Tri-Hamiltonian Duality Between Solitons and Compactons

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Abstract. A simple scaling argument shows that most integrable evolutionary systems, which are known to admit a biHamiltonian structure, are, in fact, governed by a compatible trio of Hamiltonian structures. We demonstrate how their recombination leads to new integrable hierarchies endowed with nonlinear dispersion that supports compactons, or cusped and/or peaked solitons. A general algorithm for effecting this new duality between classical solitons and their non-smooth counterparts is illustrated by the construction of dual versions of the modified Korteweg-deVries equation, the nonlinear Schrödinger equation, the integrable Boussinesq system used to model the two way propagation of shallow water waves, and the Ito system of coupled nonlinear wave equations. These new hierarchies include a remarkable variety of new, interesting integrable nonlinear differential equations.

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1. Introduction.

The discovery, [**1**], that the solitary wave solutions supported by nonlinear wave equations may compactify under nonlinear dispersion has made it clear that nonlinear dispersion is capable of causing deep qualitative changes in the nature of genuinely nonlinear phenomena. The absence of the infinite tail in the resulting solitary wave solutions, called compactons, and their genuine robustness calls for a more systematic study of nonlinearly dispersive systems. Nonlinear dispersion has been known for some time to cause wave breaking, or lead to the formation of corners or cusps, but, at least within the framework of integrable systems, with the notable exception of [**2**] was never actively pursued. The formation of cusps is, in a definite sense, dual to the process of compactification and depends on the manner of interaction between dispersion and inertia.

In an earlier work, [**3**], the second author showed that a Lagrange transform, based on changing to a locally conserved density as a new independent variable, maps solitons into compactons, which are solitary wave solutions, both stationary and traveling, having compact support. The integrable soliton equation is mapped to a new equation endowed with nonlinear dispersion that supports the propagation of compactons. Thus, for these problems, a particular duality between certain soliton equations and their compacton counterparts was established. In a recent paper relevant to the present work, using a variant of the Hamiltonian perturbation theory introduced by the first author, [**4**], Camassa and Holm, [**5**], rediscovered a new integrable model for ocean dynamics whose solitary wave solutions have a corner at their crest, i.e., a discontinuity in the first derivative and therefore were called peakons. (Interestingly, peaked solitons were obtain in an earlier study of nonlinearly elastic media by Kunin, [**6**], from a similar, but presumably non-integrable system, but their relevance to genuinely nonlinear phenomena was not realized at the time.) We also note that an elementary complex-valued transformation changes the peakon equation into the integrable compacton equation; see below. In either case, the solitary wave solution is no longer smooth, being a weak solution (in the appropriate sense) of the nonlinear system.

It is perhaps surprising that the importance of the work of Wadati et al., [**2**], was never recognized. Their discovery was catalogued as merely one more integrable system. Their work touches upon the important issue of the balance between dispersion and convection. While in nature systems are known to run out of balance, thus leading to various well known break down of waves or the formation of cusps, most of integrable systems are notorious for unconditionally preserving this balance and thus being unable to describe such phenomena. Hence the importance of the discovery by Wadati et al.; their solitons may develop a cusp and thus became nonanalytical. As such a runaway is one of a more characteristic features of large amplitude phenomena, the failure of a typical integrable system to describe it, is merely a reflection of the generic limitation of system derived through the weakly nonlinear procedure. In fact, following the derivation in [**2**] one observes that at the crucial point, where the curvature of the rope is concerned, they avoid the conventional dictum and do not expand the metric. Its expansion, though consistent with other assumptions, would have eliminated the sought after effect . It is a situation where consistency becomes its own revenge. In this connection we note that in order to regain integrability, Camassa and Holm, [**5**], had to retain a higher order term in their expansion carried on the Hamiltonian level, which made it possible to avoid the pitfalls of a direct expansion. It is quite obvious that in order to model phenomena related to wave breaking, formation of cusps, and similar wonders of nature associated with nonlinear dispersion, one has to probe deeper into the nonlinear regime, far beyond the currently attained weakly nonlinear stage. While occasionally the next order terms in dispersion, typically quadratic ones, may suffice to unfold a new phenomenon, one should not expect this to be the case in general. In a genuinely nonlinear regime, nonlinearity plays a dominant role rather then being a higher order correction.

The possibility of nonanalytical solitons, whether compactons, peakons or as yet unnamed structures, is a manifestation of nonlinear dispersion in action. The integrable example presented by Wadati et al. and the recent examples presented by Camassa and Holm and one of us, [**3**], are concrete evidences that the integrable nonlinear dispersive examples found so far, are merely an edge of the proverbial iceberg. Though, the nature of the nonanalyticity changes in each case, these differences are technical, showing merely the different facets of nonlinearity.

The approach we adopt here generalizes the Lagrange transform method, and is based on a Hamiltonian form of duality in which one rearranges the Hamiltonian operators in the original soliton system in order to produce a dual system with nonlinear dispersion. (Indeed, an interesting open question is why, when applicable, the two approaches produce the same compacton-supporting equation.) The method of rearranging the Hamiltonian operators appears in earlier work of Fokas and Fuchssteiner, [**7**]. Indeed, the peakon model for water waves can be found (modulo a slight misprint) in an earlier, neglected work of Fuchssteiner, [**8**]. Our contributions include the systematization of the initial Hamiltonians required by the dual hierarchy, the identification of associated "Casimir" flows, which include the Harry Dym equation and interesting variants thereof, and the construction and analysis of new dual hierarchies for soliton systems of physical importance. Some of these systems also appear in a recent paper of Fokas, [**9**], that came to our attention after the present paper was completed.

While the original motivation for the present work was to look for compacton carrying integrable systems, its scope is much wider; we aim to unfold new integrable systems endowed with nonlinear dispersion, hoping that their integrability will provide a handle on the understanding of and provide a valuable clue on the mathematical form(s) of such systems. One can view this as an effort to find how integrable nonlinearly dispersive systems look like. (In the case of Wadati et al., [**2**], the integrability preceded the derivation of the physical model.) Needless to say, it would be far more desirable to provide a priori a systematic derivation of such systems from physical principles, but this will await future exploration. There is a wide disparity between the available mathematical tools, that as a rule rely on expansion in a small control parameter, and the physical reality which tends to locate many of the sought after effects beyond our reach. In fact, in all known cases where this conflict was avoided, some ad hoc trick(s) were used , which by their very nature do not translate into a general method, applicable to other problems. With this in mind, we start describing the mathematical approach which generates new, integrable systems endowed with nonlinear dispersion. Hopefully, some of these will find their application in concrete physical problems.

We shall demonstrate that, from the point of view of integrability, systems with nonlinear dispersion do not, in fact, represent a different entity from conventionally integrable systems. We implement an simple explicit algorithm, based on the biHamiltonian representation of the classically integrable system, which can be used to generate a wide variety of new integrable systems. In most cases, these new, non-evolutionary nonlinear systems are endowed with nonlinear dispersion, and thus support non-smooth soliton-like structures. In the present paper, we show how to derive such systems, leaving the analysis of their integrability, solitary wave solutions, scattering problems, etc., to future publications.

Our starting point is the general biHamiltonian formulation, [**10**], [**11**], [**12**],

$$u_t = F_1[u] = J_1 \frac{\delta H_2}{\delta u} = J_2 \frac{\delta H_1}{\delta u}. \tag{1}$$

of an integrable evolution equation. If the two Hamiltonian operators J_1, J_2 are *compatible*, meaning that any constant coefficient linear combination $c_1 J_1 + c_2 J_2$ is also Hamiltonian, then Magri's Theorem, [**10**], establishes the formal existence of an infinite hierarchy of higher order commuting biHamiltonian systems,

$$u_t = F_n[u] = J_1 \frac{\delta H_{n+1}}{\delta u} = J_2 \frac{\delta H_n}{\delta u}, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

based on the higher order conservation laws $H_n[u]$ common to all members of the hierarchy. The members of the hierarchy are successively generated by the recursion operator $\mathcal{R} = J_2 J_1^{-1}$, [**11**], [**12**]. Indeed, a theorem of Fokas and Fuchssteiner, [**13**], implies that the recursion operator arising from a Hamiltonian pair is a *hereditary operator*. Consequently, if both Hamiltonian operators are translationally symmetric, i.e., do not depend explicitly on x (as they are in all examples of interest), the hereditary condition effectively means that one can take the elementary wave equation $u_t = u_x$ as the "seed" biHamiltonian system, corresponding to $n = 0$ in (2), from which the higher order systems $u_t = F_n[u] = \mathcal{R}^n[u_x]$ are generated by the usual recursion procedure. Moreover, the recursion operator criterion,

$$\mathcal{R}_t = [\mathcal{B}, \mathcal{R}], \tag{3}$$

where $\mathcal{B}$ is the Fréchet derivative of the right hand side of (1), can be interpreted as a Lax pair formulation of the integrable biHamiltonian system, (1). However, it should be noted that, in most examples, (3) does *not* represent the standard Schrödinger, or Zakharov–Shabat/Ablowitz–Kaup–Newell–Segur (AKNS–ZS) Lax pair used to solve the equation by inverse scattering, [**14**], and its analytical solution is more difficult.

Example 1. *The Korteweg-deVries (KdV) equation*

To illustrate the method, let us consider the usual Korteweg-deVries (KdV) equation

$$u_t = u_{xxx} + 3uu_x. \tag{4}$$

It is well known, [**10**], [**11**], that this equation can be written in biHamiltonian form (1), using the two compatible Hamiltonian operators [**15**]

$$J_1 = D, \qquad J_2 = D^3 + uD + Du, \tag{5}$$

and the initial two Hamiltonian functionals (or conservation laws)

$$H_1 = \int \tfrac{1}{2} u^2 \, dx, \qquad H_2 = \int \tfrac{1}{2} \left[-u_x^2 + u^3 \right] dx. \tag{6}$$

Note that the seed equation $u_t = u_x$ is biHamiltonian, with Hamiltonian functionals $H_1[u]$ and $H_0[u] = \int u \, dx$; the latter is just the Casimir functional for J_1, [**16**].

The nonlinearly dispersive counterpart of the Korteweg-deVries equation (4) is obtained by the following procedure, which shall be explained in a form that readily generalizes. We begin by transferring the leading term, D^3, from the second Hamiltonian operator to the first, thereby constructing the first of the two new Hamiltonian operators [**17**]: $\widehat{J}_1 = D \pm D^3$. We factor $\widehat{J}_1 = D \cdot S$. The self-adjoint operator $S = 1 \pm D^2$ is used to define a new field variable $\rho = Su = u \pm u_{xx}$. The second Hamiltonian operator is constructed by replacing u by ρ in the remaining part of the original Hamiltonian operator J_2, so that $\widehat{J}_2 = \rho D + D\rho$. The fact that $\widehat{J}_1$ and $\widehat{J}_2$ form a compatible Hamiltonian pair follows immediately from the compatibility of the original Hamiltonian operators (5) along with a simple scaling argument to be described below. The desired integrable compacton equation is in biHamiltonian form

$$\rho_t = \widehat{J}_1 \frac{\delta \widehat{H}_2}{\delta \rho} = \widehat{J}_2 \frac{\delta \widehat{H}_1}{\delta \rho}, \tag{7}$$

with Hamiltonian functionals

$$\widehat{H}_1 = \int \tfrac{1}{2} u\rho \, dx = \int \tfrac{1}{2} \left[u^2 \mp u_x^2 \right] dx \, dx, \qquad \widehat{H}_2 = \int \tfrac{1}{2} \left[u^3 \mp u u_x^2 \right] dx, \tag{8$\pm$}$$

and hence forms the first member of a *new biHamiltonian hierarchy*. Equations (7) take the explicit form

$$u_t \pm u_{xxt} = 3uu_x \pm \left(u u_{xx} + \tfrac{1}{2} u_x^2 \right)_x. \tag{9$\pm$}$$

The choice of plus sign, (9+), leads to an integrable equation whose solitary wave solutions have compact support, [**1**], [**18**]. On the other hand, (9−) is the peakon equation derived by Camassa and Holm, [**5**], whose solitary wave solutions have a sharp corner at the crest. Interestingly equation (9−) made its first debut a decade ago in a work by Fuchssteiner, [**8**], as a part of a general scheme, [**19**], [**7**], introduced to derive new integrable systems, but was soon laid to rest. Genuine interest in this equation started in earnest with its derivation from physical considerations in [**5**]. Note that the complex transformation $x \mapsto ix$, $t \mapsto it$ will interchange (9+) and (9−), indicating a close interconnection between compactons and peakons. Equation (9±) can be viewed as an integrable modification of the BBM or regularized long wave equation, [**20**], which is obtained by omitting the last two terms on the right hand side (which are of higher order in the original perturbation expansion), [**21**]. Although the BBM equation is not integrable — its solitary wave solutions interact inelastically, [**22**], and has only finitely many local conservation laws, [**23**], — physically it has more desirable properties than the more mathematically intriguing Korteweg-deVries equation. Note that the first and second terms both the left and right hand side of (9±)

scale differently under the rescaling ("renormalization") $(x,t) \mapsto (\lambda x, \lambda t)$. Therefore, we can decouple the scaling limit equation [**1**]

$$u_{xt} = u u_{xx} + \tfrac{1}{2} u_x^2 \tag{10}$$

which we have integrated once. Equation (10) is a particular case of a class of nonlinear wave equations which were shown to be integrable by quadrature by Calogero, [**24**]. Note that its differentiated version can be derived directly from our triHamiltonian formulation by using $\widetilde{J}_1 = D^3$, $\widetilde{J}_2 = \rho D + D\rho$, where $\rho = u_{xx}$, as the generating Hamiltonian pair, and the appropriately truncated versions of (8) as Hamiltonians, [**25**].

An interesting observation is that the second Hamiltonian operator $\widehat{J}_2$ for (9±) admits the Casimir functional $\widehat{H}_C = \int 2\sqrt{\rho}\, dx$, which is an additional conservation law for (9±). Therefore, in addition to the standard biHamiltonian hierarchy, there is an additional "Casimir equation", namely $\rho_t = \widehat{J}_1 \delta \widehat{H}_C / \delta\rho$, which turns out to be the extended Harry Dym equation

$$\rho_t = (D \pm D^3)\, \rho^{-1/2}; \tag{11±}$$

whose appearance in connection with (9±) was first noted in [**5**]. In the scaling limit, the first order differential operator D drops out, and (11±) reduces to the usual Harry Dym equation. If we set $r = 1/\rho$, then (11±) becomes

$$r_t = r^2 (D \pm D^3)\, r^{1/2}. \tag{12±}$$

Equation (12−) is known to admit solitons having an unusual amplitude/speed relation, whereas (12+) admits compactons, [**1**]. The fact that the Harry Dym equation belongs to the dual hierarchy probably explains many of its unusual properties, as compared with other integrable systems; for instance it does not satisfy the Painlevé property, [**26**].

We now explain to what extent the preceding construction can be generalized to an arbitrary biHamiltonian system (1). In most situations, the second Hamiltonian operator associated with (1) is, in fact, the sum of two distinct Hamiltonian operators: $J_2 = K_2 + K_3$. (In the KdV example, $K_2 = D^3$ and $K_3 = uD + Du$.) Usually this happens because the two summands scale differently under $x \mapsto \lambda x$ and/or $u \mapsto \mu u$. Indeed, if $J_2 = K_2 + K_3$ maps to the Hamiltonian operator $\widetilde{J}_2 = \lambda^m K_2 + \lambda^n K_3$ under scaling, and $m \neq n$, then K_2 and K_3 clearly form a compatible Hamiltonian pair. In fact, in this situation, $J_1 = K_1$, K_2, K_3 form a compatible Hamiltonian triple, meaning that each linear combination $c_1 K_1 + c_2 K_2 + c_3 K_3$ is Hamiltonian; in particular, each possible pair of these three operators is compatible, [**27**]. In such cases, we can produce a new hierarchy of integrable equations by introducing the alternative Hamiltonian pair

$$\widehat{J}_1 = K_1 \pm K_2, \qquad \widehat{J}_2 = K_3. \tag{13}$$

For simplicity, we shall assume that $K_1 = D$ and K_2 are constant coefficient skew-adjoint differential operators, and, further,

$$\widehat{J}_1 = D \cdot S, \tag{14}$$

factorizes into a product of D with a symmetric constant coefficient differential operator S. We introduce the new variable

$$\rho = Su, \tag{15}$$

to replace u, so that $\widehat{J}_2$ is obtained from K_3 by replacing u by ρ wherever it occurs. As in (7), the resulting biHamiltonian systems are written in terms of the variable ρ. The scaling limit equation is obtained by a similar procedure, omitting the K_1 component of the first Hamiltonian operator, so $\widehat{J}_1 = K_2$, $\widehat{J}_2 = K_3$; the construction of ρ proceeds as before.

Applying the resulting hereditary recursion operator $\widehat{\mathcal{R}} = \widehat{J}_2 \cdot \widehat{J}_1^{-1}$ to the seed equation $\rho_t = \rho_x$ produces a hierarchy of commuting (possibly non-local) flows $\rho_t = \widehat{\mathcal{R}}^n(\rho_x)$. These will be biHamiltonian systems, *provided* the seed equation is, i.e., we can write

$$\rho_x = \widehat{J}_1 \frac{\delta \widehat{H}_1}{\delta \rho} = \widehat{J}_2 \frac{\delta \widehat{H}_0}{\delta \rho}. \tag{16}$$

On the other hand, the original soliton hierarchy (2) also begins with the linear wave equation, so we have

$$u_x = J_1 \frac{\delta H_1}{\delta u} = J_2 \frac{\delta H_0}{\delta u}. \tag{17}$$

where

$$H_0 = \int u\, dx, \qquad H_1 = \int \tfrac{1}{2} u^2\, dx.$$

In view of the chain rule formula for variational derivatives, cf. [**11**],

$$\frac{\delta \widehat{H}}{\delta u} = S\, \frac{\delta \widehat{H}}{\delta \rho} \qquad \text{when} \qquad \rho = Su, \tag{18}$$

equation (16) will be satisfied provided

$$\rho_x = \widehat{J}_1 \frac{\delta \widehat{H}_1}{\delta \rho} = J_1 S\, \frac{\delta \widehat{H}_1}{\delta \rho} = D \frac{\delta \widehat{H}_1}{\delta u}.$$

Therefore we should choose

$$\widehat{H}_0 = \int \rho\, dx, \qquad \widehat{H}_1 = \int \tfrac{1}{2} u \rho\, dx, \tag{19}$$

as our initial Hamiltonians. The *tri-Hamiltonian dual* (or *dual* for short) of the original soliton equation will thus take the form (7), where, using (18), the next Hamiltonian functional $\widehat{H}_2$ is found by solving

$$D \frac{\delta \widehat{H}_2}{\delta u} = \widehat{J}_2 \frac{\delta \widehat{H}_1}{\delta \rho}; \tag{20}$$

the existence of a suitable Hamiltonian $\widehat{H}_2$ is guaranteed by Magri's theorem, [**10**]. Finally, we remark that, because of the homogeneity assumptions on the Hamiltonian triple K_1, K_2, K_3, the resulting Hamiltonian functionals $\widehat{H}_0$, $\widehat{H}_1$, etc., are all necessarily homogeneous functionals under rescaling $u \mapsto \mu u$ of the dependent variable.

We now illustrate the general method with four additional examples. Of the large number of known integrable equations, the examples presented next are typical members of the solitonic zoo. A wide variety of additional compacton equations can, of course, be readily constructed starting with other soliton equations and systems, thereby leading to a new, and equally interesting, compacton zoo, whose complete taxonomy awaits future investigation.

Example 2. *The modified Korteweg-deVries (mKdV) equation*

The modified Korteweg-deVries (mKdV) equation

$$u_t = u_{xxx} + \tfrac{3}{2} u^2 u_x. \tag{21}$$

can be written in the biHamiltonian form (1), using the Hamiltonian operators

$$J_1 = D, \qquad J_2 = D^3 + D u D^{-1} u D, \tag{22}$$

and the associated Hamiltonian functionals

$$H_1 = \int \tfrac{1}{2} u^2 \, dx, \qquad H_2 = \int \left[\tfrac{1}{8} u^4 - \tfrac{1}{2} u_x^2 \right] dx. \tag{23}$$

The dual version is found by moving the linear part of the second Hamiltonian operator to the first, and defining a new variable $\rho = Su = u \pm u_{xx}$ which is to replace u in the second operator. This leads to the two new Hamiltonian operators

$$\widehat{J}_1 = D \pm D^3, \qquad \widehat{J}_2 = D \rho D^{-1} \rho D. \tag{24$\pm$}$$

The dual counterpart of the modified Korteweg-deVries equation takes the explicit form

$$\rho_t = u_t \pm u_{xxt} = \tfrac{1}{2} \left[(u^2 \pm u_x^2)(u \pm u_{xx}) \right]_x, \tag{25$\pm$}$$

which is in biHamiltonian form (7) using

$$\widehat{H}_1 = \int \tfrac{1}{2} [u\rho] \, dx = \int \tfrac{1}{2} [u^2 \mp u_x^2] \, dx, \qquad \widehat{H}_2 = \int \left[\tfrac{1}{8} u^4 - \tfrac{1}{24} u_x^4 \mp \tfrac{1}{4} u^2 u_x^2 \right] dx. \tag{26$\pm$}$$

As in the KdV example, the second Hamiltonian operator $\widehat{J}_2$ admits a Casimir functional, $\widehat{H}_C = \int \rho^{-1} \, dx$, so the Casimir equation for the modified compacton hierarchy is the equation

$$\rho_t = (D \pm D^3) \, \rho^{-2}. \tag{27$\pm$}$$

According to the formal symmetry approach of Shabat, [**28**], the two Casimir equations (11), (27), are the only two integrable cases of the general class of equations $\rho_t = D(1 \pm D^2)\,\rho^k$. Interestingly, the symmetries arising via the Shabat approach are local in ρ, whereas the dual hierarchy starting with (25$\pm$) forms a new hierarchy of nonlocal symmetries and conservation laws for (27$\pm$). As with (11$\pm$), equation (27$-$) admits solitons, whereas replacing ρ by $r = 1/\rho$ in (27$+$), we obtain

$$r_t = r^2 (D + D^3) r^2, \tag{28}$$

which is a Lagrange transform of the mKdV equation (21), and admits both traveling and stationary compactons, [**3**]. In particular, the stationary compacton is a Lagrange image of the one soliton solution of the mKdV equation, while the interaction of two solitons is mapped to an interaction of two overlapping stationary compactons.

In contrast to the KdV equation, whose second Hamiltonian operator has only trivial (local) Casimirs, the second mKdV Hamiltonian operator (22) admits the semi-local Casimir functional

$$H_C = -\int \cos\left[D^{-1} u\right] dx = -\int_{-\infty}^{\infty} \cos\left[\int_{-\infty}^{x} u(\xi)\, d\xi\right] dx.$$

Ignoring integration constants, the associated Casimir equation is

$$u_t = \sin\left[\int_{-\infty}^{x} u(\xi)\, d\xi\right], \tag{29}$$

which is just the sine–Gordon equation $\psi_{xt} = \sin\psi$ for the potential function $\psi_x = u$. The mKdV hierarchy forms the associated higher order symmetries and conservation laws for the sine–Gordon equation (29), [**12**]. This leads one to interpret the Casimir equation (27) as the dual counterpart of the sine–Gordon equation. Interestingly, while the original hamiltonians (23) and their duals (26) bear an obvious resemblance, the corresponding Casimirs are strikingly different, and certainly would not be recognized as originating from the same triHamiltonian structure. This shows that the original hierarchy and its dual counterpart can be quite different in both their algebraic and analytic properties.

Example 3. *A Boussinesq System*

There is a wide variety of bidirectional Boussinesq systems that arise from the standard perturbation expansion for the free boundary problem describing the propagation of shallow water waves, [**29**], [**30**]. From an algebraic standpoint, the most interesting of these is the version proposed by Whitham, [**31**], which was shown to be integrable by inverse scattering by Kaup, [**32**]. Subsequently, Kupershmidt, [**33**], rewrote the physical system in the form

$$v_t = v v_x + w_x - v_{xx}, \qquad w_t = (vw)_x + w_{xx}, \tag{30}$$

and showed that this system is, in fact, tri-Hamiltonian. Concentrating on the simpler Hamiltonian pair, (30) appears in the biHamiltonian form (1) with

$$J_1 = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 2D & -2D^2 + Dv \\ 2D^2 + vD & wD + Dw \end{pmatrix}, \tag{31}$$

$$H_1 = \int \tfrac{1}{2} vw \, dx, \qquad H_2 = \int \left[-v_x w + \tfrac{1}{2} v^2 w + \tfrac{1}{2} w^2 \right] dx. \tag{32}$$

The dual version of (30) relies on the Hamiltonian operators [**34**]

$$\widehat{J}_1 = \begin{pmatrix} 2D & -2D^2 + D \\ 2D^2 + D & 0 \end{pmatrix} = D \cdot \mathcal{S}, \qquad \widehat{J}_2 = \begin{pmatrix} 0 & D\sigma \\ \sigma D & \tau D + D\tau \end{pmatrix}, \tag{33}$$

where $\mathcal{S} = \begin{pmatrix} 2 & -2D+1 \\ 2D+1 & 0 \end{pmatrix}$. We therefore define the new variables

$$\begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \mathcal{S} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 2v + w - 2w_x \\ v + 2v_x \end{pmatrix}. \tag{34}$$

The associated Hamiltonian functionals are

$$\begin{aligned} \widehat{H}_1 &= \int \tfrac{1}{2} \left[v\sigma + w\tau \right] dx = \int \left[v^2 + vw - 2vw_x \right] dx, \\ \widehat{H}_2 &= \int \left[v^2 w + vw^2 - 2vww_x \right] dx. \end{aligned} \tag{35}$$

Using the analogue of (18) to compute variational derivatives with respect to σ, τ, we deduce that the associated biHamiltonian system has the explicit form

$$\sigma_t = (w\sigma)_x, \qquad \tau_t = (w\tau + v^2 + vw)_x,$$

or, in full detail,

$$2v_t + w_t - 2w_{xt} = -(w^2)_{xx} + (2vw + w^2)_x, \qquad v_t + 2v_{xt} = (2v_x w + v^2 + 2vw)_x. \tag{36}$$

The dispersion relation for the linear terms in (36) are found by setting the right hand side to zero. Eliminating v, we find $w_{tt} - 4w_{xxtt}$, which, up to scaling, is the linear dispersion relation for the modification of the Boussinesq equation considered in [**35**] as a model for ion-acoustic waves in plasma and longitudinal waves in an elastic rod. In particular, if we take $v = 0$ we find the interesting equation

$$w_t - 2w_{xt} = D(1 - D)(w^2).$$

The second Hamiltonian operator $\widehat{J}_2$ admits a Casimir functional, $\widehat{H}_C = \int [\tau/\sigma] \, dx$, so the dual Casimir equation for (36) is the unusual bidirectional equation:

$$\sigma_t = -2(\sigma^{-2}\tau)_x + (\sigma^{-1})_x - 2(\sigma^{-1})_{xx}, \qquad \tau_t = -(\sigma^{-2}\tau)_x - 2(\sigma^{-2}\tau)_{xx}. \tag{37}$$

Interestingly, when $\tau = 0$, using a Lagrange transformation the first equation can be mapped into a Burgers' equation for $z = 1/\sigma$. (Similarly, setting $w = 0$ in (30) reduces it to Burgers' equation.)

Example 4. *The Ito system*

Inspired by the symmetry approach, Ito, [**36**], proposed an integrable, coupled nonlinear wave equation

$$u_t = u_{xxx} + 3uu_x + vv_x, \qquad v_t = (uv)_x, \tag{38}$$

that extends the Korteweg-deVries equation for u by an additional "enslaved" field variable v. The biHamiltonian form for (38) requires

$$J_1 = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \qquad J_2 = \begin{pmatrix} D^3 + uD + Du & vD \\ Dv & 0 \end{pmatrix}, \tag{39}$$

$$H_1 = \int \tfrac{1}{2}\left[u^2 + v^2\right] dx, \qquad H_2 = \int \tfrac{1}{2}\left[u^3 + uv^2 - u_x^2\right] dx. \tag{40}$$

Introduce $\rho = Su = u \pm u_{xx}$ and $\sigma = v$ as new variables, whose forms are governed by the dual Hamiltonian operators

$$\widehat{J}_1 = \begin{pmatrix} D \pm D^3 & 0 \\ 0 & D \end{pmatrix} = D \cdot \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}, \qquad \widehat{J}_2 = \begin{pmatrix} \rho D + D\rho & vD \\ Dv & 0 \end{pmatrix}. \tag{41$\pm$}$$

Setting

$$\begin{aligned} \widehat{H}_1 &= \int \tfrac{1}{2}\left[\rho u + v^2\right] dx = \int \tfrac{1}{2}\left[u^2 + v^2 \mp u_x^2\right] dx, \\ \widehat{H}_2 &= \int \tfrac{1}{2}\left[u^3 + uv^2 \mp uu_x^2\right] dx, \end{aligned} \tag{42$\pm$}$$

the dual biHamiltonian system takes the form

$$u_t \pm u_{xxt} = 3uu_x + vv_x \pm \left(uu_{xx} + \tfrac{1}{2}u_x^2\right)_x, \qquad v_t = (uv)_x, \tag{43$\pm$}$$

which effectively defines an integrable enslavement of v to the compacton/peakon equations (9$\pm$). Again, the second Hamiltonian operator admits a Casimir functional, $H_C = \int (\rho/v)\, dx = \int (u \pm u_{xx})/v\, dx$, leading to an associated Casimir equation

$$u_t \pm u_{xxt} = (v^{-1})_x \pm (v^{-1})_{xxx}, \qquad v_t = -[v^{-2}(u \pm u_{xx})]_x. \tag{44$\pm$}$$

Interestingly, in the above derivation, except for the Casimir equation (44$\pm$), we can set $v = 0$ and reduce back to the KdV hierarchy or its dual counterpart. However, in (44$\pm$) the variables ρ and v become coupled in an essential manner. Let us set $w = 1/v$ and invert the operator $1 \pm D^2$ in the first equation; then (44$\pm$) reduces to the new integrable nonlinearly dispersive system

$$u_t = w_x, \qquad w_t = w^2[w^2(u \pm u_{xx})]_x. \tag{45$\pm$}$$

If we define the "stream function" so that $\psi_x = u$, $\psi_t = w$, then ψ satisfies the unusual fourth order equation

$$\psi_{tt} = \psi_t^2[\psi_t^2(\psi_x \pm \psi_{xxx})]_x. \tag{46$\pm$}$$

In the scaling limit, and integrating once, (43$\pm$) reduces to

$$u_{xt} = \mp\tfrac{1}{2}v^2 + uu_{xx} + \tfrac{1}{2}u_x^2, \qquad v_t = (uv)_x. \tag{47$\pm$}$$

The corresponding reduced Casimir equation (46) has scaling limit

$$\psi_{tt} = \psi_t^2(\psi_t^2\psi_{xxx})_x, \tag{48}$$

which can be viewed as an integrable bi-directional version of the Harry Dym equation!

Example 5. *The Nonlinear Schrödinger Equation*

The nonlinear Schrödinger equation

$$u_t = i(u_{xx} + |u|^2 u) \tag{49}$$

can also be treated by the general method, although its dual "compacton" version is perhaps a curiosity. The two Hamiltonian operators are

$$J_1(F) = iF, \qquad J_2(F) = DF + uD^{-1}(\bar{u}F - u\overline{F}), \tag{50}$$

and

$$H_1 = \int \left[-iu\bar{u}_x \right] dx, \qquad H_2 = \int \left[-|u_x|^2 + \tfrac{1}{2}|u|^4 \right] dx, \tag{51}$$

are the required conservation laws, [**10**]. To verify this, it is important to note that the variational derivative is to be computed based on the Hermitian inner product $\langle u\,; v \rangle = \int \left[u\bar{v} + \bar{u}v \right] dx$, so $\delta H/\delta u = \mathrm{E}_{\bar{u}}(H)$ where $\mathrm{E}_{\bar{u}} = \partial_{\bar{u}} - D\partial_{\bar{u}_x} + D^2\partial_{\bar{u}_{xx}} + \cdots$ denotes the Euler operator with respect to the complex conjugate variable $\bar{u}$, [**37**].

In accordance with the general method, we introduce the two new Hamiltonian operators

$$\widehat{J}_1(F) = (D + i)F, \qquad \widehat{J}_2(F) = \rho D^{-1}(\bar{\rho}F - \rho\overline{F}), \tag{52}$$

leading to the new field variable $\rho = Su = -iu_x + u$, whose form is dictated by the factorization $\widehat{J}_1 = D + i = (-iD + 1)i = \mathcal{S} \cdot J_1$. The resulting biHamiltonian system

$$\rho_t = u_t - iu_{xt} = |u|^2(u_x + iu). \tag{53}$$

uses the dual Hamiltonian functionals

$$\widehat{H}_1 = \int \left[-i\bar{u}u_x + |u|^2 \right] dx = \int \left[\bar{u}\rho \right] dx, \qquad \widehat{H}_2 = \int \tfrac{1}{2}\left[-i|u|^2\bar{u}u_x + |u|^4 \right] dx. \tag{54}$$

The triHamiltonian dual to the nonlinear Schrödinger equation (53) is particularly trivial, since replacing u by $v = ue^{ix}$, we find

$$-iv_{xt} = |v|^2 v_x. \tag{55}$$

This equation has a first integral $|v_x|^2$. Here, in contrast to the compacton version of the KdV equation, the dispersion remains linear; this is because, in contrast to the previous two cases, the Hamiltonian operator $\widehat{J}_2$ is a pure integral operator. The construction of an associated hierarchy is more problematic in this case due to nonlocalities.

Discussion. In this paper, we have shown how a simple scaling argument leads to a triHamiltonian structure for standard integrable soliton equations. Rearranging the Hamiltonian operators in an algorithmic manner leads to dual integrable systems which, in most instances, have nonlinear dispersion and thus admit non-smooth solitons, either compactly supported or with cusps or corners. Our general method can be readily applied to all of the known soliton hierarchies, and, as we have demonstrated with a few of the more standard examples, immediately leads to new and interesting integrable systems. The mathematical and physical properties of the new hierarchies remains to be developed. Topics that will be under investigation include the properties of the non-smooth solitary wave solutions, the analysis of the associated scattering problems, which may be based on the recursion operators as in (3), the locality or nonlocality of the associated hierarchies of symmetries and conservation laws, and, of course, physical applications of these new systems. In addition, we anticipate a large number of new and interesting compacton equations can be generated by other soliton hierarchies, such as the general AKNS system, the three wave interaction equations, and others.

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[**15**] Here $D = D_x$ denotes derivative with respect to x.

[16] A *Casimir* or *distinguished functional* is one whose variational derivative is annihilated the Hamiltonian operator: $J(\delta H/\delta u) = 0$. It thus forms a conservation law for any Hamiltonian system having J as the Hamiltonian operator; see [**11**], [**12**], for details.

[17] We can, of course, use other linear combinations of D and D^3. However, a simple rescaling reduces them to $D \pm D^3$.

[18] Equation (9+) is not well posed, but higher order members of its hierarchy are.

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[37] Alternatively, one can rewrite (49) as a system for the real and imaginary parts of the field variable $u = v + iw$.