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Euler operators and conservation laws of the BBM equation

BY PETER J. OLVER
University of Oxford

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Abstract. The BBM or Regularized Long Wave Equation is shown to possess only three non-trivial independent conservation laws. In order to prove this result, a new theory of Euler-type operators in the formal calculus of variations will be developed in detail.

1. *Introduction.* The nonlinear partial differential equation to be studied in this paper was derived for the description of the unidirectional propagation of long waves in certain nonlinear dispersive systems. In their pioneering work on this problem (8), Korteweg and de Vries were led to the equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1·1}$$

as an approximation to the behaviour of long waves in shallow water. Much recent research has been concerned with the many strange and intriguing properties of the KdV equation (1·1); see, for instance, (11) and (12) for an introduction to this subject. From the viewpoint of the present paper the most important of these properties are the existence of an infinite series of independent conservation laws, which was first proved in (13), and the 'soliton' solutions, whose properties are rigorously derived in (10). It has generally been supposed that these two properties are complementary, in the sense that either one implies the other one. However, this intuitive guess has yet to be rigorously formulated, much less proved.

In 1972, Benjamin, Bona and Mahony (2) proposed that, given the same approximations and assumptions that originally led Korteweg and de Vries to their equation, the partial differential equation

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1·2}$$

could equally well be justified as a model of the same phenomena. Equation (1·2) is called the BBM equation or regularized long wave (RLW) equation. The word 'regularized' refers to the fact that from the standpoint of existence, uniqueness and stability theory, Equation (1·2) offers considerable technical advantages over (1·1), as is demonstrated in the aforementioned reference. However, many of the more sophisticated mathematical properties of (1·1) are not known or fail to hold for (1·2). In this paper it will be proved that (1·2) possesses only three independent conservation laws, in contrast with (1·1). Perhaps before stating a theorem to this effect, the notion of a conservation law should be defined.

Definition 1·1. Given a partial differential equation

$$\Delta(x, t, u, u_x, u_t, \ldots) = 0$$

involving two independent variables x, t and one dependent variable u, a *conservation law* is an equation of the form

$$T_t + X_x = 0 \tag{1·3}$$

(the subscripts denoting derivatives), which is satisfied for all solutions of the equation $\Delta = 0$. Here T is called the conserved *density* and X is called the conserved *flux*. Note that for any conservation law, the quantity

$$I = \int_{-\infty}^{\infty} T\,dx,$$

for solutions such that the integral converges, is a constant of motion, i.e. independent of time.

Note first that if $T = P_x$ and $X = -P_t$ for some P, then the conservation law (1·3) is trivially satisfied. In general, we shall be interested in non-trivial conservation laws. If $T_1, \ldots, T_n$ are the densities of n different conservation laws, then these laws are called *dependent* if there exist constants $c_1, \ldots, c_n$ such that

$$c_1 T_1 + \ldots + c_n T_n = P_x$$

for some P; otherwise the laws are called *independent*. In this paper we shall restrict the notion of conservation law somewhat by requiring the density T to depend only on $x, u, u_x, u_{xx}, \ldots$.

If u is replaced by $-u-1$ in Equation (1·2), we are led to the somewhat simpler equation

$$u_t - u_{xxt} = uu_x. \tag{1·4}$$

We will find it easier to work with Equation (1·4) in lieu of (1·3). Note that the conservation laws of these two equations are in one-to-one correspondence under the above transformation. The main result of this paper is:

THEOREM 1·2. *The only non-trivial, independent conservation laws of* (1·4) *in which* $T(x, u, u_x, u_{xx}, \ldots)$ *depends smoothly on* x, u *and the various spatial derivatives of* u *are*

$$u_t - (u_{xt} + \tfrac{1}{2}u^2)_x = 0, \tag{1·5}$$

$$(\tfrac{1}{2}u^2 + \tfrac{1}{2}u_x^2)_t - (uu_{xt} + \tfrac{1}{3}u^3)_x = 0, \tag{1·6}$$

$$(\tfrac{1}{3}u^3)_t + (u_t^2 - u_{xt}^2 - u^2u_{xt} - \tfrac{1}{4}u^4)_x = 0. \tag{1·7}$$

It is routine to check that (1·5, 1·6, 1·7) are indeed conservation laws for equation (1·4); these were already discovered in (2). The only new information is that these are the *only* non-trivial conservation laws. Note that the conserved density T is not restricted to be of polynomial form for the theorem to hold.

The methods developed here to prove Theorem 1·2 constitute the beginnings of a comprehensive algebraic machinery for use in the investigation of conservation laws of partial differential equations. (See (14) for some preliminary applications to other equations.) These techniques were inspired by the recent work of Gel'fand and Dikiĭ on the formal calculus of variations (6, 7), and also by that of Kruskal, Miura, Gardner and Zabusky on the KdV equation (9).

In Section 2 we generalize the notion of an Euler operator or variational derivative, cf. (15). The goal of this section is to solve equations of the form

$$E(P) = \mathscr{D}Q, \tag{1·8}$$

where E is the ordinary Euler operator and $\mathscr{D}$ is a constant coefficient linear differential operator. A recursive procedure is developed for this purpose. The main results from this section are the notion of a substitution map, given in Definition 2·10; and Theorem 2·19, which solves (1·8) in the special case $\mathscr{D} = 1 - d^2/dx^2$. These techniques have more widespread application than just to the BBM equation; this will be reported in subsequent publications.

In Section 3 the general results of the preceding section are specialized to study the BBM equation and ultimately prove Theorem 1·2. The main step is to show that if any conservation laws other than those listed in Theorem 1·2 exist, the conserved density T must be equivalent to a density of the form

$$T' = T'(x, u - u_{xx}, u_x - u_{xxx}, \ldots).$$

This can be inferred from the arguments immediately after Lemma 3·1, and is a direct consequence of Theorem 2·19. T' must also satisfy fairly stringent requirements, and the remainder of Section 3 proves that these requirements can never be satisfied.

In view of Theorem 1·2, it is of interest to investigate whether or not the BBM equation possesses soliton solutions. It is quite easy to verify that solitary wave solutions of (1·2) have the form

$$u(x,t) = 3\,\frac{c^2}{1-c^2}\operatorname{sech}^2 \frac{1}{2}\left(cx - \frac{ct}{1-c^2} + \delta\right),$$

(cf. (5)). Preliminary numerical evidence on the interaction of these solutions was contradictory, with some studies indicating that the waves were indeed solitons (4, 5), whereas others claimed that a small rarefaction wave appeared after interaction (1). More recent numerical studies of J. Bona, W. Pritchard and R. Scott (3) have shown conclusively that the two solitary waves do not emerge from the interaction unscathed, and are therefore not solitons. Combining this result with Theorem 1·2 lends additional credence to the general connexion between solitons and conservation laws alluded to above. What is clearly lacking is a general procedure for relating these two concepts. Although it was remarked by Lax (10) that the eigenspeeds of solitons do provide constants of motion, it is not altogether clear how these are related to conservation laws of the form (1·3). I hope that the future will provide insight into this intriguing question.

2. *A calculus of Euler operators.* The Euler operator arises in the calculus of variations as the operator which to each Lagrangian of a variational problem assigns the Euler equation associated with that problem. Gel'fand and Dikiĭ's work in the formal calculus of variations (6, 7) demonstrates the importance of this operator in the algebraic theory of differential equations. In this section a parametrized family of Euler-type operators is introduced, and some impoıtant properties of this family are derived.

First we need to make our notation precise. Let $I \subset \mathbb{R}$ be an open subinterval, and let $\mathbf{A} = \mathbf{A}_I\{u, x\}$ denote the algebra of all C^∞ complex-valued functions of the form $L(x, u, u_1, \ldots, u_n)$, where $x \in I$, $u, u_1, \ldots, u_n \in \mathbb{R}$, and n is arbitrary (but finite). The algebra $\mathbf{A}$ represents the collection of 'smooth' differential equations on I, where x is viewed as the independent variable, u the dependent variable, and u_i represents the derivative d^iu/dx^i. For convenience, the derivative $\partial/\partial u_i$, acting on $\mathbf{A}$, will be denoted by ∂_i. (Also $\partial_0 = \partial/\partial u$.) The (total) derivative operator for $\mathbf{A}$ is

$$D = \frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1}\,\partial_i.$$

The *Euler operator* on $\mathbf{A}$ is

$$E = \sum_{i=0}^{\infty} (-D)^i \partial_i.$$

(Note that in the expressions for D and E, for any fixed $L \in A$ only finitely many terms in the sums are needed.) The key property of E is:

THEOREM 2·1. *Given $L \in \mathbf{A}$, $L = DP$ for some $P \in \mathbf{A}$ if and only if $E(L) = 0$.*

Proof. The proof that $ED = 0$ is left to the reader.

Let $\phi : I \to \mathbb{R}$, $\phi \in C^\infty$. Then for any closed subinterval $[a, b] \subset I$, integration by parts shows that

$$\frac{d}{d\epsilon}\int_a^b L(x, \epsilon\phi, \epsilon\phi', \ldots, \epsilon\phi^{(n)})\,dx = \int_a^b E[L(x, \epsilon\phi, \ldots, \epsilon\phi^{(n)})]\,\phi(x)\,dx + B_\phi(x)\Big|_a^b$$

for some function B_ϕ depending on L and ϕ. Therefore

$$\int_a^b L(x, \phi, \ldots, \phi^{(n)})\,dx = B_\phi(x)\Big|_a^b + C_{a,b}$$

for some constant $C_{a,b}$ depending only on a, b. Hence

$$L(x, \phi, \ldots, \phi^{(n)}) - L(x, 0, \ldots, 0) = dB_\phi(x)/dx,$$

for any ϕ. Given $x, u, u_1, \ldots, u_n$, choose ϕ with $\phi(x) = u$, $\phi'(x) = u_1$, $\ldots$, $\phi^{(n)}(x) = u_n$, and let

$$P(x, u, \ldots, u_n) = B_\phi(x).$$

This is the required $P \in \mathbf{A}$.

Definition 2·2. Given $\lambda \in \mathbb{C}$, define the operators

$$E(\lambda) = \sum_{i=0}^{\infty} (\lambda - D)^i \partial_i, \qquad (2·1)$$

$$\partial(\lambda) = \sum_{i=0}^{\infty} \lambda^i \partial_i. \qquad (2·2)$$

(Note that $E(0) = E$, $\partial(0) = \partial_0$.)

THEOREM 2·1′. *Given $\lambda \in \mathbb{C}$, $L \in \mathbf{A}$, then $L = (D - \lambda)P$ for some $P \in \mathbf{A}$ if and only if*

$$E(\lambda)P = 0.$$

Proof. It suffices to notice that

$$D-\lambda = e^{\lambda x} D\, e^{-\lambda x},$$

hence

$$E(\lambda) = e^{\lambda x} E\, e^{-\lambda x}.$$

This reduces Theorem 2·1′ to Theorem 2·1.

COROLLARY 2·3. *Suppose $\lambda_1, \ldots, \lambda_n$ are distinct complex numbers. Let*

$$\mathscr{D} = \prod_{j=1}^{n} (D-\lambda_j).$$

Then $L = \mathscr{D}P$ for some P if and only if $E(\lambda_j)\, L = 0$ for $j = 1, \ldots, n$.

Proof. Suppose $\lambda \neq \mu$ and $E(\lambda)\, L = 0 = E(\mu)\, L$. Then $L = (D-\lambda)\, Q$ for some $Q \in \mathbf{A}$. Moreover,

$$\begin{aligned} 0 = E(\mu)\, L &= E(\mu)\,(D-\lambda)\, Q \\ &= (\mu-\lambda)\, E(\mu)\, Q. \end{aligned}$$

Therefore $Q = (D-\mu)\, P$ for some $P \in \mathbf{A}$. The general case follows by induction.

For the sake of completeness, we now describe the analogue of Corollary 2·3 in the case some of the factors in the constant coefficient differential operator $\mathscr{D}$ might be repeated.

Definition 2·4. For each non-negative integer k and each $\lambda \in \mathbb{C}$, define the operators

$$E^{(k)}(\lambda) = \sum_{i=k}^{\infty} \binom{i}{k} (\lambda - D)^{i-k} \partial_i, \tag{2·3}$$

$$\partial_k(\lambda) = \sum_{i=k}^{\infty} \binom{i}{k} \lambda^{i-k} \partial_i. \tag{2·4}$$

(The operators $E^{(k)} = E^{(k)}(0)$ were originally defined in (9).)

Using the formula

$$\partial_i D = D \partial_i + \partial_{i-1}, \tag{2·5}$$

we conclude that

$$E^{(k)}(\lambda)\,(D-\lambda) = E^{(k-1)}(\lambda). \tag{2·6}$$

Suppose

$$\begin{aligned} p(z) &= z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0 \\ &= \prod_{j=1}^{n} (z-\lambda_j)^{m_j} \end{aligned}$$

is a complex coefficient polynomial, so $a_0, \ldots, a_{N-1}, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, the λ_j's are distinct, and $m_1, \ldots, m_n$ are positive integers. Associated with p there is a constant coefficient differential operator:

$$\begin{aligned} \mathscr{D} = p(D) &= D^N + a_{N-1} D^{N-1} + \ldots + a_1 D + a_0 \\ &= \prod_{j=1}^{n} (D-\lambda_j)^{m_j}. \end{aligned} \tag{2·7}$$

Corollary 2·3 and formula (2·6) prove:

THEOREM 2·5. *Let $\mathscr{D}$ be as in* (2·7). *Then $L = \mathscr{D}P$ for some $P \in \mathbf{A}$ if and only if*

$$E^{(i)}(\lambda_j)\, L = 0 \quad (j = 1, \ldots, n, \quad i = 1, \ldots, m_j).$$

PROPOSITION 2·6. *For $P, Q \in \mathbf{A}$, we have*

$$E(PQ) = \sum_{k=0}^{\infty} [E^{(k)}(P)\,(-D)^k Q + E^{(k)}(Q)\,(-D)^k P],$$

where $E^{(k)} = E^{(k)}(0)$.

Proof. We compute, using Leibnitz' formula:

$$\begin{aligned} E(PQ) &= \sum_{i=0}^{\infty} (-D)^i \partial_i (PQ) \\ &= \sum_{i=0}^{\infty} (-D)^i [\partial_i P . Q + \partial_i Q . P] \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left[\binom{i}{k} (-D)^{i-k} \partial_i P . (-D)^k Q + \binom{i}{k} (-D)^{i-k} \partial_i Q . (-D)^k P \right] \\ &= \sum_{k=0}^{\infty} \sum_{i=k}^{\infty} \left[\binom{i}{k} (-D)^{i-k} \partial_i P . (-D)^k Q + \binom{i}{k} (-D)^{i-k} \partial_i Q . (-D)^k P \right]. \end{aligned}$$

This proves the lemma. (There are generalizations of this formula for $E^{(k)}(\lambda)$, but these will not be used.)

In the sequel, we shall be interested in solving equations of the form

$$E(\mu)\, P = \mathscr{D}Q \tag{2·8}$$

for P, $Q \in \mathbf{A}$, where $\mathscr{D}$ is as in (2·7). The reason for using complex valued coefficients for the polynomials in $\mathbf{A}$ now becomes clear, since $\mathscr{D}$ can then be factored into a product of first order differential operators, $D - \lambda_j$.

LEMMA 2·7. *Suppose λ, $\mu \in \mathbb{C}$. Then*

$$E(\lambda)\, E(\mu) = \partial(\mu - \lambda)\, E(\mu) = E(\lambda)\, \partial(\mu - \lambda). \tag{2·9}$$

Furthermore, if k is a non-negative integer, then

$$E^{(k)}(\lambda)\, E(\mu) = (-1)^k \partial_k (\mu - \lambda)\, E(\mu). \tag{2·10}$$

Proof. To show (2·9),

$$\begin{aligned} E(\lambda)\, E(\mu) &= E(\lambda) . \Sigma(\mu - D)^i \partial_i \\ &= E(\lambda) . \Sigma[(\mu - \lambda) + (\lambda - D)]^i \partial_i \\ &= E(\lambda) . \Sigma(\mu - \lambda)^i \partial_i . \end{aligned}$$

Moreover, (2·5) implies

$$\partial(\mu - \lambda)\,(D - \mu) = (D - \lambda)\, \partial(\mu - \lambda). \tag{2·11}$$

This yields the second equation of (2·9).

To prove (2·10), we need the following lemma.

LEMMA 2·8. *Let λ, μ, k be as in Lemma* 2·6, *then*

$$E^{(k)}(\lambda) = \sum_{j=k}^{\infty} \binom{j}{k} (\lambda - \mu)^{j-k} E^{(j)}(\mu). \tag{2·12}$$

Proof.

$$\begin{aligned} E^{(k)}(\lambda) &= \sum_{i=k}^{\infty} \binom{i}{k} [(\lambda-\mu)+(\mu-D)]^{i-k}\partial_i \\ &= \sum_{i=k}^{\infty}\sum_{j=0}^{i-k} \binom{i}{k}\binom{i-k}{j} (\lambda-\mu)^j(\mu-D)^{i-k-j}\partial_i \\ &= \sum_{j=0}^{\infty}\binom{k+j}{k}(\lambda-\mu)^j \sum_{i=k+j}^{\infty}\binom{i}{k+j}(\mu-D)^{i-k-j}\partial_i \\ &= \sum_{j=0}^{\infty}\binom{k+j}{k}(\lambda-\mu)^j E^{(j+k)}(\mu), \end{aligned}$$

from which (2·12) follows. Note that we are justified in interchanging the order of summation since to apply either sum to any fixed $P \in \mathbf{A}$ only finitely many terms are necessary.

Next note that

$$\partial_j(\mu-D)^i = \sum_{l=0}^{\min\{i,j\}} (-1)^l \binom{i}{l} (\mu-D)^{i-l}\partial_{j-l}, \tag{2·13}$$

which may be proved by induction using (2·5). Therefore,

$$\begin{aligned} \partial_j E(\mu) &= \sum_{i=0}^{\infty}\sum_{l=0}^{\min\{i,j\}} (-1)^l\binom{i}{l}(\mu-D)^{i-l}\partial_i\,\partial_{j-l} \\ &= \sum_{l=0}^{j}\sum_{i=l}^{\infty}(-1)^l\binom{i}{l}(\mu-D)^{i-l}\partial_i\,\partial_{j-l} \\ &= \sum_{l=0}^{j}(-1)^l E^{(l)}(\mu)\,\partial_{j-l}. \end{aligned} \tag{2·14}$$

Hence, by Lemma 2·8, (2·6) and (2·14),

$$\begin{aligned} E^{(k)}(\lambda)\,E(\mu) &= \sum_{j=k}^{\infty}\sum_{i=0}^{\infty}\binom{j}{k}(\lambda-\mu)^{j-k}E^{(j)}(\mu)\,(\mu-D)^i\partial_i \\ &= \sum_{j=k}^{\infty}\binom{j}{k}(\lambda-\mu)^{j-k}\sum_{i=0}^{j}(-1)^i E^{(j-i)}(\mu)\,\partial_i \\ &= \sum_{j=k}^{\infty}\binom{j}{k}(\lambda-\mu)^{j-k}(-1)^j\,\partial_j\,E(\mu) \\ &= (-1)^k\partial_k(\mu-\lambda)\,E(\mu). \end{aligned}$$

Using Theorem 2·5 and Lemma 2·7, we immediately prove

THEOREM 2·9. *Suppose $\mathscr{D}$ is a constant coefficient differential operator, which factors as in* (2·7). *If $P, Q, R \in \mathbf{A}$ satisfy the equation*

$$E(\mu)P = \mathscr{D}Q = R$$

then $$\partial_i(\mu-\lambda_j)\,R = 0 \quad (j = 1,\dots,n, \quad i = 1,\dots,m_j).$$

Definition 2·10. Suppose $K \in \mathbf{A}$. The *substitution map* associated with K is the map $\mathscr{S}_K\colon \mathbf{A} \to \mathbf{A}$ such that

$$\mathscr{S}_K P(x,u,u_1,\dots,u_n) = P(x,K,DK,\dots,D^nK).$$

Note that $\mathscr{S}_K$ is a differential algebra morphism, i.e.

$$\begin{aligned}\mathscr{S}_K(P+Q) &= \mathscr{S}_K P+\mathscr{S}_K Q,\\ \mathscr{S}_K(P.Q) &= \mathscr{S}_K P.\mathscr{S}_K Q,\\ \mathscr{S}_K DP &= D\mathscr{S}_K P.\end{aligned} \tag{2·15}$$

Given $\lambda\in\mathbb{C}$, let $\mathscr{S}_\lambda$ be the substitution map associated with $u_1-\lambda u = (D-\lambda)u$.

LEMMA 2·11. *Let $\lambda\in\mathbb{C}$. Given $P\in\mathbf{A}$, $\partial(\lambda)P = 0$ if and only if $P\in\operatorname{Im}\mathscr{S}_\lambda$. In other words, $\partial(\lambda)P = 0$ if and only if P can be written as a polynomial in $u_1-\lambda u$, $u_2-\lambda u_1$, $u_3-\lambda u_2, \ldots$.*

THEOREM 2·12. *Suppose $\mathscr{D}$ is a constant coefficient differential operator, as given by (2·7). Let $\mathscr{S}$ be the substitution map associated with $K = \mathscr{D}(u)$. Then $P\in\operatorname{Im}\mathscr{S}$ if and only if*

$$\partial_i(\lambda_j)P = 0 \quad (j = 1,\ldots,n, \quad i = 1,\ldots,m_j).$$

Combining Theorems 2·9 and 2·12 yields:

THEOREM 2·13. *Suppose $p(z)$ is a complex polynomial with $\mathscr{D} = p(D)$ the corresponding constant coefficient differential operator. Let $\mathscr{S}$ be the substitution map associated with $K = p(\mu-D)u$. If P, Q, $R\in\mathbf{A}$ satisfy*

$$E(\mu)P = \mathscr{D}Q = R$$

then

$$R\in\operatorname{Im}\mathscr{S}.$$

This theorem partially resolves our problem. It would be nicer if we could assert that P and Q were both in $\operatorname{Im}\mathscr{S}$ and could find some way to relate $\mathscr{S}^{-1}P$ to $\mathscr{S}^{-1}Q$. In general this is not true, but this is only because of the presence of extraneous linear differential polynomials, as we will soon see.

LEMMA 2·14. *Suppose λ, $\mu\in\mathbb{C}$. Then*

$$E(\mu)\mathscr{S}_\lambda = \mathscr{S}_\lambda(\mu-\lambda-D)E(\mu). \tag{2·16}$$

(As above, $\mathscr{S}_\lambda$ is the substitution map associated with $u_1-\lambda u$.)

Proof. First note that

$$\begin{aligned}\partial_i\mathscr{S}_\lambda &= \mathscr{S}_\lambda(\partial_{i-1}-\lambda\partial_i) \quad (i>0),\\ \partial_0\mathscr{S}_\lambda &= \mathscr{S}_\lambda(-\lambda\partial_0).\end{aligned} \tag{2·17}$$

Therefore

$$\begin{aligned}E(\mu)\mathscr{S}_\lambda &= \mathscr{S}_\lambda\left[-\lambda\partial_0+\sum_{i=1}^{\infty}(\mu-D)^i(\partial_{i-1}-\lambda\partial_i)\right]\\ &= \mathscr{S}_\lambda.(\mu-\lambda-D).\sum_{j=0}^{\infty}(\mu-D)^j\partial_j,\end{aligned}$$

which proves (2·16).

Next we choose for once and all a right inverse to the operator $\partial(\lambda)$, which means an operator $\partial^{-1}(\lambda)\colon \mathbf{A}\to\mathbf{A}$ such that $\partial(\lambda)\partial^{-1}(\lambda)P = P$ for all $P\in\mathbf{A}$. For instance, if we let $v_1 = u_1-\lambda u, \ldots, v_n = u_n-\lambda u_{n-1}$, then any $P\in\mathbf{A}$ can be uniquely written as

$$P = P(x,u,v_1,\ldots,v_n),$$

and one candidate for the right inverse is

$$\partial^{-1}(\lambda)\,P = \int_0^u P(x,u,v_1,\ldots,v_n)\,du. \tag{2·18}$$

LEMMA 2·15. *Given $P\in \mathbf{A}$, we have $E(\mu)\,P\in \operatorname{Im}\mathscr{S}_\lambda$ if and only if there exists $P'\in\operatorname{Im}\mathscr{S}_\lambda$ with $E(\mu)\,P' = E(\mu)\,P$.*

Proof. The hypothesis is equivalent to $\partial(\lambda)\,E(\mu)\,P = 0$, so by Lemma 2·7 and Theorem 2·1',

$$\partial(\lambda)\,P = (D-\mu+\lambda)\,R,$$

for some $R\in\mathbf{A}$. Let

$$P' = P-(D-\mu)\,\partial^{-1}(\lambda)\,R,$$

so $E(\mu)\,P' = E(\mu)\,P$. Then (2·11) shows that

$$\partial(\lambda)\,P' = \partial(\lambda)\,P-(D-\mu+\lambda)\,R = 0,$$

which proves the result.

COROLLARY 2·16. *If $P,Q\in A$ and $\partial(\lambda)\,P = Q$, then $Q = \partial^{-1}(\lambda)\,P+R$ for some $R\in\operatorname{Im}\mathscr{S}_\lambda$.*

We now return to the problem of finding the general solution to Equation (2·8). For reasons that will become apparent later, we generalize this problem and try to find the general solution of

$$E(\mu)\,P = \mathscr{D}Q+\sum_{j=0}^{m}\alpha_j(x)\,u_j, \tag{2·19}$$

where $\mathscr{D} = p(D)$ for some complex polynomial p and $\alpha_j(x)$ are C^∞ complex-valued functions of x. The solution to this problem seems to be too complicated to write down explicitly. Instead, we shall devise a reasonably straightforward recursive procedure for reducing the solution of (2·19) to that of finding the general solution to a 'reduced' equation of the same type involving $\tilde{\mathscr{D}} = q(D)$ and $\tilde{\alpha}_j(x)$, but where $\tilde{\mathscr{D}}$ has degree $n-1$. Iteration of this procedure $n-1$ times reduces (2·19) to an equation of the form

$$E(\mu)\,\tilde{P} = a\tilde{Q}+\sum_{j=0}^{\tilde{m}}\tilde{\alpha}_j(x)\,u_j, \tag{2·20}$$

where a is a constant, hence $\tilde{P}$ is now an arbitrary element of A. The reason for the introduction of the α's is that even when the original α_j's are all zero, the recursive procedure may introduce non-zero $\tilde{\alpha}_j$'s in the reduced equation. After deriving the recursive procedure, we will illustrate its application to two concrete examples, which will prove to be of use in the subsequent section.

First suppose that λ is a root of the polynomial p. Let $q(z) = p(z)/(z-\lambda)$. If we define the functions $\gamma = \beta_0,\beta_1,\ldots,\beta_n$ by the equations

$$\beta_j = \sum_{i=0}^{n-j}(\lambda-D)^i\alpha_{n-i}, \tag{2·21}$$

then

$$p(D)\,Q+\sum_{j=0}^{n}\alpha_j u_j = (D-\lambda)\left[q(D)\,Q+\sum_{j=0}^{n-1}\beta_{j+1}u_j\right]+\gamma u. \tag{2·22}$$

Substituting (2·22) into (2·19) yields

$$E(\mu)\,(P-\tfrac{1}{2}\gamma u^2) = (D-\lambda)\,[q(D)\,Q+\Sigma\beta_{j+1}u_j].$$

Theorem 2·13 along with Lemma 2·15 now imply

$$P-\tfrac{1}{2}\gamma u^2 = \mathscr{S}_{\mu-\lambda}\tilde{P}+(D-\mu)R \tag{2·23}$$

for some $\tilde{P}, R\in \mathbf{A}$. Since P is determined only up to addition of elements of $\operatorname{Im}(D-\mu)$ anyway, we will always identify $P-\tfrac{1}{2}\gamma u^2$ and $\mathscr{S}_{\mu-\lambda}\tilde{P}$. Lemma 2·14 shows that

$$-\mathscr{S}_{\mu-\lambda}E\tilde{P} = q(D)Q+\sum_{j=0}^{n-1}\beta_{j+1}u_j+c\,e^{\lambda x}, \tag{2·24}$$

for some $c\in\mathbb{C}$. Without loss of generality we may take $c=0$, since this will not affect the final form of P and Q. Applying $\partial(\mu-\lambda)$ to (2·24), and using the identity

$$\partial(\mu-\lambda)\,q(D) = q(D+\mu-\lambda)\,\partial(\mu-\lambda),$$

we find that

$$q(D+\mu-\lambda)\,[\partial(\mu-\lambda)\,Q]+\Sigma\beta_{j+1}(\mu-\lambda)^j = 0.$$

Therefore $\partial(\mu-\lambda)\,Q = -\phi(x)$ where ϕ is an arbitrary solution of the inhomogeneous linear ordinary differential equation

$$q(D+\mu-\lambda)\,[\phi] = \Sigma\beta_{j+1}(\mu-\lambda)^j. \tag{2·25}$$

From Corollary 2·16 we find

$$Q = -\phi(x)\,u-\mathscr{S}_{\mu-\lambda}\tilde{Q}$$

for some $\tilde{Q}\in\mathbf{A}$. Moreover, we now have

$$q(D)\,Q+\Sigma\beta_{j+1}u_j = -\mathscr{S}_{\mu-\lambda}[q(D)\,\tilde{Q}+\Sigma\tilde{\alpha}_j u_j] \tag{2·26}$$

for suitable functions $\tilde{\alpha}_j(x)$. Combining equations (2·21–2·26), we are led to the following recursive procedure for solving (2·19):

(1) Let λ be a root of $p(z)$ and let $q(z) = p(z)/(z-\lambda)$.

(2) Find $\gamma = \beta_0, \beta_1, \ldots, \beta_n$ from Equations (2·21).

(3) Let

$$\delta(x) = \sum_{j=0}^{n-1}(\mu-\lambda)^j\beta_{j+1}(x), \tag{2·27}$$

and let $\phi(x)$ be an arbitrary solution of the differential equation

$$q(D+\mu-\lambda)\,\phi(x) = \delta(x) \tag{2·28}$$

(4) Solve the equation

$$\sum_{j=0}^{\tilde{m}}\tilde{\alpha}_j[u_{j+1}+(\mu-\lambda)\,u_j] = q(D)\,[\phi u]+\sum_{j=0}^{n-1}\beta_{j+1}u_j \tag{2·29}$$

for the functions $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_{\tilde{m}}$. (This is always possible.)

(5) Set

$$P = \mathscr{S}_{\mu-\lambda}\tilde{P}+\tfrac{1}{2}\gamma u^2, \tag{2·30}$$

$$Q = -\mathscr{S}_{\mu-\lambda}\tilde{Q}-\phi u.$$

Then $\tilde{P}$ and $\tilde{Q}$ are solutions of

$$E(\mu)\,(\tilde{P}) = q(D)\,\tilde{Q}+\Sigma\tilde{\alpha}_j u_j. \tag{$\widetilde{2·19}$}$$

Iteration of this procedure $n-1$ times will yield the general solution of (2·19).

As a first application, consider the equation

$$E(\mu)P = (D-\lambda)Q.$$

In this case $\alpha_j, \beta_j, \gamma, \delta, \phi, \tilde{\alpha}_j$ are all zero, so we have the following theorem:

THEOREM 2·17. *Suppose $\lambda, \mu \in \mathbb{C}$. Then $P, Q \in \mathbf{A}$ satisfy*

$$E(\mu)P = (D-\lambda)Q$$

if and only if there exist $\tilde{P}, \tilde{Q}, R \in \mathbf{A}$ such that

$$P = \mathcal{S}_{\mu-\lambda}\tilde{P} + DR$$
$$Q = -\mathcal{S}_{\mu-\lambda}\tilde{Q}$$

and
$$E(\tilde{P}) = \tilde{Q}$$

Example 2·18. Consider the case $\lambda = \mu = 0$. Then Theorem 2·17 implies that if P is the Lagrangian of some variational problem, then $E(P) = DQ$ for some Q if and only if $Q = Q'(x, u_1, u_2, \ldots, u_n)$ and there is an equivalent Lagrangian $P' = P'(x, u_1, u_2, \ldots, u_n)$ to P, such that

$$E[P'(x, u, \ldots, u_{n-1})] = -Q'(x, u, \ldots, u_{n-1}).$$

For instance if $P = uu_2 - 2uu_1u_2$, then

$$EP = 2u_2 - 6u_1u_2 = D(2u_1 - 3u_1^2).$$

Then $Q' = 2u - 3u^2$ and $P'(u) = -u^2 + u^3$. These satisfy $EP' = -Q'$. Moreover,

$$P'(u_1) = -u_1^2 + u_1^3$$
$$= P + D(uu_1 - uu_1^2),$$

as was guaranteed by the theorem.

As a second application of the recursive procedure, we consider the equation

$$E(P) = (D^2 - \lambda^2)Q. \tag{2·31}$$

Here $p(z) = z^2 - \lambda^2$ and, taking $+\lambda$ as the root, $q(z) = z + \lambda$. Since $\alpha_j = 0$, we get $\beta_j = 0$ for all j, so $\delta = 0$. Step three implies that $\phi(x)$ solves the equation $D\phi = 0$, hence $\phi = c$ for some constant c. Equation (2·29) shows that $\tilde{\alpha}_0 = c$. Step five now shows that $P \equiv \mathcal{S}_{-\lambda}\tilde{P}$, $Q = -\mathcal{S}_{-\lambda}\tilde{Q} - cu$ and

$$E\tilde{P} = (D+\lambda)\tilde{Q} + cu. \tag{2·32}$$

This example shows the necessity of allowing α_j's to appear in the general formulation of the problem. To solve (2·32), we again use the recursive procedure. Now $\tilde{\beta}_0 = 0$, $\tilde{\gamma} = c$ and we find $\tilde{\delta} = \tilde{\phi} = 0$. Therefore

$$\tilde{P} = \mathcal{S}_{\lambda}\hat{P} + \tfrac{1}{2}cu^2$$
$$\tilde{Q} = -\mathcal{S}_{\lambda}\hat{Q}$$
$$E\hat{P} = \hat{Q}.$$

Combining these results, we have proved:

THEOREM 2·19. *Suppose* $\lambda \in \mathbb{C}$. *Let* $\mathscr{S}$ *be the substitution map associated with* $u_2 - \lambda^2 u$. *Then for* $P, Q \in \mathbf{A}$,

$$E(P) = (D^2 - \lambda^2) Q$$

if and only if there exist $\hat{P}, \hat{Q}, R \in \mathbf{A}$ *with*

$$P = \mathscr{S}\hat{P} + \tfrac{1}{2}c(u_1 + \lambda u)^2,$$

$$Q = \mathscr{S}\hat{Q} - cu$$

and

$$E\hat{P} = \hat{Q}.$$

Notice that in this theorem an extraneous term not in the image of the substitution appears. Indeed

$$E(\tfrac{1}{2}(u_1 + \lambda u)^2) = -u_2 + \lambda^2 u = (D^2 - \lambda^2)(-u).$$

It can be seen that in general, except for the appearance of some of these extraneous quadratic terms, we could write $P = \mathscr{S}\hat{P}$, $Q = (-1)^n \mathscr{S}\hat{Q}$, $E\hat{P} = \hat{Q}$, where $\mathscr{S}$ is the substitution map associated with $p(\mu - D)u$.

3. *Non-existence of conservation laws.* In this section the methods developed in the preceding section will be applied to study the conservation laws of the BBM equation

$$u_t - u_{xxt} = uu_x. \tag{3·1}$$

The ultimate goal is to prove Theorem 1·2, which states that there are only three independent conservation laws. We shall use the notation $\mathbf{A} = \mathbf{A}_I\{u, x\}$ as in Section 2, and the notation $^*\mathbf{A} = \mathbf{A}^*_{I \times J}\{u; x, t\}$ to denote the differential algebra consisting of all C^∞ functions of x, t, u and the various derivatives of u with respect to both x and t, defined for $x \in I$ and $t \in J$. Note that $\mathbf{A} \subset \mathbf{A}^*$. We have two total derivative operators on $\mathbf{A}^*$: D, the total derivative with respect to x, and D_t, the total derivative with respect to t. Using this notation, Theorem 1·2 states that the only conservation laws, $D_t T + DX = 0$, with $T \in \mathbf{A}$ are those given by (1·5, 1·6, 1·7). Note first that integration by parts shows that

$$D_t T = E(T) u_t + DS$$

for some $S \in \mathbf{A}^*$. Therefore we can replace (1·3) by the equivalent condition

$$E(T) u_t = DX, \tag{3·2}$$

for some (different) $X \in \mathbf{A}^*$. This has the effect of eliminating the trivial conservation laws when $T = DP$ for some $P \in \mathbf{A}$. Let us abbreviate

$$v = u_t, \quad w = u_{xt}.$$

Note that the only t-derivatives of u which can occur in X are of the form

$$\partial^{n+1} u / \partial x^n \partial t;$$

moreover, since (3·2) holds only when u is a solution of (3·1), we can replace these derivatives for $n \geqslant 2$ by an expression only involving $u, u_1, u_2, \ldots, u_n, v, w$; for example $u_{xxxt} = uu_{xx} + u_x^2 - u_{xt}$. Therefore we can assume that X only depends on x, u, u_1, ..., u_n, v, w and

$$E(T) v = D^* X \tag{3·3}$$

holds identically, where

$$D^* = w\partial_v + (v - uu_x)\partial_w + D_u,$$

$$D_u = \partial_x + u_1\partial_0 + u_2\partial_1 + \dots. \tag{3·4}$$

(Here $\partial_v = \partial/\partial v$, etc.)

LEMMA 3·1. *If X and T satisfy* (3·3), *then*

$$X = c[v^2 - w^2 - u^2 w - \tfrac{1}{4}u^4] + Pv + Qw + R, \tag{3·5}$$

and

$$T = -\frac{c}{3}u^3 + S,$$

where $c \in \mathbb{C}$, $P, Q, R, S \in \mathrm{A}$, and satisfy

$$E(S) = (1 - D^2)Q, \tag{3·6}$$

$$DR = Quu_x. \tag{3·7}$$

Proof. Applying the operators ∂_v^2, $\partial_v\partial_w$ and ∂_w^2 to (3·3) (and using subscripts to denote partial derivatives), results in the equations

$$\begin{aligned} wX_{vvv} + (v - uu_x)X_{vvw} + 2X_{vw} + D_u X_{vv} &= 0,\\ X_{vv} + wX_{vvw} + (v - uu_x)X_{vww} + X_{ww} + D_u X_{vw} &= 0,\\ 2X_{vw} + wX_{vww} + (v - uu_x)X_{www} + D_u X_{ww} &= 0. \end{aligned} \tag{3·8}$$

Now if the highest derivative of u occurring in X_{ww} were u_n for some $n \geqslant 1$, the third equation would imply that X_{vw} must depend on u_{n+1}, otherwise the term $u_{n+1}\partial_n X_{ww}$ would not cancel. Similar reasoning applied to the second equation shows that X_{vv} must depend on u_{n+2}, but the first equation shows that this is absurd. Similar arguments allow us to conclude that X_{vv}, X_{vw} and X_{ww} can only depend on x, u, v, w.

Next, define the vector fields

$$\left.\begin{aligned} \pi &= -u\partial_w + \partial_u,\\ \rho &= v\partial_w + w\partial_v + \partial_x. \end{aligned}\right\} \tag{3·9}$$

If the terms involving u_x and the terms not involving u_x in (3·8) are separated, we find

$$\partial_v^2 \pi X = \partial_v\partial_w \pi X = \partial_w^2 \pi X = 0,$$

$$\partial_v^2 \rho X = \partial_v\partial_w \rho X = \partial_w^2 \rho X = 0.$$

Therefore

$$\pi X = A'v + B'w + C', \tag{3·10}$$

$$\rho X = A''v + B''w + C'',$$

where A', B', C', A'', B'', C'' do not depend on v or w. Since $[\rho, \pi] = u\partial_v$, we have

$$\partial_v X = A^*v + B^*w + C^*, \tag{3·10$'$}$$

where A^*, B^*, C^* do not depend on v or w. A similar application of the formula $[\partial_v, \rho] = \partial_w$ shows

$$\partial_w X = A^{**}v + B^{**}w + C^{**}. \tag{3·10$''$}$$

Integrating (3·10′) and (3·10″) yields

$$X = Av^2 + Bvw + Cw^2 + Pv + Qw + R, \tag{3·11}$$

where A, B, C depend only on x and u, and P, Q, $R \in \mathbf{A}$. Substituting (3·11) in (3·3) and equating the coefficients of the various powers of v and w, yields

$$\left.\begin{aligned} DA+B &= 0,\\ DB+2A+2C &= 0,\\ DC+B &= 0.\end{aligned}\right\} \tag{3·12}$$

and

$$\left.\begin{aligned} DP+Q-uu_x B &= E(T),\\ DQ+P-2uu_x C &= 0,\\ DR-uu_x Q &= 0.\end{aligned}\right\} \tag{3·13}$$

The solution of (3·12) is

$$\left.\begin{aligned} A &= c_1 e^{-2x}+c_2 e^{2x}+c_0,\\ B &= 2c_1 e^{-2x}-2c_2 e^{2x},\\ C &= c_1 e^{-2x}+c_2 e^{2x}-c_0\end{aligned}\right\} \tag{3·14}$$

for constants c_0, c_1, $c_2 \in \mathbb{C}$. Elimination of P from (3·13) gives

$$\left.\begin{aligned} ET &= (1-D^2)\,Q-3Buu_x+2C(u_x^2+uu_{xx}),\\ DR &= Quu_x.\end{aligned}\right\} \tag{3·15}$$

Finally, to eliminate the exponential terms in A, B and C, we apply $E(1)$ and $E(-1)$ to the first equation of (3·15), and use (2·9) and Theorem 2·1′. This yields

$$\partial(-1)\,ET = (3DB-3B+2D^2C-4DC+2C)\,u,$$
$$\partial(+1)\,ET = (3DB+3B+2D^2C+4DC+2C)\,u.$$

Applying $\partial(+1)$ to the first and $\partial(-1)$ to the second on these, and equating the results shows that

$$6B+8DC = 0.$$

Comparing this with (3·14) demonstrates that $c_1 = 0 = c_2$, which completes the proof of the lemma.

Note that the leading terms of X and T in (3·5) correspond to the conservation law (1·7). Thus we can without loss of generality take $c = 0$, and concentrate on determining all P, Q, R and S which satisfy (3·6) and (3·7). The purpose behind the extensive investigations of Section 2 is now revealed. Let $\mathscr{S}$ denote the substitution map associated with $\omega = u-u_2$. From Theorem 2·19 we conclude that

$$\left.\begin{aligned} Q &= \mathscr{S}Q'+c'u,\\ S &= \mathscr{S}S'+\tfrac{1}{2}c'(u_x-u)^2+DZ\\ ES' &= Q'.\end{aligned}\right\} \tag{3·16}$$

and

The terms involving $c' \in \mathbb{C}$ in the expressions for Q and S can be seen to form the second conservation law, (1·6). We now leave these aside, and assume that $Q = \mathscr{S}Q'$, $S = \mathscr{S}S'$. Note that from the assumption on S, we have proved that aside from the conservation laws (1·6) and (1·7), all nontrivial conserved densities must be equivalent to one of the form $T = T(x, u-u_2, u_1-u_3, \ldots, u_n-u_{n+2})$. Let $\mathbf{A}_0 = \operatorname{Im}\mathscr{S} \subset \mathbf{A}$. Define the operator $\partial_k'\colon \mathbf{A}_0 \to \mathbf{A}_0$, for k a nonnegative integer, from the equation

$$\partial_k'\mathscr{S} = \mathscr{S}\partial_k.$$

Note that on $\mathbf{A}_0$

$$\partial'_k = \sum_{i=0}^{[k/2]} \partial_{k-2i}. \tag{3·17}$$

LEMMA 3·2. *Define the operator*

$$\sigma = \sum_{k=0}^{\infty} D^k(uu_x)\,\partial'_k : \mathbf{A}_0 \to \mathbf{A}. \tag{3·18}$$

If $Q, S \in \mathbf{A}_0$ *satisfy* (3·6–3·7), *then*

$$\sigma.(1-D^2)\,Q = uDQ. \tag{3·19}$$

Proof. Apply the Euler operator E to (3·7) and use Proposition 2·6. This shows

$$\sum_{k=0}^{\infty} E^{(k)}Q(-D)^k(uu_x) = uDQ.$$

Therefore it suffices to show that

$$(-1)^k E^{(k)}Q = \partial'_k(1-D^2)\,Q.$$

Applying $\mathscr{S}$ to the last equation in (3·16) and using the definition of ∂'_i shows that

$$Q = \sum_{i=0}^{\infty} (-D)^i \partial'_i S$$

Therefore, Equations (2·6), (2·14), (3·6) and (3·17) imply

$$\begin{aligned} E^{(k)}Q &= \sum_{i=0}^{k} (-1)^i E^{(k-i)}\partial'_i S \\ &= \sum_{i=0}^{k}\sum_{l=0}^{[i/2]} (-1)^i E^{(k-i)}\partial_{i-2l} S \\ &= (-1)^k \sum_{l=0}^{[k/2]}\sum_{j=0}^{k-2l} (-1)^j E^{(j)}\partial_{k-2l-j} S \\ &= (-1)^k \sum_{l=0}^{[k/2]} \partial_{k-2l} ES \\ &= (-1)^k \partial'_k(1-D^2)\,Q. \end{aligned}$$

This proves the lemma.

LEMMA 3·3. *Suppose* $P \in \mathbf{A}_0$ *and*

$$\sigma P = f(x)\,u_x \tag{3·20}$$

for some smooth f. *Then* $f \equiv 0$ *and* P *is just a function of* x.

Proof. Assume that P depends on $x, u, u_1, \ldots, u_n$. Given $\lambda \in \mathbb{C}$, define the operators

$$\partial_e(\lambda) = \sum_{i=0}^{\infty} \lambda^{2i}\partial_{2i}, \quad \partial_0(\lambda) = \sum_{i=0}^{\infty} \lambda^{2i+1}\partial_{2i+1},$$

$$\partial'_e(\lambda) = \sum_{i=0}^{\infty} \lambda^{2i}\partial'_{2i}, \quad \partial'_0(\lambda) = \sum_{i=0}^{\infty} \lambda^{2i+1}\partial'_{2i+1}.$$

Note that $P \in \mathrm{A}_0$ if and only if $\partial_e(1)P = 0 = \partial_0(1)P$. A short computation shows that for any $\lambda \neq \pm 1$, and any integer m,

$$\sum_{i=0}^{m} \lambda^{2i}\partial_{2i} = (1-\lambda^2)\sum_{i=0}^{m} \lambda^{2i}\partial'_{2i} + \lambda^{2m+2}\partial'_{2m},$$

$$\sum_{i=0}^{m} \lambda^{2i+1}\partial_{2i+1} = (1-\lambda^2)\sum_{i=0}^{m} \lambda^{2i+1}\partial'_{2i+1} + \lambda^{2m+3}\partial'_{2m+1}.$$

Choosing $2m > n$ shows that for any $P \in \mathrm{A}_0$,

$$\left.\begin{aligned} \partial_e(\lambda)P &= (1-\lambda^2)\,\partial'_e(\lambda)P, \\ \partial_0(\lambda)P &= (1-\lambda^2)\,\partial'_0(\lambda)P. \end{aligned}\right\} \tag{3·21}$$

(These also trivially hold for $\lambda = \pm 1$.) Next define the operators

$$\theta_k = [\partial_k, \sigma] = \sum_{i=0}^{\infty} \binom{k+i}{k} u_i\,\partial'_{k+i-1}.$$

(For $k = 0$, omit the term $i = 0$.) Define

$$\left.\begin{aligned} \theta_e(\lambda) &= \sum_{i=0}^{\infty} \lambda^{2i}\theta_{2i} = [\partial_e(\lambda), \sigma], \\ \theta_0(\lambda) &= \sum_{i=0}^{\infty} \lambda^{2i+1}\theta_{2i+1} = [\partial_0(\lambda), \sigma] \end{aligned}\right\} \quad (\lambda \in \mathbb{C}). \tag{3·22}$$

Using the binomial theorem, we can show that

$$\begin{aligned} [\partial_e(\lambda), \theta_0(\mu)] + [\partial_e(\lambda), \theta_0(\mu)] &= (\lambda+\mu)\,\partial'_e(\lambda+\mu) \\ [\partial_e(\lambda), \theta_e(\mu)] + [\partial_0(\lambda), \theta_0(\mu)] &= (\lambda+\mu)\,\partial'_0(\lambda+\mu). \end{aligned} \tag{3·23}$$

for any λ, $\mu \in \mathbb{C}$.

Now given $P \in \mathrm{A}_0$ satisfying (3·20), Equations (3·22) imply

$$\theta_e(1)P = 0 = \theta_0(1)P.$$

Next, Equations (3·23) imply

$$\partial'_e(2)P = 0 = \partial'_0(2)P.$$

But finally (3·21) shows that

$$\partial_e(2)P = 0 = \partial_0(2)P.$$

By induction we can show that

$$\partial_e(m)P = 0 = \partial_0(m)P$$

for any positive integer m. However, this shows $\partial_j P = 0$ for $j = 0, \ldots, n$, hence P is just a function of x. This completes the proof of the lemma.

LEMMA 3·4. *If $Q \in \mathrm{A}_0$ and satisfies* (3·19), *then Q is a constant.*

Proof. We abbreviate $\omega_n = u_n - u_{n+2}$. Suppose Q depends on $x, \omega, \omega_1, \ldots, \omega_n$ with $\partial'_n Q \equiv Q'_n \neq 0$ for some $n \geqslant 1$. The highest-order term in uDQ is $Q'_n u u_{n+3}$. Similarly the highest order term in $(1-D^2)Q$ is $-Q'_n \omega_{n+2}$, hence the highest order term in $\sigma(1-D^2)Q$ is $\sigma(Q'_n)u_{n+4}$. Therefore $\sigma(Q'_n) = 0$, and Lemma 3·3 implies $Q'_n = q(x)$ for some nonzero function q. Now we have $Q = q(x)\,\omega_n + \tilde{Q}$, where $\partial'_n \tilde{Q} = 0$. As before, the highest-order terms in $\sigma(1-D^2)\,Q$ are

$$-quu_{n+3} - \sigma(\tilde{Q}'_{n-1})\,u_{n+3},$$

where $\tilde{Q}'_{n-1} = \partial'_{n-1}\tilde{Q}$. Comparison with (3·19), and again using Lemma 3·3, shows

$$\tilde{Q} = r(x)\,\omega_{n-1} + \bar{Q},$$

where $\partial'_n \bar{Q} = 0 = \partial'_{n-1}\bar{Q}$. Finally, the highest order terms in $\sigma(1-D^2)\,Q$ are now

$$-quu_{n+3} - (n+2)\,qu_1\,u_{n+2} - (r+2q')\,uu_{n+2} - \sigma(\bar{Q}'_{n-2})\,u_{n+2}.$$

We conclude that

$$\sigma(\bar{Q}'_{n-2}) + (n+2)\,qu_1 = 0.$$

A final application of Lemma 3·3 shows that $q = 0$, which is a contradiction. Therefore Q only depends on x and ω, but a short calculation shows that (3·19) is only satisfied when Q is a constant. This completes the proof of Lemma 3·4.

If we take Q to be a constant, and solve for P, R, S in Lemma 3·1, we just rederive the first conservation law (1·5) of the BBM equation. This completes the proof of the main theorem.

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