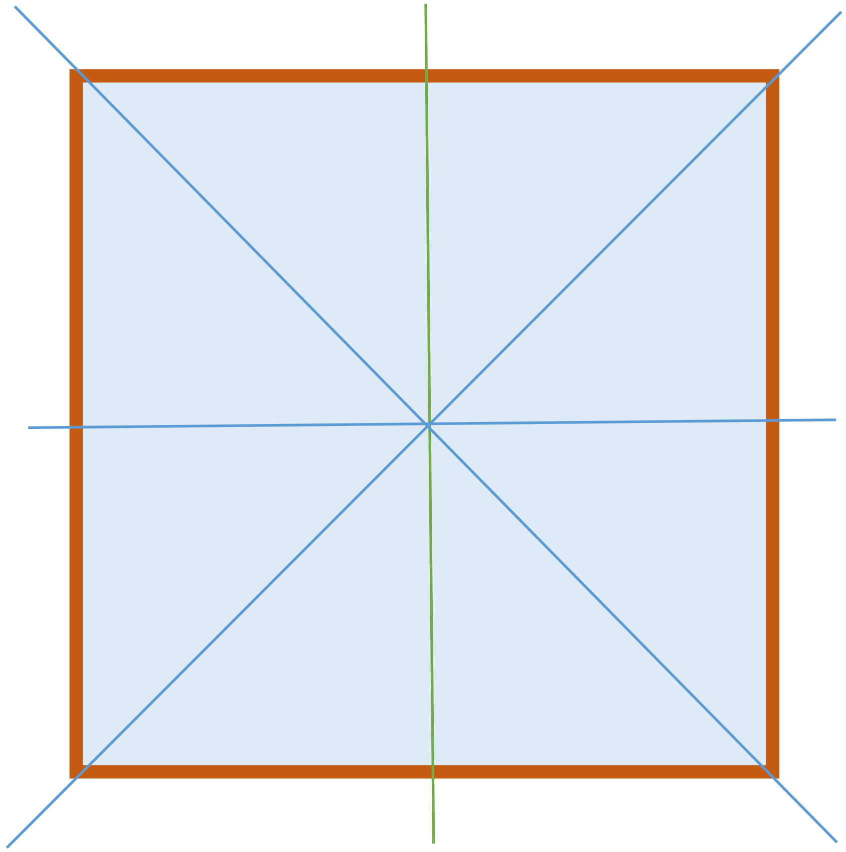
Discrete Symmetry Group



Rotations by 90º, 180º, 270º

and 0º (identity)

Discrete Symmetry Group



Rotations by 90º, 180º, 270º

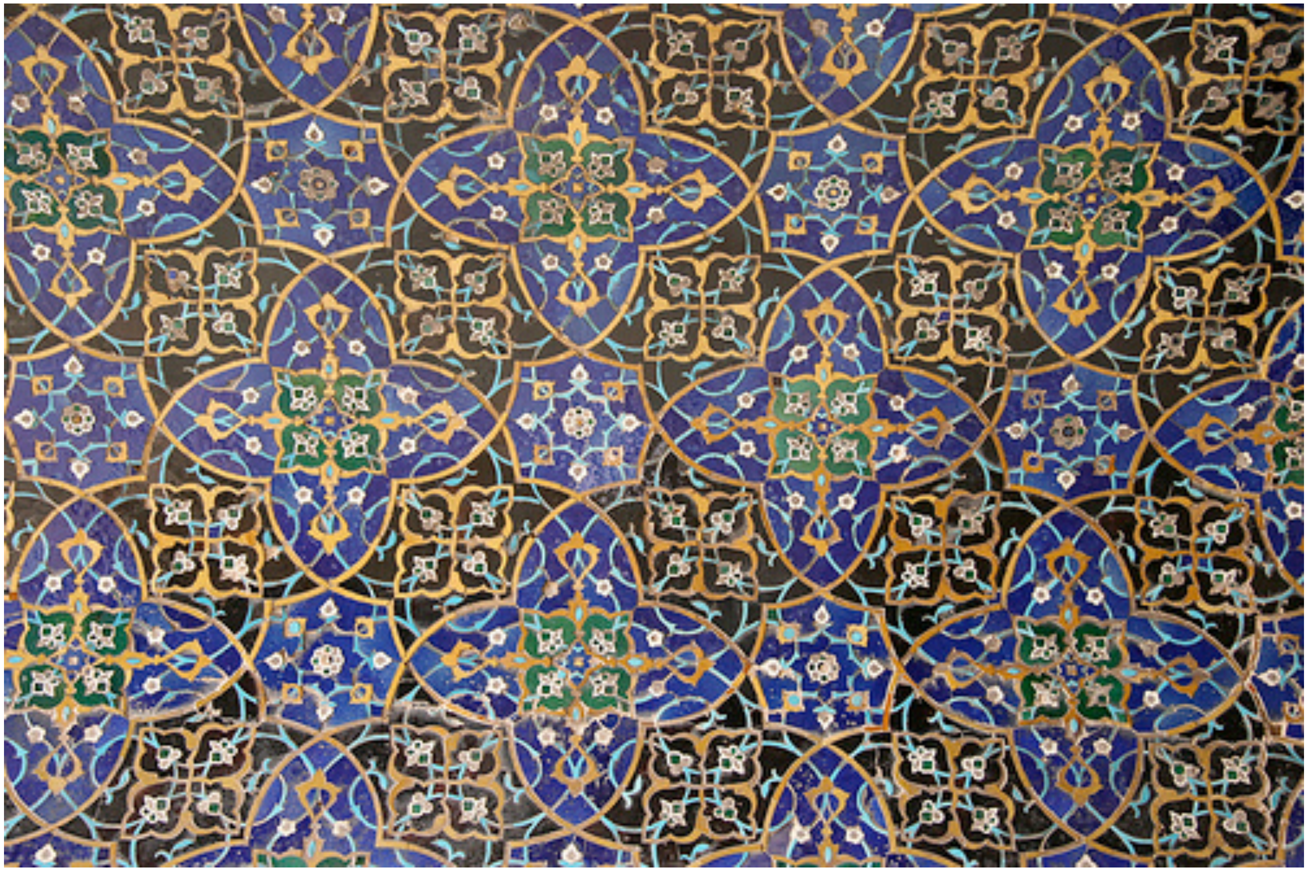
and 0º (identity)

… and 4 reflections
(mirror image)

Wallpaper patterns

17 symmetry types

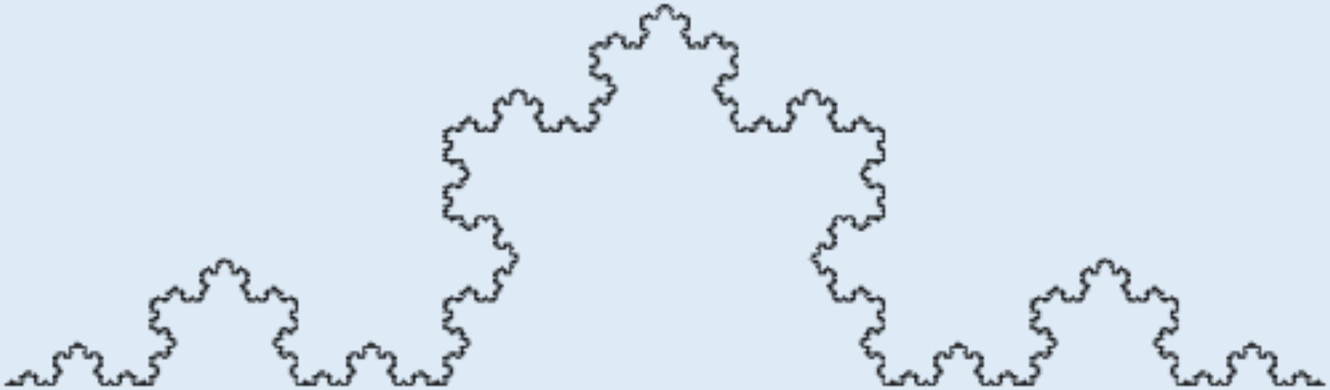
Tilings — *Jameh Mosque, Esfahan, Iran*



Crystallography

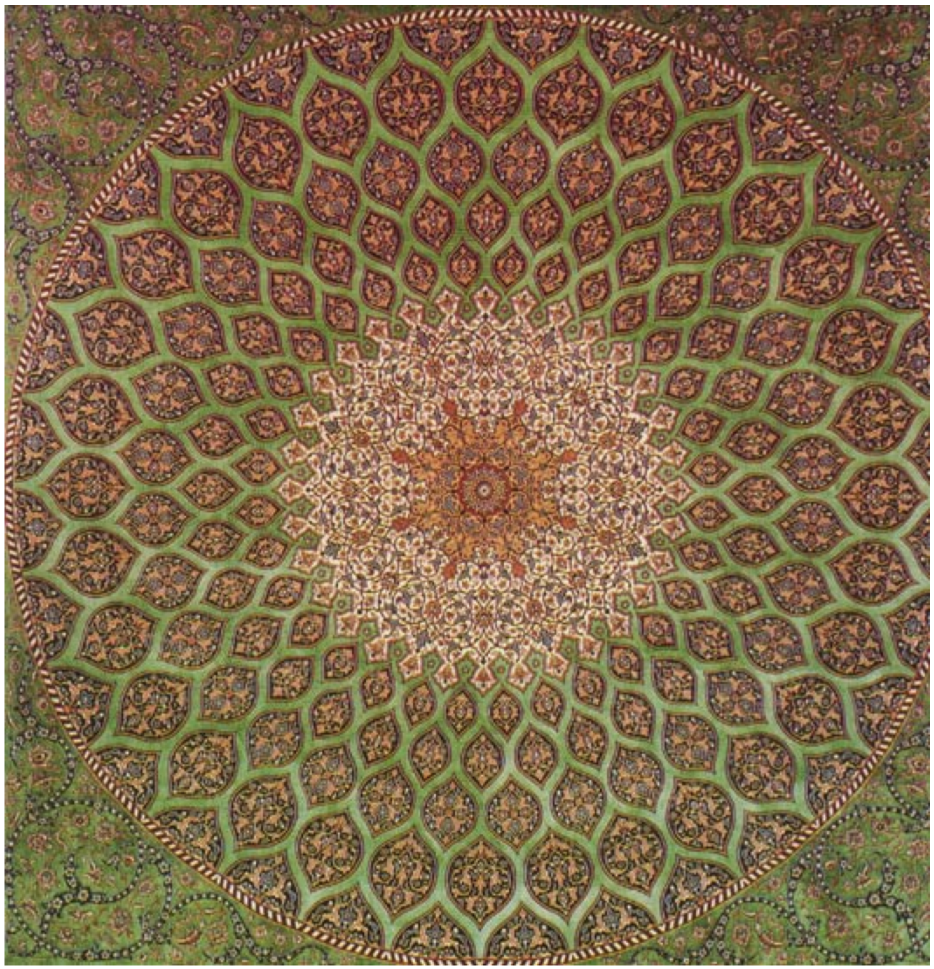
❄ 230 groups

The Koch snowflake — a fractal curve

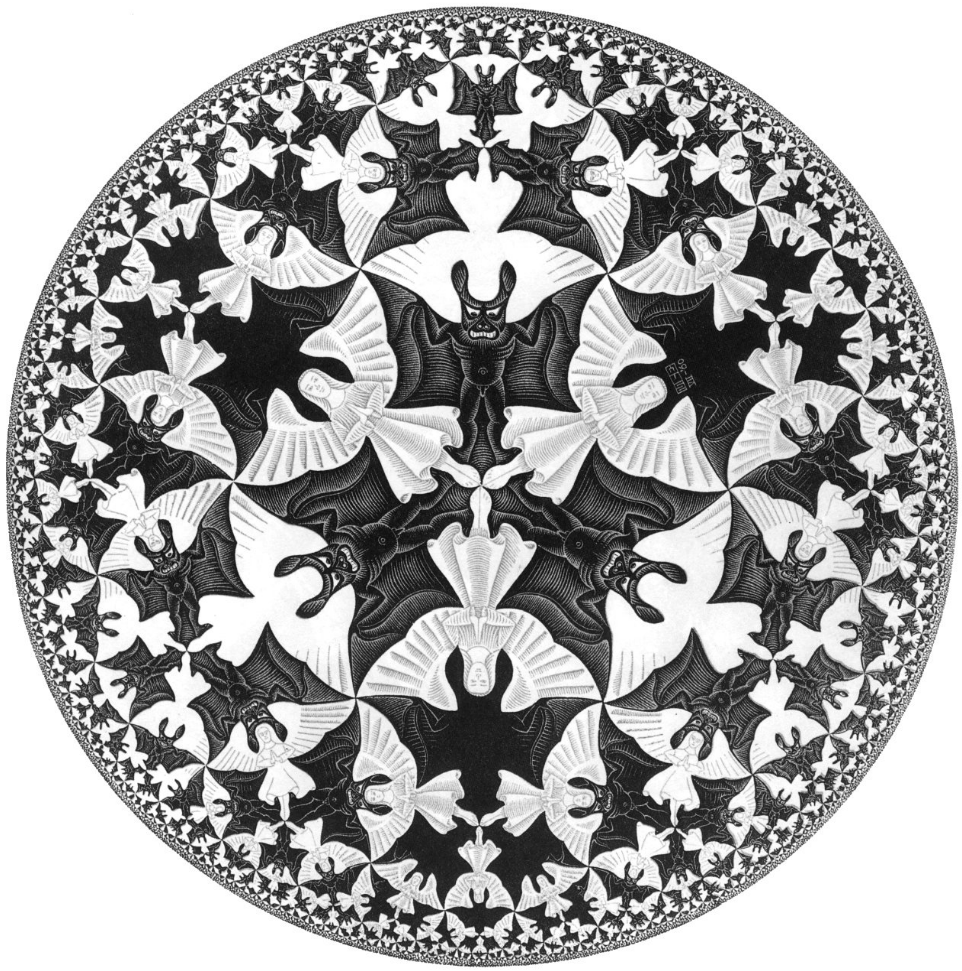


❋❋ Scaling symmetry

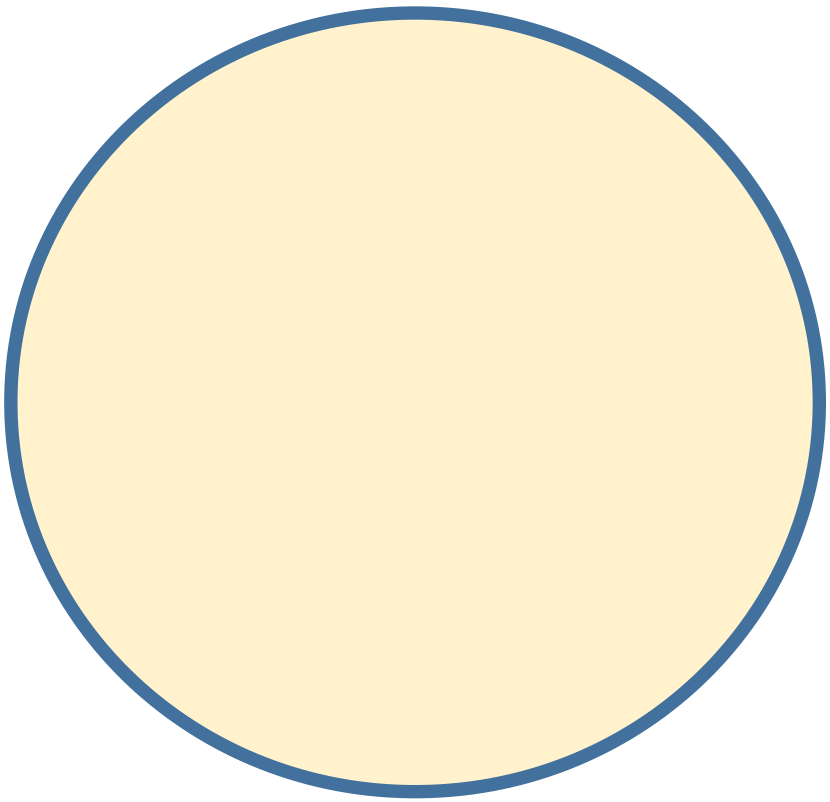
Dome of the Sheikh Lotfollah Mosque – Isfahan, Iran



M.C. Escher — Circle Limit IV

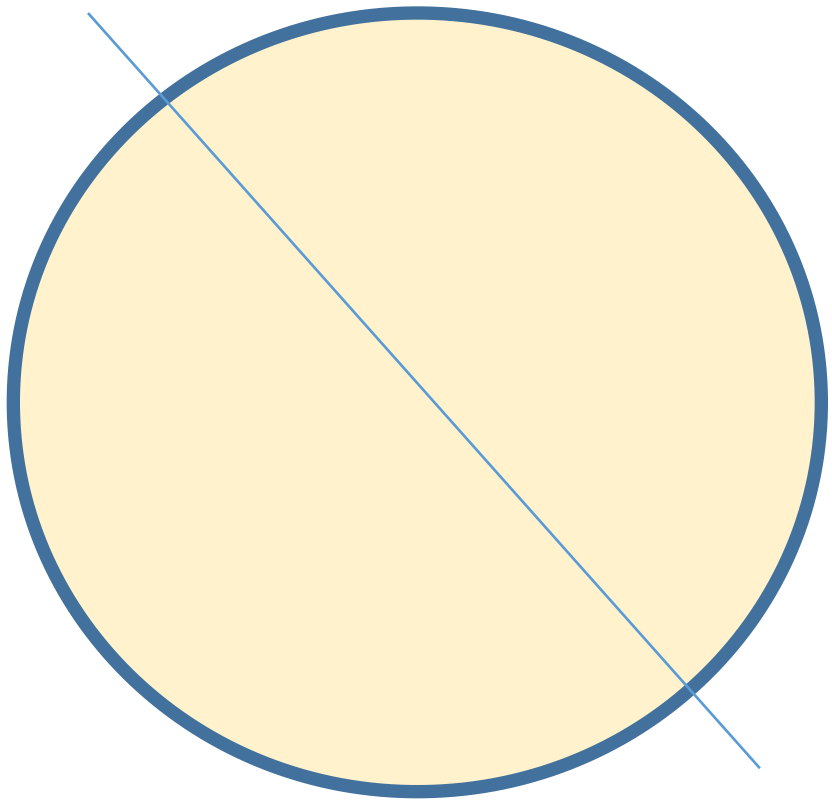


Continuous Symmetry Group



Rotations through any angle

Continuous Symmetry Group



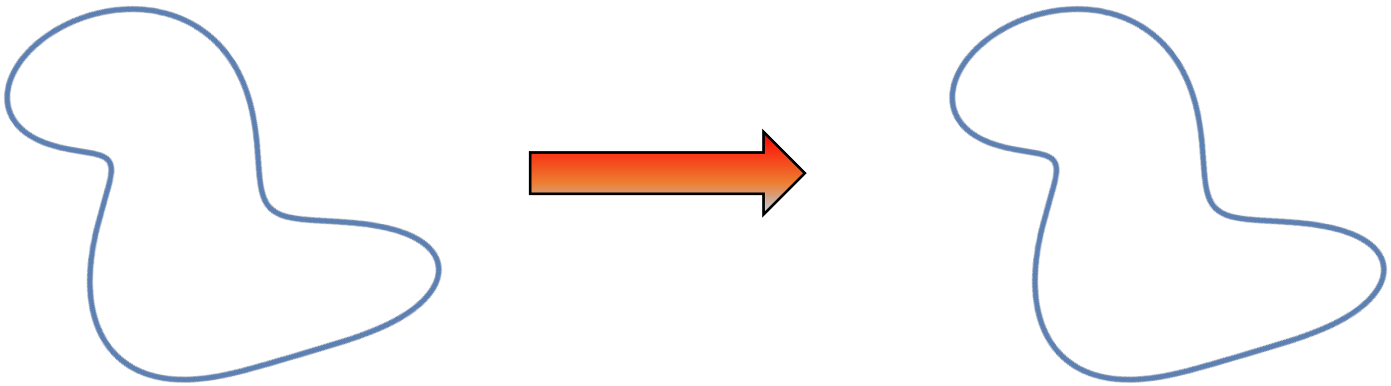
Rotations through any angle

and reflections

A continuous symmetry group is known as a Lie group in honor of the nineteenth century Norwegian mathematician Sophus Lie (Lee)

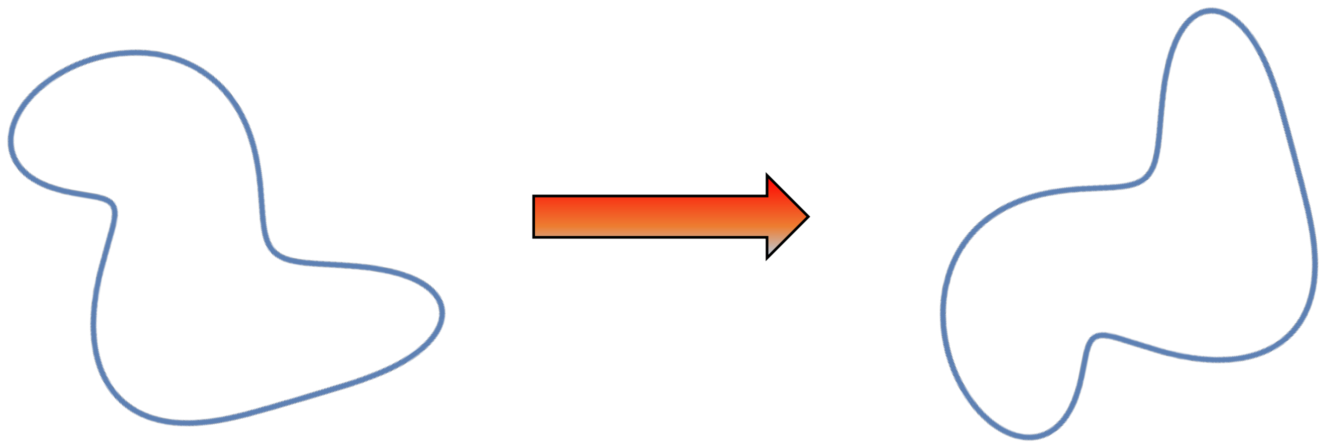
Transformation groups

Translations

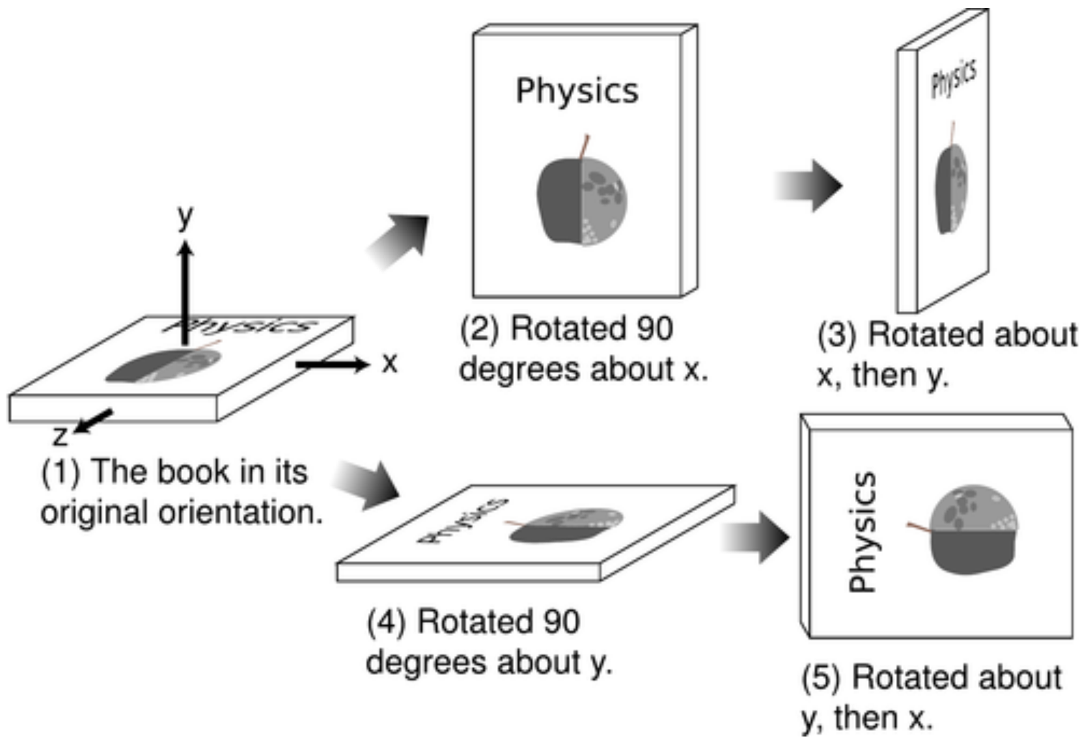


Transformation groups

Rotations

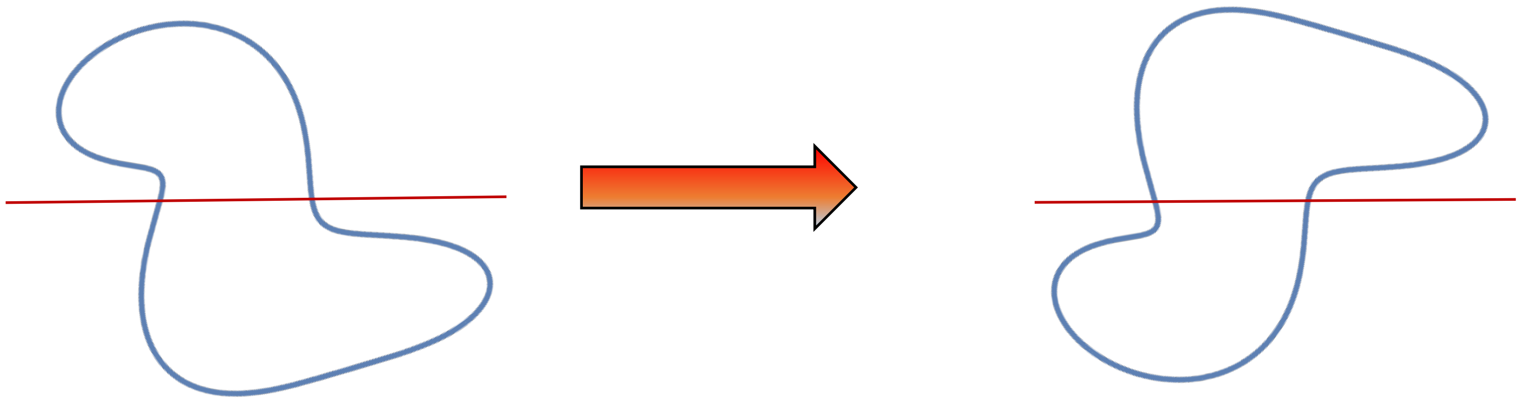


Noncommutativity of 3D rotations — order matters!



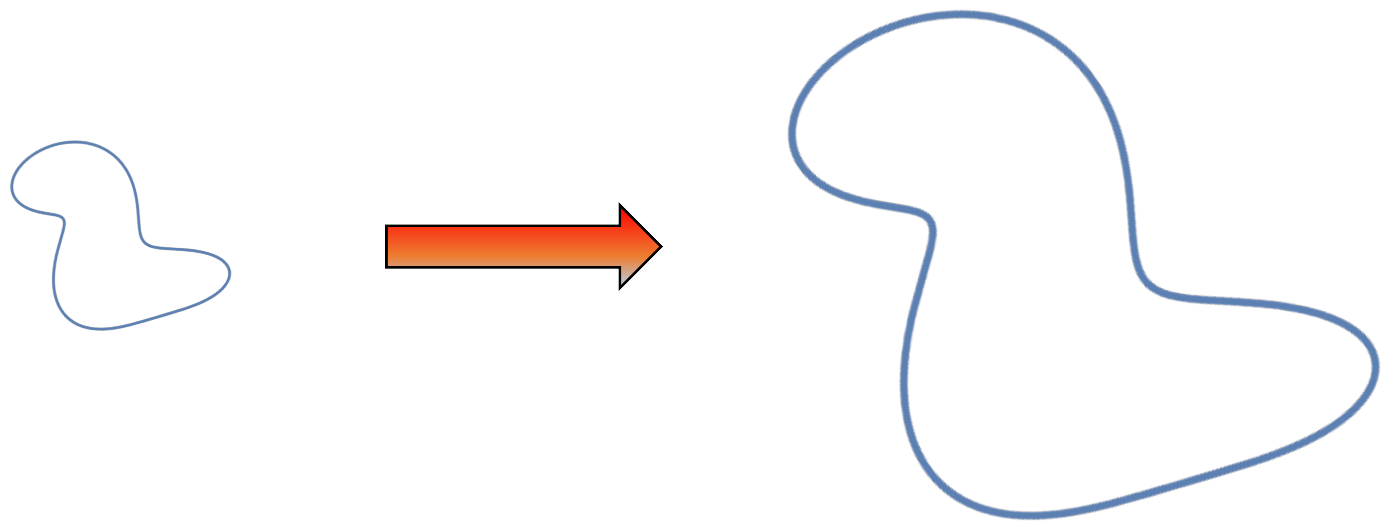
Transformation groups

Reflections



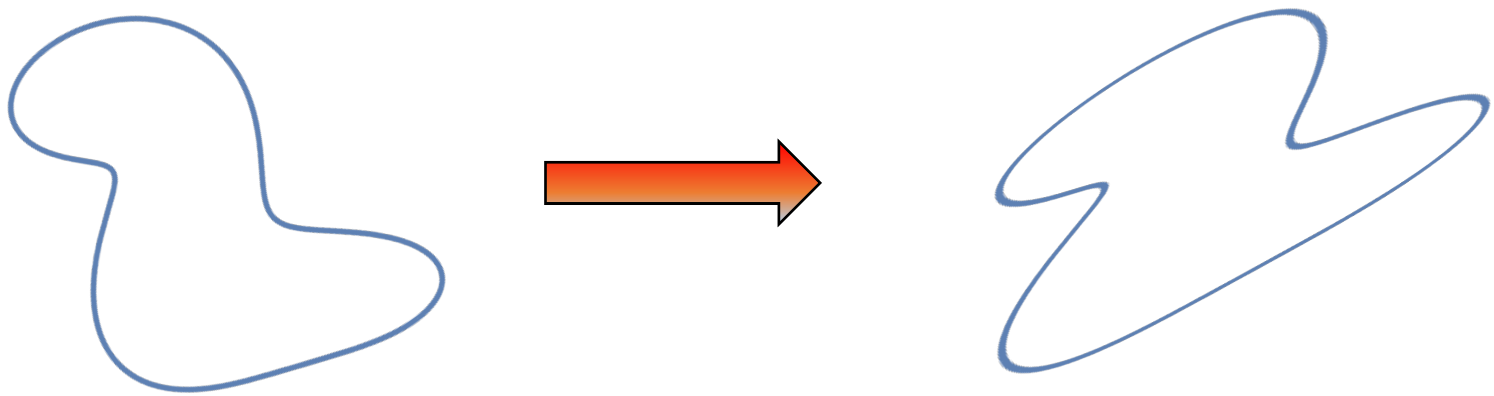
Transformation groups

Scaling (similarity)



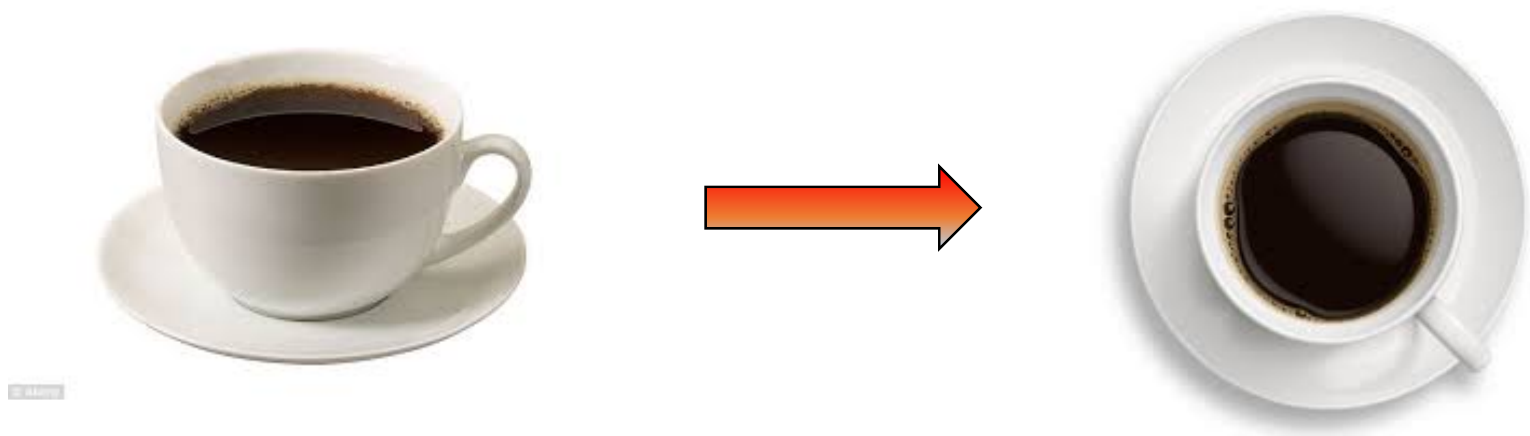
Transformation groups

Projective Transformation

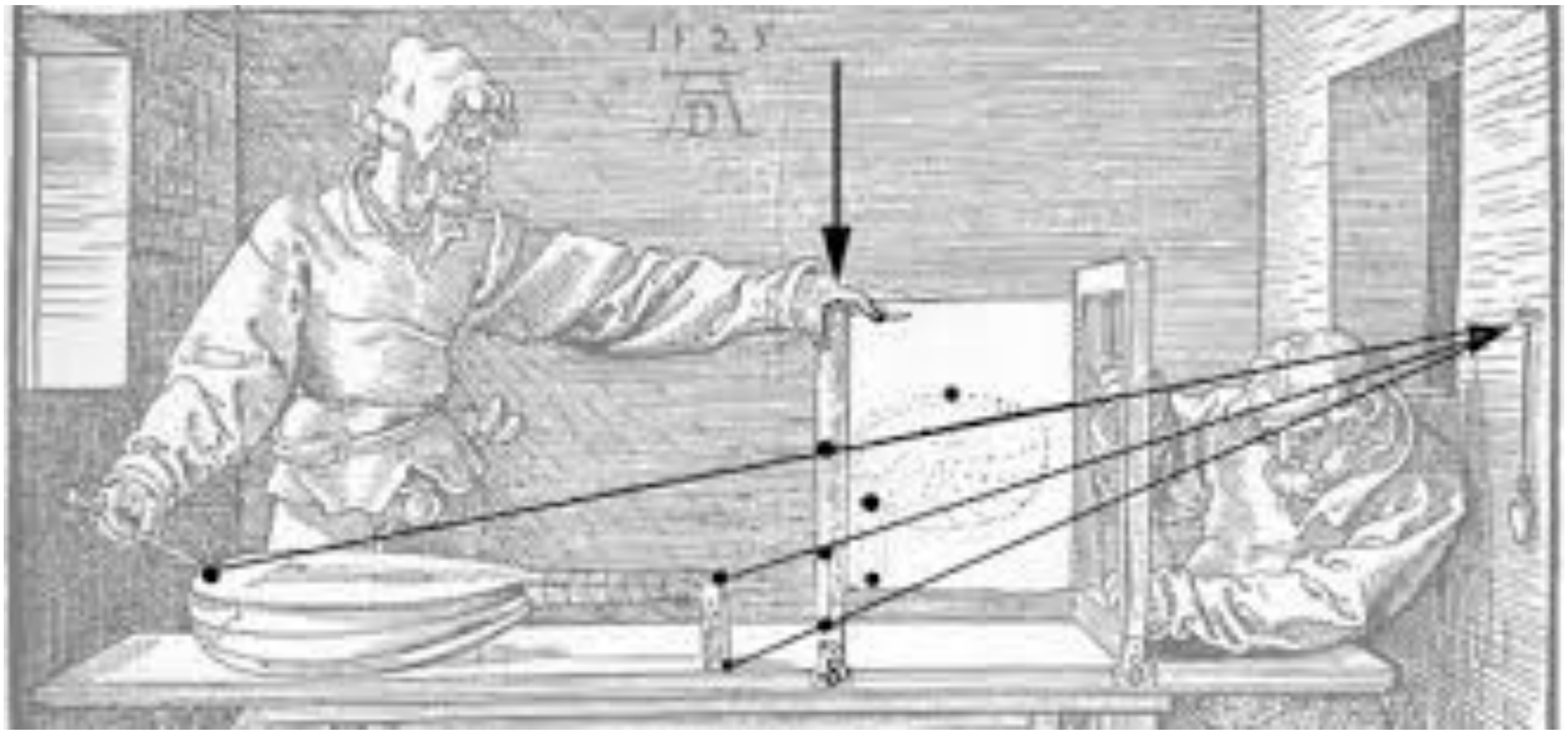


Transformation groups

Projective Transformation

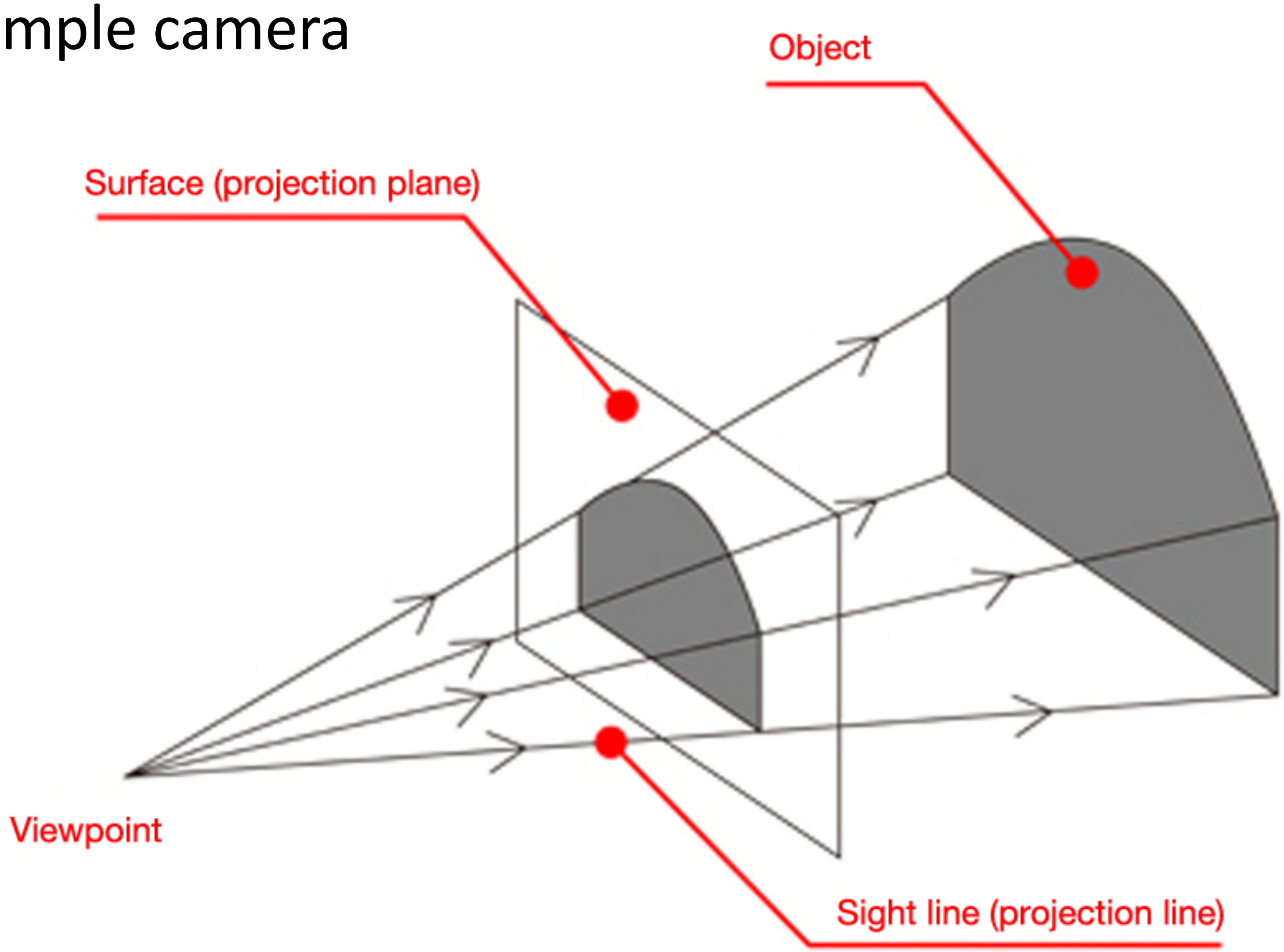


Projective transformations in art and photography



Albrecht Durer — 1500

A simple camera



Musashino Art University

Geometry = Group Theory

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on a corresponding transformation group.

Euclidean geometry:	rigid motions (translations and rotations)
"Mirror" geometry:	translations, rotations, and reflections
Similarity geometry:	translations, rotations, reflections, and scalings
Projective geometry:	all projective transformations

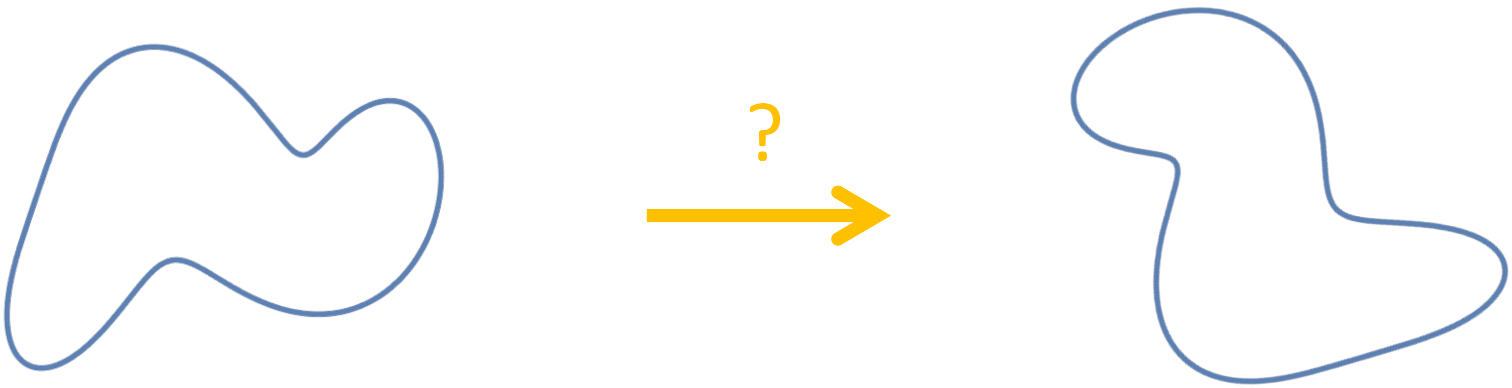
The Equivalence Problem

When are two shapes related by a group transformation?

- Rigid (Euclidean) equivalence
- Similarity equivalence
- Projective equivalence
- etc.

Rigid equivalence

When are two shapes related by a rigid motion?

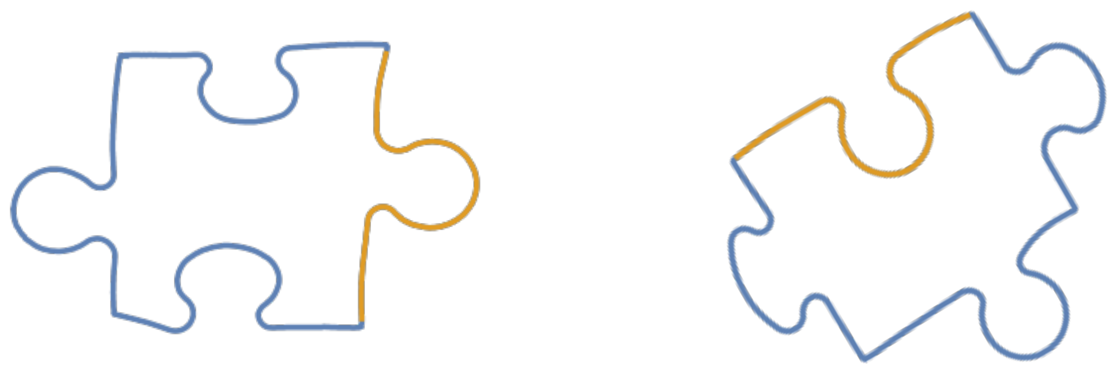


Tennis, anyone?

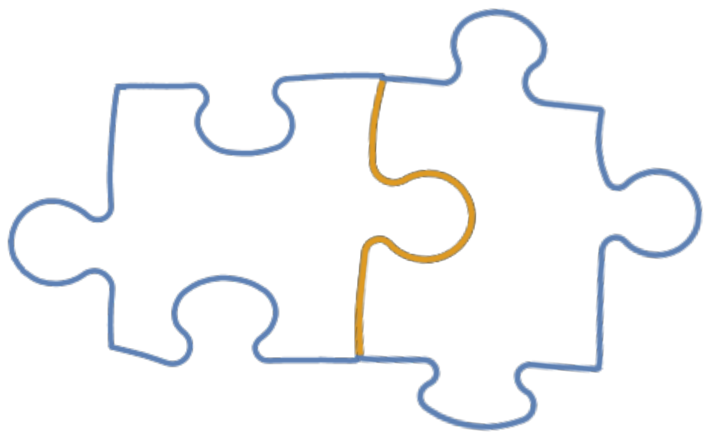
?

☞ Projective equivalence & symmetry

Equivalence of puzzle pieces



Equivalence of puzzle pieces



The Equivalence Problem

When are two shapes related by a group transformation?

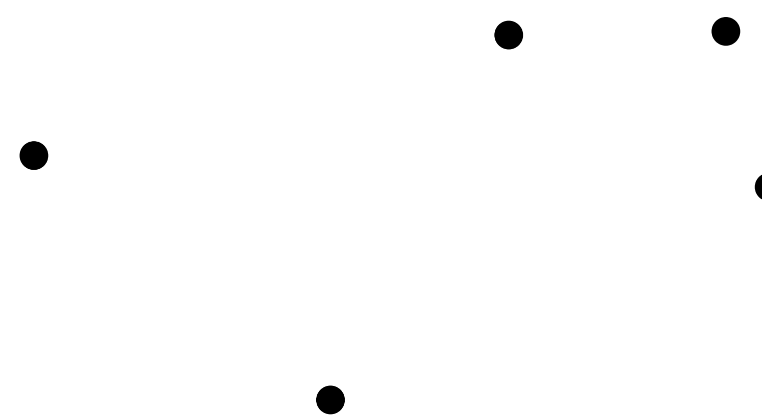
Invariants

✮✮ Solving the equivalence problem requires knowing enough invariants — quantities that are unchanged by the group transformations

Invariants are quantities that are unchanged by the group transformations

★ If two shapes are equivalent, they must have the same invariants.

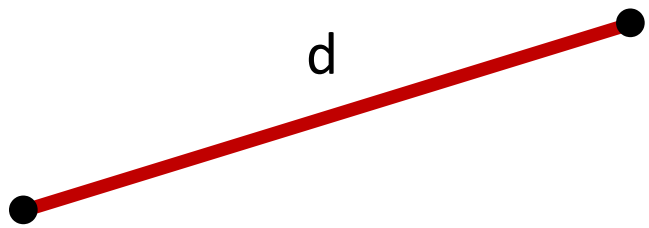
Joint invariants



An invariant that depends on several points is known as a joint invariant

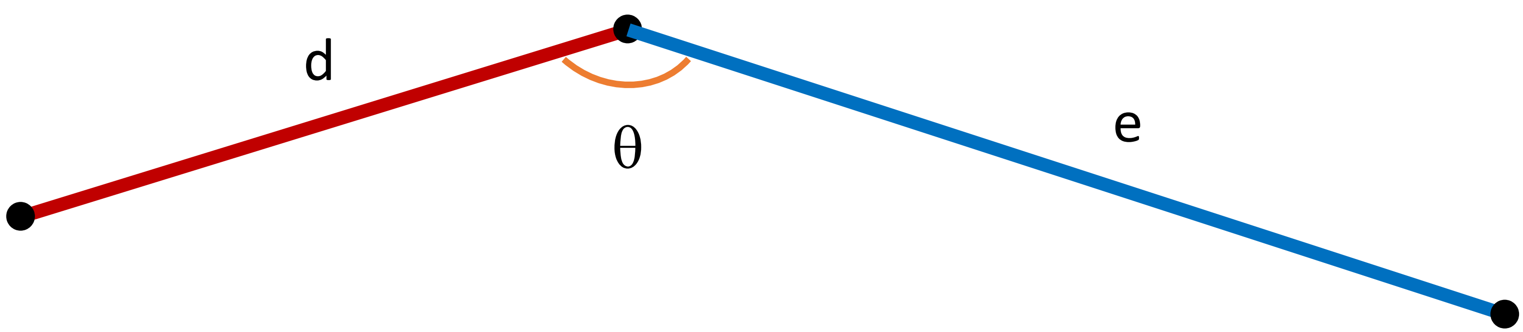
Joint invariants

Rigid motions: distance between two points



Joint invariants

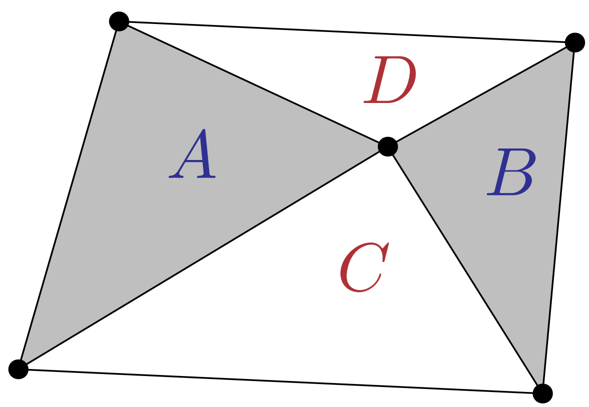
Similarity group: ratios of distances R and angles θ



$R = d/e$

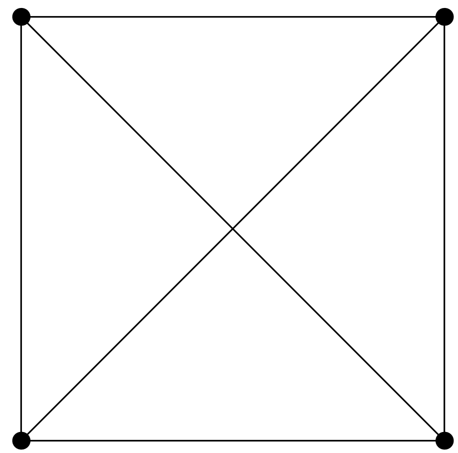
Joint invariants

Projective group: ratios of 4 areas



$$\frac{A\,B}{C\,D}$$

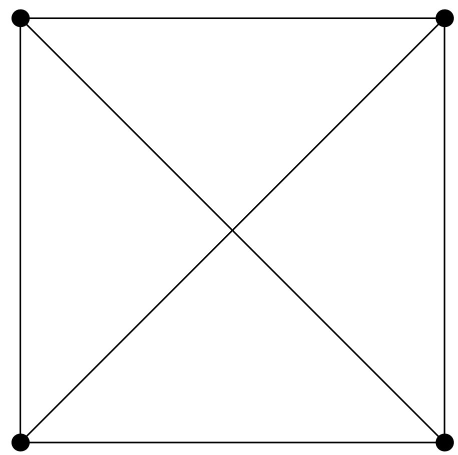
Distances between multiple points



$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

The Distance Histogram —

invariant under rigid motions



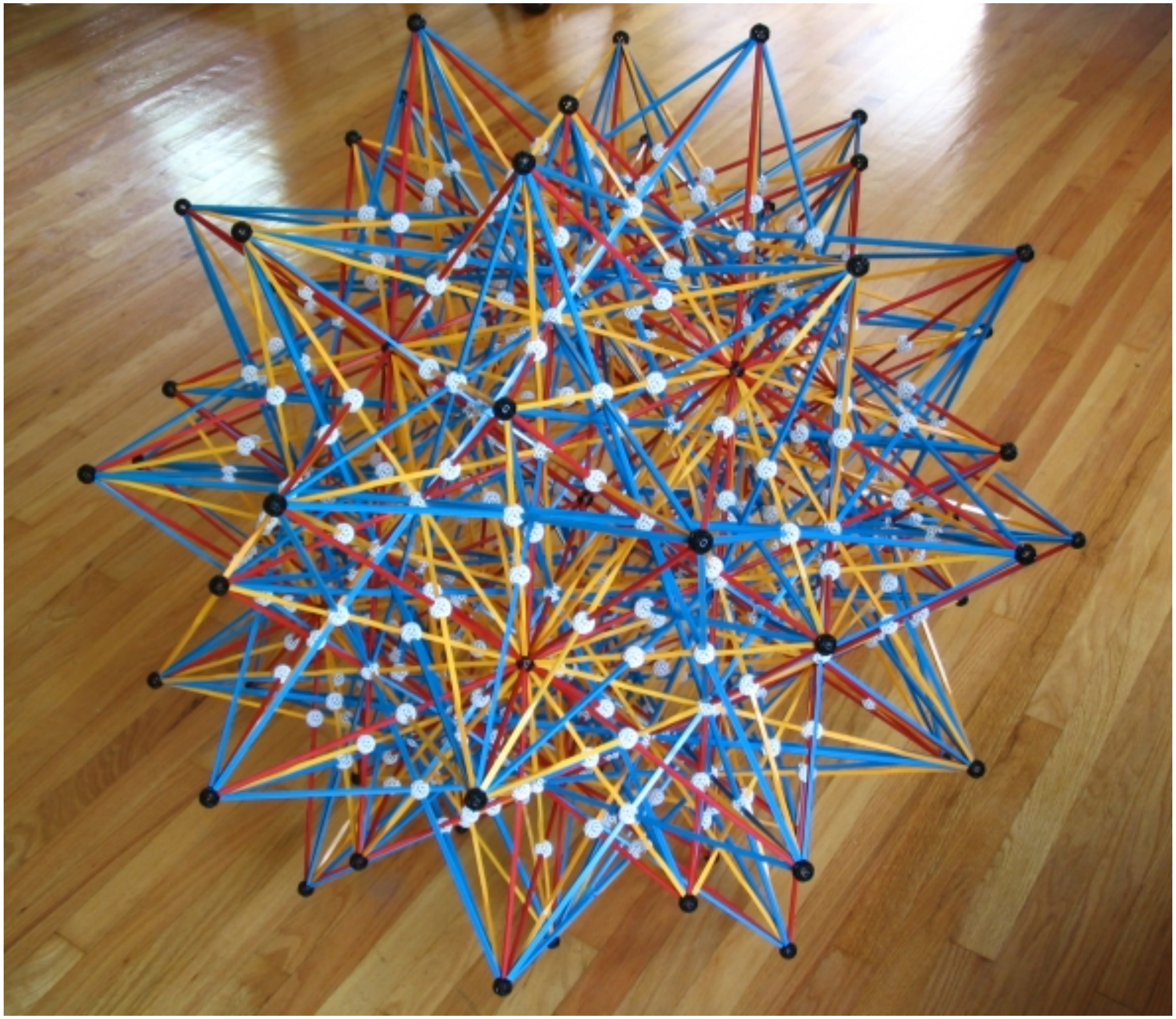
$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

If two sets of points are equivalent up to rigid motion, they have the same distance histogram

Does the distance histogram uniquely determine a set of points up to rigid motion?

The Tinkertoy Problem

The Tinkertoy Problem



Zome System

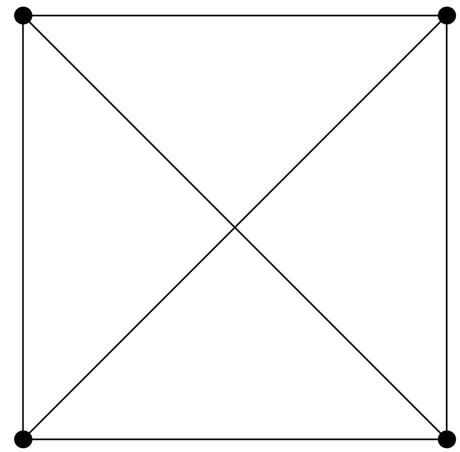
✷✷ David Richter

Does the distance histogram uniquely determine a set of points up to rigid motion?

Answer: Yes for most sets of points, but there are some exceptions!

☆☆ Mireille (Mimi) Boutin and Gregor Kemper (2004)

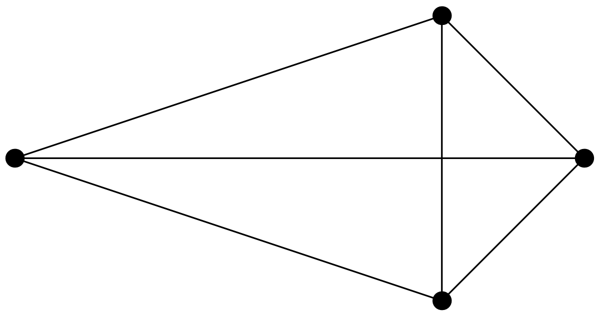
Yes:



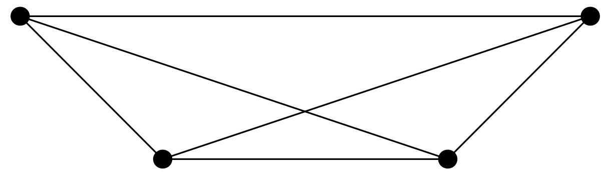
$$1, \quad 1, \quad 1, \quad 1, \quad \sqrt{2}, \quad \sqrt{2}.$$

No:

Kite



Trapezoid



$$\sqrt{2}, \quad \sqrt{2}, \quad 2, \quad \sqrt{10}, \quad \sqrt{10}, \quad 4.$$

Distance histogram for points on a line

Does the distance histogram uniquely determine a set of points up to rigid motion?

Distance histogram for points on a line

No:

$$P = \{0, 1, 4, 10, 12, 17\}$$

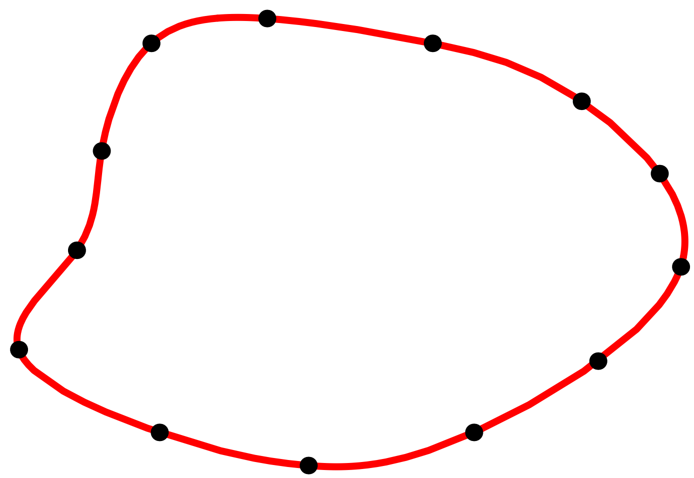
$$Q = \{0, 1, 8, 11, 13, 17\}$$

$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

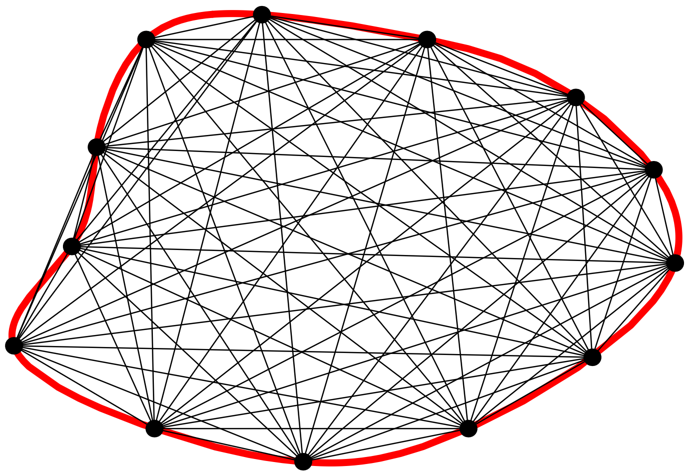
$\Longrightarrow$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Limiting Curve Histogram

Limiting Curve Histogram

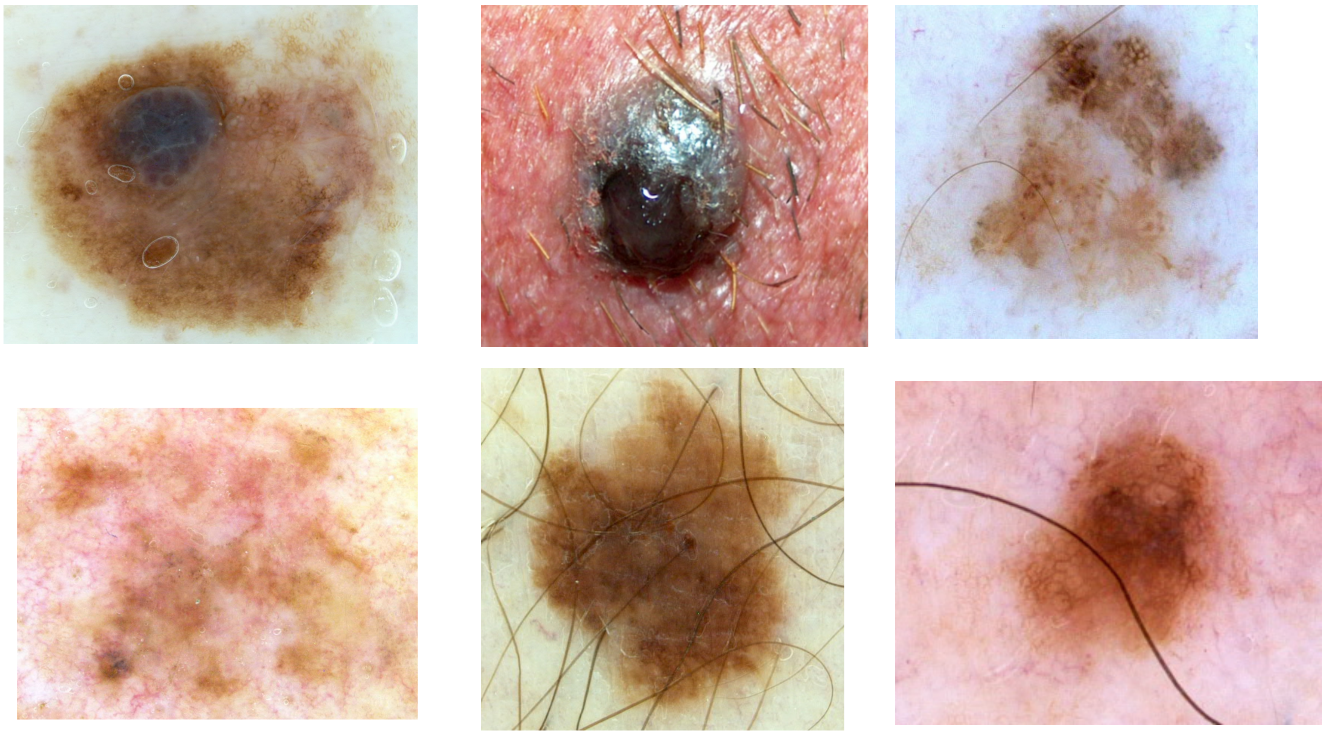


Limiting Curve Histogram



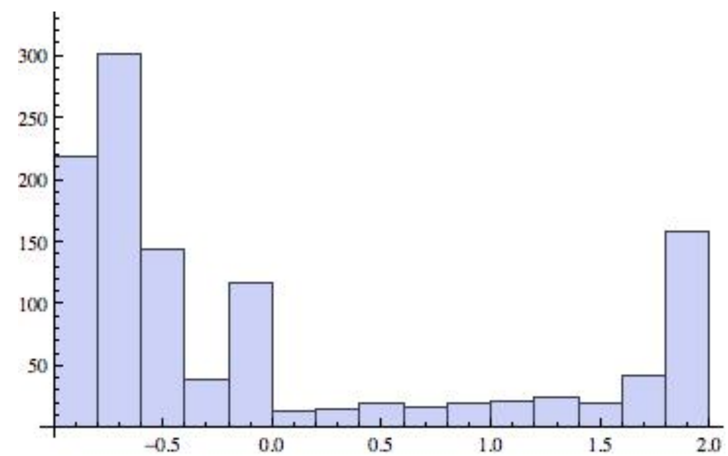
Brinkman, D., and Olver, P.J., Invariant histograms, Amer. Math. Monthly 119 (2012), 4-24

Distinguishing Moles from Melanomas

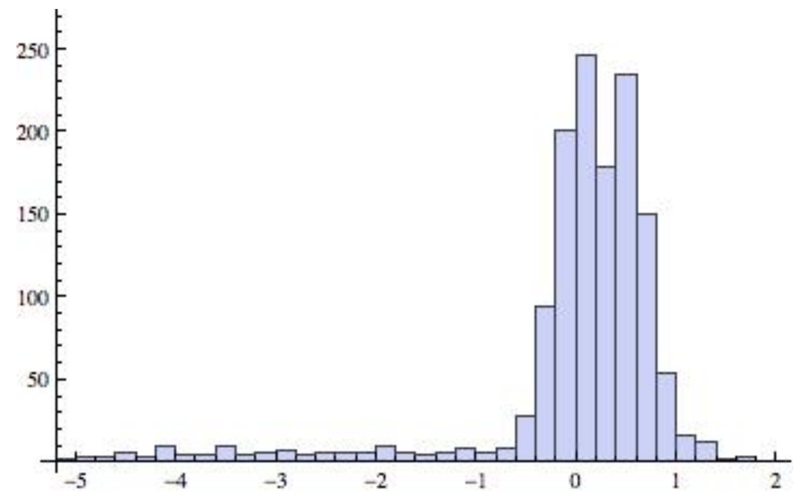


- Anna Grim and Cheri Shakiban, 2015

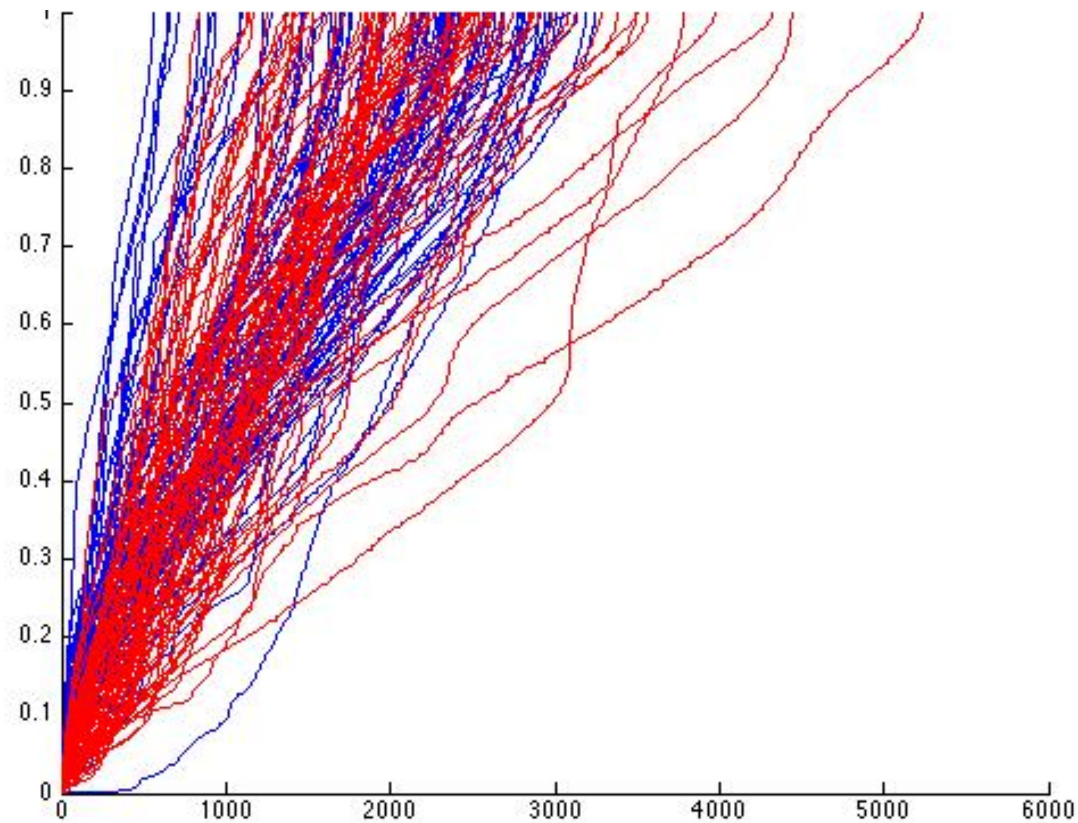
Distance Histogram — Melanoma



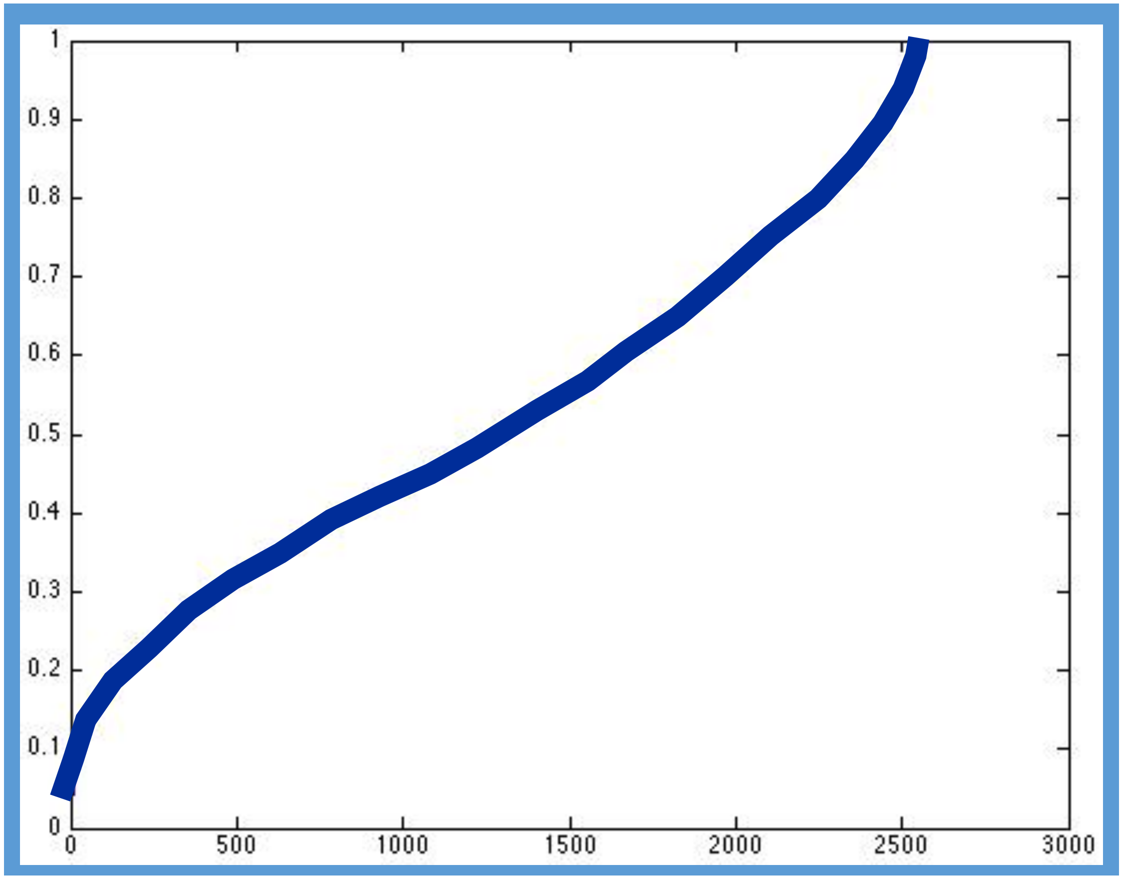
Distance Histogram — Mole



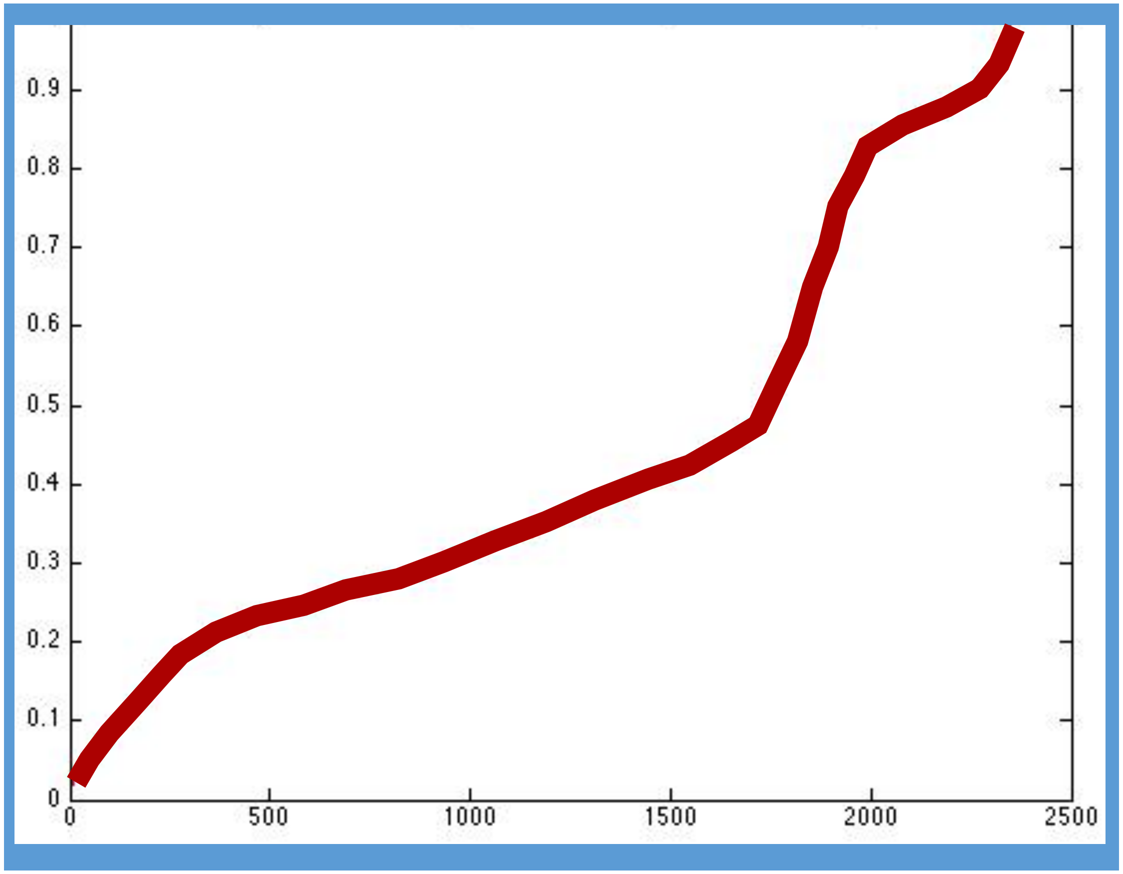
CUMULATIVE HISTOGRAM:
Mole versus Melanoma



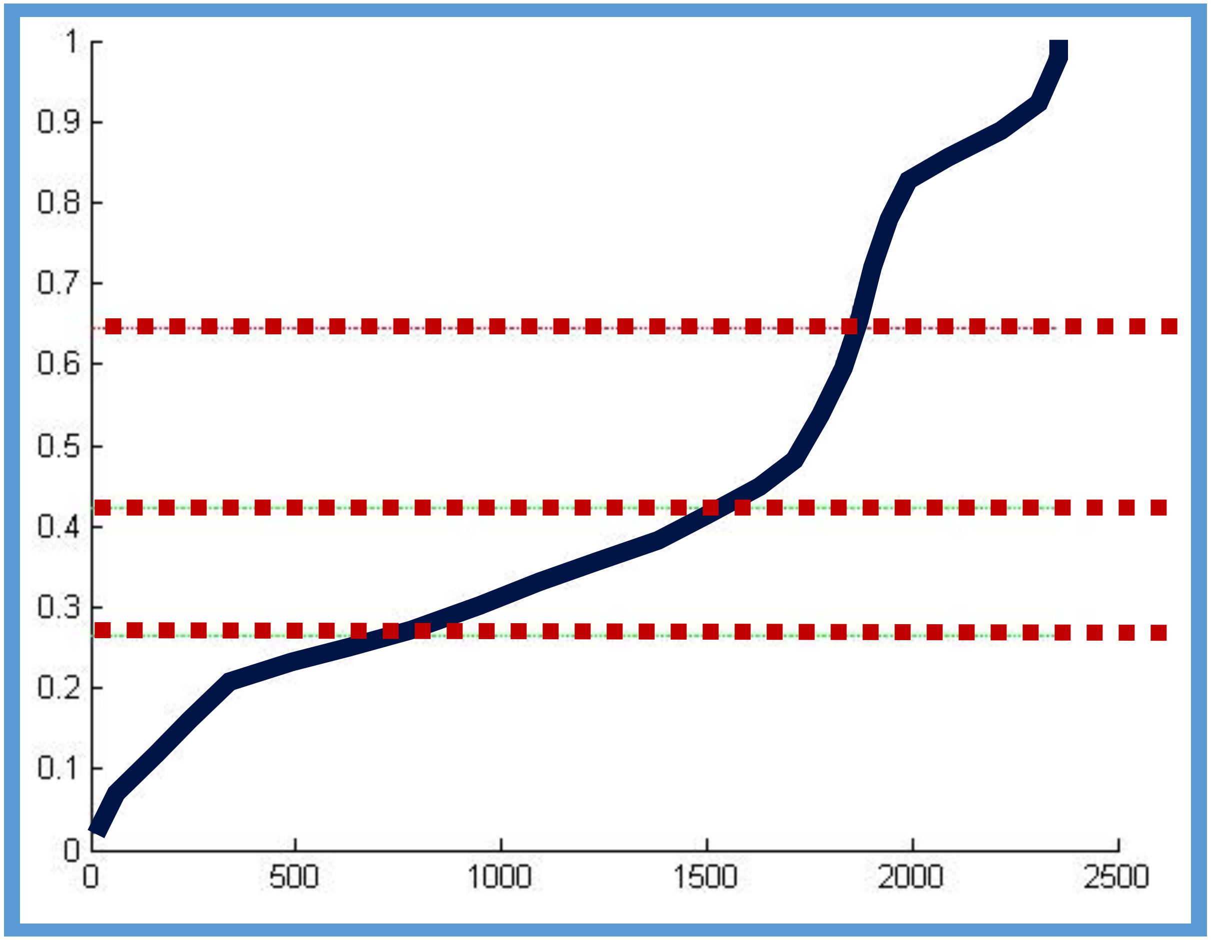
TYPICAL MOLE CUMULATIVE HISTOGRAM

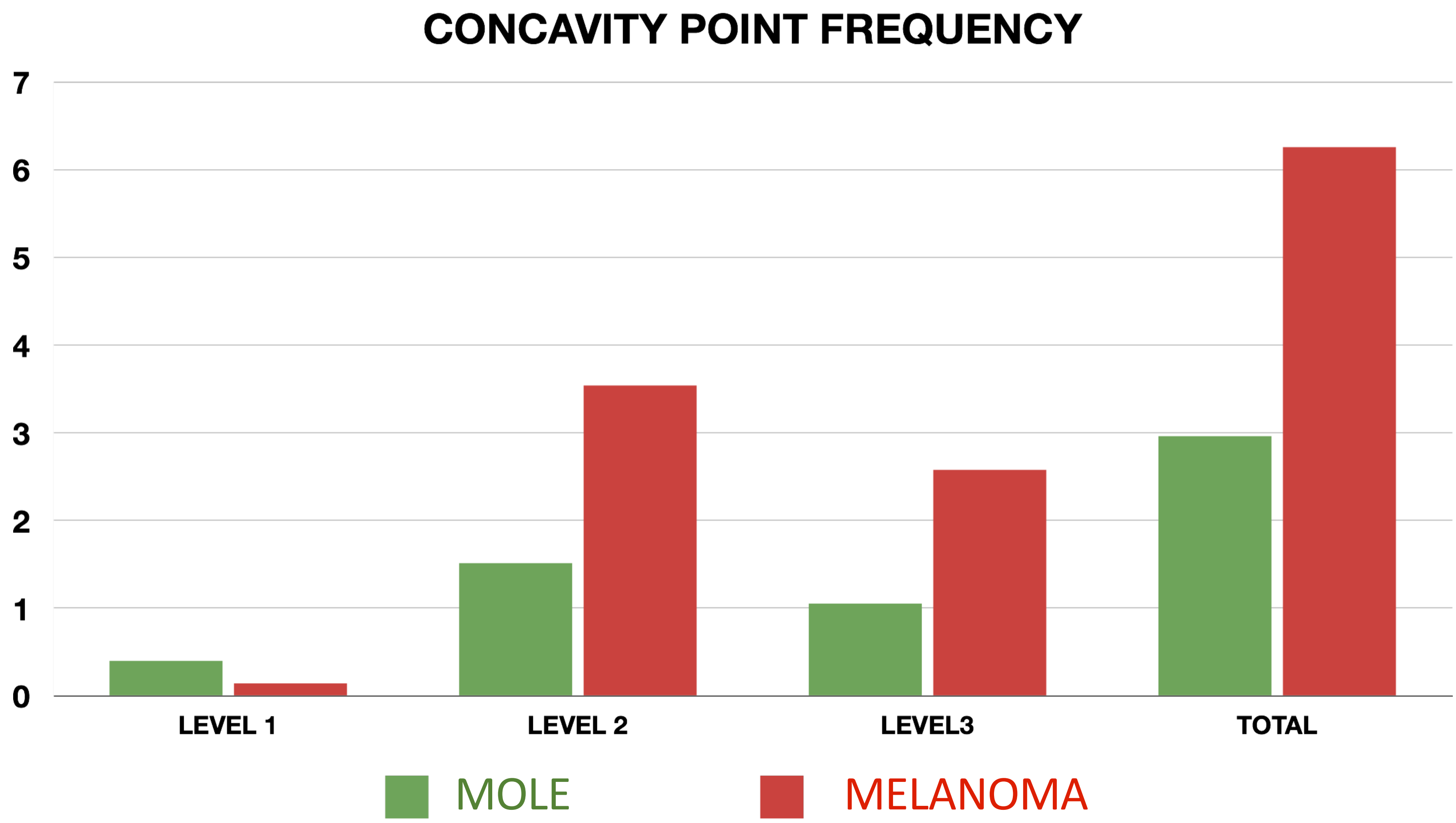


TYPICAL MELANOMA CUMULATIVE HISTOGRAM



CONCAVITY POINT ANALYSIS




CONCAVITY POINT FREQUENCY
7
6
5
4
3
2
1
0
LEVEL 1
LEVEL 2
LEVEL3
TOTAL
MOLE
MELANOMA

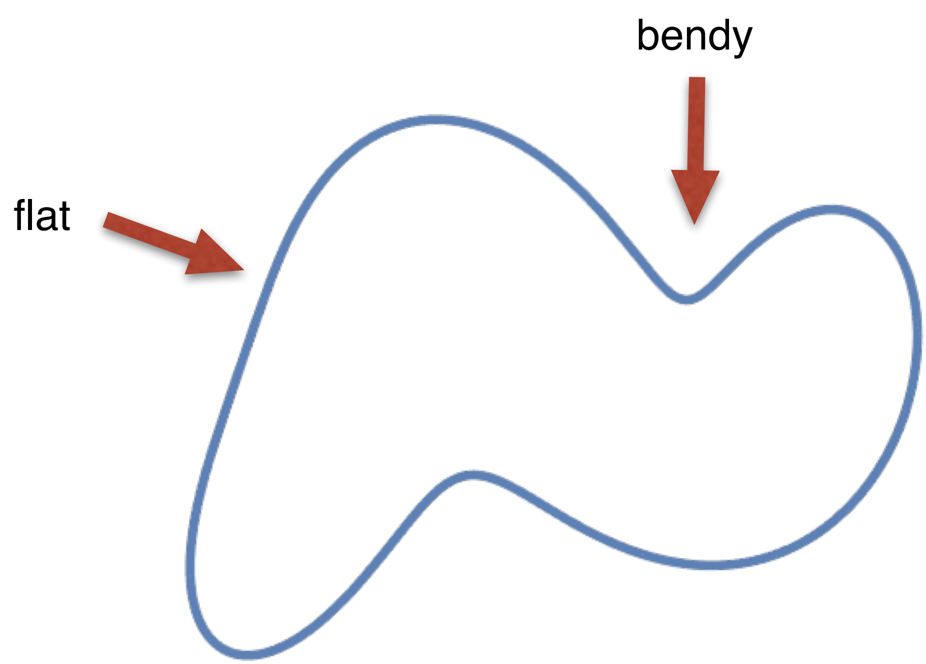
For smooth objects — curves, surfaces, etc.,

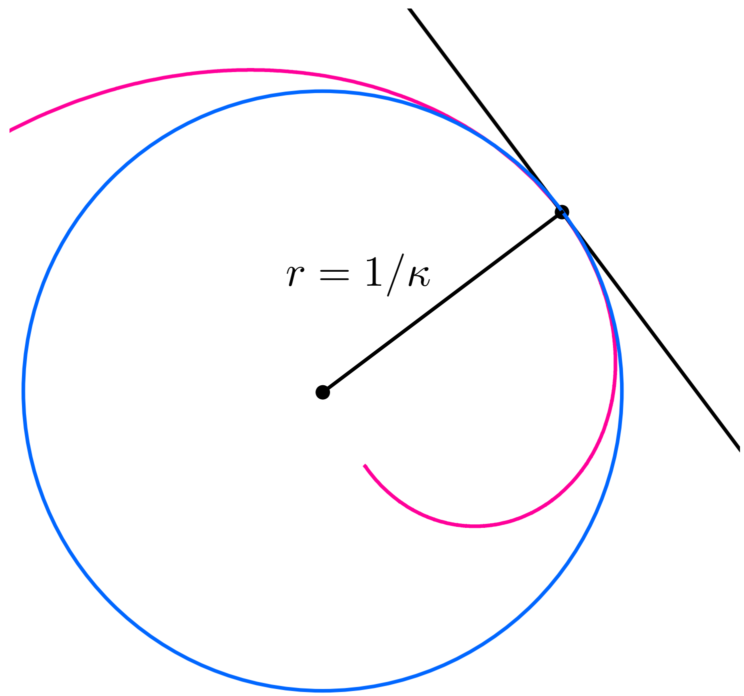
we need to use calculus to find

Differential Invariants

A Differential Invariant

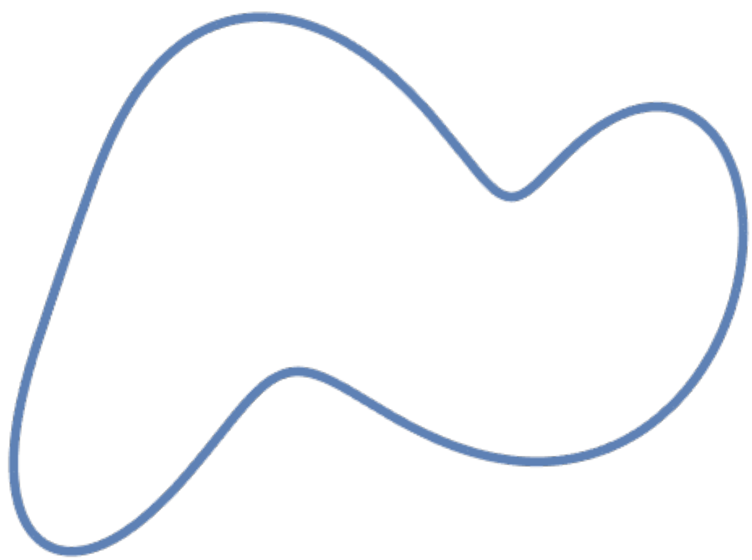
Curvature is a measure of “bendiness”.



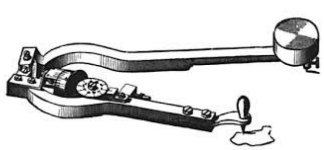


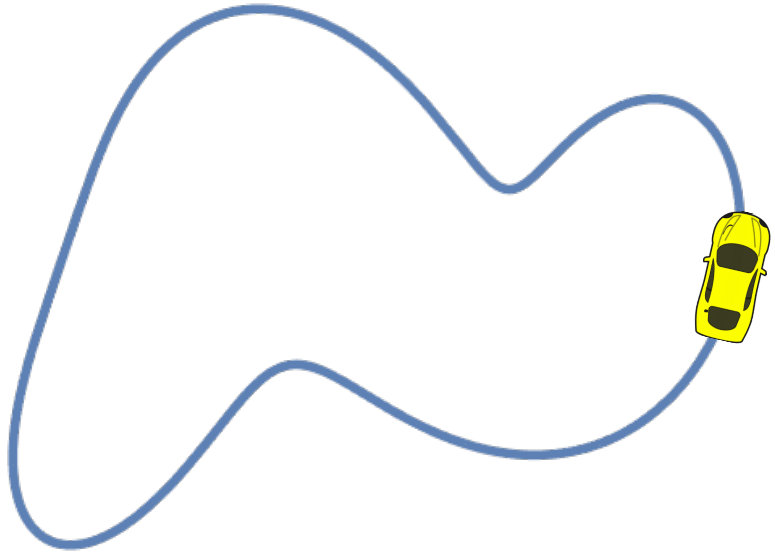
Curvature = reciprocal of radius of osculating circle

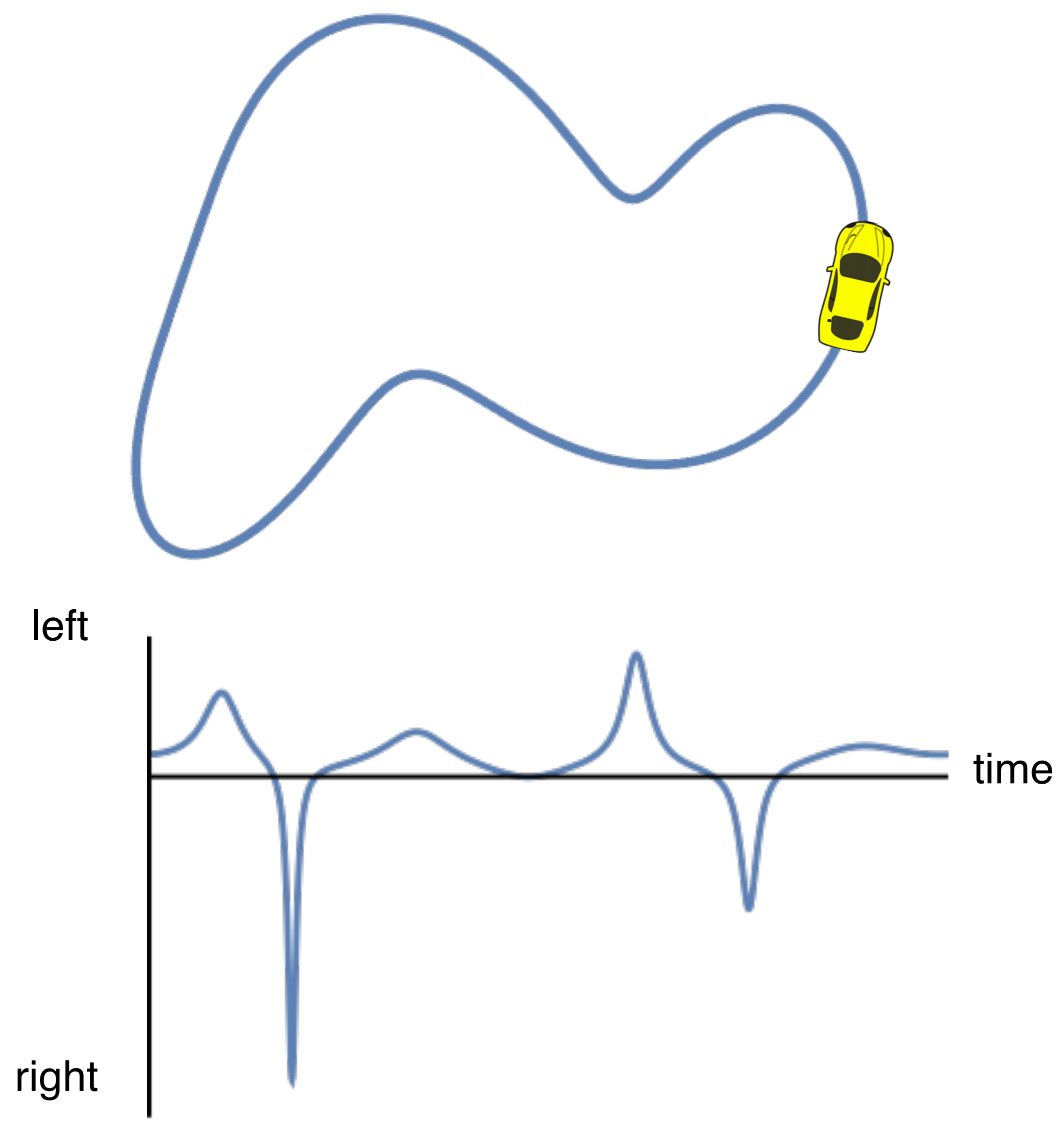
Curvature is a measure of "bendiness".



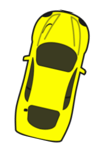
What everyday device can measure curvature?

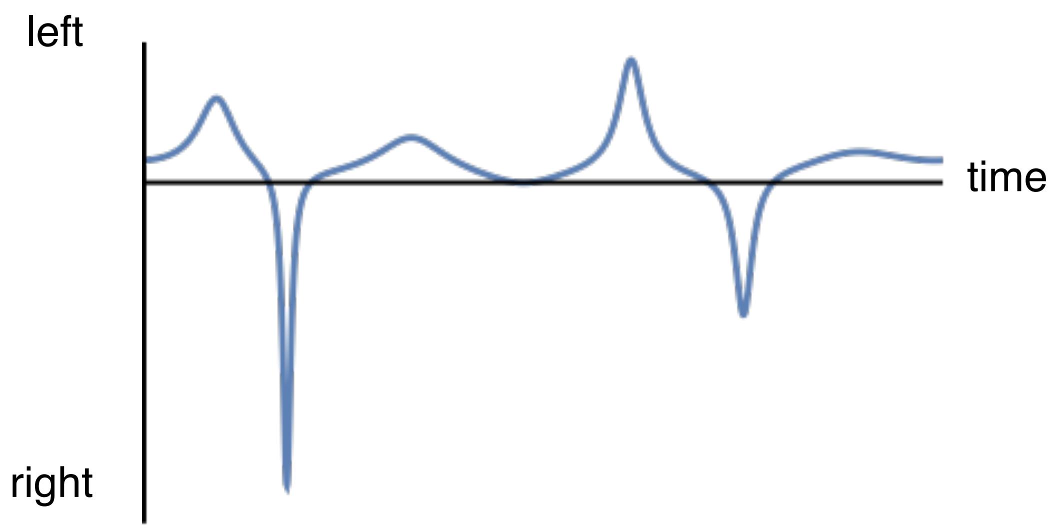





left
right
time

Can you reconstruct the racetrack?





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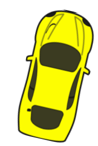
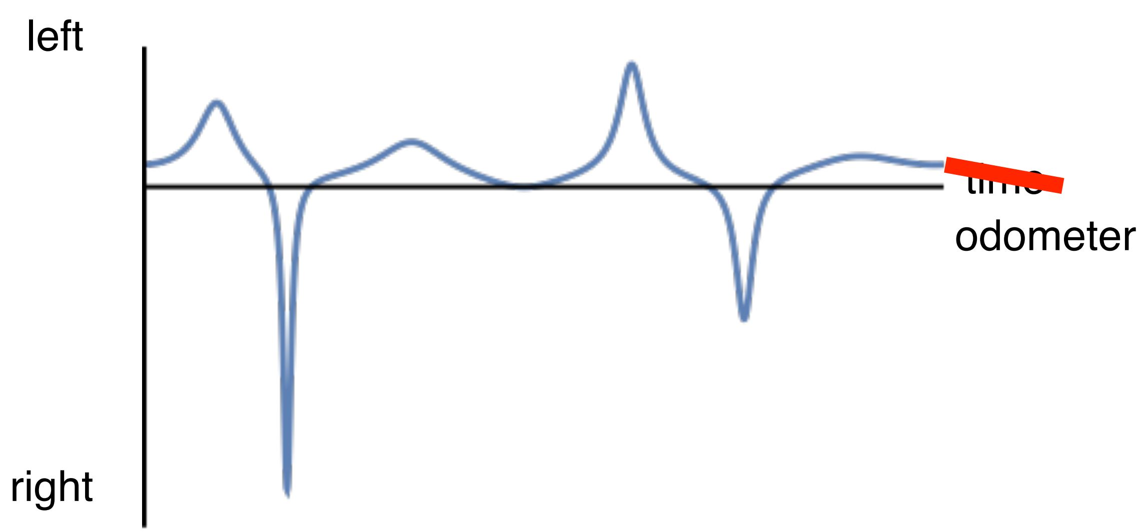
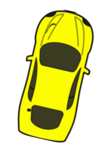


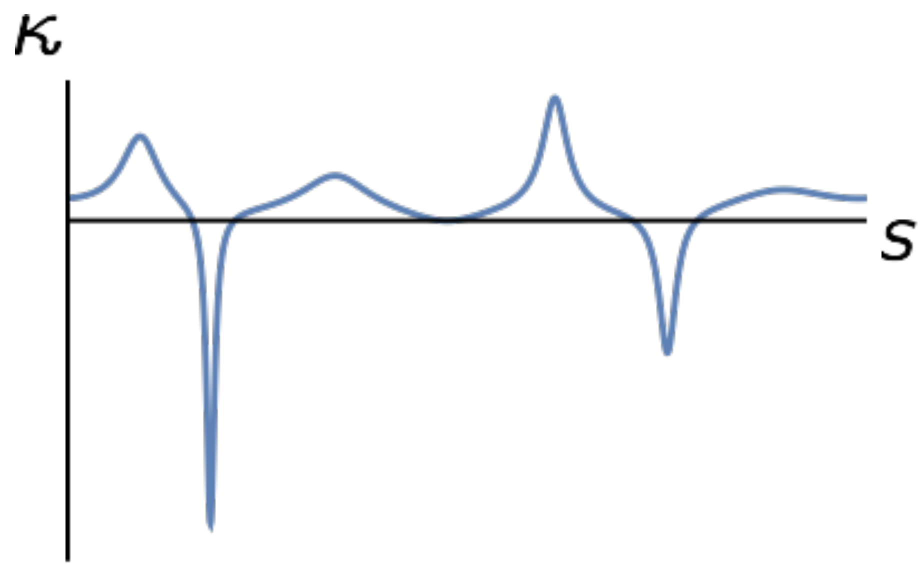


Can you reconstruct the racetrack?

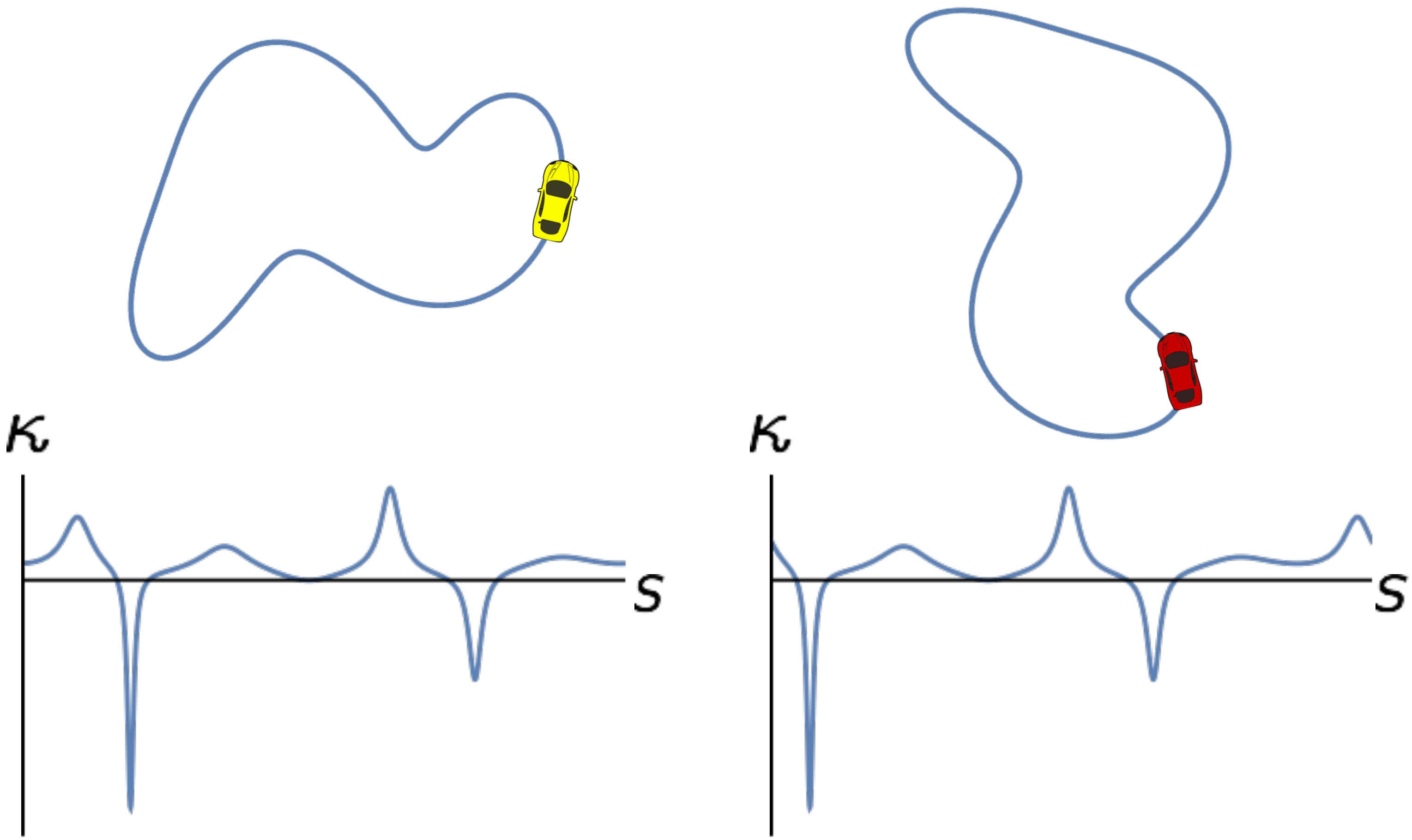


κ is (Euclidean) curvature

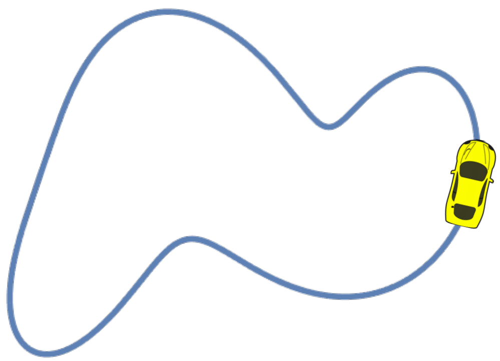
s is (Euclidean) arclength

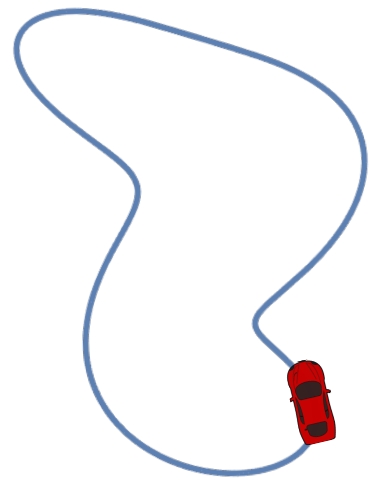


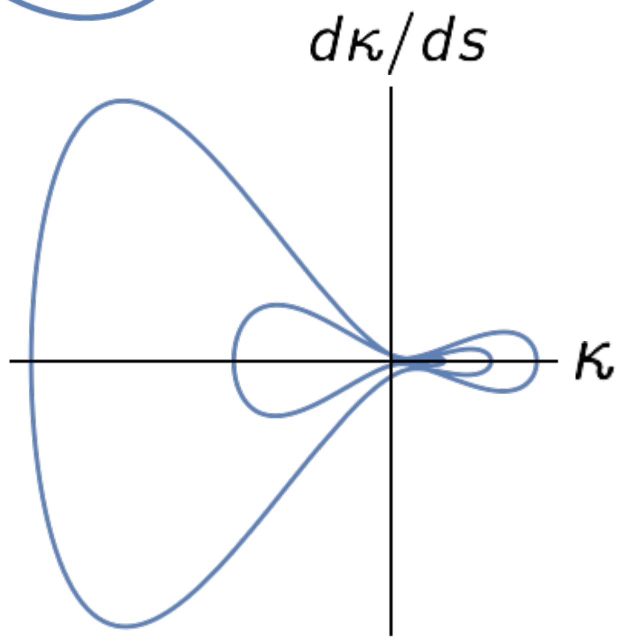
Racetrack comparison problem



Racetrack comparison problem





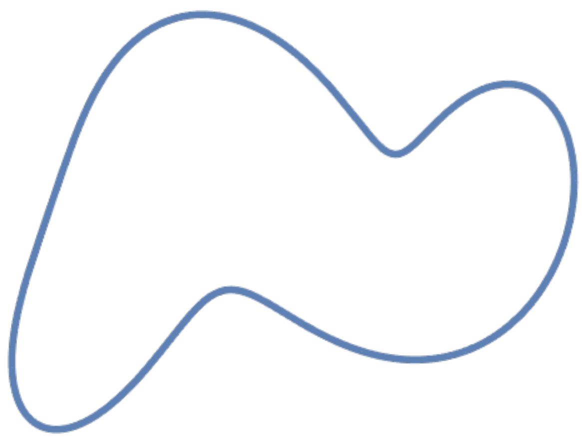


$d\kappa/ds$

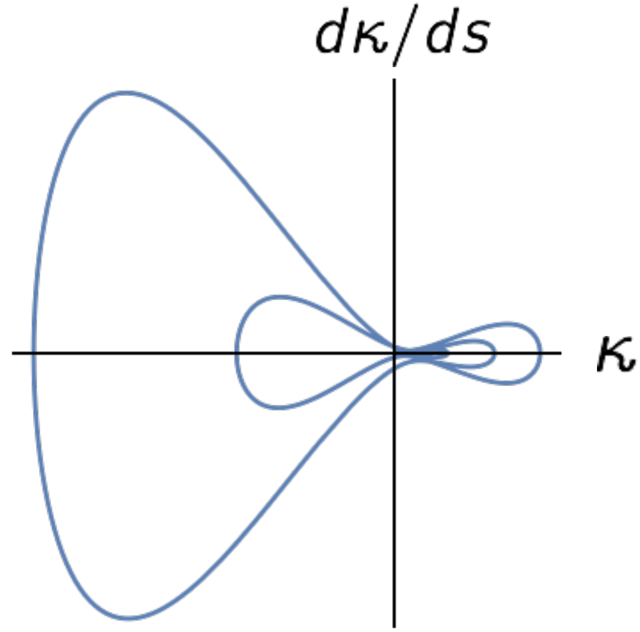
= rate of change (derivative) of curvature

The Invariant Signature

The invariant signature of a planar curve is the set traced out by curvature and the rate of change of curvature (its arclength derivative).



original curve



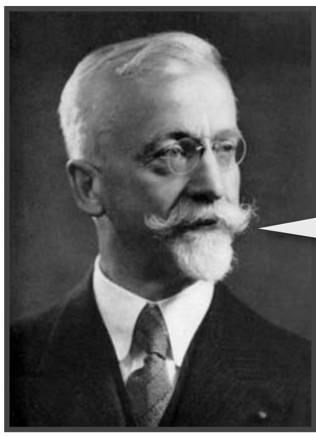
invariant signature

The invariant signature

Theorem

Two curves are related by a rotation and translation if* and only if they have the same invariant signatures.

Proof idea



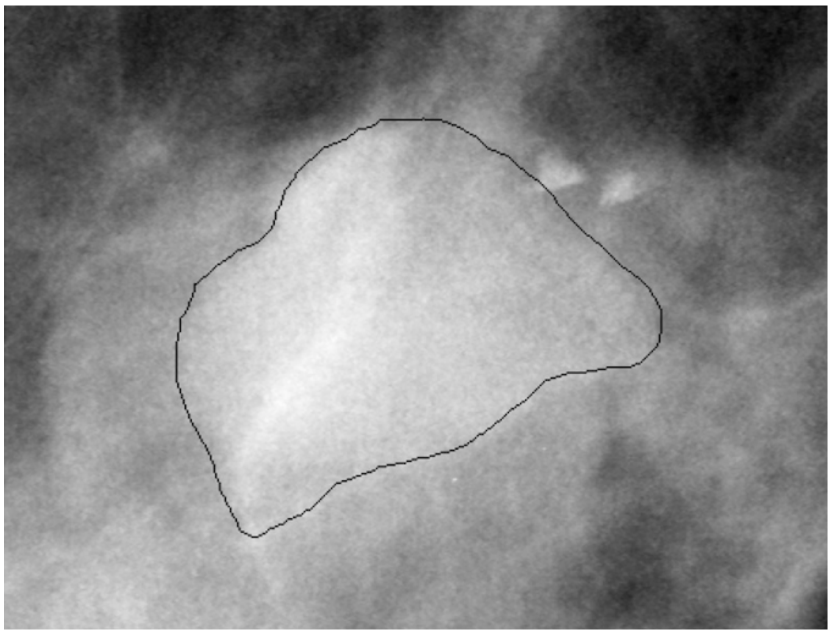
Theorem (Élie Cartan 1908)

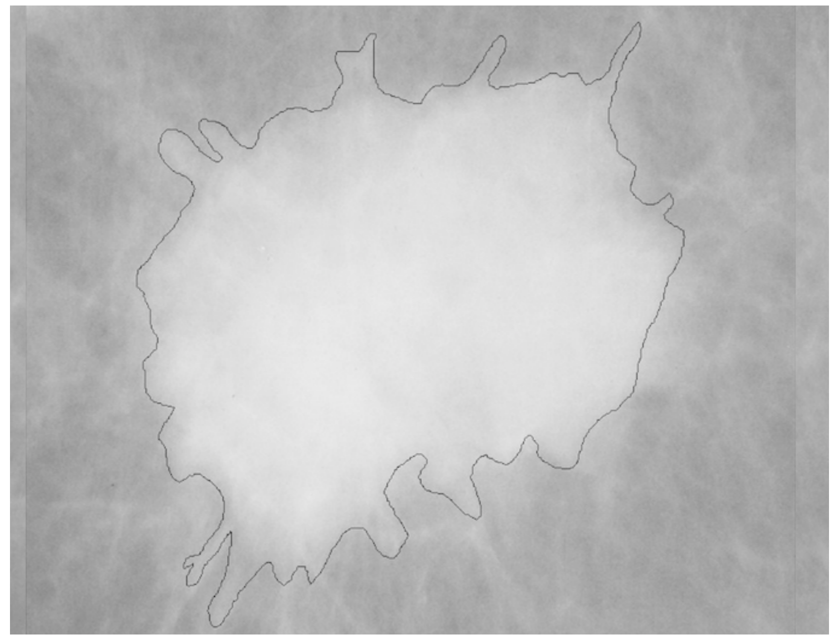
Shapes are related if and only if they have the same relationships among their **differential invariants**.

(Calabi, Haker, Olver, Shakiban, Tannenbaum 1998)

Diagnosing breast tumors

Anna Grim, Cheri Shakiban



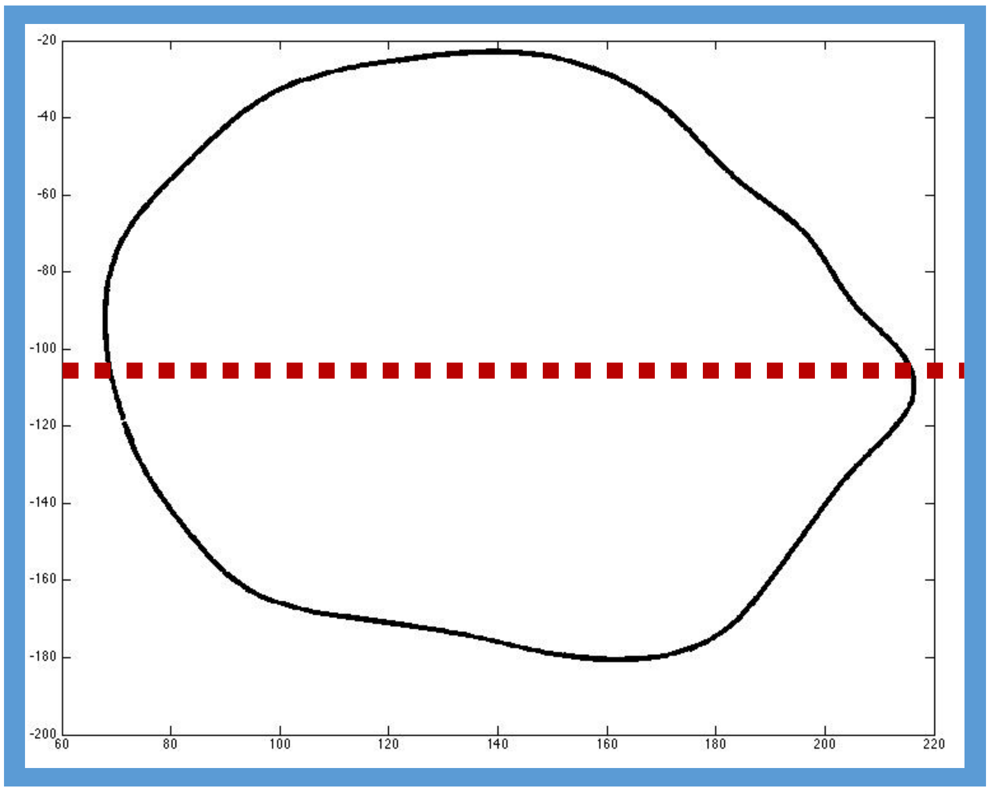


Benign — cyst

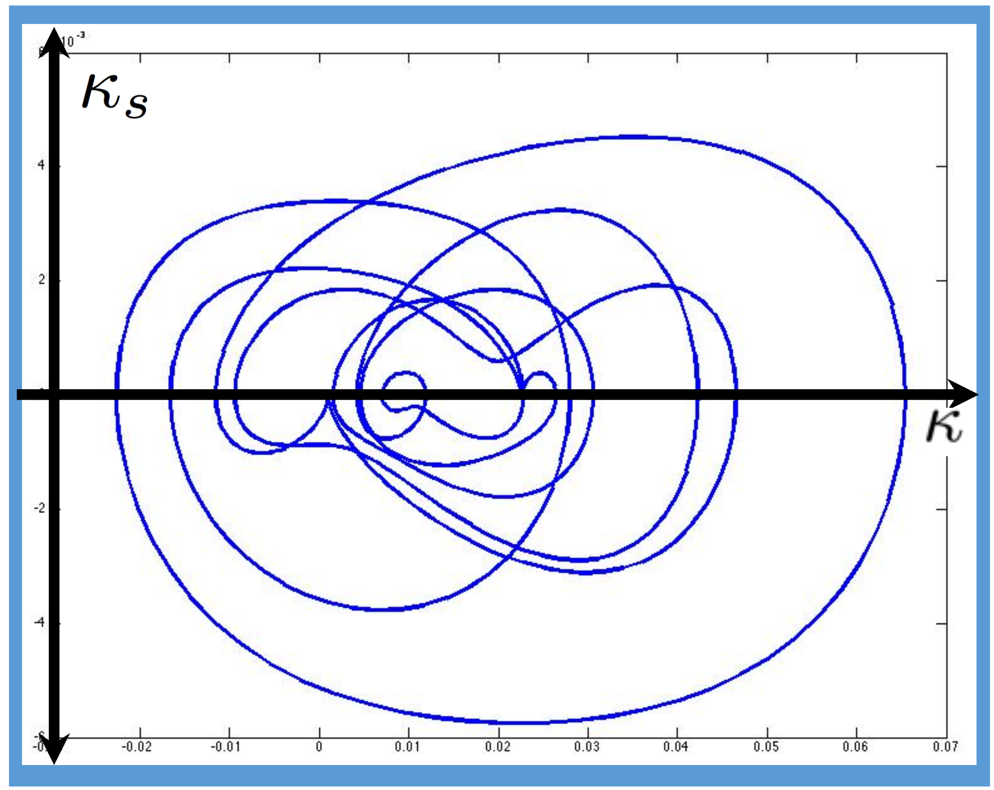
Malignant — cancerous

A BENIGN TUMOR

Contour

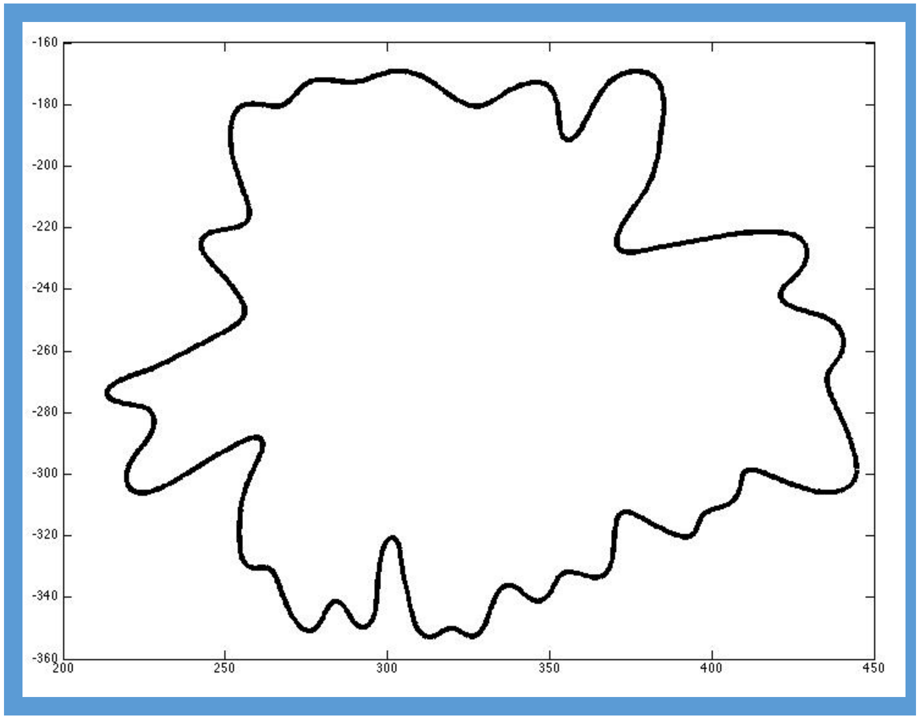


Signature Curve

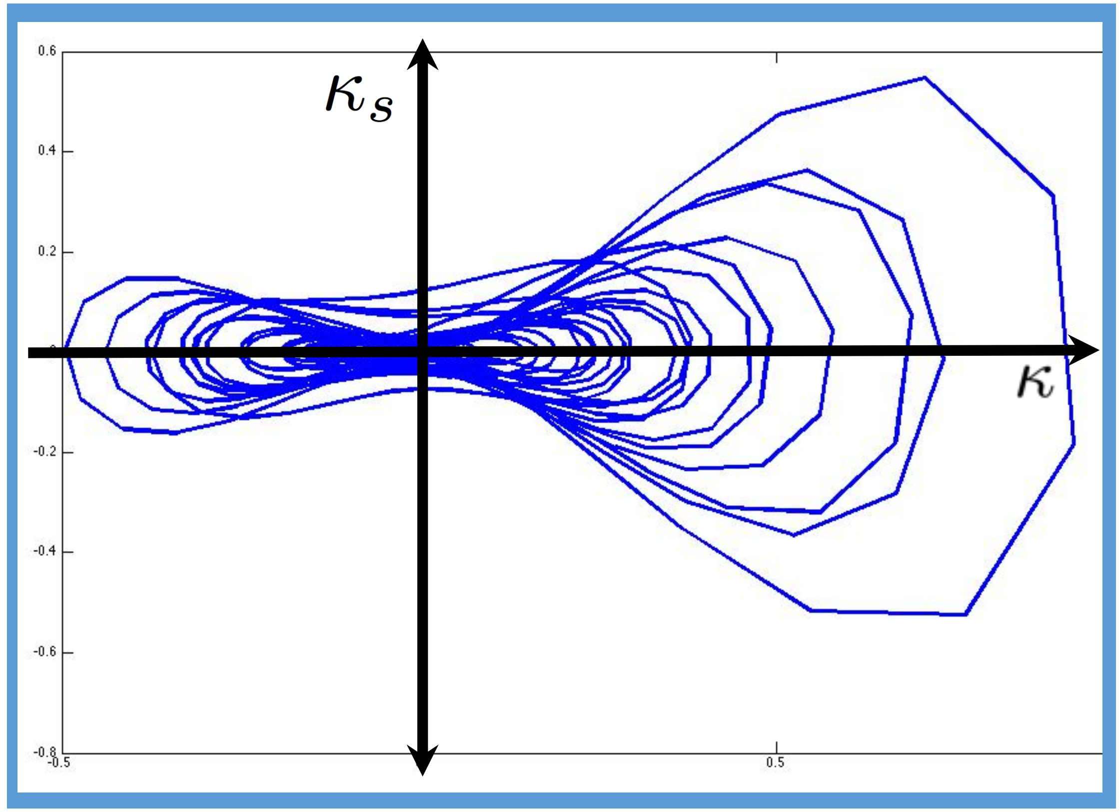


A MALIGNANT TUMOR

Contour

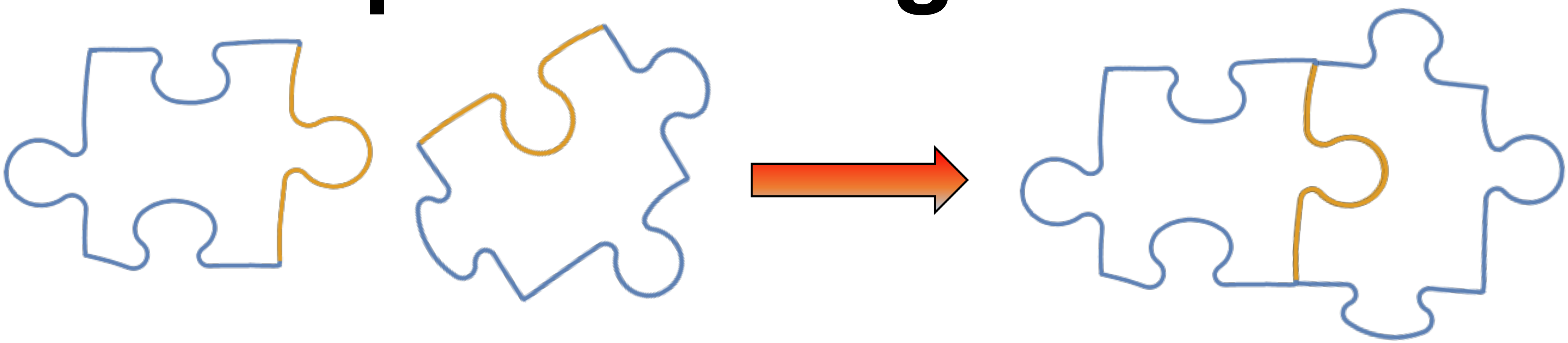


Signature Curve



Applications to Jigsaw Puzzles

A practical algorithm



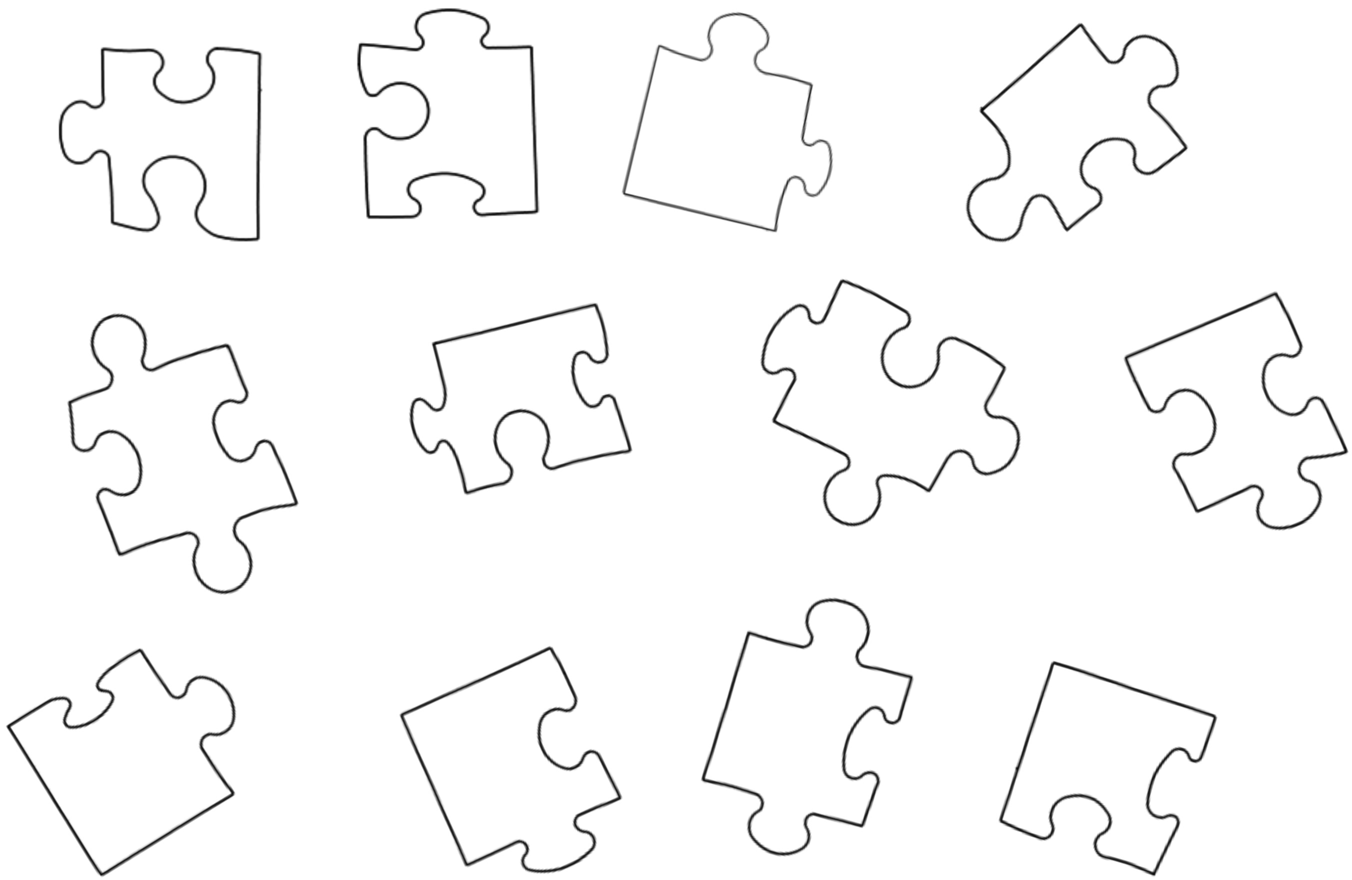
Step 1. Compute invariant signatures of both pieces.

Step 2. Compare signatures to find potential fits.

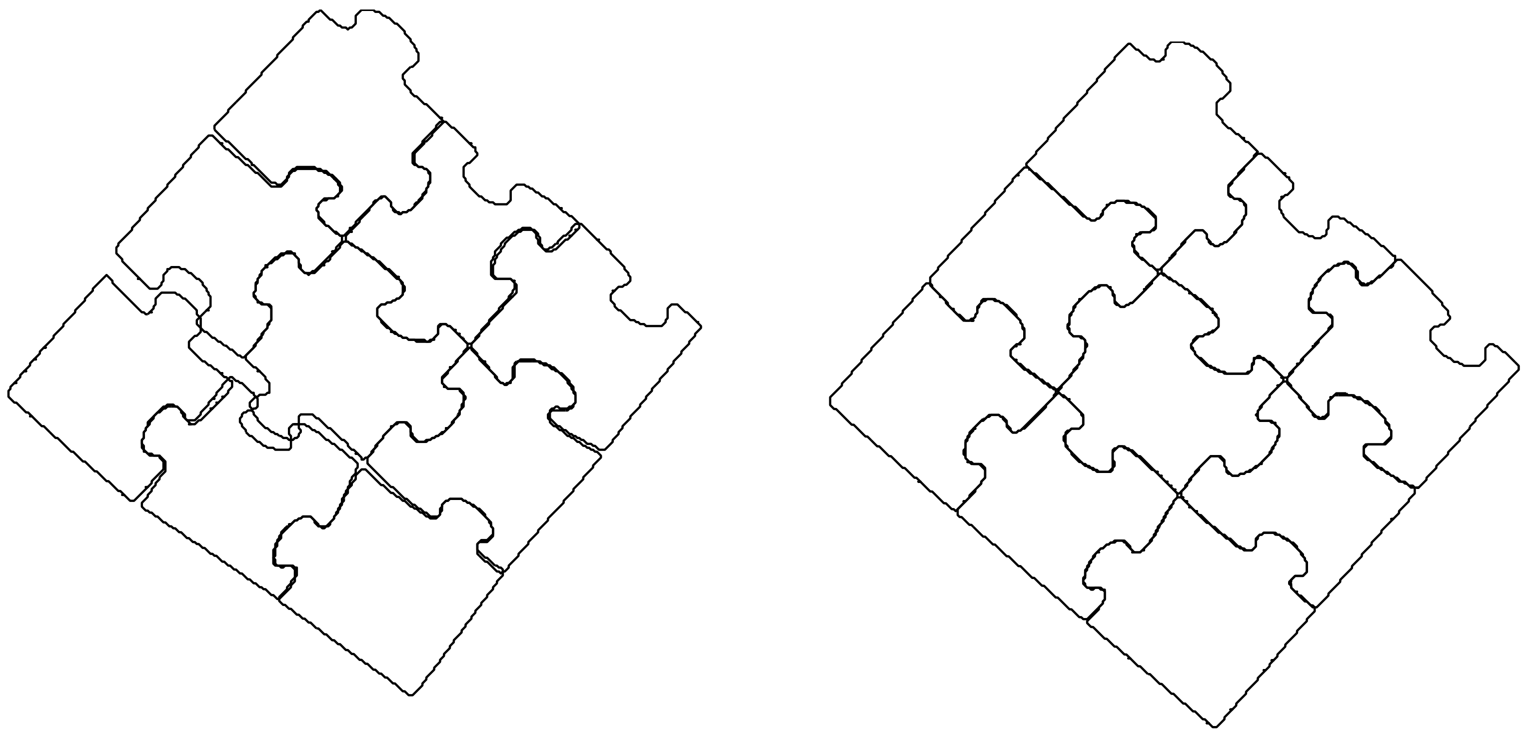
Step 3. Put them together, if they fit.

Repeat until puzzle is assembled….

Assembling the puzzle…

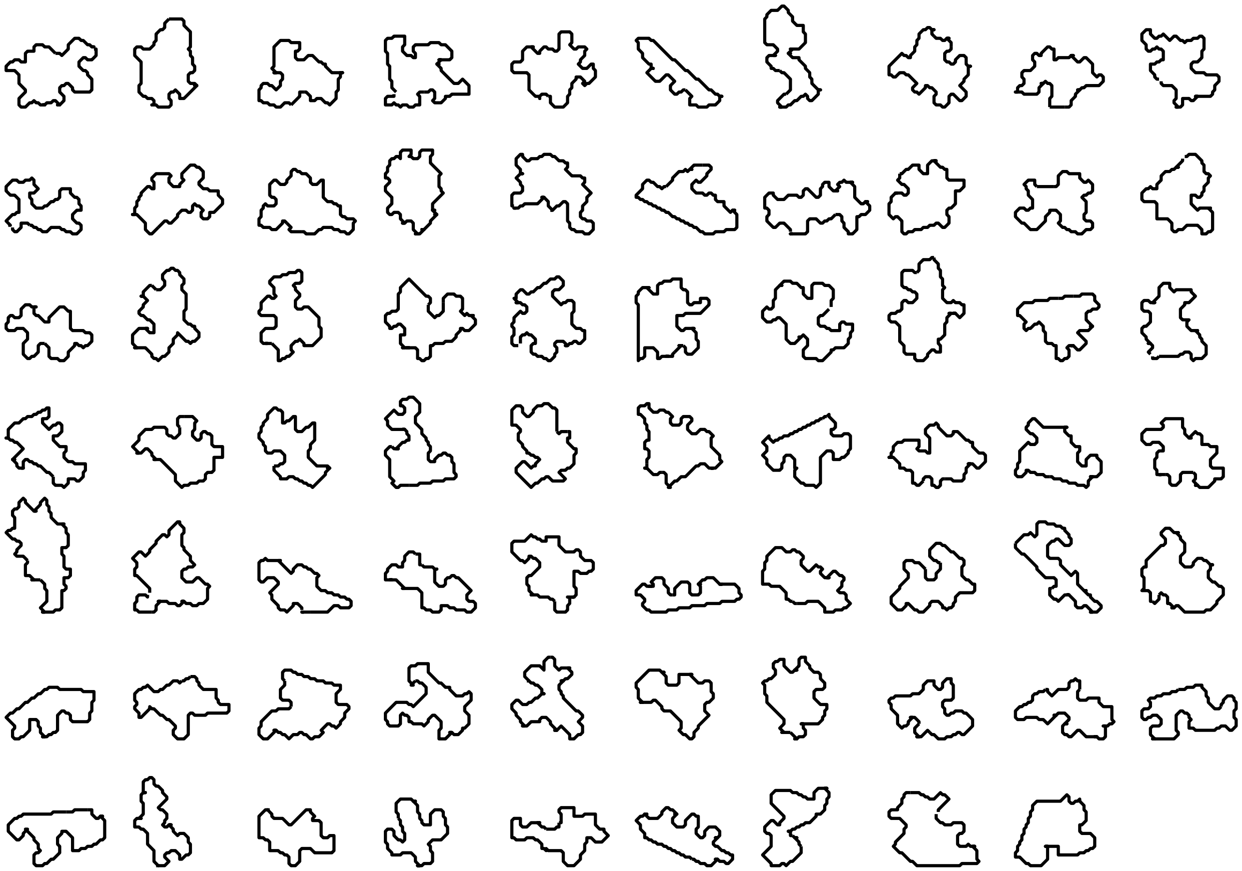


Piece Locking

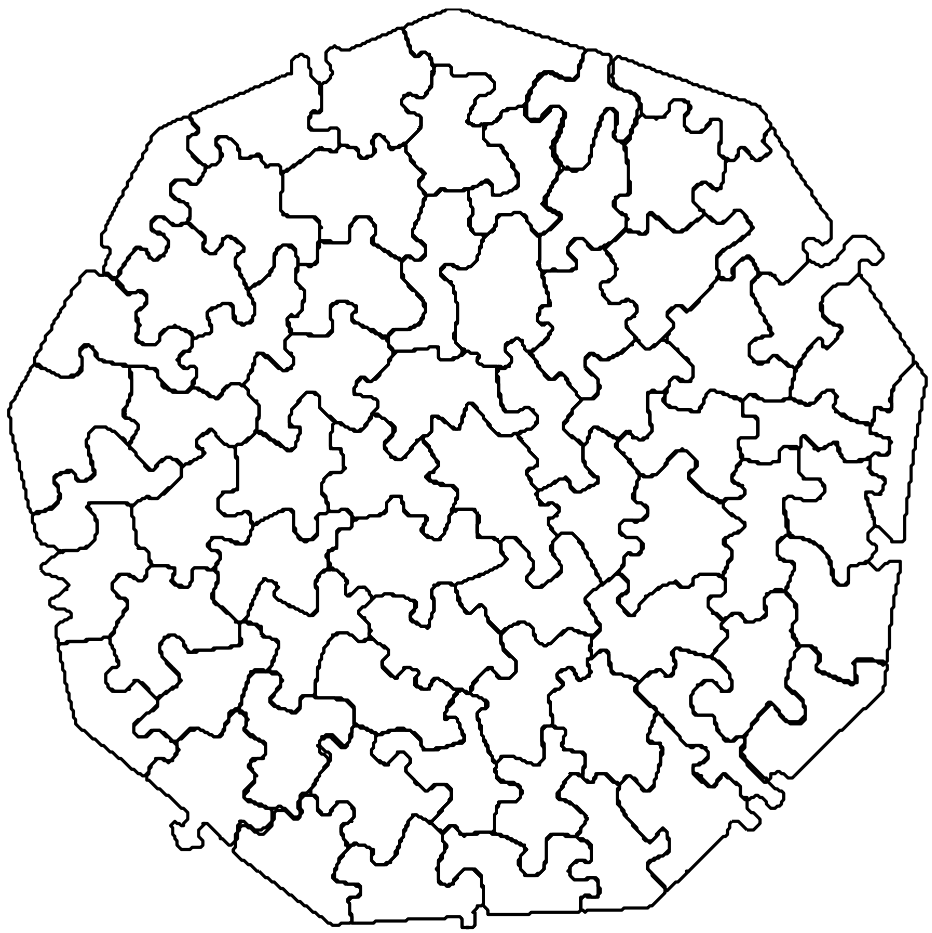


⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.

The Baffler Nonagon



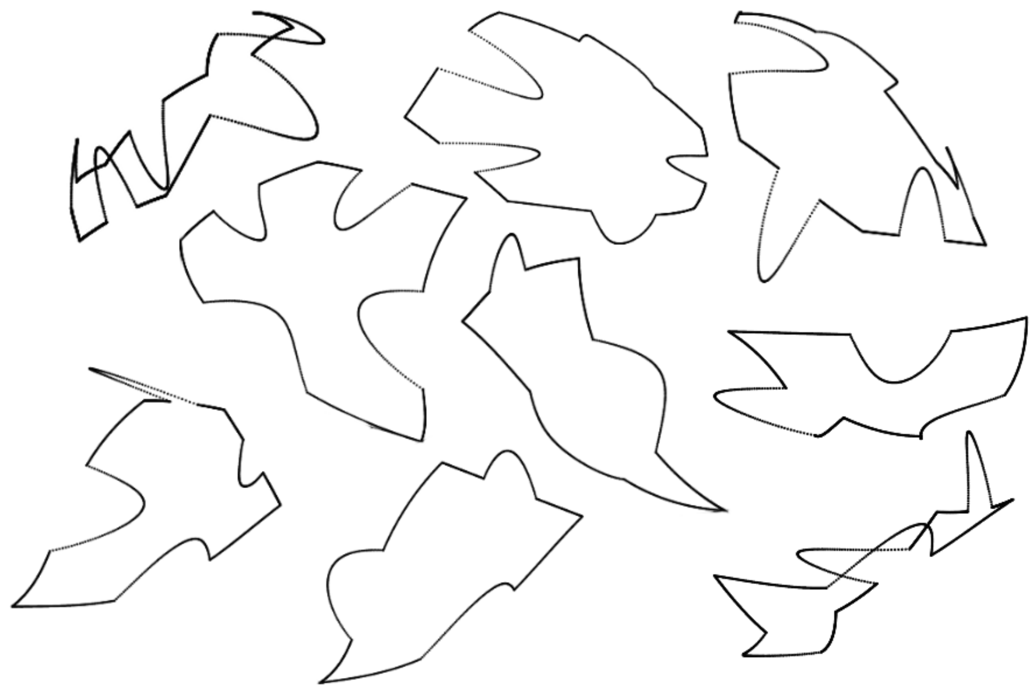
The Baffler Nonagon — Solved



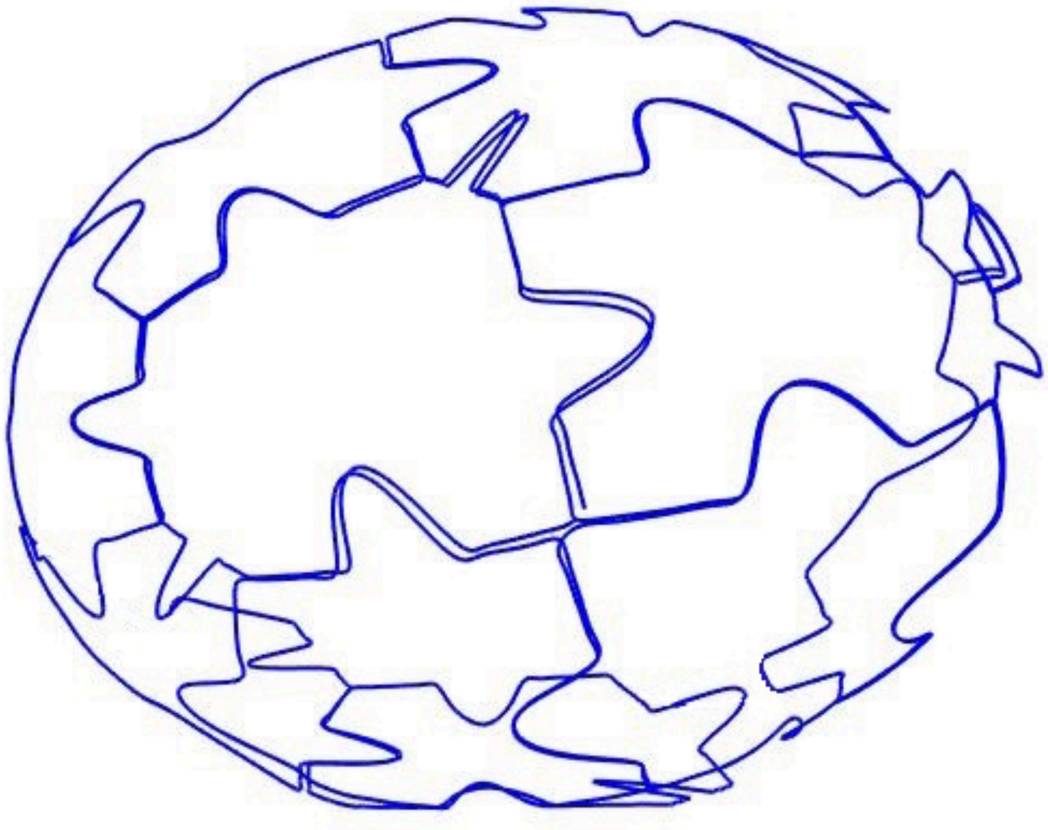
Putting Humpty Dumpty Together Again

→ Anna Grim, Ryan Slechta, Tim O'Connor, Rob Thompson, Cheri Shakiban, Peter Olver

A synthetic 3d jigsaw puzzle

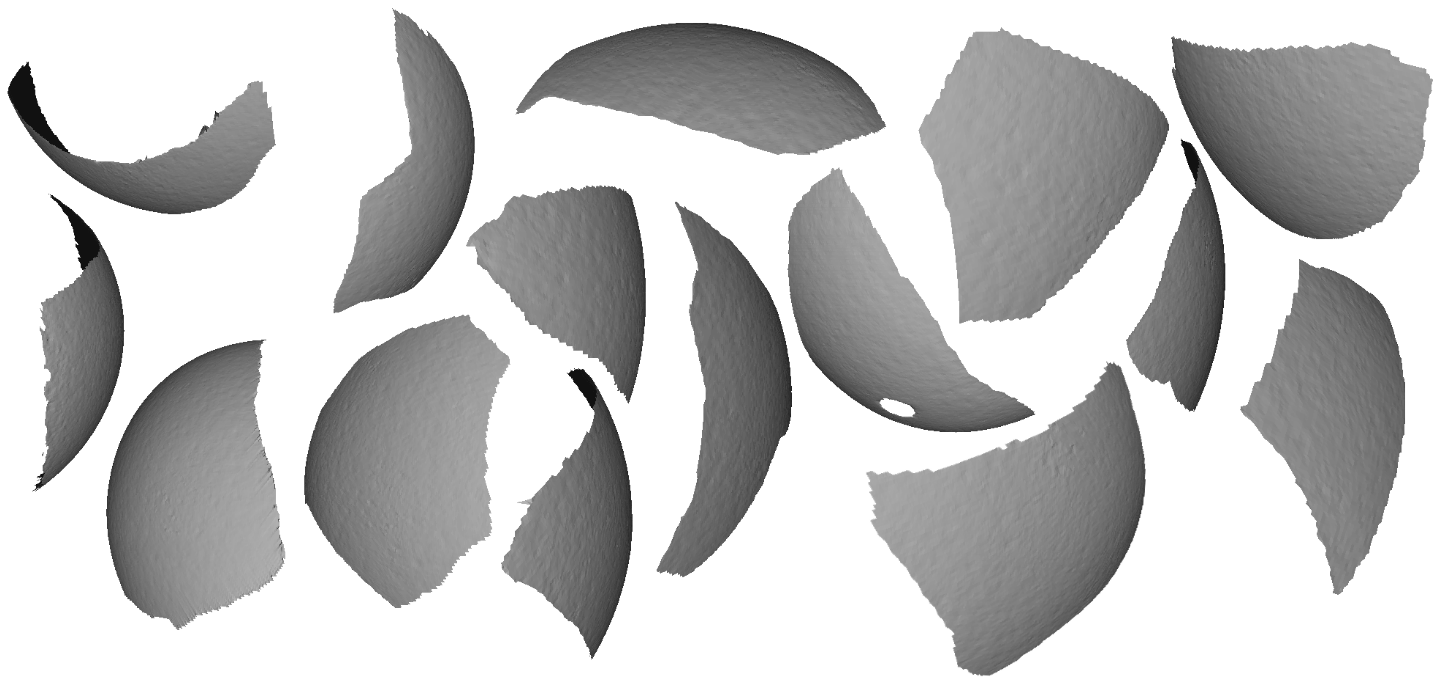


Assembly of synthetic spherical puzzle



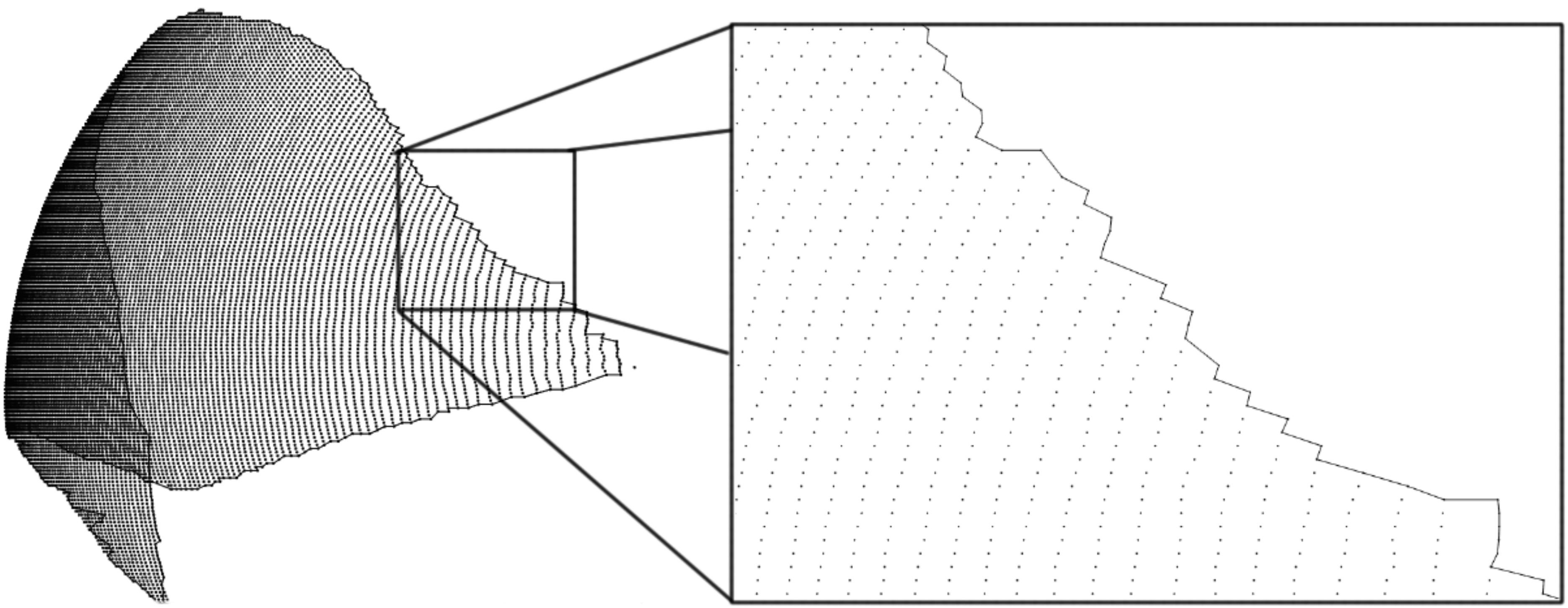
- Uses curvature and torsion invariants

A broken ostrich egg

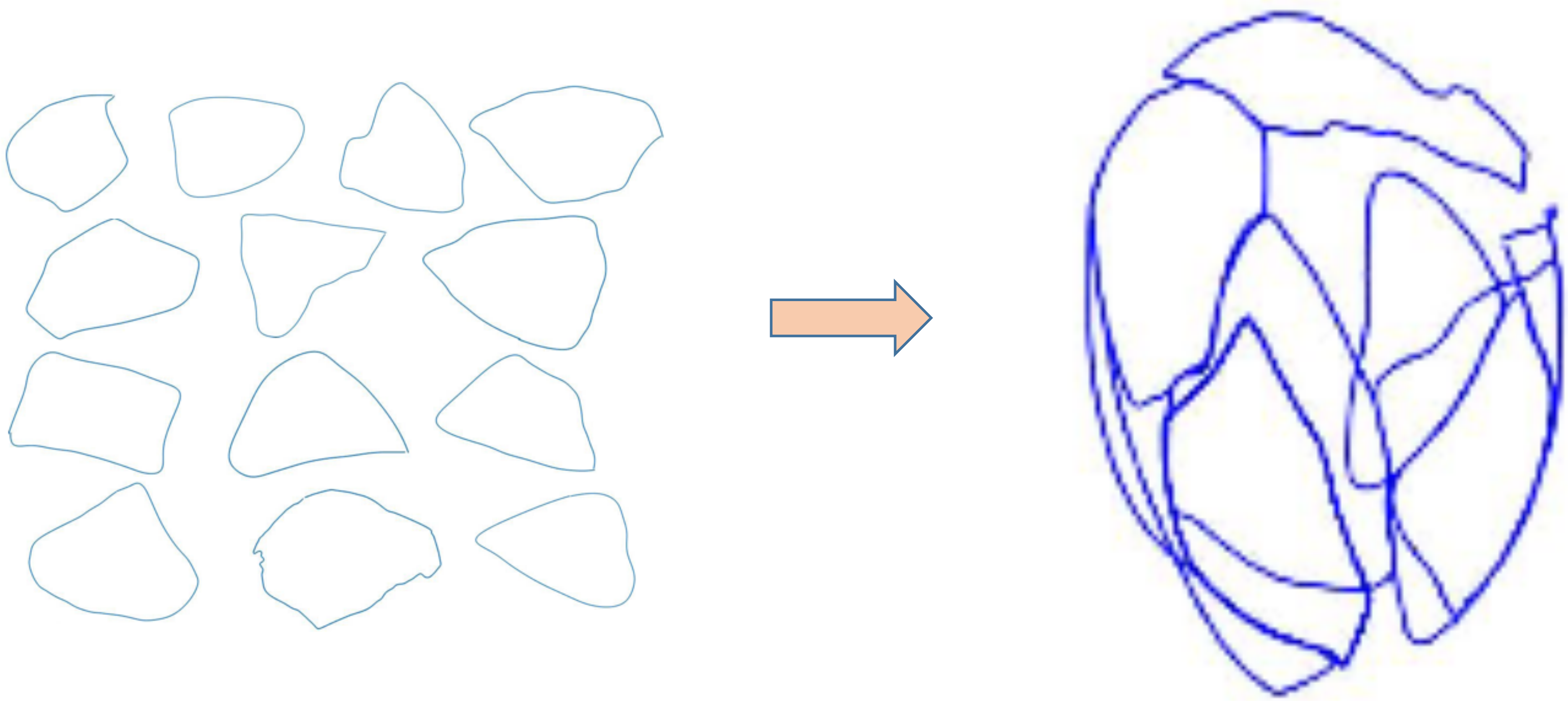


(Scanned by M. Bern, Xerox PARC)

An egg piece

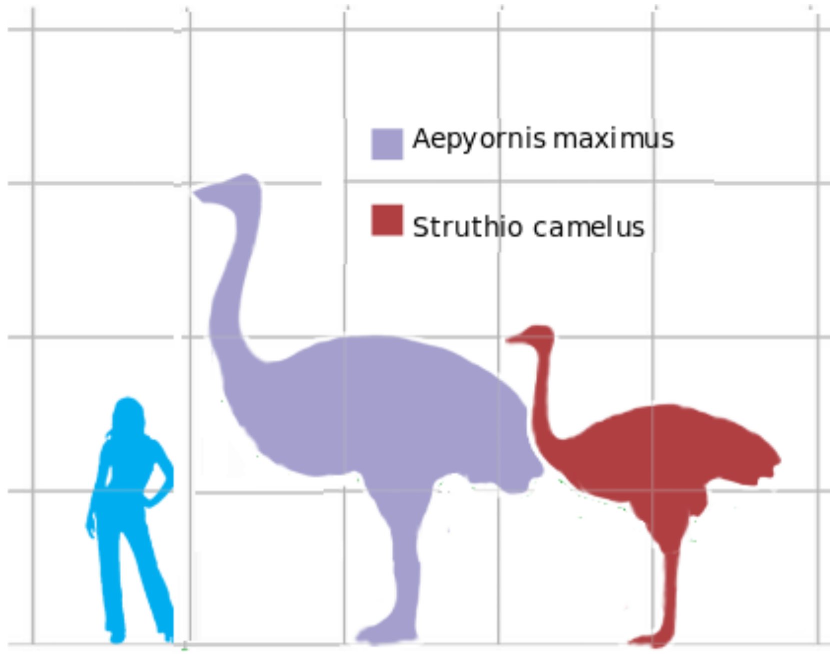


All the king's horses and men



The elephant bird business plan

The elephant bird of Madagascar



(Image from wikipedia.org)

- more than 3 meters tall
- extinct by the 1700's
- one egg could make about 160 omelets

Elephant bird egg shells

(Extract from “Zoo Quest to Madagascar”, BBC 1961)

The elephant bird of Madagascar

(Image from Tennant's Auctioneers)

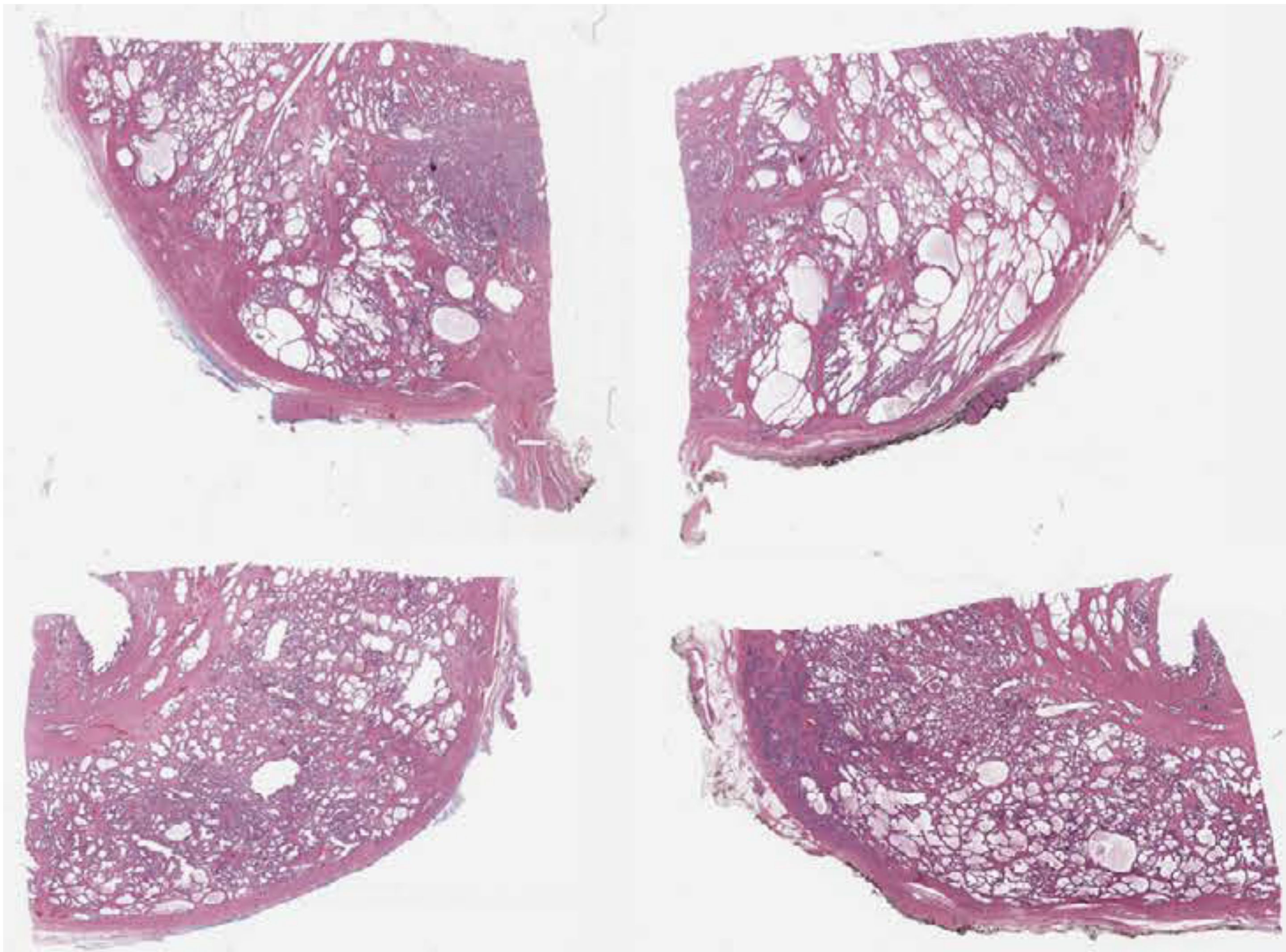
- pictured egg is 70% complete
- complete egg recently sold for $100,000

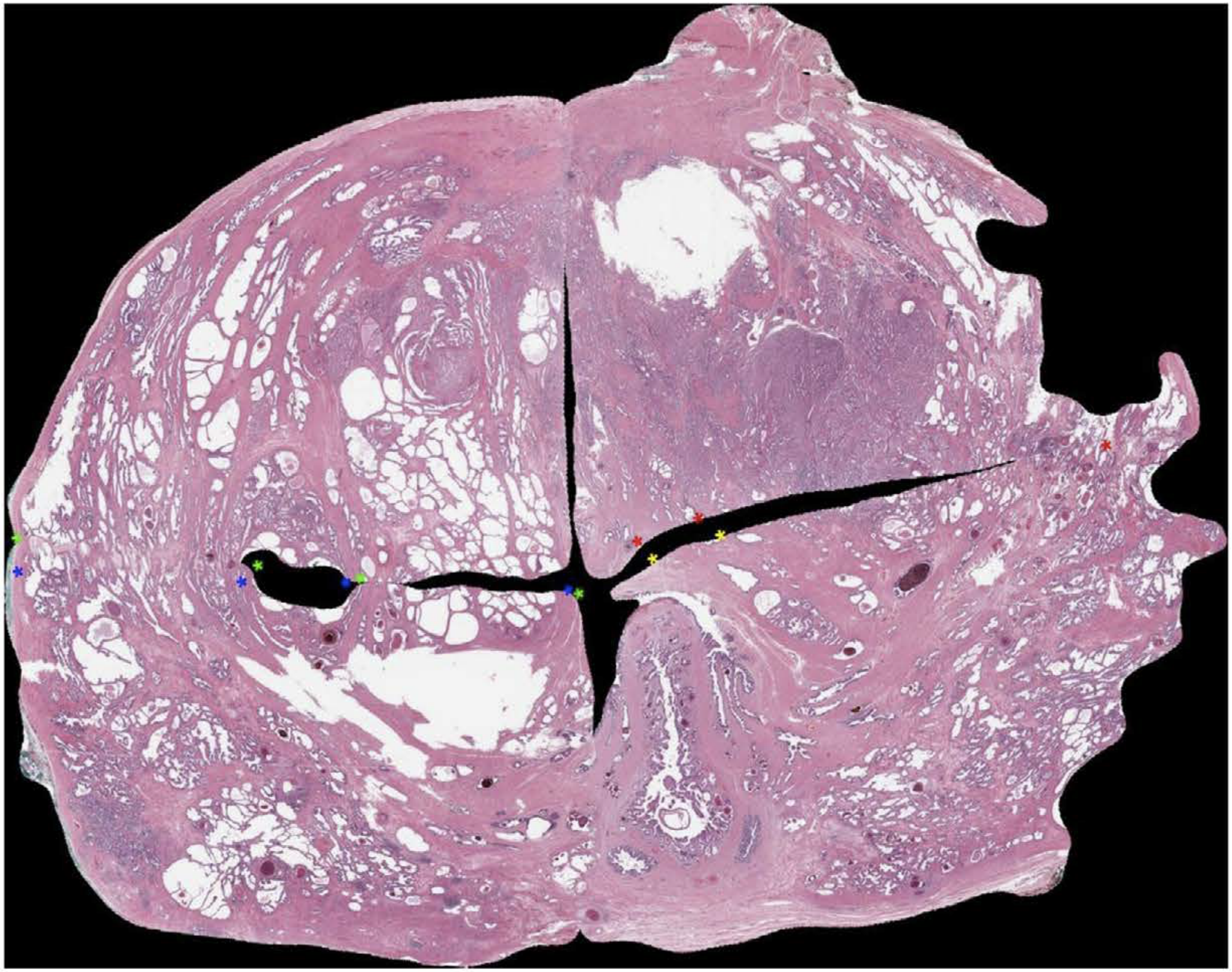
Puzzles in archaeology

Puzzles in archaeology

Puzzles in surgery

Puzzles in biology — Autostitcher for histological sections





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