ORBITAL STABILITY OF PEAKONS FOR A GENERALIZATION OF THE MODIFIED CAMASSA-HOLM EQUATION

XIAOCHUAN LIU, YUE LIU, PETER J. OLVER, AND CHANGZHENG QU

Abstract. The orbital stability of the peaked solitary-wave solutions for a generalization of the modified Camassa-Holm equation with both cubic and quadratic nonlinearities is investigated. The equation is a model of asymptotic shallow-water wave approximations to the incompressible Euler equations. It is also formally integrable in the sense of the existence of a Lax formulation and bi-Hamiltonian structure. It is demonstrated that, when the Camassa-Holm energy counteracts the effect of the modified Camassa-Holm energy, the peakon and periodic peakon solutions are orbitally stable under small perturbations in the energy space.

Key words and phrases: Camassa-Holm equation; modified Camassa-Holm equation; peakon; integrable system; orbital stability.

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1. Introduction

The well-studied *Camassa-Holm* (CH) *equation*

$$m_t + 2u_x m + u m_x = 0, \qquad m = u - u_{xx}, \tag{1.1}$$

was originally proposed as a nonlinear model for the unidirectional propagation of the shallow water waves over a flat bottom [1, 9, 18, 22]. The CH equation can also be derived by applying the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the Korteweg-de Vries (KdV) equation, thus justifying its status as a dual integrable bi-Hamiltonian system [17, 29]. Tri-Hamiltonian duality is based on the observation that most compatible pairs of Hamiltonian operators are, in fact, linear combinations of three mutually compatible Hamiltonian operators, and reconfiguring the operators in question will yield interesting new bi-Hamiltonian systems.

Applying tri-Hamiltonian duality to the modified Korteweg–deVries (mKdV) equation, leads to the *modified Camassa-Holm* (mCH) *equation* with cubic nonlinearity

$$m_t + \big((u^2 - u_x^2)m\big)_x = 0, \qquad m = u - u_{xx}. \tag{1.2}$$

As a consequence of duality, the mCH equation (1.2) is formally integrable in the sense that it admits a bi-Hamiltonian structure [29] and was later shown to admit a Lax formulation [30]. Moreover, the mCH equation (1.2) exhibits new features, including wave breaking and blowup criteria [19] that do not appear in the original CH equation (1.1) [3, 5, 6, 7, 24]. On the other hand, since the mCH equation (1.2) also arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [19], it can be regarded as a Euclidean-invariant version of the CH equation (1.1), just as the mKdV equation is a Euclidean-invariant counterpart to the KdV equation from the viewpoint of curve flows in Klein geometries [2, 20]. It is worth mentioning that besides the application of the mCH equation in the modeling of nonlinear water waves pointed out by Fokas [15], the present authors [19] showed that the scaling limit equation of (1.2), when combined with the first-order term γu_x, satisfies the *short-pulse equation*

$$v_{xt} = \frac{1}{3}\,(v^3)_{xx} + \gamma\, v, \tag{1.3}$$

which is a model for the propagation of ultra-short light pulses in silica optical fibers [34].

More generally, applying tri-Hamiltonian duality to the bi-Hamiltonian *Gardner equation*

$$u_t + u_{xxx} + k_1 u^2 u_x + k_2 u u_x = 0, \tag{1.4}$$

the resulting dual system is the following *generalized modified Camassa-Holm* (gmCH) *equation* with both cubic and quadratic nonlinearities [16, 17]:

$$m_t + k_1\left((u^2 - u_x^2)m\right)_x + k_2(2u_x m + u m_x) = 0, \qquad m = u - u_{xx}. \tag{1.5}$$

This equation, posed on the real line $x \in \mathbb{R}$ and also on the circle $x \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, i.e. subject to periodic boundary conditions, is the object of study in the present paper. It models the unidirectional propagation of the shallow-water waves over a flat bottom, where the function u represents the free surface elevation, and was derived from the two-dimensional hydrodynamical equations for surface waves by Fokas [16]. Note that equation (1.5) reduces to the CH equation (1.1) when $k_1 = 0$, $k_2 = 1$, and to the mCH equation (1.2) when $k_1 = 1$, $k_2 = 0$, respectively.

The derivation of the gmCH (1.5) via the method of tri-Hamiltonian duality reveals its status as an integrable system. Indeed, it can be written in the bi-Hamiltonian form [28, 31]

$$m_t = J\,\frac{\delta H_1}{\delta m} = K\,\frac{\delta H_2}{\delta m}, \tag{1.6}$$

where

$$J = -k_1 \partial_x\, m\, \partial_x^{-1}\, m\, \partial_x - \tfrac{1}{2}k_2(m\,\partial_x + \partial_x\, m) \qquad \text{and} \qquad K = -\tfrac{1}{4}\left(\partial_x - \partial_x^3\right)$$

are compatible Hamiltonian operators, while the corresponding Hamiltonian functionals are given by

$$H_1[u] = \int_X \left(u^2 + u_x^2\right)\, dx, \qquad \text{and} \qquad H_2[u] = k_1 I_1[u] + 2k_2 I_2[u], \tag{1.7}$$

where

$$I_1[u] = \int_X \left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right)\, dx, \qquad I_2[u] = \int_X \left(u^3 + u u_x^2\right)\, dx. \tag{1.8}$$

Throughout, $X = \mathbb{R}$ in the real line case, while $X = \mathbb{S}^1$ in the periodic case. Furthermore, a Lax pair for (1.5) was established in [31].

The gmCH equation (1.5) belongs to a novel class of physically important integrable equations. Indeed, consider the motion of a 2-dimensional, inviscid, incompressible and irrotational fluid (e.g. water) on a horizontal flat bottom located at $y = -h_0$, with h_0 a positive constant, and air above the free surface, whose displacement from equilibrium is represented by $\eta(t, x)$. The system of such a motion is characterized by two parameters, $\alpha = a/h_0$ and $\beta = h_0^2/l^2$, where a and l are typical values of the amplitude and of the wavelength of the waves. For unidirectional wave propagation, applying the physically meaningful asymptotic analysis in the shallow water regime, that is, neglecting the terms of $\mathcal{O}(\alpha^4, \alpha^3\beta, \beta^2)$, it is shown that the free surface $\eta(t, x)$ satisfies [16]

$$\begin{aligned}\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} &+ \rho_1\alpha^2\eta^2\eta_x + \alpha\beta\left(\rho_2\eta\eta_{xxx} + \rho_3\eta_x\eta_{xx}\right)\\ &+ \rho_4\alpha^3\eta^3\eta_x + \alpha^2\beta\left(\rho_5\eta^2\eta_{xxx} + \rho_6\eta\eta_x\eta_{xx} + \rho_7\eta_x^3\right) = 0,\end{aligned} \tag{1.9}$$

where $\rho_1, \ldots, \rho_7$, are constants. Employing the following Kodama transformation

$$\eta = u + \lambda_1\alpha u^2 + \lambda_2\beta u_{xx} + \lambda_3\alpha^2 u^3 + \alpha\beta\left(\lambda_4 u u_{xx} + \lambda_5 u_x^2\right),$$

where $\lambda_1, \ldots, \lambda_5$ are certain combinations of the ρ_j, Fokas [16] pointed out that the asymptotic model (1.9) is equivalent to the integrable gmCH equation (1.5). The appearance of an integrable equation in a class of equivalent models can reveal a great amount of information about the underlying physical system. In addition, it was argued by Fokas [15] that equation (1.5) describes the physics more accurately than its celebrated counterpart, the Gardner equation (1.4).

Furthermore, the gmCH equation (1.5) admits the following scaling-limit version

$$v_{xt} - k_1 v_x^2 v_{xx} + k_2 \left(v v_{xx} + \frac{1}{2} v_x^2 \right) = 0, \tag{1.10}$$

which is also integrable and models the asymptotic dynamics of a short capillary-gravity wave with $v(t,x)$ denoting the fluid velocity on the surface [14]. Notably, the integrable model that, in a sense, lies midway between (1.5) and its limiting version (1.10), known as the *generalized μ-CH equation*, was introduced in [33]:

$$m_t + k_1 \left((2\mu(u)\, u - u_x^2) m \right)_x + k_2 (2u_x m + u m_x) = 0, \qquad m = \mu(u) - u_{xx}, \tag{1.11}$$

where $u(t,x)$ is a real-valued spatially periodic function, with

$$\mu(u) = \int_{\mathbb{S}^1} u(t,x) dx \tag{1.12}$$

denoting its total integral.

Dual integrable nonlinear systems, such as the CH equation (1.1), the mCH equation (1.2) and the gmCH equation (1.5), exhibit nonlinear dispersion, and, in most cases, admit a remarkable variety of non-smooth soliton-like solutions, including peakons, compactons, tipons, rampons, mesaons, and so on [25]. For the CH equation (1.1), its single peakon has the exponential form

$$\varphi_c(t,x) = a\, e^{-|x-ct|}, \tag{1.13}$$

with amplitude equal to the wave speed:

$$a = c, \qquad c \in \mathbb{R}.$$

On the other hand, the periodic peakons are given by

$$\psi_c(t,x) = a \cosh\left(\tfrac{1}{2} - (x - c\,t) + [x - c\,t] \right), \tag{1.14}$$

where the notation $[x]$ denotes the largest integer part of the real number $x \in \mathbb{R}$; the amplitude is given by

$$a = \frac{c}{\cosh \frac{1}{2}}, \qquad c \in \mathbb{R}.$$

The mCH equation (1.2) admits the single peakon of the same exponential form (1.13), but with only positive wave speeds and corresponding amplitude [19]

$$a = \sqrt{\frac{3c}{2}}, \qquad c > 0.$$

Similarly, its periodic peakon is given by (1.14), again with positive wave speed and amplitude [32]

$$a = \sqrt{\frac{3c}{1 + 2\cosh^2 \frac{1}{2}}}, \qquad c > 0.$$

Both equations (1.1) and (1.2) also admit multi-peakon solutions of the form

$$u(t,x) = \sum_{i=1}^{N} p_i(t)\, e^{-|x - q_i(t)|},$$

although the dynamical behavior of the coefficients $p_i(t)$, $q_i(t)$ is rather different in each instance. Indeed, for the CH equation they satisfy the dynamical system [1, 12, 21]

$$\dot{p}_i = \sum_{j\neq i} p_i p_j \,\mathrm{sign}\,(q_i - q_j)\, e^{-|q_i-q_j|}, \qquad \dot{q}_i = \sum_{j} p_j \, e^{-|q_i-q_j|}, \qquad i = 1,\ldots,N,$$

whereas for the mCH equation [19] they satisfy

$$\dot{p}_i = 0, \qquad \dot{q}_i = \tfrac{2}{3}p_i^2 + 2\sum_{j=1}^{N} p_j p_i e^{-|q_i-q_j|} + 4 \sum_{1\leq k<i,i<j\leq N} p_k p_j e^{-|q_k-q_j|}, \qquad i = 1,\ldots,N.$$

Thus, interestingly, unlike the CH equation, the multi-peakon amplitudes for the mCH equation are independent of time [19].

Recently, it was found [31] that, for $k_1 \neq 0$, the gmCH equation (1.5) admits a single peakon of the form (1.13) with the following amplitude and restriction on the wave speed:

$$a = \frac{3}{4}\,\frac{-k_2 \pm \sqrt{k_2^2 + \frac{8}{3}k_1 c}}{k_1}, \qquad k_2^2 + \tfrac{8}{3}k_1 c \geq 0. \tag{1.15}$$

In this paper, we shall prove that equation (1.5) also possesses periodic peakons of the form (1.14) with

$$a = \frac{3}{2}\,\frac{-k_2 \cosh\frac{1}{2} \pm \sqrt{k_2^2 \cosh^2\frac{1}{2} + \frac{4}{3}k_1 c\left(1 + 2\cosh^2\frac{1}{2}\right)}}{k_1\left(1 + 2\cosh^2\frac{1}{2}\right)}, \qquad k_2^2 \cosh^2\tfrac{1}{2} + \tfrac{4}{3}k_1 c\left(1 + 2\cosh^2\tfrac{1}{2}\right) \geq 0. \tag{1.16}$$

It is worth noting that the periodic peakons of the μ-integrable equation are of a manifestly different character. For example, in [33], the authors showed that the periodic peakons of the generalized μ-CH equation (1.11) take the following form

$$u(t,x) = \chi_c(x - ct) = a\chi(x - ct), \tag{1.17}$$

where

$$\chi(x) = x^2 + \frac{23}{12}, \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

extended periodically to the real line and

$$a = \frac{-13k_2 \pm \sqrt{169k_2^2 + 1200ck_1}}{50k_1}$$

with $169k_2^2 + 1200ck_1 \geq 0$.

Physically, the principal feature of the preceding peakons that their profile is smooth, except at the crest where it is continuous but the lateral tangents differ, is similar to that of the well known Stokes waves of greatest height — the traveling waves of maximum possible amplitude that are solutions to the governing equations for irrotational water waves [4, 8, 35]. In our case, choosing different signs of the parameters k_1 and k_2 enables us to better understand how the peakons and anti-peakons interact in propagation of waves, recovering their shape and speed after a nonlinear interaction. However, if they are to be validated as physically relevant solutions, they must be dynamically stable under small perturbations. Since a small change in the height of a peakon yields another one traveling at a different speed, the appropriate notion of stability here is that of orbital stability: a wave with an initial profile close to a peakon remains close to some translate of it for all later times. That is, the shape of the wave remains approximately the same for all times.

In an innovative paper [11], using the known conservation laws of the CH equation and underlying features of the peakons, Constantin and Strauss proved that the single peakon solutions of the CH equation (1.1) are orbitally stable. Their key argument is to establish an inequality relating the conserved densities with the maximal values of the perturbed solutions. The Constantin–Strauss approach was recently extended to study the orbital stability of single peakons for the Degasperis–Procesi (DP) equation [26], and the mCH equation [32]. A variational approach for establishing the orbital stability of the CH peakons was introduced by Constantin and Molinet [10]. The orbital stability of trains of peakons of the CH equation and the mCH equation was explored in [13] and in [27], respectively. Stability of the periodic peakons of the CH equation was established by Lenells [23]. Very recently, Lenells' approach was further extended to prove the orbital stability of the periodic peakons for the mCH equation [32]. In [33], we were able to find an analytical method that could deal with the interaction between the different copmponents in the energy and obtained the orbital stability of periodic peakons (1.17) of the generalized μ-CH equation (1.11). However, for reasons explained in detail below, this analysis does not extend to the gmCH equation, which requires a new approach.

The goal of this paper is to establish the orbital stability of single peakons and periodic peakons of the gmCH equation (1.5). We will establish the following respective stability results.

Theorem 1.1. *Assume that* $k_1 > 0$, $k_2 > 0$ *and* $c \geq -\frac{3}{8}k_2^2/k_1$, *or* $k_1 > 0$, $k_2 \leq 0$ *and* $c > \frac{2}{3}k_2^2/k_1$. *Under either of these conditions, the peakon solutions of the gmCH equation* (1.5) *are orbitally stable in the energy space* $H^1(\mathbb{R})$.

Theorem 1.2. *Assume that either*

$$k_1 > 0, \quad k_2 > 0, \quad \textit{and} \quad c \geq -\frac{3k_2^2 \cosh^2\frac{1}{2}}{4k_1\left(1+2\cosh^2\frac{1}{2}\right)}, \tag{1.18}$$

or

$$k_1 > 0, \quad k_2 \leq 0, \quad \textit{and} \quad c > \frac{3k_2^2}{4k_1}\left(2\cosh^2\tfrac{1}{2} - 2\cosh\tfrac{1}{2} + 1\right). \tag{1.19}$$

Then the periodic peakon solutions of the gmCH equation (1.5) *are orbitally stable in the energy space* $H^1(\mathbb{S}^1)$.

For the gmCH equation (1.5), the three conservation laws $H_1[u], H_2[u]$ in (1.7), along with

$$H_0[u] = \int_X u\,dx, \tag{1.20}$$

will play a major role in our analysis. From the conservation law $H_1[u]$, it is reasonable to expect the orbital stability of (periodic) peakons for (1.5) in the sense of the energy space H^1 norm. The approach used here is motivated by the recent works [11, 23, 32]. The key issue is to establish a suitable inequality relating the maximum (and minimum) of the perturbed solution with the conserved densities, and this will rely on the introduction of a suitably constructed auxiliary function. Moreover, the corresponding equality is required to hold at the (periodic) peakons; this condition is crucial since stable (periodic) peakons must be critical points of the energy functional with the momentum constraint, and satisfy the corresponding Euler-Lagrangian equations. Since the gmCH equation (1.5) consists of two parts — the cubic mCH terms and the quadratic CH terms — this suggests choosing the required auxiliary function in the form

$$h(t,x) = k_1\left(u^2(t,x) \mp \tfrac{2}{3}u(t,x)u_x(t,x) - \tfrac{1}{3}u_x^2(t,x)\right) + 2k_2u(t,x)$$

for the non-periodic case $X = \mathbb{R}$, and

$$h(t,x) = k_1\left(u^2(t,x) \pm \tfrac{2}{3}u_x(t,x)\sqrt{u^2(t,x)-L^2} - \tfrac{1}{3}u_x^2(t,x) - L^2\right) + 2k_2u(t,x)$$

with $L = \min u(t,x)$, for the periodic case $X = \mathbb{S}^1$, respectively.

As for the signs of k_1 and k_2, we shall consider two possibilities: (i) $k_1 > 0$ and $k_2 > 0$; (ii) $k_1 > 0$ and $k_2 \le 0$. In the first case, $h(t,x)$ can be estimated by the maximum (and minimum) of $u(t,x)$ using the approach in [23, 32], and the stability results follow similarly. The second case is more delicate to deal with due to the interaction between two components (1.8) of the energy H_2 that have opposite signs. For peakons on the line, motivated by the idea in [33], the key observation is that the mCH part $I_1[u]$ of $H_2[u]$ can be dominated by the CH part $I_2[u]$ of $H_2[u]$, as in (1.8), in a subtle way. More precisely, the following inequality is derived:

$$I_1[u] \le \tfrac{4}{3} M\, I_2[u], \qquad \text{where} \qquad M = \max u(t,x).$$

While a slight different relationship was established in [33] in the periodic case for the corresponding periodic peakons, the detailed proofs are, however, rather different and neither can be deduced from the other. It is noted that the stability results in [33] rely on the construction of a concave function associated to the maximum and minimum of the perturbed solution with certain properties of periodic peakons. Our argument in the present paper, however, requires only estimating the maximum of the perturbed solutions.

In particular, in the periodic case, the generalized μ-CH equation (1.11) admits the integral $\int_{\mathbb{S}^1} u_x^2\, dx$ as a conservation law, which is crucial in the argument to establish the orbital stability of the periodic peakons (1.17) of (1.11). But this approach does not work for the gmCH equation (1.5), due to a lack of such a conserved quantity, and thus a new approach is be required. A key observation to our case is how to estimate a uniform lower bound for the minimum of the perturbed solution $u(t,x)$. It is worth mentioning that the required lower bound depends only on the appropriate conserved densities. As a result of these insights, the orbital stability of periodic peakons in case (ii) can be established.

The outline of the paper is as follows. Section 2 is a short review on the well-posedness of the gmCH equation (1.5). In addition, the existence of periodic peakons is demonstrated rigorously in Theorem 2.1, whose proof is relegated to the Appendix. The orbital stability of the peakon solutions in the Sobolev space $H^1(\mathbb{R})$ is established in Section 3. In Section 4, it is shown that the periodic peakons are dynamically stable under small perturbations in the energy space $H^1(\mathbb{S}^1)$.

Notation. Throughout the paper, the norm of a Banach space Z is denoted by $\|\cdot\|_Z$, while $C([0,T), Z)$ denotes the class of continuous functions from the interval $[0,T)$ to Z. In the periodic case, we denote $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ as the unit circle and regard functions on $\mathbb{S}^1$ as periodic on the entire line with period one. Given $T > 0$, let $C_c^\infty([0,T) \times X)$ denote the space of all smooth functions with compact support on $[0,T) \times X$, which can also be viewed as the space of smooth functions on $\mathbb{R} \times X$ having compact support contained in $[0,T) \times X$. For $1 \le p < \infty$, $L^p(X)$ denotes the space of equivalence classes of Lebesgue measurable, $p^{th}-$power integrable, real-valued functions defined on X. The usual modification is in effect for $p = \infty$. The norm on $L^p(X)$ is written as $\|\cdot\|_{L^p(X)}$. For $s \ge 0$, the L^2-based Sobolev space H^s is the subspace of those L^2 functions whose derivatives up to order s all lie in L^2. The associated norm is denoted as $\|\cdot\|_{H^s(X)}$.

2. Preliminaries

In this paper, we are concerned with the Cauchy problem for the gmCH equation on both the line and the unit circle:

$$\begin{cases} m_t + k_1\big((u^2 - u_x^2)\, m\big)_x + k_2(2u_x m + u m_x) = 0, & m(t,x) = u(t,x) - u_{xx}(t,x), \\ u(0,x) = u_0(x), & t > 0, \quad x \in X = \mathbb{R} \ \text{ or } \ \mathbb{S}^1. \end{cases} \tag{2.1}$$

We first formalize the notion of a strong solution.

Definition 2.1. *If $u \in C([0,T), H^s(X)) \cap C^1([0,T), H^{s-1}(X))$, with $s > \frac{5}{2}$ and some $T > 0$, satisfies* (2.1)*, then u is called a* strong solution *on $[0,T)$. If u is a strong solution on $[0,T)$ for every $T > 0$, then it is called a global strong solution.*

The following local well-posedness result and properties for strong solutions on both the line and the unit circle can be established using the same approach as in [19]. The proofs are thus omitted.

Proposition 2.1. *Let $u_0 \in H^s(X)$ with $s > \frac{5}{2}$. Then there exists a time $T > 0$ such that the initial value problem* (2.1) *has a unique strong solution $u \in C([0,T), H^s(X)) \cap C^1([0,T), H^{s-1}(X))$. Moreover, the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $H^s(X)$ into $C([0,T), H^s(X)) \cap C^1([0,T), H^{s-1}(X))$.*

Proposition 2.2. *The Hamiltonian functionals* (1.7)*,* (1.20) *are all conserved for the strong solution $u(t,x)$ obtained in Proposition 2.1, that is*

$$\frac{d}{dt}H_0[u] = \frac{d}{dt}H_1[u] = \frac{d}{dt}H_2[u] = 0, \qquad \text{for all} \quad t \in [0,T).$$

Furthermore, if the initial data $m_0(x) = (1-\partial_x^2)u_0(x)$ does not change sign, then $m(t,x)$ will not change sign for any $t \in [0,T)$. It follows that if $m_0(x) \geq 0$ (≤ 0), then the corresponding solution $u(t,x)$ remains positive (negative) for $(t,x) \in [0,T) \times X$.

Note that the inverse operator $(1-\partial_x^2)^{-1}$ can be obtained by convolution with the corresponding Green's function, so that

$$u = (1-\partial_x^2)^{-1} m = G * m, \tag{2.2}$$

where

$$G(x) = \tfrac{1}{2}\, e^{-|x|} \qquad \text{for the non-periodic case} \quad X = \mathbb{R}, \tag{2.3}$$

while

$$G(x) = \frac{\cosh\left(\frac{1}{2} - x + [x]\right)}{2\sinh\frac{1}{2}} \qquad \text{for the periodic case} \quad X = \mathbb{S}^1, \tag{2.4}$$

and the convolution product is defined by

$$f * g(x) = \int_X f(y)\, g(x-y)\, dy.$$

Applying the operator $(1-\partial_x^2)^{-1}$ to equation (2.1), we obtain the following nonlocal nonlinear equation

$$\begin{aligned} u_t + k_1\left(u^2 - \tfrac{1}{3}u_x^2\right)u_x + k_1(1-\partial_x^2)^{-1}\partial_x\left(\tfrac{2}{3}u^3 + uu_x^2\right) + \tfrac{1}{3}k_1(1-\partial_x^2)^{-1}\left(u_x^3\right) \\ + k_2 uu_x + k_2(1-\partial_x^2)^{-1}\partial_x\left(u^2 + \tfrac{1}{2}u_x^2\right) = 0. \end{aligned} \tag{2.5}$$

The formulation (2.5) allows us to define the notion of a weak solution as follows.

Definition 2.2. *Given initial data $u_0 \in W^{1,3}(X)$, a function $u \in L^\infty_{loc}([0,T), W^{1,3}_{loc}(X))$ is said to be a* weak solution *to the initial value problem* (2.1) *if it satisfies the following identity:*

$$\begin{aligned} \int_0^T \int_X \Big\{ u\partial_t\phi + \tfrac{1}{3}k_1u^3\partial_x\phi + \tfrac{1}{3}k_1u_x^3\phi + \tfrac{1}{2}k_2u^2\partial_x\phi + k_1\left[G * \left(\tfrac{2}{3}u^3 + u\,u_x^2\right)\right]\partial_x\phi \\ - \tfrac{1}{3}k_1\left[G * u_x^3\right]\phi + k_2\left[G * \left(u^2 + \tfrac{1}{2}u_x^2\right)\right]\partial_x\phi \Big\}\, dxdt + \int_X u_0(x)\phi(0,x)\, dx = 0, \end{aligned} \tag{2.6}$$

for any smooth test function $\phi(t,x) \in C_c^\infty([0,T) \times X)$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

The following theorem deals with the existence of periodic peakons for the gmCH equation (1.5) over a range of wave speeds. The proof's details can be found in the appendix.

Theorem 2.1. *For wave speeds c satisfying the inequality in* (1.16), *the gmCH equation* (1.5) *with $k_1 \neq 0$ possesses periodic peaked traveling-wave solutions of the form* (1.14). *These periodic peakons are global weak solutions to* (2.1) *in the sense of Definition 2.2.*

Remark 2.1. *Since the gmCH equation* (1.5) *is invariant under the transformation*

$$u \longmapsto -u, \qquad k_2 \longmapsto -k_2,$$

it suffices to consider the peakon (1.13) *and periodic peakon* (1.14) *with the + sign in their amplitude formulas* (1.15), (1.16), *an assumption that, we emphasize, will hold for the remainder of the paper.*

3. Stability of Peakons on the line

The purpose of this section is to establish the orbital stability for the single peakon solution (1.13), (1.15) to the gmCH equation. Here, we will only consider two cases: (i) $k_1 > 0$, $k_2 > 0$ and (ii) $k_1 > 0$, $k_2 \leq 0$. On the one hand, it is easy to check that for cases (i) with $c > 0$ and (ii), the amplitude $a > 0$. That is, there are only peakons, and no anti-peakons. On the other hand, for case (i) with $-\frac{3}{8}k_2^2/k_1 \leq c < 0$, the amplitude $a < 0$, which implies φ_c is an anti-peakon traveling from right to left.

Clearly,

$$\max_{x\in\mathbb{R}}\{|\varphi_c(x)|\} = |\varphi_c(0)| = |a|. \tag{3.1}$$

In addition, a direct computation leads to

$$H_1[\varphi_c] = \|\varphi_c\|_{H^1}^2 = 2\,a^2 > 0, \tag{3.2}$$

and, using (1.7), (1.8),

$$H_2[\varphi_c] = k_1 I_1[\varphi_c] + 2k_2 I_2[\varphi_c] = \tfrac{4}{3}a^3(k_1 a + 2k_2). \tag{3.3}$$

Moreover, it is worth mentioning that in case (ii), $H_2[\varphi_c] > 0$ is equivalent to the condition that the wave speed c satisfies

$$c > \frac{2k_2^2}{3k_1}. \tag{3.4}$$

We are now in a position to precisely formulate Theorem 1.1. For brevity, we will concentrate our attention on case (ii).

Theorem 3.1. *Let $k_1 > 0$ and $k_2 \leq 0$. Let φ_c be the peaked soliton defined in* (1.13), (1.15), *with wave speed satisfying* (3.4). *Assume that $u_0 \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, satisfies $0 \not\equiv m_0(x) = (1-\partial_x^2)u_0(x) \geq 0$. Then there exists $\delta_0 > 0$, depending on k_1, k_2, c, and $\|u_0\|_{H^s(\mathbb{R})}$, such that if*

$$\|u_0 - \varphi_c\|_{H^1(\mathbb{R})} < \delta, \qquad 0 < \delta < \delta_0,$$

then the corresponding positive solution $u(t,x)$ of the Cauchy problem for the gmCH equation (2.1) *with initial data $u(0,x) = u_0(x)$ satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < A\,\delta^{1/4},$$

where $T > 0$ is the maximal existence time, $\xi(t) \in \mathbb{R}$ is the point at which the solution $u(t,\cdot)$ achieves its maximum, and the constant $A > 0$ depends on k_1, k_2, the wave speed c, and the norm $\|u_0\|_{H^s(\mathbb{R})}$.

Remark 3.1. *As a consequence of the convolution formula* (2.2), *the assumptions on the initial profile m_0 imply that u_0 is strictly positive, and therefore, according to Proposition 2.2, the resulting solution $u(t,x)$ is also positive.*

The proof of this theorem is based on a series of lemmas. The first two are elementary, and their proofs can be found in [11, 32].

Lemma 3.1. *For any $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$, we have*

$$H_1[u] - H_1[\varphi_c] = \|u - \varphi_c(\cdot - \xi)\|^2_{H^1(\mathbb{R})} + 4a(u(\xi) - a). \tag{3.5}$$

Lemma 3.2. *Let $u \in H^s(\mathbb{R})$, $s > \frac{5}{2}$. Assume $\|u - \varphi_c\|_{H^1(\mathbb{R})} < \delta$ with $0 < \delta \ll 1$. Then*

$$\big|H_1[u] - H_1[\varphi_c]\big| \le B\,\delta \qquad \text{and} \qquad \big|H_2[u] - H_2[\varphi_c]\big| \le B\,\delta, \tag{3.6}$$

where $B > 0$ is a constant depending on k_1, k_2, c and the norm $\|u\|_{H^s(\mathbb{R})}$.

Let us apply Lemmas 3.1 and 3.2 to the positive function $u(x)$ and the peakon φ_c with positive amplitude a. Since (3.5) holds for any $\xi \in \mathbb{R}$, one can choose ξ such that

$$u(\xi) = \max_{x\in\mathbb{R}}\{u(x)\} = M, \tag{3.7}$$

and thus

$$\|u - \varphi_c(\cdot - \xi)\|^2_{H^1(\mathbb{R})} = (H_1[u] - H_1[\varphi_c]) - 4a(M - a).$$

Since the functional $H_1[u]$ represents the kinetic energy of the wave profile $u \in H^1(\mathbb{R})$, Lemma 3.1 tells that if the energy $H_1[u]$ and height M of a wave $u \in H^1(\mathbb{R})$ are close to the peakon's energy and height, then the entire shape of u is close to that of the peakon. Furthermore, the peakon has maximal height among all waves of fixed energy. The same remarks also apply to the CH equation [11] and the mCH equation [32]. On the other hand, in view of Lemma 3.1 and 3.2, the key task for proving the orbital stability of peakon is to control the error term which represents the difference of the maximum of the perturbed solution and the maximum of the peakon. The following Lemma is crucial to establish the estimate of such a difference.

Lemma 3.3. *Suppose that $k_1 > 0$, $k_2 \le 0$ and c satisfies (3.4). For $0 < u(x) \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, let $M = \max_{x\in\mathbb{R}}\{u(x)\}$. Then there exists a $\delta_0 > 0$ depending on k_1, k_2, c, and $\|u\|_{H^s(\mathbb{R})}$, such that if $\|u - \varphi_c\|_{H^1(\mathbb{R})} < \delta$ for $0 < \delta < \delta_0$, then*

$$0 < H_2[u] \le \left(\tfrac{4}{3}k_1 M^2 + 2k_2 M\right) H_1[u] - \tfrac{4}{3}k_1 M^4 - \tfrac{4}{3}k_2 M^3. \tag{3.8}$$

Proof. First, we define the real-valued function

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi, \\ u(x) + u_x(x), & x > \xi, \end{cases}$$

where ξ satisfies (3.7). It then follows from the proof of Lemma 2 in [11] that

$$\int_{\mathbb{R}} g^2(x)\,dx = H_1[u] - 2u^2(\xi) = H_1[u] - 2M^2. \tag{3.9}$$

Inspired by the approach in [32], we define the function

$$h(x) = \begin{cases} k_1\big(u^2(x) - \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x)\big) + 2k_2 u(x), & x < \xi, \\ k_1\big(u^2(x) + \frac{2}{3}u(x)u_x(x) - \frac{1}{3}u_x^2(x)\big) + 2k_2 u(x), & x > \xi. \end{cases}$$

A direct computation then shows that

$$
\begin{aligned}
&\int_{\mathbb{R}} h(x)\, g^2(x)\, dx \\
&\qquad = \int_{-\infty}^{\xi} \left[k_1\left(u^2 - \tfrac{2}{3}uu_x - \tfrac{1}{3}u_x^2\right) + 2k_2 u\right] \cdot \left(u^2 - 2uu_x + u_x^2\right)\, dx \\
&\qquad\quad + \int_{\xi}^{\infty} \left[k_1\left(u^2 + \tfrac{2}{3}uu_x - \tfrac{1}{3}u_x^2\right) + 2k_2 u\right] \cdot \left(u^2 + 2uu_x + u_x^2\right)\, dx \\
&\qquad = \int_{-\infty}^{\xi} \left[k_1\left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right) + 2k_2\left(u^3 + uu_x^2\right) - \tfrac{8}{3}k_1 u^3u_x - 4k_2u^2u_x\right]\, dx \\
&\qquad\quad + \int_{\xi}^{\infty} \left[k_1\left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right) + 2k_2\left(u^3 + uu_x^2\right) + \tfrac{8}{3}k_1 u^3u_x + 4k_2u^2u_x\right]\, dx \\
&\qquad = k_1 \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right)\, dx + 2k_2 \int_{\mathbb{R}} \left(u^3 + uu_x^2\right)\, dx \\
&\qquad\quad - \tfrac{8}{3}k_1 \int_{-\infty}^{\xi} u^3u_x\, dx + \tfrac{8}{3}k_1 \int_{\xi}^{\infty} u^3u_x\, dx - 8k_2 \int_{-\infty}^{\xi} u^2u_x\, dx + 8k_2 \int_{\xi}^{\infty} u^2u_x\, dx.
\end{aligned}
$$

Thus, by (1.7), (1.8),

$$
\int_{\mathbb{R}} h(x)\, g^2(x)\, dx = (k_1 I_1[u] + 2k_2 I_2[u]) - \tfrac{4}{3}k_1M^4 - \tfrac{8}{3}k_2M^3 = H_2[u] - \tfrac{4}{3}k_1M^4 - \tfrac{8}{3}k_2M^3. \tag{3.10}
$$

By our assumption, $k_1 > 0$ and $k_2 \le 0$, hence

$$
\begin{aligned}
h(x) &= k_1\left[u^2(x) \mp \tfrac{2}{3}u(x)u_x(x) - \tfrac{1}{3}u_x^2(x)\right] + 2k_2u(x) \\
&\le \tfrac{4}{3}k_1u^2(x) + 2k_2u(x) = \left(\tfrac{4}{3}k_1u(x) + 2k_2\right)u(x).
\end{aligned}
$$

Noting that $u(x) > 0$, we argue that if $M = \max_{x\in\mathbb{R}}\{u(x)\}$ satisfies

$$
\tfrac{4}{3}k_1M + 2k_2 > 0, \tag{3.11}
$$

then, for $x \in \mathbb{R}$,

$$
h(x) \le u(x)\left(\tfrac{4}{3}k_1u(x) + 2k_2\right) \le M\left(\tfrac{4}{3}k_1M + 2k_2\right) = \tfrac{4}{3}k_1M^2 + 2k_2M, \tag{3.12}
$$

which is the crucial observation for the stability analysis of peakons. Indeed, in this case, plugging (3.9) and (3.12) into (3.10) produces

$$
\begin{aligned}
H_2[u] - \tfrac{4}{3}k_1M^4 - \tfrac{8}{3}k_2M^3 = \int_{\mathbb{R}} h(x)g^2(x)\, dx &\le \left(\tfrac{4}{3}k_1M^2 + 2k_2M\right)\int_{\mathbb{R}} g^2(x)\, dx \\
&= \left(\tfrac{4}{3}k_1M^2 + 2k_2M\right)H_1[u] - \tfrac{8}{3}k_1M^4 - 4k_2M^3.
\end{aligned}
$$

Therefore, (3.8) follows and the proof of the lemma will be complete.

To justify that (3.11) holds for u sufficiently close to φ_c, we first note that, since $u(x) > 0$, by the Cauchy–Schwarz inequality,

$$
I_1[u] = \int_{\mathbb{R}} \left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right)\, dx \le \tfrac{4}{3}\int_{\mathbb{R}} (u^4 + u^2u_x^2)\, dx \le \tfrac{4}{3}\, M \int_{\mathbb{R}} (u^3 + uu_x^2)\, dx = \tfrac{4}{3}\, M\, I_2[u].
$$

Coupled with the fact that $I_2[u] > 0$, we deduce that

$$
H_2[u] \le \left(\tfrac{4}{3}k_1M + 2k_2\right)\, I_2[u].
$$

On the other hand, in view of Lemma 3.2, if $\|u-\varphi_c\|_{H^1(\mathbb{R})}$ is small, then $H_2[u]$ is near $H_2[\varphi_c]$. We now conclude that if $H_2[\varphi_c] > 0$, i.e. $c > \frac{2}{3}k_2^2/k_1$, there exists a $\delta_0 > 0$ depending on k_1, k_2, c and $\|u\|_{H^s(\mathbb{R})}$ such that, for any $0 < u(x) \in H^s(\mathbb{R})$, $s > \frac{5}{2}$, with $\|u - \varphi_c\|_{H^1(\mathbb{R})} < \delta$ for $0 < \delta < \delta_0$, we have $H_2[u] > 0$ and thus the condition (3.11) is verified. □

Lemma 3.4. *Suppose that $k_1 > 0$, $k_2 \le 0$, and c satisfies (3.4). Given $0 < u(x) \in H^s(\mathbb{R})$ with $s > \frac{5}{2}$, let $M = \max_{x\in\mathbb{R}}\{u(x)\}$. Then there exists a $\delta_0 > 0$ depending on k_1, k_2, c, and $\|u\|_{H^s(\mathbb{R})}$, such that if $\|u - \varphi_c\|_{H^1(\mathbb{R})} < \delta$, $0 < \delta < \delta_0$, then*

$$|M - a\,| < A\,\delta^{1/2},$$

where A is a constant depending on k_1, k_2, c and $\|u\|_{H^s(\mathbb{R})}$.

Proof. Due to (3.8) and formula (1.7) for $H_2[u] = k_1 I_1[u] + 2k_2 I_2[u]$, we have

$$k_1\left(M^4 - H_1[u]\,M^2 + \tfrac{3}{4}I_1[u]\right) + k_2\left(M^3 - \tfrac{3}{2}H_1[u]M + \tfrac{3}{2}I_2[u]\right) \le 0. \tag{3.13}$$

Define the quartic polynomial

$$P(z) = k_1\left(z^4 - H_1[u]\,z^2 + \tfrac{3}{4}I_1[u]\right) + k_2\left(z^3 - \tfrac{3}{2}H_1[u]\,z + \tfrac{3}{2}I_2[u]\right). \tag{3.14}$$

Noticing that

$$H_1[\varphi_c] = 2a^2, \qquad I_1[\varphi_c] = \tfrac{4}{3}a^4, \qquad I_2[\varphi_c] = \tfrac{4}{3}a^3,$$

one can define another quartic polynomial

$$\begin{aligned} P_0(z) =& k_1\left(z^4 - H_1[\varphi_c]\,z^2 + \tfrac{3}{4}I_1[\varphi_c]\right) + k_2\left(z^3 - \tfrac{3}{2}H_1[\varphi_c]\,z + \tfrac{3}{2}I_2[\varphi_c]\right) \\ =& \left[k_1 z^2 + (2k_1 a + k_2)z + a(k_1 a + 2k_2)\right](z-a)^2. \end{aligned} \tag{3.15}$$

A direct calculation using (3.14) and (3.15) shows that

$$P_0(M) = P(M) + k_1 M^2(H_1[u] - H_1[\varphi_c]) + \tfrac{3}{2}k_2 M\left(H_1[u] - H_1[\varphi_c]\right) - \tfrac{3}{4}(H_2[u] - H_2[\varphi_c])\right),$$

which together with (3.4), (3.11), (3.13), and (3.15) implies

$$\begin{aligned} a(k_1 a + 2k_2)(M-a)^2 &\le \left(k_1 M^2 + \tfrac{3}{2}k_2 M\right)(H_1[u] - H_1[\varphi_c]) - \tfrac{3}{4}\left(H_2[u] - H_2[\varphi_c]\right) \\ &\le k_1 M^2\big|H_1[u] - H_1[\varphi_c]\big| + \tfrac{3}{4}\big|H_2[u] - H_2[\varphi_c]\big|. \end{aligned} \tag{3.16}$$

On the other hand, since

$$H_1[u] - 2M^2 = \int_{\mathbb{R}} g^2(x)\,dx \ge 0, \qquad \text{we have} \qquad 0 < M^2 \le \tfrac{1}{2}H_1[u] \le A, \tag{3.17}$$

where A depends on k_1, k_2, c and $\|u\|_{H^s(\mathbb{R})}$. Hence, in view of (3.16), (3.17), and Lemma 3.2, we conclude that

$$|M - a| \le A\sqrt{A\left|H_1[u] - H_1[\varphi_c]\right| + \left|H_2[u] - H_2[\varphi_c]\right|} \le A\,\delta^{1/2}.$$

This completes the proof of Lemma 3.4. □

Proof of Theorem 3.1. Given initial data $u_0(x) \in H^s(\mathbb{R})$ satisfying the hypotheses, let $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ be the corresponding positive solution of the Cauchy problem (2.1) on the line, with maximal existence time $T > 0$. Since H_1 and H_2 are both conserved, that means

$$H_1[u(t,\cdot)] = H_1[u_0] \quad \text{and} \quad H_2[u(t,\cdot)] = H_2[u_0], \qquad t \in [0,T). \tag{3.18}$$

Since the assumptions of Lemma 3.4 are satisfied for $u(t,\cdot)$, $t \in [0,T)$, with a positive constant A depending on k_1, k_2, c and $\|u_0\|_{H^s(\mathbb{R})}$, we have

$$|u(t,\xi(t)) - a| < A\,\delta^{1/2}, \qquad t \in [0,T),$$

where $u(t,\xi(t)) = M(t) = \max_{x\in\mathbb{R}}\{u(t,x)\}$. Due to (3.18) and Lemma 3.1,

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|^2_{H^1(\mathbb{R})} = H_1[u_0] - H_1[\varphi_c] - 4a(u(t,\xi(t)) - a), \qquad t \in [0,T).$$

Combining the above estimates, we conclude that

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t)\|_{H^1(\mathbb{R})} \le \sqrt{|H_1[u_0] - H_1[\varphi_c]| + 4a|u(t,\xi(t)) - a|} < A\delta^{1/4}, \quad t \in [0,T),$$

which thus completes the proof of Theorem 3.1. □

Remark 3.2. *In the proof, it follows from the estimate* (3.16) *that the peakons are energy minimizers with a fixed invariant* H_2*, which reveals their stability. In fact, if* $H_2[u] = H_2[\varphi_c]$*, we deduce from* (3.16) *that* $H_1[u] \geq H_1[\varphi_c]$*. The same remark applies to the CH, DP, and mCH equations and demonstrates that their peakons are also energy minimizers with corresponding fixed invariants* [11, 26, 27, 32].

Next, for case (i) with positive wave speed c, we have the corresponding stability result.

Theorem 3.2. *Assume that* $k_1 > 0$ *and* $k_2 > 0$*. Let* φ_c *be the peaked soliton defined in* (1.13), (1.15)*, traveling with speed* $c > 0$*. Then* φ_c *is orbitally stable in the following sense. Given initial data* $u_0 \in H^s(\mathbb{R})$*,* $s > \frac{5}{2}$*, with* $0 \not\equiv m_0(x) = (1 - \partial_x^2)u_0(x) \geq 0$*, there exists* $\delta_0 > 0$*, depending on* k_1*,* k_2*,* c*, and* $\|u_0\|_{H^s(\mathbb{R})}$*, such that if*

$$\|u_0 - \varphi_c\|_{H^1(\mathbb{R})} < \delta, \qquad 0 < \delta < \delta_0,$$

then the corresponding solution $u(t,x)$ *of the Cauchy problem for the gmCH equation* (2.1) *satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < A\,\delta^{1/4},$$

where $T > 0$ *is the maximal existence time,* $\xi(t) \in \mathbb{R}$ *is the maximum point of the function* $u(t,\cdot)$*, and the constant* $A > 0$ *depends on* k_1*,* k_2*, the wave speed* c*, and the norm* $\|u_0\|_{H^s(\mathbb{R})}$*.*

We first claim that, since $k_1 > 0$ and $k_2 > 0$, for the positive solution $u(t,x)$, the following key inequality holds:

$$h(x) = k_1\left(u(x)^2 \mp \tfrac{2}{3}u(x)u_x(x) - \tfrac{1}{3}u_x^2(x)\right) + 2k_2u(x) \leq \tfrac{4}{3}k_1u^2(x) + 2k_2u(x) \leq \tfrac{4}{3}k_1M^2 + 2k_2M.$$

With this in hand, the remainder of the proof is similar to that in our previous paper [32], and we omit it here for the sake of brevity.

Finally, invariance of the gmCH equation under the transformation $u \to -u$ immediately implies the corresponding stability result for anti-peakons.

Theorem 3.3. *Let* $k_1 > 0$ *and* $k_2 > 0$*. Let* φ_c *be the peaked soliton defined in* (1.13), (1.15)*, traveling with speed* $-\frac{3}{8}k_2^2/k_1 \leq c < 0$*. Then* φ_c *is orbitally stable in the following sense. Given initial data* $u_0 \in H^s(\mathbb{R})$*,* $s > \frac{5}{2}$*, satisfying* $0 \not\equiv m_0(x) = (1 - \partial_x^2)u_0(x) \leq 0$*, there exists* $\delta_0 > 0$*, depending on* k_1*,* k_2*,* c*, and* $\|u_0\|_{H^s(\mathbb{R})}$*, such that if*

$$\|u_0 - \varphi_c\|_{H^1(\mathbb{R})} < \delta, \qquad 0 < \delta < \delta_0,$$

then the corresponding solution $u(t,x)$ *of the Cauchy problem for the gmCH equation* (2.1) *satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < A\,\delta^{1/4},$$

where $T > 0$ *is the maximal existence time,* $\xi(t) \in \mathbb{R}$ *is the minimum point of the solution* $u(t,\cdot)$*, and the constant* $A > 0$ *depends on* k_1*,* k_2*, the wave speed* c*, and the norm* $\|u_0\|_{H^s(\mathbb{R})}$*.*

4. Stability of Periodic Peakons

In this section, we study the orbital stability of the periodic peakons (1.14), (1.16) of the gmCH equation (2.1). We will identify functions on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ as periodic functions on the whole line of period one. Hence, we write the periodic peakon in the form

$$\psi_c(t,x) = a\,\psi(x - c\,t), \tag{4.1}$$

where, in view of Remark 2.1, a is defined by (1.16) with the + sign, while $\psi(x)$ represents the 1–periodic function such that

$$\psi(x) = \cosh\left(\tfrac{1}{2} - x\right) \qquad \text{for} \qquad 0 \leq x < 1. \tag{4.2}$$

As before, we will consider two cases: (i) $k_1 > 0$, $k_2 > 0$, and (ii) $k_1 > 0$, $k_2 \le 0$. Note that the case (i) contains periodic peakons with both positive amplitude and negative amplitude, whereas in case (ii), they have only positive amplitude.

According to (4.2), $\psi(x)$ is continuous on $\mathbb{S}^1$ with peak at $x = 0$, and hence, for the periodic peakon with positive amplitude a, its maximum and minimum values are, respectively,

$$M_{\psi_c} = \max_{x\in\mathbb{S}^1}\{\psi_c(x)\} = a\,\psi(0) = a\cosh\tfrac{1}{2}, \qquad L_{\psi_c} = \min_{x\in\mathbb{S}^1}\{\psi_c(x)\} = a\,\psi\left(\tfrac{1}{2}\right) = a. \tag{4.3}$$

On the other hand, $\psi(x)$ is smooth on $(0,1)$, and satisfies

$$\psi'(x) = -\sinh\left(\tfrac{1}{2} - x\right), \qquad \psi''(x) = \psi(x) - 2\sinh\tfrac{1}{2}\,\delta(x), \tag{4.4}$$

where δ denotes the Dirac distribution. Thus, by direct evaluation, the values of the conservation laws for the periodic peakon (4.1) are given by

$$\begin{aligned} H_0[\psi_c] &= a\int_{\mathbb{S}^1}\cosh\left(\tfrac{1}{2} - x\right)\,dx = 2a\sinh\tfrac{1}{2},\\ H_1[\psi_c] &= a^2\int_{\mathbb{S}^1}\left(\psi^2 + \psi_x^2\right)\,dx = a^2\int_{\mathbb{S}^1}\left(\psi^2 - \psi_{xx}\psi\right)\,dx = 2a^2\sinh\tfrac{1}{2}\,\cosh\tfrac{1}{2}. \end{aligned} \tag{4.5}$$

Furthermore,

$$\begin{aligned} I_1[\psi_c] &= a^4\int_{\mathbb{S}^1}\left(\psi^4 + 2\psi^2\psi_x^2 - \tfrac{1}{3}\psi_x^4\right)dx = a^4\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\cosh^4 x + 2\cosh^2 x\,\sinh^2 x - \tfrac{1}{3}\sinh^4 x\right)dx\\ &= a^4\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\tfrac{1}{3}\cosh(4x) + \tfrac{2}{3}\cosh(2x)\right)\,dx = \tfrac{2}{3}a^4\sinh\tfrac{1}{2}\,\left(2\cosh^3\tfrac{1}{2} + \cosh\tfrac{1}{2}\right),\\ I_2[\psi_c] &= a^3\int_{\mathbb{S}^1}\left(\psi^3 + \psi\psi_x^2\right)\,dx = a^3\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\cosh(3x) - 2\cosh x\,\sinh^2 x\right)\,dx\\ &= 2a^3\left(\tfrac{2}{3}\sinh^3\tfrac{1}{2} + \sinh\tfrac{1}{2}\right), \end{aligned}$$

and hence

$$\begin{aligned} H_2[\psi_c] &= k_1 I_1[\psi_c] + 2k_2 I_2[\psi_c]\\ &= \tfrac{2}{3}k_1 a^4\sinh\tfrac{1}{2}\,\left(2\cosh^3\tfrac{1}{2} + \cosh\tfrac{1}{2}\right) + 4k_2 a^3\left(\tfrac{2}{3}\sinh^3\tfrac{1}{2} + \sinh\tfrac{1}{2}\right). \end{aligned} \tag{4.6}$$

The following result is a reformulated version of the second case (1.19) of Theorem 1.2.

Theorem 4.1. *Assume that $k_1 > 0$ and $k_2 \le 0$. Let ψ_c be the periodic peaked soliton defined in* (4.1) *with the traveling wave speed c satisfying the inequality in* (1.19). *Then ψ_c is orbitally stable in the following sense. Suppose that $u_0 \in H^s(\mathbb{S}^1)$, $s > \frac{5}{2}$, with $0 \not\equiv m_0 = (1 - \partial_x^2)u_0(x) \ge 0$. Let $T > 0$ be the maximal existence time of the corresponding periodic solution $u(t,x) \in C([0,T), H^s(\mathbb{S}^1))$ with initial data u_0. Then, for every $\epsilon > 0$, there is a $\delta > 0$ such that if*

$$\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta,$$

then

$$\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{S}^1)} < \epsilon \qquad \textit{for} \qquad t \in [0,T),$$

where $\xi(t)$ is any point where the function $u(t,\cdot)$ attains its maximum.

Remark 4.1. *Due to the convolution formula* (2.2) *with the Green's function* (2.4), *the assumption on m_0 implies that u_0 is strictly positive, and hence, by Proposition 2.2, the corresponding solution $u(t,x)$ is also strictly positive.*

The proof of Theorem 4.1 will be carried out through a series of lemmas. We first consider the expansion of the conservation law H_1 around the peakon ψ_c in the $H^1(\mathbb{S}^1)$-norm.

Lemma 4.1. *For any $u \in H^1(\mathbb{S}^1)$ and $\xi \in \mathbb{R}$, we have*

$$H_1[u] - H_1[\psi_c] = \|u - \psi_c(\cdot - \xi)\|^2_{H^1(\mathbb{S}^1)} + 4a \sinh\left(\tfrac{1}{2}\right)\left(u(\xi) - M_{\psi_c}\right). \tag{4.7}$$

Proof. Using the second formulas in (4.4) and (4.5), we calculate

$$\begin{aligned}
\|u - \psi_c(\cdot - \xi)\|^2_{H^1(\mathbb{S}^1)} &= \int_{\mathbb{S}^1} \left(u(x) - \psi_c(x-\xi)\right)^2 dx + \int_{\mathbb{S}^1} \left(u_x(x) - \partial_x\psi_c(x-\xi)\right)^2 dx \\
&= H_1[u] + H_1[\psi_c] - 2a \int_{\mathbb{S}^1} u(x+\xi)\psi(x)\,dx + 2a \int_{\mathbb{S}^1} u(x+\xi)\psi_{xx}(x)\,dx \\
&= H_1[u] + H_1[\psi_c] - 4a \sinh\left(\tfrac{1}{2}\right) u(\xi) \\
&= H_1[u] - H_1[\psi_c] - 4a \sinh\left(\tfrac{1}{2}\right)\left(u(\xi) - a\cosh\tfrac{1}{2}\right) \\
&= H_1[u] - H_1[\psi_c] - 4a \sinh\left(\tfrac{1}{2}\right)\left(u(\xi) - M_{\psi_c}\right),
\end{aligned}$$

which establishes (4.7). □

Next, we state, without proof, a periodic version of Lemma 3.2, establishing the continuity of the three conservation laws (1.7), (1.20), in the H^1-norm.

Lemma 4.2. *Let $u \in H^s(\mathbb{S}^1)$, $s > \frac{5}{2}$. Assume $\|u - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$ with $0 < \delta \ll 1$. Then*

$$\left|H_i[u] - H_i[\psi_c]\right| \le C\,\delta, \qquad i = 0, 1, 2,$$

where $C > 0$ is a constant depending on k_1, k_2, $c > 0$ and $\|u\|_{H^s(\mathbb{S}^1)}$.

To proceed, it is observed from (4.4) that the periodic peakon ψ_c satisfies the following differential equation

$$\partial_x\psi_c(x) = \begin{cases} -\sqrt{\psi_c^2 - L^2_{\psi_c}}\,, & 0 < x \le \frac{1}{2}, \\ \sqrt{\psi_c^2 - L^2_{\psi_c}}\,, & \frac{1}{2} \le x < 1. \end{cases} \tag{4.8}$$

Let $0 < u(x) \in H^s(\mathbb{S}^1) \subset C^2(\mathbb{S}^1)$, $s > \frac{5}{2}$, and write

$$M = M_u = \max_{x\in\mathbb{S}^1}\{u(x)\} = u(\xi), \qquad L = L_u = \min_{x\in\mathbb{S}^1}\{u(x)\} = u(\eta),$$

for some ξ, $\eta \in \mathbb{S}^1$. We now define the real function

$$g(x) = \begin{cases} u_x + \sqrt{u^2 - L^2}, & \xi < x \le \eta, \\ u_x - \sqrt{u^2 - L^2}, & \eta \le x < \xi + 1, \end{cases} \tag{4.9}$$

which is extended periodically to the entire line. A direct computation [23] yields

$$\int_{\mathbb{S}^1} g^2(x)\,dx = 2\,L^2 \log\left(\frac{M + \sqrt{M^2 - L^2}}{L}\right) - 2M\sqrt{M^2 - L^2} - L^2 + H_1[u]. \tag{4.10}$$

In addition, motivated by [32], we define the another auxiliary real function

$$h(x) = \begin{cases} k_1\left(u^2 + \frac{2}{3}u_x\sqrt{u^2 - L^2} - \frac{1}{3}u_x^2 - L^2\right) + 2k_2 u, & \xi < x \le \eta, \\ k_1\left(u^2 - \frac{2}{3}u_x\sqrt{u^2 - L^2} - \frac{1}{3}u_x^2 - L^2\right) + 2k_2 u, & \eta \le x < \xi + 1 \end{cases} \tag{4.11}$$

and extend it periodically to the entire line. Then

$$\begin{aligned}
&\int_{\mathbb{S}^1} h(x)g^2(x)\,dx \\
&\quad= \int_\xi^\eta \Big(k_1\big(u^2+\tfrac{2}{3}u_x\sqrt{u^2-L^2}-\tfrac{1}{3}u_x^2-L^2\big)+2k_2u\Big)\big(u_x+\sqrt{u^2-L^2}\big)^2\,dx \\
&\qquad+\int_\eta^{\xi+1}\Big(k_1(u^2-\tfrac{2}{3}u_x\sqrt{u^2-L^2}-\tfrac{1}{3}u_x^2-L^2)+2k_2u\Big)\big(u_x-\sqrt{u^2-L^2}\big)^2\,dx \\
&\quad= J_1+J_2.
\end{aligned}$$

A straightforward computation leads to

$$\begin{aligned}
J_1 &= \int_\xi^\eta k_1\Big(u^2+\tfrac{2}{3}u_x\sqrt{u^2-L^2}-\tfrac{1}{3}u_x^2-L^2\Big)\big(u_x^2+2u_x\sqrt{u^2-L^2}+u^2-L^2\big)\,dx \\
&\quad+\int_\xi^\eta 2k_2u\big(u_x^2+2u_x\sqrt{u^2-L^2}+u^2-L^2\big)\,dx \\
&= k_1\int_\xi^\eta \big(u^4+2u^2u_x^2-\tfrac{1}{3}u_x^4\big)\,dx-k_1L^2\int_\xi^\eta\big(u^2+u_x^2\big)\,dx-k_1L^2\int_\xi^\eta g^2(x)\,dx \\
&\quad+\tfrac{8}{3}k_1\int_\xi^\eta u^2u_x\sqrt{u^2-L^2}\,dx-\tfrac{2}{3}k_1L^2\int_\xi^\eta u_x\sqrt{u^2-L^2}\,dx \\
&\quad+2k_2\int_\xi^\eta\big(u^3+uu_x^2\big)\,dx+4k_2\int_\xi^\eta uu_x\sqrt{u^2-L^2}\,dx-2k_2L^2\int_\xi^\eta u\,dx.
\end{aligned}$$

Using the identities

$$\begin{aligned}
u^2u_x\sqrt{u^2-L^2} &= \frac{1}{4}\frac{d}{dx}\Big(u(u^2-L^2)^{3/2}\Big)+\frac{1}{4}L^2u_x\sqrt{u^2-L^2}, \\
uu_x\sqrt{u^2-L^2} &= \frac{1}{3}\frac{d}{dx}(u^2-L^2)^{3/2},
\end{aligned}$$

we find

$$\begin{aligned}
J_1 =& k_1\int_\xi^\eta \big(u^4+2u^2u_x^2-\tfrac{1}{3}u_x^4\big)\,dx-k_1L^2\int_\xi^\eta\big(u^2+u_x^2\big)\,dx-k_1L^2\int_\xi^\eta g^2(x)\,dx \\
&-\tfrac{2}{3}k_1M(M^2-L^2)^{3/2}+2k_2\int_\xi^\eta\big(u^3+uu_x^2\big)\,dx-\tfrac{4}{3}k_2(M^2-L^2)^{3/2}-2k_2L^2\int_\xi^\eta u\,dx.
\end{aligned}$$

In a similar manner, we have

$$\begin{aligned}
J_2 =& k_1\int_\eta^{\xi+1} \big(u^4+2u^2u_x^2-\tfrac{1}{3}u_x^4\big)\,dx-k_1L^2\int_\eta^{\xi+1}\big(u^2+u_x^2\big)\,dx-k_1L^2\int_\eta^{\xi+1} g^2(x)\,dx \\
&-\tfrac{2}{3}k_1M(M^2-L^2)^{3/2}+2k_2\int_\eta^{\xi+1}\big(u^3+uu_x^2\big)\,dx-\tfrac{4}{3}k_2(M^2-L^2)^{3/2}-2k_2L^2\int_\eta^{\xi+1} u\,dx.
\end{aligned}$$

Recalling the definitions (1.7) of $H_1[u], H_2[u]$,

$$\begin{aligned}
\int_{\mathbb{S}^1} h(x)g^2(x)\,dx =& k_1I_1[u]-k_1L^2H_1[u]-k_1L^2\int_{\mathbb{S}^1}g^2(x)\,dx-\tfrac{4}{3}k_1M(M^2-L^2)^{3/2} \\
&+2k_2I_2[u]-\tfrac{8}{3}k_2(M^2-L^2)^{3/2}-2k_2L^2\int_{\mathbb{S}^1}u\,dx \\
=&-k_1L^2\int_{\mathbb{S}^1}g^2(x)\,dx-\tfrac{4}{3}k_1M(M^2-L^2)^{3/2}-\tfrac{8}{3}k_2(M^2-L^2)^{3/2} \\
&+H_2[u]-k_1L^2H_1[u]-2k_2L^2H_0[u].
\end{aligned} \tag{4.12}$$

The assumption $k_1 > 0$ and $k_2 \le 0$ implies

$$\begin{aligned} h(x) &= k_1 \left(u^2(x) \pm \tfrac{2}{3} u_x(x) \sqrt{u^2(x) - L^2} - \tfrac{1}{3} u_x^2(x) - L^2 \right) + 2k_2 u(x) \\ &\le k_1 \left(\tfrac{4}{3} u^2(x) - \tfrac{4}{3} L^2 \right) + 2k_2 u(x) = u(x) \left(\tfrac{4}{3} k_1 u(x) + 2k_2 \right) - \tfrac{4}{3} k_1 L^2. \end{aligned}$$

This, if the positive function $u(x)$ with minimum L satisfies

$$\tfrac{4}{3} k_1 L + 2k_2 > 0, \tag{4.13}$$

then

$$h(x) \le M \left(\tfrac{4}{3} k_1 M + 2k_2 \right) - \tfrac{4}{3} k_1 L^2 = \tfrac{4}{3} k_1 (M^2 - L^2) + 2k_2 M. \tag{4.14}$$

Since $u(x) > 0$, the convolution formula (2.2), (2.4) implies that

$$L = \min_{x \in \mathbb{S}^1} \{u(x)\} \ge \frac{1}{2 \sinh \frac{1}{2}} \int_{\mathbb{S}^1} m(z)\, dz = \frac{H_0[u]}{2 \sinh \frac{1}{2}}. \tag{4.15}$$

Note that the equation holds at $u(x) = \psi_c(x)$. On the other hand, thanks to Lemma 4.2, we argue that if

$$\frac{2}{3 \sinh \frac{1}{2}} k_1 H_0[\psi_c] + 2k_2 > 0, \tag{4.16}$$

then there is a $\delta > 0$ small enough, such that for $u(x)$ satisfying $\|u - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, one has

$$\frac{2}{3 \sinh \frac{1}{2}} k_1 H_0[u] + 2k_2 > 0,$$

which, along with (4.15), leads to (4.13) and

$$\tfrac{4}{3} k_1 u(x) + 2k_2 > 0, \qquad x \in \mathbb{S}^1,$$

and this establishes the inequality (4.14). In view of (4.5), the inequality (4.16) is equivalent to the following inequality about the amplitude a:

$$\tfrac{4}{3} k_1 a + 2k_2 > 0,$$

which implies that the wave speed c satisfies

$$c > \frac{3k_2^2}{4k_1} \left(2 \cosh^2 \tfrac{1}{2} - 2 \cosh \tfrac{1}{2} + 1 \right).$$

In view of (4.12) and (4.14), we obtain

$$\begin{aligned} &- k_1 L^2 \int_{\mathbb{S}^1} g^2(x)\, dx - \tfrac{4}{3} k_1 M (M^2 - L^2)^{3/2} - \tfrac{8}{3} k_2 (M^2 - L^2)^{3/2} + H_2[u] - k_1 L^2 H_1[u] \\ &\qquad - 2k_2 L^2 H_0[u] \le \tfrac{4}{3} k_1 (M^2 - L^2) \int_{\mathbb{S}^1} g^2(x)\, dx + 2k_2 M \int_{\mathbb{S}^1} g^2(x)\, dx, \end{aligned}$$

which, combined with (4.10), yields the following inequality:

$$\begin{aligned} 0 \le F_u(M, L) &= \left(k_1 \left(\tfrac{4}{3} M^2 - \tfrac{1}{3} L^2 \right) + 2k_2 M \right) \left(2\, L^2 \log \frac{M + \sqrt{M^2 - L^2}}{L} \right. \\ &\quad \left. - 2M \sqrt{M^2 - L^2} - L^2 + \tfrac{4}{3} k_1 M (M^2 - L^2)^{3/2} + \tfrac{8}{3} k_2 (M^2 - L^2)^{3/2} \right. \\ &\quad - H_2[u] + \left(k_1 \left(\tfrac{4}{3} M^2 + \tfrac{2}{3} L^2 \right) + 2k_2 M \right) H_1[u] + 2k_2 L^2 H_0[u]. \end{aligned} \tag{4.17}$$

Note that the function F_u depends on u only through the values of the three conservation laws $H_0[u]$, $H_1[u]$, $H_2[u]$. This establishes the following lemma.

Lemma 4.3. *Assume that $k_1 > 0$ and $k_2 \le 0$. Then, for wave speed c satisfying* (1.19), *there exists $\delta > 0$ such that if $\|u - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, then*

$$k_1 H_0[u] + 3k_2 \sinh\tfrac{1}{2} > 0 \qquad \text{and} \qquad F_u(M_u, L_u) \ge 0,$$

where $M_u = \max_{x\in\mathbb{S}^1}\{u(x)\}$, $L_u = \min_{x\in\mathbb{S}^1}\{u(x)\}$.

The next lemma summarizes some properties of the function (4.17) when u is the periodic peakon ψ_c.

Lemma 4.4. *Assume that $k_1 > 0$, $k_2 \le 0$ and the wave speed c satisfies* (1.19). *Then*

$$\begin{aligned} &F_{\psi_c}(M_{\psi_c}, L_{\psi_c}) = 0, \qquad \frac{\partial F_{\psi_c}}{\partial M}(M_{\psi_c}, L_{\psi_c}) = 0, \qquad \frac{\partial F_{\psi_c}}{\partial L}(M_{\psi_c}, L_{\psi_c}) = 0,\\ &\frac{\partial^2 F_{\psi_c}}{\partial M^2}(M_{\psi_c}, L_{\psi_c}) = -\tfrac{32}{3}k_1 a^2 \sinh\tfrac{1}{2}\cosh\tfrac{1}{2} - 8k_2 a \sinh\tfrac{1}{2},\\ &\frac{\partial^2 F_{\psi_c}}{\partial M\,\partial L}(M_{\psi_c}, L_{\psi_c}) = 0,\\ &\frac{\partial^2 F_{\psi_c}}{\partial L^2}(M_{\psi_c}, L_{\psi_c}) = -\tfrac{16}{3}k_1 a^2 \sinh\tfrac{1}{2}\cosh\tfrac{1}{2} - 8k_2 a \sinh\tfrac{1}{2}, \end{aligned} \tag{4.18}$$

where $M_{\psi_c} = \max_{x\in\mathbb{S}^1}\{\psi_c(x)\}$ and $L_{\psi_c} = \min_{x\in\mathbb{S}^1}\{\psi_c(x)\}$. Moreover, (M_{ψ_c}, L_{ψ_c}) is an isolated local maximum of F_{ψ_c}.

Proof. The expressions in (4.18) are obtained by a straightforward computation. The wave speed condition (1.19) is equivalent to the amplitude inequality $a > -\frac{3}{2}k_2/k_1$, which in turn implies that the Hessian matrix of F_{ψ_c} at the critical point (M_{ψ_c}, L_{ψ_c}) is diagonal and negative definite, which establishes its status as an isolated local maximum. □

Lemma 4.5. [23] *For any $v \in H^1(\mathbb{S}^1)$, we have*

$$\max_{x\in\mathbb{S}^1} |v(x)| \le \sqrt{\tfrac{1}{2}\coth\tfrac{1}{2}}\ \|v\|_{H^1(\mathbb{S}^1)}. \tag{4.19}$$

Moreover, $\sqrt{\frac{1}{2}\coth\frac{1}{2}}$ is the best constant, and equality holds if and only if $v = \psi_c(\cdot - \xi)$ for some $c > 0$ and $\xi \in \mathbb{R}$, that is, if and only if v has the profile of a peakon.

Lemma 4.6. [23] *If $u \in C([0,T), H^1(\mathbb{S}^1))$, then its spatial maximum and minimum values,*

$$M_{u(t)} = \max_{x\in\mathbb{S}^1}\{u(t,x)\}, \qquad L_{u(t)} = \min_{x\in\mathbb{S}^1}\{u(t,x)\},$$

are continuous functions of $t \in [0,T)$.

Lemma 4.7. *Assume that $k_1 > 0$, $k_2 \le 0$, and the wave speed c satisfies* (1.19). *Then there exist a small neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of (M_{ψ_c}, L_{ψ_c}) and a $\delta > 0$ such that whenever $u \in C([0,T), H^s(\mathbb{S}^1))$, $s > \frac{5}{2}$, is a periodic solution of the gmCH equation* (2.1) *with initial data $u_0 = u(0,\cdot)$ satisfying $m_0 = (1-\partial_x^2)u_0(x) \ge 0$ and $\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, then*

$$(M_{u(t)}, L_{u(t)}) \in \mathcal{U}, \qquad \text{for all} \qquad t \in [0,T). \tag{4.20}$$

Proof. According to Lemma 4.4, (M_{ψ_c}, L_{ψ_c}) is an isolated local maximum of $F_{\psi_c}(M,L)$, and hence there exist neighborhoods $\mathcal{U} \subset \mathcal{V} \subset \mathbb{R}^2$ of (M_{ψ_c}, L_{ψ_c}) and a constant $\alpha > 0$ such that

$$F_{\psi_c}(M,L) \le -\alpha < 0, \qquad \text{for} \qquad (M,L) \in \mathcal{V}\setminus\mathcal{U}.$$

Lemma 4.2 implies continuity of the conserved functionals H_0, H_1, H_2 with respect to the $H^1(\mathbb{S}^1)$-norm. Thus, if $u_0 \in H^s(\mathbb{S}^1)$, $s > \frac{5}{2}$, is a small perturbation of ψ_c in the $H^1(\mathbb{S}^1)$-norm such that $H_j[u_0] = H_j[\psi_c] + \epsilon_j$ for $j = 0,1,2$, then the corresponding function (4.17), namely

$$F_{u_0}(M,L) = F_{\psi_c}(M,L) + 2k_2 L^2 \epsilon_0 + \left(k_1\left(\tfrac{4}{3}M^2 - \tfrac{1}{3}L^2\right) + 2k_2 M\right)\epsilon_1 - \epsilon_2,$$

is a small perturbation of F_{ψ_c}. Thus, by choosing $\epsilon_0, \epsilon_1, \epsilon_2$ small enough, one ensures that

$$\big\{(M,L) \in \mathcal{V} \,|\, F_{u_0}(M,L) \geq 0\big\} \subset \mathcal{U}. \tag{4.21}$$

Hence, for $\delta > 0$ sufficiently small, if $\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, then $(M_{\psi_c}, L_{\psi_c}) \in \mathcal{U}$. Let $u(t,x)$ be the corresponding periodic solution with initial data u_0. Lemmas 4.6 and 4.3 imply that $M_{u(t)}$ and $L_{u(t)}$ are continuous functions of $t \in [0,T)$ and that $F_{u(t)}(M_{u(t)}, L_{u(t)}) \geq 0$ for $t \in [0,T)$. This immediately implies that (4.20) holds, completing the proof. □

Proof of Theorem 4.1. Let $u \in C([0,T), H^s(\mathbb{S}^1))$, $s \geq \frac{5}{2}$, be a periodic solution of the gmCH equation (2.1) with the initial data $m_0(x) = (1-\partial_x^2)u_0(x) \geq 0$. Given $\epsilon > 0$, let $\mathcal{U}$ be a sufficiently small neighborhood of (M_{ψ_c}, L_{ψ_c}) such that

$$|M - M_{\psi_c}| < \frac{\epsilon^2}{8a}, \qquad \text{whenever} \qquad (M,L) \in \mathcal{U}.$$

Choose $\delta > 0$ as in Lemma 4.7 so that (4.20) holds. Decreasing δ as necessary, we may also assume that

$$\big|H_1[u] - H_1[\psi_c]\big| < \tfrac{1}{2}\,\epsilon^2, \qquad \text{provided} \qquad \|u(0,\cdot) - \psi_c\|_{H^1(S^1)} < \delta.$$

Using Lemma 4.1, we conclude that, for $t \in [0,T)$,

$$\begin{aligned}\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|^2_{H^1(\mathbb{S}^1)} &= H_1[u] - H_1[\psi_c] - 4a\,(u(t,\xi(t)) - M_{\psi_c}) \\ &\leq \big|H_1[u] - H_1[\psi_c]\big| + 4a\,\big|M_{u(t)} - M_{\psi_c}\big| < \epsilon^2,\end{aligned}$$

where $\xi(t) \in \mathbb{R}$ is any point where $u(t,\xi(t)) = M_{u(t)}$. □

Finally, we state the corresponding results in the case $k_1 > 0$ and $k_2 > 0$, with positive and negative wave speeds, respectively. Theorems 4.2 and 4.3 are proved in a similar manner to the approach given in [23, 32], so we omit the details here.

Theorem 4.2. *Assume that $k_1 > 0$ and $k_2 > 0$. Let ψ_c be the peaked soliton defined in* (4.1) *with the traveling wave speed $c > 0$. Then ψ_c is orbitally stable in the following sense. Suppose that $u_0 \in H^s(\mathbb{S}^1)$, $s > \frac{5}{2}$, with $0 \not\equiv m_0(x) = (1-\partial_x^2)u_0(x) \geq 0$. Let $T > 0$ be the maximal existence time of the corresponding periodic solution $u(t,x) \in C([0,T), H^s(\mathbb{S}^1))$ to the initial data u_0. For every $\epsilon > 0$, there is a $\delta > 0$ such that if $\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, then*

$$\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{S}^1)} < \epsilon \qquad \text{for} \qquad t \in [0,T),$$

where $\xi(t) \in \mathbb{R}$ is any point where the function $u(t,\cdot)$ attains its maximum.

Theorem 4.3. *Assume that $k_1 > 0$ and $k_2 > 0$. Let ψ_c be the peaked soliton defined in* (4.1) *with amplitude and traveling wave speed $c < 0$ satisfying* (1.16). *Then ψ_c is orbitally stable in the following sense. Suppose that $u_0 \in H^s(\mathbb{S}^1)$, $s > \frac{5}{2}$, with $0 \not\equiv m_0(x) = (1-\partial_x^2)u_0(x) \leq 0$. Let $T > 0$ be the maximal existence time of the corresponding periodic solution $u(t,x) \in C([0,T), H^s(\mathbb{S}^1))$ to the initial data u_0. For every $\epsilon > 0$, there is a $\delta > 0$ such that if $\|u_0 - \psi_c\|_{H^1(\mathbb{S}^1)} < \delta$, then*

$$\|u(t,\cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{S}^1)} < \epsilon \qquad \text{for} \qquad t \in [0,T),$$

where $\xi(t) \in \mathbb{R}$ is any point where the function $u(t,\cdot)$ attains its minimum.

5. Appendix

In this appendix, we provide the details of the proof of Theorem 2.1.

Proof of Theorem 2.1. As always, we identify $\mathbb{S}^1$ with $[0,1)$ and consider $\psi_c(t,x)$ as a spatially periodic function on $\mathbb{R}$ with period one. Note first that ψ_c is continuous on $\mathbb{S}^1$ with peak at $x=0$. Moreover, ψ_c is smooth on $(0,1)$ and for $t>0$, its first order partial derivatives

$$\partial_x\psi_c(t,x) = -a\sinh\zeta, \quad \partial_t\psi_c(t,x) = ac\sinh\zeta, \quad \text{where} \quad \zeta = \tfrac{1}{2} - (x-c\,t) + [x-c\,t], \tag{5.1}$$

are both in $L^\infty(\mathbb{S}^1)$. Hence, if one denotes $\psi_{c,0}(x) = \psi_c(0,x)$, then

$$\lim_{t\to0^+} \|\psi_c(t,\cdot) - \psi_{c,0}(\cdot)\|_{W^{1,\infty}(\mathbb{S}^1)} = 0. \tag{5.2}$$

Using (5.1), (5.2), and integration by parts, for any $\phi(t,x)\in C_c^\infty([0,\infty)\times\mathbb{S}^1)$, we have

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{S}^1} \left(\psi_c\partial_t\phi + \tfrac{1}{3}k_1\psi_c^3\partial_x\phi + \tfrac{1}{3}k_1(\partial_x\psi_c)^3\phi + \tfrac{1}{2}k_2\psi_c^2\partial_x\phi\right)\,dxdt + \int_{\mathbb{S}^1}\psi_{c,0}(x)\phi(0,x)\,dx\\
&\quad= -\int_0^\infty\int_{\mathbb{S}^1}\phi\,\left(\partial_t\psi_c + k_1\psi_c^2\partial_x\psi_c - \tfrac{1}{3}k_1(\partial_x\psi_c)^3 + k_2\psi_c\partial_x\psi_c\right)\,dxdt\\
&\quad= \int_0^\infty\int_{\mathbb{S}^1}\phi\,\left((k_1a^3 - ac)\sinh\zeta + \tfrac{2}{3}k_1a^3\sinh^3\zeta + k_2a^2\sinh\zeta\,\cosh\zeta\right)\,dxdt.
\end{aligned}\tag{5.3}$$

In addition, the explicit form (2.4) of the Green's function $G(x)$ for the periodic case, along with (5.1) and the proof of Theorem 4.1 in [32], implies that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{S}^1}\left[k_1G(x)*\left(\tfrac{2}{3}\psi_c^3 + \psi_c(\partial_x\psi_c)^2\right)\partial_x\phi - \tfrac{1}{3}k_1G(x)*(\partial_x\psi_c)^3\phi\right]\,dxdt\\
&\quad= k_1a^3\int_0^\infty\int_{\mathbb{S}^1}\phi\,G(x)*\left(2\sinh\zeta + \tfrac{7}{3}\sinh^3\zeta\right)\,dxdt\\
&\qquad\qquad - k_1a^3\int_0^\infty\int_{\mathbb{S}^1}\phi\,G_x(x)*\left(\cosh\zeta\sinh^2\zeta\right)\,dxdt\\
&\quad= \tfrac{2}{3}k_1a^3\int_0^\infty\int_{\mathbb{S}^1}\phi\,\left(\sinh^2\tfrac{1}{2}\,\sinh\zeta - \sinh^3\zeta\right)\,dxdt.
\end{aligned}\tag{5.4}$$

Now, we compute directly

$$\begin{aligned}
\int_0^\infty\int_{\mathbb{S}^1}k_2G*\left(\psi_c^2 + \tfrac{1}{2}(\partial_x\psi_c)^2\right)\partial_x\phi\,dxdt &= \int_0^\infty\int_{\mathbb{S}^1}k_2\,\phi\,G*\partial_x\left(\psi_c^2 + \tfrac{1}{2}(\partial_x\psi_c)^2\right)\,dxdt\\
&= -\tfrac{3}{2}k_2a^2\int_0^\infty\int_{\mathbb{S}^1}\phi\,G*\sinh(2\zeta)\,dxdt.
\end{aligned}\tag{5.5}$$

Thus, when $x>ct$,

$$\begin{aligned}
&G*\sinh(2\zeta)(t,x)\\
&\quad= \frac{1}{2\sinh\frac{1}{2}}\int_{\mathbb{S}^1}\cosh\left(\tfrac{1}{2} - (x-z) + [x-z]\right)\cdot\sinh\left(1 - 2(z-c\,t) + 2[z-c\,t]\right)\,dz\\
&\quad= \tfrac{2}{3}\left[\cosh\left(\tfrac{1}{2}\right)\,\sinh\left(\tfrac{1}{2} - (x-c\,t)\right) - \sinh\left(\tfrac{1}{2} - (x-c\,t)\right)\,\cosh\left(\tfrac{1}{2} - (x-c\,t)\right)\right].
\end{aligned}\tag{5.6}$$

In a similar manner, for $x<ct$, we have

$$\begin{aligned}
&G*\sinh(2\zeta)(t,x)\\
&\quad= \tfrac{2}{3}\left[-\cosh\left(\tfrac{1}{2}\right)\,\sinh\left(\tfrac{1}{2} + (x-c\,t)\right) + \sinh\left(\tfrac{1}{2} + (x-c\,t)\right)\cosh\left(\tfrac{1}{2} + (x-c\,t)\right)\right].
\end{aligned}\tag{5.7}$$

Plugging (5.6) and (5.7) into (5.5) yields

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{S}^1}k_2G*\left(\psi_c^2 + \tfrac{1}{2}(\partial_x\psi_c)^2\right)\,\partial_x\phi\,dxdt\\
&\qquad= k_2a^2\int_0^\infty\int_{\mathbb{S}^1}\phi\sinh\zeta\,\left(\cosh\zeta - \cosh\tfrac{1}{2}\right)\,dxdt.
\end{aligned}\tag{5.8}$$

In view of (5.3), (5.4), and (5.8), we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{S}^1} \Big[&\psi_c \partial_t \phi + \tfrac{1}{3} k_1 \psi_c^3 \partial_x \phi + \tfrac{1}{3} k_1 (\partial_x \psi_c)^3 \phi + \tfrac{1}{2} k_2 \psi_c^2 \partial_x \phi \\ &+ k_1 G(x) * \left(\tfrac{2}{3} \psi_c^3 + \psi_c \left(\partial_x \psi_c\right)^2 \right) \partial_x \phi - \tfrac{1}{3} k_1 G(x) * (\partial_x \psi_c)^3 \phi \\ &+ k_2 G * \left(\psi_c^2 + \tfrac{1}{2} (\partial_x \psi_c)^2 \right) \partial_x \phi \Big] \, dx dt + \int_{\mathbb{S}^1} \psi_{c,0}(x) \phi(0,x) \, dx \\ = \int_0^\infty \int_{\mathbb{S}^1} &\phi \Big[(k_1 a^3 - ac) \sinh \zeta + \tfrac{2}{3} k_1 a^3 \sinh^2 \tfrac{1}{2} \cdot \sinh \zeta + k_2 a^2 \cosh \tfrac{1}{2} \sinh \zeta \Big] \, dx dt \\ = \int_0^\infty \int_{\mathbb{S}^1} &\phi \, a \left[\tfrac{1}{3} k_1 \left(1 + 2 \cosh^2 \tfrac{1}{2} \right) a^2 + k_2 a \cosh \tfrac{1}{2} - c \right] \sinh \zeta \, dx dt.\end{aligned}$$

If a takes the value given in (1.16), then

$$\tfrac{1}{3} k_1 \left(1 + 2 \cosh^2 \tfrac{1}{2} \right) a^2 + k_2 a \cosh \tfrac{1}{2} - c = 0,$$

which implies that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{S}^1} \Big[&\psi_c \partial_t \phi + \tfrac{1}{3} k_1 \psi_c^3 \partial_x \phi + \tfrac{1}{3} k_1 (\partial_x \psi_c)^3 \phi + \tfrac{1}{2} k_2 \psi_c^2 \partial_x \phi \\ &+ k_1 G(x) * \left(\tfrac{2}{3} \psi_c^3 + \psi_c \left(\partial_x \psi_c\right)^2 \right) \partial_x \phi - \tfrac{1}{3} k_1 G(x) * (\partial_x \psi_c)^3 \phi \\ &+ k_2 G(x) * \left(\psi_c^2 + \tfrac{1}{2} (\partial_x \psi_c)^2 \right) \partial_x \phi \Big] \, dx dt + \int_{\mathbb{S}^1} \psi_{c,0}(x) \phi(0,x) \, dx = 0,\end{aligned}$$

for any $\phi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{S}^1)$. This completes the proof. □

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References

[1] R. Camassa and D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661-1664.
[2] K.S. Chou and C.Z. Qu, Integrable equations arising from motions of plane curves I, *Physica D*, **162** (2002), 9-33.
[3] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321-362.
[4] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.*, **166** (2006), 523-535.
[5] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229-243.
[6] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa*, **26** (1998), 303-328.
[7] A. Constantin and J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, **233** (2000), 75-91.
[8] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, *Ann. Math.*, **173** (2011), 559-568.

[9] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rational Mech. Anal.*, **192** (2009), 165–186.
[10] A. Constantin and L. Molinet, Orbital stability of solitary waves for a shallow water equation, *Physica D*, **157** (2001), 75-89.
[11] A. Constantin and W. Strauss, Stability of peakons, *Commun. Pure Appl. Math.*, **53** (2000), 603-610.
[12] A. Degasperis, D.D. Holm, and A.N.W. Hone, Integrable and non-integrable equations with peakons, Nonlinear physics: theory and experiment, **II** (Gallipoli, 2002), 37 (World Sci. Publ., River Edge, NJ, 2003).
[13] K. El Dika and L. Molinet, Stability of multipeakons, *Ann. Inst. Henri Poincaré Anal. Non Lin'eaire*, **18** (2009), 1517-1532.
[14] M. Faquir, M.A. Manna, and A. Neveu, An integrable equation governing short waves in a long-wave model, *Proc. R. Soc. A*, **463** (2007), 1939-1954.
[15] A.S. Fokas, The Korteweg-de Vries equation and beyond, *Acta Appl. Math.*, **39** (1995), 295-305.
[16] A.S. Fokas, On a class of physically important integrable equations, *Physica D*, **87** (1995), 145-150.
[17] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D*, **95** (1996), 229-243.
[18] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4** (1981/1982), 47-66.
[19] G.L. Gui, Y. Liu, P.J. Olver, and C.Z. Qu, Wave-breaking and peakons for a modified Camassa-Holm equation, *Commun. Math. Phys.*, **319** (2013), 731-759.
[20] R.E. Goldstein and D.M. Petrich, The Korteweg-de Vries hierarchy as dynamics of closed curves in the plane, *Phys. Rev. Lett.*, **67** (1991), 3203-3206.
[21] H. Holden and X. Raynaud, A convergent numerical scheme for the Camassa-Holm equation based on multipeakons, *Discrete Cont. Dyn. Syst. A*, **14** (2006), 505–523.
[22] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, **455** (2002), 63-82.
[23] J. Lenells, Stability of periodic peakons, *Int. Math. Res. Not.*, **10** (2004), 485-499.
[24] Y.A. Li and P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Diff. Eq.*, **162** (2000), 27-63.
[25] Y.A. Li, P.J. Olver, and P. Rosenau, Non-analytic solutions of nonlinear wave models, in: *Nonlinear Theory of Generalized Functions*, M. Grosser, G. Höormann, M. Kunzinger, and M. Oberguggenberger, eds., Research Notes in Mathematics, vol. 401, Chapman and Hall/CRC, New York, 1999, pp. 129-145.
[26] Z.W. Lin and Y. Liu, Stability of peakons for the Degasperis-Procesi equation, *Commun. Pure Appl. Math.*, **62** (2009), 125-146.
[27] X.C. Liu, Y. Liu, and C.Z. Qu, Orbital stability of the train of peakons for an integrable modified Camassa-Holm equation, *Adv. Math.*, **255** (2014), 1-37.
[28] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, Springer–Verlag, New York, 1993.
[29] P.J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E*, **53** (1996), 1900-1906.
[30] Z. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, *J. Math. Phys.*, **47** (2006), 112701, 9pp.
[31] Z. Qiao, B. Xia, and J.B. Li, A generalized Camassa-Holm equation and its peakon solutions, http://xxx. lanl. gov/abs/1205. 2028.
[32] C.Z. Qu, X.C. Liu, and Y. Liu, Stability of peakons for an integrable modified Camassa-Holm equation, *Commun. Math. Phys.*, **322** (2013), 967-997.
[33] C.Z. Qu, Y. Zhang, X.C. Liu, and Y. Liu, Orbital stability of periodic peakons to a generalized μ-Camassa-Holm equation, *Arch. Rational Mech. Anal.*, **211** (2014), 593-617.
[34] T. Schäfer and C.E. Wayne, Propagation of ultra-short optical pulses in cubic nonlinear media, *Physica D*, **196** (2004), 90-105.

[35] J. F. Toland, Stokes waves, *Topol. Methods Nonlinear Anal.,* **7** (1996), 1-48.

Xiaochuan Liu
Department of Mathematics, Northwest University, Xi'an 710069, P. R. China
E-mail address: liuxc@nwu.edu.cn

Yue Liu
Department of Mathematics, University of Texas, Arlington, TX 76019, USA
E-mail address: yliu@uta.edu

Peter J. Olver
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: olver@umn.edu

Changzheng Qu
Department of Mathematics, Ningbo University, Ningbo 315211, P. R. China
E-mail address: quchangzheng@nbu.edu.cn