LIOUVILLE CORRESPONDENCE BETWEEN THE MODIFIED KDV HIERARCHY AND ITS DUAL INTEGRABLE HIERARCHY

JING KANG, XIAOCHUAN LIU, PETER J. OLVER, AND CHANGZHENG QU

Abstract. We study an explicit correspondence between the integrable modified KdV hierarchy and its dual integrable modified Camassa-Holm hierarchy. A Liouville transformation between the isospectral problems of the two hierarchies also relates their respective recursion operators, and serves to establish the Liouville correspondence between their flows and Hamiltonian conservation laws. In addition, a novel transformation mapping the modified Camassa-Holm equation to the Camassa-Holm equation is found. Furthermore, it is shown that the Hamiltonian conservation laws in the negative direction of the modified Camassa-Holm hierarchy are both local in the field variables and homogeneous under rescaling.

Key words and phrases: Liouville transformation; modified Camassa-Holm hierarchy; modified KdV hierarchy; tri-Hamiltonian duality; Hamiltonian conservation law; local conservation law; scaling homogeneity.

2000 *Mathematics Subject Classification* : 37K05, 37K10.

1. Introduction

In this paper, we investigate the correspondence between the modified KdV (mKdV) hierarchy, that is initiated with the (focusing) mKdV equation

$$Q_\tau + Q_{yyy} + 6\,Q^2\,Q_y = 0, \tag{1.1}$$

and its dual integrable hierarchy, initiated with the following nonlinear evolution equation

$$m_t + \big((u^2 - u_x^2)\,m\big)_x = 0, \qquad m = u - u_{xx}, \tag{1.2}$$

known as the *modified Camassa-Holm* (mCH) *equation* [18, 21, 44].

Recent years have seen the appearance of a large number of papers devoted to equations of Camassa-Holm type. The Camassa-Holm (CH) equation — see (1.4) below — and the mCH equation (1.2) both support nonlinear dispersion, and can describe qualitative properties in the fully nonlinear regime. In particular, these equations describe the breakdown of nonlinear waves and support a notable variety of non-smooth soliton-like solutions. For instance, they possess peakon and multi-peakon solutions, and can model wave-breaking phenomena for appropriate initial data.

Several methods have been employed to construct integrable equations endowed with nonlinear dispersion. In particular, a theory of tri-Hamiltonian duality was developed systematically in the references [21, 22, 44]. This approach starts from the basic observation that most standard integrable soliton equations, which are known to exhibit a bi-Hamiltonian structure, actually support a compatible trio of Hamiltonian structures through a particular scaling argument. Different recombinations of the operators in such a compatible Hamiltonian triple yield different types of bi-Hamiltonian integrable systems, which are recognized to admit a dual relationship. In [44], an explicit algorithm to construct dual integrable systems is provided. Typically, these dual systems are endowed with nonlinear dispersion and thus admit non-smooth solitons including compactons, cuspons, peakons, and more exotic species, [32].

Indeed, applying tri-Hamiltonian duality to the bi-Hamiltonian representation of the KdV equation

$$P_\tau + P_{yyy} - 6\,PP_y = 0 \tag{1.3}$$

leads to the well-studied CH equation

$$\rho_t + 2v_x\rho + v\rho_x = 0, \qquad \rho = v - v_{xx}, \tag{1.4}$$

and justifies its status as a dual integrable bi-Hamiltonian system [21, 22, 44]. Equation (1.4) originally appears as an abstract integrable system with an infinite number of higher-order symmetries using the method of recursion operators [23]. The CH equation has attracted enormous attention in the last two decades because of its many remarkable properties: complete integrability [4, 5, 9, 12, 14], the existence of an infinite number of nontrivial local conservation laws and a full Lie algebra of nonlocal symmetries [17, 25, 30, 48], physical relevance of the nonlinear shallow-water waves [4, 13, 26], non-smooth soliton structures of peaked solitons and multi-peakons [2, 4, 5, 15, 16, 27] and the presence of breaking waves [8, 10, 31]. Furthermore, the CH equation (1.4) has nice geometric formulations: it describes a geodesic flow on the diffeomorphism group on the circle $\mathbb{S}^1$ [28] and arises naturally from a non-stretching invariant planar curve flow in centro-equiaffine geometry [7].

Similarly, if one applies tri-Hamiltonian duality to the mKdV equation (1.1), then the corresponding dual system is exactly the mCH equation (1.2) with cubic nonlinearity [18, 21, 44]. Physically, the mCH equation (1.2) models the unidirectional propagation of shallow-water waves over a flat bottom, where the function u represents the free surface elevation; it was derived from the two-dimensional hydrodynamical equations for surface waves by Fokas [19]. Geometrically, the mCH equation is shown to arise from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry [24]. It also admits non-smooth peakons and multi-peakon solutions, which are, in fact, stable configurations [34, 47]. Moreover, it is worth mentioning that due to the higher-order nonlinearity, the mCH equation (1.2) exhibits new features, including wave breaking and multi-peakon dynamics [24, 35], that differ from the well-known properties of the CH equation (1.4).

In view of the dual relationship between the KdV equation and the CH equation, and between the mKdV equation and the mCH equation, it is anticipated that the original soliton hierarchies should be related to their dual counterparts in a certain manner. For instance, in [29] and [37], the correspondence between the CH hierarchy and the KdV hierarchy is established by the Liouville transformation; see also [1, 2]. More precisely, the positive and negative flows of the CH hierarchy are generated by the negative and positive flows of the KdV hierarchy respectively. The correspondence between the Hamiltonian conservation laws of the CH hierarchy and the original KdV hierarchy is also derived in [29]. Such a relationship was also explored by using the loop group approach [52]. And yet there were some questions that remained unanswered. For instance, can one construct such a correspondence between the mCH hierarchy and mKdV hierarchy? Is it possible to relate the conservation laws of the mCH hierarchy to the mKdV hierarchy? The associated conservation laws play a crucial role in the study of well-posedness, wave breaking and stability of peakons — see [8, 10, 11, 15] for the CH equation and [24, 33, 34, 35, 47] for the mCH equation. Hence, the induced relationship between the conservation laws for the dual integrable systems and their original soliton counterparts turns out to be of value for revealing the relevant structure of the former conservation laws.

The first topic of this paper is to study the Liouville correspondence between the integrable mKdV and the mCH hierarchies. In [29] and [37], the analysis for the KdV-CH setting depends in a large part on the well-established relation between one of the original KdV Hamiltonian operators and one of the dual CH Hamiltonian operators. For the mKdV-mCH setting considered here, it seems difficult to derive a similar relation linking a Hamiltonian operator of the mKdV hierarchy with a dual Hamiltonian operator of the mCH hierarchy. Nevertheless, based on the Liouville transformation between the isospectral problems of the mKdV hierarchy and the mCH hierarchy, we are able to establish certain nontrivial relations between the respective recursion operators. In contrast to the approach used in [29] and [37], we combine a reciprocal transformation [49], which adheres to the conservative structure of the mCH flows, with certain operator identities to establish a one-to-one Liouville correspondence between the two integrable hierarchies. In addition, we investigate the relationship between the Hamiltonian conservation laws for the mCH hierarchy and those for the mKdV hierarchy, as well as their local nature and homogeneous character.

The outline of the paper is as follows. In Section 2, we recall some known results on integrability of the mKdV and mCH equations and the corresponding hierarchies. The main results in this paper are also presented. In Section 3, we first derive the Liouville transformation relating the isospectral problems of the mCH hierarchy and the mKdV hierarchy in Section 3.1. Next in Section 3.2, based on the conservative structures of flows in the mCH hierarchy, we exploit the Liouville transformation to establish the one-to-one correspondence between the flows in the mCH and the mKdV hierarchies. Furthermore in Section 3.3, we give a novel transformation which maps the mCH equation (1.2) into the CH equation (1.4). Section 4 deals with the hierarchy of the Hamiltonian conservation laws of the mCH equation. It is proved in Section 4.1 that the Liouville transformation establishes the correspondence between the series of Hamiltonian conservation laws of the mCH equation and the mKdV equation. Finally in Section 4.2, we establish the homogeneity and the local nature for the Hamiltonian conservation laws in the negative direction admitted by the mCH equation.

2. Preliminaries and main results

The modified Camassa-Holm equation (1.2) can be written in bi-Hamiltonian form [44]

$$m_t = \mathcal{K}\,\frac{\delta \mathcal{H}_1}{\delta m} = \mathcal{J}\,\frac{\delta \mathcal{H}_2}{\delta m}, \qquad m = u - u_{xx}, \tag{2.1}$$

where

$$\mathcal{K} = -\partial_x\, m\, \partial_x^{-1}\, m\, \partial_x \qquad \text{and} \qquad \mathcal{J} = -\left(\partial_x - \partial_x^3\right) \tag{2.2}$$

are compatible Hamiltonian operators, while the corresponding Hamiltonian functionals are given by

$$\mathcal{H}_1[m] = \int m\, u \,\mathrm{d}x = \int \left(u^2 + u_x^2\right)\,\mathrm{d}x \quad \text{and} \quad \mathcal{H}_2[m] = \tfrac{1}{4}\int \left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right)\,\mathrm{d}x. \tag{2.3}$$

In general, for an integrable bi-Hamiltonian equation with two compatible Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$, Magri's Theorem [36, 42] establishes the formal existence of an infinite hierarchy

$$m_t = K_n = \mathcal{K}\,\frac{\delta \mathcal{H}_{n-1}}{\delta m} = \mathcal{J}\,\frac{\delta \mathcal{H}_n}{\delta m}, \qquad n = 1, 2, \ldots, \tag{2.4}$$

of higher-order commuting bi-Hamiltonian systems, based on the higher-order Hamiltonian conservation laws $\mathcal{H}_n$, $n = 0, 1, 2, \ldots$, common to all members of the hierarchy. The members in the hierarchy (2.4) are obtained by applying successively the recursion operator $\mathcal{R} = \mathcal{K}\,\mathcal{J}^{-1}$ to a seed symmetry [41], which in the mCH setting takes the following form:

$$m_t = K_1[m] = -2m_x, \qquad \text{with} \qquad \mathcal{H}_0[m] = \int m\,\mathrm{d}x.$$

Clearly, the mCH equation (1.2) in this hierarchy is exactly

$$m_t = K_2 = -\left((u^2 - u_x^2)\, m\right)_x = \mathcal{R}K_1[m].$$

Similarly, one can also construct an infinite number of higher-order commuting bi-Hamiltonian systems in the negative direction:

$$m_t = K_{-n} = \mathcal{K}\,\frac{\delta \mathcal{H}_{-(n+1)}}{\delta m} = \mathcal{J}\,\frac{\delta \mathcal{H}_{-n}}{\delta m}, \qquad n = 1, 2, \ldots, \tag{2.5}$$

starting from the variational derivative of the Casimir functional associated with the Hamiltonian operator $\mathcal{K}$, which, in the mCH setting, is given, up to constant multiple, by

$$\mathcal{H}_C[m] = \int \frac{1}{m}\,\mathrm{d}x \qquad \text{with variational derivative} \qquad \frac{\delta \mathcal{H}_C}{\delta m} = -\frac{1}{m^2}\,. \tag{2.6}$$

As the original soliton equation in the duality relationship with the mCH equation (1.2), the mKdV equation (1.1) also admits a hierarchy consisting of an infinite number of integrable equations in both the positive and negative directions. Each member in the positive direction takes the form

$$Q_\tau = \bar{K}_n = \bar{\mathcal{K}}\,\frac{\delta \bar{\mathcal{H}}_{n-1}}{\delta Q} = \bar{\mathcal{J}}\,\frac{\delta \bar{\mathcal{H}}_n}{\delta Q}, \qquad n = 1, 2, \ldots, \tag{2.7}$$

where

$$\bar{\mathcal{K}} = -\tfrac{1}{4}\partial_y^3 - \partial_y Q \partial_y^{-1} Q \partial_y, \qquad \bar{\mathcal{J}} = -\partial_y, \tag{2.8}$$

are the compatible Hamiltonian operators, and $\bar{\mathcal{H}}_n$, $n = 0, 1, 2, \ldots$, are the corresponding Hamiltonian conservation laws. However, in the negative direction, we exploit a similar argument used to construct the negative KdV hierarchy in [22], and thus use the following equivalence

$$Q_\tau = \bar{K}_{-n} = \bar{\mathcal{K}}\frac{\delta \bar{\mathcal{H}}_{-(n+1)}}{\delta Q} = \bar{\mathcal{J}}\frac{\delta \bar{\mathcal{H}}_{-n}}{\delta Q} \quad \Longleftrightarrow \quad \bar{\mathcal{R}}^n Q_\tau = 0, \qquad n = 1, 2, \ldots, \tag{2.9}$$

where $\bar{\mathcal{R}} = \bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}$ is the recursion operator of the mKdV hierarchy.

The bi-Hamiltonian structures (2.1)-(2.2) and (2.7)-(2.8) reveal the underlying duality between the mCH hierarchy and the mKdV hierarchy. Schiff [51] established a different kind of duality by exploiting the zero curvature formulations of dual hierarchies and the standard soliton equation hierarchies. From this perspective, we focus our attention on the isospectral problems for the mCH hierarchy [46, 51]

$$\mathbf{\Psi}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda\, m \\ -\frac{1}{2}\lambda\, m & \frac{1}{2} \end{pmatrix} \mathbf{\Psi}, \tag{2.10}$$

and the mKdV hierarchy [39]

$$\mathbf{\Phi}_y = \begin{pmatrix} -\,\mathrm{i}\,\mu & Q \\ -Q & \mathrm{i}\,\mu \end{pmatrix} \mathbf{\Phi}, \tag{2.11}$$

where λ and μ are the respective spectral parameters. First, we show that the Liouville transformation

$$Q(y) = \frac{\mathrm{i}}{2m(x)}, \qquad y = \int^x m(\xi)\,\mathrm{d}\xi, \qquad m = u - u_{xx}, \tag{2.12}$$

relating the isospectral problems (2.10) and (2.11) establishes the one-to-one correspondence between the flows in the mCH hierarchy and the mKdV hierarchy. Usually, the process of going from one spectral problem to another one by means of a change of variables has been recognized as a form of the classical Liouville transformation, which arises naturally in the context of the so-called WKB approximation; see [38, 40]. Note further that the second equation in (2.12) has the form of a reciprocal transformation [49]. More precisely, we establish the following theorem.

Theorem 2.1. *For any integer n, the $(n+1)$-st equation $m_t = K_{n+1}$ in the mCH hierarchy is mapped into $(-n)$-th equation $Q_\tau = \bar{K}_{-n}$ in the mKdV hierarchy under the Liouville transformation* (2.12)*, and conversely.*

Remark 2.1. *The first equation $m_t = K_{-1}[m]$ in the negative direction of the mCH hierarchy takes the explicit form*

$$m_t = \mathcal{J}\,\frac{\delta\mathcal{H}_{-1}}{\delta m} = \mathcal{J}\,\frac{\delta\mathcal{H}_C}{\delta m} = \left(\frac{1}{m^2}\right)_x - \left(\frac{1}{m^2}\right)_{xxx}, \tag{2.13}$$

which is called the Casimir equation in [44]*; see also* [6]*. It was noted in* [44] *that equation* (2.13) *has the form of a Lagrange transformation, modulo an appropriate complex transformation, of the mKdV equation* (1.1)*. We remark that the transformation* (2.12) *used in this paper can be regarded as a complex version of the Lagrange transformation proposed by Rosenau in* [50].

Remark 2.2. *The negative flows of the mKdV hierarchy are also generated from the Casimir equation*

$$Q_\tau = \bar{\mathcal{J}}\,\frac{\delta\bar{\mathcal{H}}_C}{\delta Q}, \tag{2.14}$$

where $\bar{\mathcal{H}}_C$ is the Casimir functional of the Hamiltonian operator $\bar{\mathcal{K}}$. Due to the form of the Hamiltonian operators (2.8)*, equation* (2.14) *can be rewritten as*

$$Q_\tau = -\bar{\mathcal{J}}\partial_y^{-1}\left(\tfrac{1}{4}\partial_y + Q\partial_y^{-1}Q\right)^{-1}\partial_y^{-1}0 = \left(\tfrac{1}{4}\partial_y + Q\,\partial_y^{-1}Q\right)^{-1}\bar{C}_{-1} \tag{2.15}$$

with $\bar{C}_{-1}$ being the corresponding constant of integration. When $\bar{C}_{-1}=0$, the kernel of the operator $\frac{1}{4}\partial_y + Q\partial_y^{-1}Q$ is spanned by $\sin(2\,\partial_y^{-1}Q)$, and so Q satisfies

$$Q_\tau = \sin(2\,\partial_y^{-1}Q), \tag{2.16}$$

which is just the potential form of the sine-Gordon equation

$$U_{y\tau} = \sin(2\,U) \tag{2.17}$$

for the potential function $U = \partial_y^{-1}Q$. Observe that the corresponding Casimir functional is

$$\bar{\mathcal{H}}_S = -\frac{1}{2}\int\cos(2\,\partial_y^{-1}Q)\,\mathrm{d}y, \quad \textit{with variational derivative} \quad \frac{\delta\bar{\mathcal{H}}_S}{\delta Q} = -\,\partial_y^{-1}\sin(2\,\partial_y^{-1}Q). \tag{2.18}$$

However, we will see that the mCH equation is related, via the Liouville correspondence, to (2.15) *when $\bar{C}_{-1}\neq 0$, and not to the sine-Gordon hierarchy coming from $\bar{C}_{-1}=0$. In the case $\bar{C}_{-1}\neq 0$, we do not know explicit elementary formulas for the corresponding Casimir functional and flow.*

Remark 2.3. *More interestingly, combining Theorem 2.1 and the given results on the correspondence between the CH equation and the first negative flow of the KdV hierarchy, and the usual Miura transformation linking the KdV equation and the mKdV equation* [42]*, we are able to establish a non-obvious transformation mapping the mCH equation* (1.2) *into the CH equation* (1.4)*. The detailed analysis will be presented in Section 3.3.*

For the CH equation (1.4), the bi-Hamiltonian structure takes the following form

$$\rho_t = \mathcal{L}\,\frac{\delta E_1}{\delta\rho} = \mathcal{J}\,\frac{\delta E_2}{\delta\rho}, \tag{2.19}$$

where the compatible Hamiltonian operators are given by

$$\mathcal{L} = -\left(\partial_x\rho + \rho\,\partial_x\right) \qquad \text{and} \qquad \mathcal{J} = -\left(\partial_x - \partial_x^3\right),$$

which are related to the Hamiltonian pair

$$\bar{\mathcal{L}} = \tfrac{1}{4}\partial_y^3 - \tfrac{1}{2}\left(P\partial_y + \partial_y P\right) \qquad \text{and} \qquad \bar{\mathcal{D}} = \partial_y$$

of the KdV equation (1.3). It was proved in [29] and [37] that the corresponding Liouville transformation relating the isospectral problems of the CH hierarchy and the KdV hierarchy transforms one hierarchy into the other. The following identities

$$\mathcal{L}^{-1} = -\frac{1}{2\sqrt{\rho}}\,\bar{\mathcal{D}}^{-1}\frac{1}{\rho} \qquad \text{and} \qquad \bar{\mathcal{L}} = \frac{1}{4\rho}\,\mathcal{J}\frac{1}{\sqrt{\rho}} \tag{2.20}$$

relating the Hamiltonian operators under the Liouville transformation play an important role in the analysis used in [29] and [37]. In the present paper, we establish new identities

$$\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^n \left(1-\partial_x^2\right) = \frac{1}{(-4)^n}\left(1 - \frac{Q_y}{4Q^3}\partial_y + \frac{1}{4Q^2}\partial_y^2\right)\left(\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\right)^n, \qquad n = 1, 2, \ldots, \tag{2.21}$$

relating the recursion operators for the mCH and the mKdV hierarchies under the Liouville transformation (2.12). Theorem 2.1 is then a consequence of formula (2.21).

The compatible bi-Hamiltonian structure (2.4)-(2.5) produces the recursively constructed bi-infinite sequence of functionals in both the negative and positive directions:

$$\ldots, \mathcal{H}_{-2},\, \mathcal{H}_{-1},\, \mathcal{H}_0,\, \mathcal{H}_1,\, \mathcal{H}_2,\, \ldots, \tag{2.22}$$

which are all conserved densities of the mCH equation (1.2). Similarly, the recursive formula

$$\bar{\mathcal{K}}\frac{\delta\bar{\mathcal{H}}_n}{\delta Q} = \bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_{n+1}}{\delta Q}, \qquad n \in \mathbb{Z},$$

where $\bar{\mathcal{K}}$ and $\bar{\mathcal{J}}$ are given by (2.8), gives rise to an infinite sequence of Hamiltonian functionals

$$\ldots, \bar{\mathcal{H}}_{-2},\, \bar{\mathcal{H}}_{-1},\, \bar{\mathcal{H}}_0,\, \bar{\mathcal{H}}_1,\, \bar{\mathcal{H}}_2,\, \ldots, \tag{2.23}$$

conserved under the mKdV flow (1.1) [36, 42]. We will study the correspondence between the conserved quantities in the hierarchies (2.22) and (2.23), and prove that the Liouville transformation (2.12) not only links the integrable flows in the respective hierarchies but also relates the corresponding Hamiltonian conservation laws of the original soliton equation and its dual counterpart. The following theorem is thus established to illustrate the preceding claim.

Theorem 2.2. *For any non-zero integer n, each Hamiltonian conservation law $\bar{\mathcal{H}}_n(Q)$ of the mKdV equation in* (2.23) *yields the Hamiltonian conservation law $\mathcal{H}_{-n}(m)$ of the mCH equation in* (2.22)*, under the Liouville transformation* (2.12)*, according to the following identity*

$$\mathcal{H}_{-n}(m) = (-1)^n 2^{2n-1}\,\bar{\mathcal{H}}_n(Q), \qquad 0 \neq n \in \mathbb{Z}. \tag{2.24}$$

Remark 2.4. *A direct application of relation* (2.24) *is to derive another Casimir functional, in addition to the Hamitonian functional* (2.18) *of the sine-Gordon equation* (2.17)*, for the Hamiltonian operator $\bar{\mathcal{K}}$ given by* (2.8) *in the mKdV bi-Hamiltonian structure. Indeed, with Q and m related by* (2.12)*, one exploits* (2.3) *and* (2.24) *to obtain*

$$\bar{\mathcal{H}}_{-1}(Q) = -8\,\bar{\mathcal{H}}_C(Q), \quad \textit{where} \quad \bar{\mathcal{H}}_C(Q) = \int m\left(1-\partial_x^2\right)^{-1} m\,\mathrm{d}x$$

is another Casimir functional for $\bar{\mathcal{K}}$ that differs from $\bar{\mathcal{H}}_S$ given in (2.18).

In general, for an evolution equation or system, possession of a suitable collection of local conservation laws is a key ingredient in the analysis of the qualitative properties such as blow-up criteria, asymptotic behavior, and stability. Although, the mCH equation admits infinitely many Hamiltonian conservation laws (2.22), their recursive construction requires inverting either $\mathcal{K}$ or $\mathcal{J}$ in (2.4) and (2.5). Due to the specific forms of the operators $\mathcal{K}$ and $\mathcal{J}$ defined in (2.2), nonlocal terms might appear and their structure and utility is generally unclear.

Furthermore, it is remarked that tri-Hamiltonian duality is based on the scaling properties of the original Hamiltonian operators. This means that, after recombination, the resulting dual Hamiltonian operator exhibits a particular homogeneity under the scaling transformations $x \mapsto \nu x$ and/or $u \mapsto \sigma u$. Because of the homogeneous nature of the Hamiltonian operator triple, the recombined pair of compatible Hamiltonian operators and the resulting Hamiltonian functionals admitted by the tri-Hamiltonian duality systems, such as the mCH equation, the CH equation, etc., will maintain homogeneity.

In light of these considerations, we are led to establish the following theorem concerning the locality and homogeneity of the Hamiltonian conservation laws (2.22) of the mCH equation.

Theorem 2.3. *For the mCH equation* (1.2)*, each $\mathcal{H}_{-n}[m]$, $n \geq 1$, in* (2.22) *is a local functional, meaning that it is the integral of a differential function depending on m and its x-derivatives. Furthermore, $\mathcal{H}_{-n}$ is homogenous of weight $1-2n$ with respect to the rescaling $m \mapsto \sigma m$:*

$$\mathcal{H}_{-n}[\sigma m] = \sigma^{1-2n}\, \mathcal{H}_{-n}[m], \qquad 0 \neq \sigma \in \mathbb{R}, \quad n = 1, 2, \ldots.$$

By introducing an associated CH equation, it was proved that the CH equation (1.4) possesses an infinite sequence of conservation laws, both local and nonlocal [17]. In addition, Reyes [48] proved, using a geometrical procedure based on the relationship between the CH equation and pseudo-spherical surfaces, that all the Hamiltonian conservation laws in the negative direction are local. Later, a more direct verification was given by Lenells [30] using the bi-Hamiltonian structure (2.19). Our proof of Theorem 2.3 will be based on the bi-Hamiltonian structure (2.1). In [3], the existence of an infinite hierarchy of local conservation laws for the mCH equation (1.2) was constructed through a geometric reformulation of the system.

Remark 2.5. *Note that the inverse differentiation operator ∂_x^{-1} occurs both in the expressions for $\mathcal{K} = -\partial_x\, m\, \partial_x^{-1}\, m\, \partial_x$ and its inverse $\mathcal{K}^{-1} = -\partial_x^{-1} \frac{1}{m} \partial_x \frac{1}{m} \partial_x^{-1}$. For simplicity, we usually prescribe the choice of the integration constant to be zero whenever the operator ∂_x^{-1} is applied. However, in the nonlocal setting, one needs to be quite careful, owing to the possible appearance of ghost symmetries,* [43, 45]*, which can produce apparent paradoxes in the calculus of nonlocal symmetries and nonlocal equations.*

3. The correspondence between the mCH and mKdV hierarchies

3.1. **A Liouville transformation between the isospectral problems of the mCH and mKdV equations.** Our starting point is the isospectral problems associated to the mCH equation (1.2) and the mKdV equation (1.1). The mCH equation (1.2) can be expressed as the compatibility condition $\partial_t(\mathbf{\Psi}_x) = \partial_x(\mathbf{\Psi}_t)$ between

$$\mathbf{\Psi}_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda\, m \\ -\frac{1}{2}\lambda\, m & \frac{1}{2} \end{pmatrix} \mathbf{\Psi}, \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.1}$$

and

$$\mathbf{\Psi}_t = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}\lambda\, m(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}\lambda\, m(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix} \mathbf{\Psi}.$$

On the other hand, the zero curvature formulation for the mKdV equation (1.1) comes from the compatibility condition $\partial_\tau(\mathbf{\Phi}_y) = \partial_y(\mathbf{\Phi}_\tau)$ between

$$\mathbf{\Phi}_y = \begin{pmatrix} -\mathrm{i}\,\mu & Q \\ -Q & \mathrm{i}\,\mu \end{pmatrix} \mathbf{\Phi}, \qquad \mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3.2}$$

and

$$\mathbf{\Phi}_\tau = \begin{pmatrix} -4\,\mathrm{i}\,\mu^3 + 2\,\mathrm{i}\,\mu Q^2 & 4\mu^2 Q - 2\,Q^3 + 2\,\mathrm{i}\,\mu Q_y - Q_{yy} \\ -4\mu^2 Q + 2\,Q^3 + 2\,\mathrm{i}\,\mu Q_y + Q_{yy} & 4\,\mathrm{i}\,\mu^3 - 2\,\mathrm{i}\,\mu Q^2 \end{pmatrix} \mathbf{\Phi}.$$

One can verify that the following Liouville transformation

$$\mathbf{\Phi} = \begin{pmatrix} -1 & \mathrm{i} \\ -\,\mathrm{i} & 1 \end{pmatrix} \mathbf{\Psi}, \qquad y = \int^x m(\xi)\,\mathrm{d}\xi,$$

will convert the isospectral problem (3.1) into the isospectral problem (3.2), with

$$Q = \frac{\mathrm{i}}{2m} \qquad \text{and} \qquad \lambda = -2\mu.$$

Motivated by this and the form of mCH equation (1.2), we introduce the following coordinate transformations:

$$y = \int^x m(t,\xi)\,\mathrm{d}\xi, \qquad \tau = t \tag{3.3}$$

and

$$Q(\tau, y) = \frac{\mathrm{i}}{2m(t,x)}. \tag{3.4}$$

The following subsection will investigate how the transformations (3.3) and (3.4) affect the underlying correspondence between the flows of the mCH and mKdV hierarchies.

3.2. **The correspondence between the mCH and mKdV hierarchies.** Let us now consider the mCH and mKdV hierarchies. First of all, the positive flows in the mCH hierarchy (2.4) are generated successively by applying the recursion operator $\mathcal{K}\,\mathcal{J}^{-1}$ to the seed symmetry $-2m_x$, namely,

$$m_t + \left(\mathcal{K}\,\mathcal{J}^{-1}\right)^n (2\,m_x) = 0, \qquad n = 0, 1, \dots. \tag{3.5}$$

On the other hand, since the Hamiltonian operator $\mathcal{K}$ admits the local Casimir functional (2.6), the negative flows (2.5) begin with the corresponding Casimir equation (2.13). Successively applying the inverse recursion operator $\mathcal{J}\,\mathcal{K}^{-1}$ produces the hierarchy of negative flows, in which the n-th member takes the form

$$m_t = K_{-n} = -\left(\mathcal{J}\,\mathcal{K}^{-1}\right)^{n-1} \mathcal{J}\frac{1}{m^2}, \quad n = 1, 2, \dots. \tag{3.6}$$

Similarly, for the mKdV hierarchy, the positive flows in (2.7) take the form

$$Q_\tau + \left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n (4\,Q_y) = 0, \qquad n = 0, 1, \dots, \tag{3.7}$$

where $\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}$ is the recursion operator, whereas the negative flows

$$\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n Q_\tau = 0, \qquad n = 1, 2, \dots,$$

can be rewritten as

$$\partial_y \left(\tfrac{1}{4}\partial_y + Q\partial_y^{-1}Q\right) \left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = 0,$$

due to the forms of the Hamiltonian operators (2.8). Thus, for each $n \geq 1$,

$$\left(\tfrac{1}{4}\partial_y + Q\partial_y^{-1}Q\right) \left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1} Q_\tau = \bar{C}_{-n}, \tag{3.8}$$

with $\bar{C}_{-n}$ being the corresponding constant of integration.

Remark 3.1. *In view of* (2.16)*, when the integration constant $\bar{C}_{-n} = 0$, the negative mKdV flows* (3.8) *have the form*

$$Q_\tau = \left(\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\right)^{n-1} \sin(2\,\partial_y^{-1}Q). \tag{3.9}$$

The associated potential function $U = \partial_y^{-1}Q$ satisfies

$$\widetilde{R}^{n-1} U_\tau = \partial_y^{-1} \sin(2\,U),$$

where

$$\widetilde{R} = \tfrac{1}{4}\partial_y^2 + U_y^2 - U_y\partial_y^{-1}U_{yy}$$

is the recursion operator of the sine-Gordon equation [41]. *Furthermore, the equations*

$$U_\tau + \widetilde{R}^{n-1}(4\,U_y) = 0, \quad \textit{for} \quad n = 1, 2, \ldots,$$

are the positive flows in the potential mKdV hierarchy, which starts, at $n = 2$, *with*

$$U_\tau + U_{yyy} + 2\,U_y^3 = 0.$$

However, as mentioned in Remark 2.2, the mCH hierarchy is not related to the hierarchy (3.9) *under the Liouville transformations* (3.3) *and* (3.4). *Therefore, in what follows, we only consider the negative mKdV hierarchy* (3.8) *with the integration constants* $\bar{C}_{-n} \neq 0$.

Hereafter, for convenience, for each non-negative integer n, we denote the n-th equation in the positive and negative directions of the mCH hierarchy by $(\mathrm{mCH})_n$ and $(\mathrm{mCH})_{-n}$, respectively, while the n-th positive and negative flows in the mKdV hierarchy are denoted by $(\mathrm{mKdV})_n$ and $(\mathrm{mKdV})_{-n}$ respectively. With this notation, we are now in a position to restate Theorem 2.1, establishing the correspondence between the two hierarchies.

Theorem 3.1. *Under the transformations* (3.3) *and* (3.4), *for each* $l \in \mathbb{Z}$, *the* $(mCH)_{l+1}$ *equation is related to the* $(mKdV)_{-l}$ *equation. More precisely, for each integer* $n \geq 0$,

(i) m *solves the equation* (3.5) *if and only if* Q *satisfies* $Q_\tau = 0$ *for* $n = 0$ *or* (3.8) *for* $n \geq 1$, *with* $\bar{C}_{-n} = \mathrm{i}\,/(-4)^n$;

(ii) *For* $n \geq 1$, m *is a solution of the following rescaled version of* (3.6),

$$m_t = K_{-n} = \frac{(-1)^{n+1}}{2^{2n-1}}\left(\mathcal{J}\,\mathcal{K}^{-1}\right)^{n-1}\mathcal{J}\frac{1}{m^2}, \qquad n = 1, 2, \ldots, \tag{3.10}$$

if and only if Q *satisfies the equation* (3.7). *In addition, for* $n = 0$, *the corresponding equation* $m_t = 0$ *is equivalent to* $Q_\tau + 4Q_y = 0$.

Let us begin with some preliminary lemmas.

Lemma 3.1. *Let* $m(t,x)$ *and* $Q(\tau,y)$ *be related by the transformations* (3.3), (3.4). *Then*

$$1 - \partial_x^2 = 1 - \frac{Q_y}{4Q^3}\partial_y + \frac{1}{4Q^2}\partial_y^2. \tag{3.11}$$

Proof. First, it is easy to see from (3.3) that $\partial_x = m\partial_y$. It follows that

$$\partial_x^2 = m\,\partial_y\,m\,\partial_y = m\,(m_y\partial_y + m\partial_y^2). \tag{3.12}$$

Next, using (3.4), we have $m_y = -\,\mathrm{i}\,Q_y/(2\,Q^2)$. Plugging the formulae for m and m_y into (3.12) verifies (3.11). □

Lemma 3.2. *Let* $\mathcal{K}$, $\mathcal{J}$ *be the two compatible Hamiltonian operators* (2.2) *for the mCH equation* (1.2), *and* $\bar{\mathcal{K}}$, $\bar{\mathcal{J}}$ *the two compatible Hamiltonian operators* (2.8) *for the mKdV equation* (1.1). *Then, for each integer* $n \geq 1$,

$$\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^n\left(1 - \partial_x^2\right) = \frac{1}{(-4)^n}\left(1 - \frac{Q_y}{4Q^3}\partial_y + \frac{1}{4Q^2}\partial_y^2\right)\left(\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\right)^n \tag{3.13}$$

under the transformations (3.3), (3.4).

Proof. Note first, in view of (3.11), that (3.13) is equivalent to the identity

$$\left(\mathcal{J}\,\mathcal{K}^{-1}\right)^n(1 - \partial_x^2) = (-4)^n(1 - \partial_x^2)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n. \tag{3.14}$$

We prove (3.14) by induction on n. For the case $n=1$, on the one hand, using (3.11) and the transformations (3.3), (3.4), we obtain

$$\begin{aligned}\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1} &= \tfrac{1}{4}\partial_y^2 + Q_y\partial_y^{-1}Q + Q^2 = Q^2\left(1-\partial_x^2 - \frac{\mathrm{i}\,Q_y}{2\,Q^2}\partial_x + \frac{\mathrm{i}\,Q_y}{2\,Q^2}\partial_x^{-1}\right)\\ &= \left(Q^2 + \frac{\mathrm{i}\,Q_y}{2}\partial_x^{-1}\right)(1-\partial_x^2) = \partial_y Q\partial_y^{-1}Q(1-\partial_x^2).\end{aligned} \tag{3.15}$$

On the other hand, the identity

$$\mathcal{J}\,\mathcal{K}^{-1} = (1-\partial_x^2)\frac{1}{m}\partial_x\frac{1}{m}\partial_x^{-1} = -4(1-\partial_x^2)\partial_y Q\partial_y^{-1}Q$$

implies that (3.14) holds for $n=1$. For the general case, we assume that (3.14) holds for $n=k$. Then for $n=k+1$, the result when $n=1$ readily leads to

$$\left(\mathcal{J}\,\mathcal{K}^{-1}\right)^{k+1}(1-\partial_x^2) = (-4)^k\left(\mathcal{J}\,\mathcal{K}^{-1}\right)(1-\partial_x^2)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^k = (-4)^{k+1}(1-\partial_x^2)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{k+1},$$

verifying that (3.13) holds for each $n \geq 1$. □

Lemma 3.3. *Under the transformations* (3.3) *and* (3.4)*, the following formulae hold:*

(i) *For each $n \geq 1$,*

$$\partial_x\left(\mathcal{K}^{-1}\mathcal{J}\right)^n\partial_x^{-1} = (-4)^n\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n. \tag{3.16}$$

(ii) *For each $n \geq 0$,*

$$(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^n = (-4)^n\frac{1}{Q}\left(\tfrac{1}{4}\partial_y + Q\partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n\frac{1}{Q}\partial_y. \tag{3.17}$$

Proof. **(i)** Referring back to the forms of the Hamiltonian operators $\mathcal{K}$ and $\mathcal{J}$, and using the transformations (3.3) and (3.4), it is easy to verify that

$$\partial_x\mathcal{K}^{-1}\mathcal{J}\partial_x^{-1} = -4\partial_y Q\partial_y^{-1}Q(1-\partial_x^2).$$

When combined with (3.15), this shows that (3.16) holds for $n=1$. A straightforward induction verifies (3.16) for any $n \geq 1$.

(ii) The identity

$$\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1} = \tfrac{1}{4}\partial_y^2 + \partial_y Q\partial_y^{-1}Q = \partial_y\left(\tfrac{1}{4}\partial_y + Q\partial_y^{-1}Q\right)$$

implies that

$$\tfrac{1}{4}\partial_y + Q\,\partial_y^{-1}Q = \partial_y^{-1}\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1},$$

which shows that (3.17) is equivalent to

$$\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n+1} = \frac{1}{(-4)^n}\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^n\partial_y^{-1}Q. \tag{3.18}$$

Hence, it suffices to prove (3.18). In view of (3.16) with $n=1$, the case $n=0$ is obvious. Next, we assume (3.18) holds for $n=k-1$, say

$$\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^k = \frac{1}{(-4)^{k-1}}\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{k-1}\partial_y^{-1}Q.$$

Then,

$$\begin{aligned}\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{k+1} =& \frac{1}{(-4)^{k-1}}\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{k-1}\partial_y^{-1}Q\\ =& \frac{1}{(-4)^{k-1}}\partial_y Q(1-\partial_x^2)\partial_y^{-1}Q\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{k-1}\partial_y^{-1}Q\\ =& \frac{1}{(-4)^k}\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^k\partial_y^{-1}Q,\end{aligned}$$

where the formulae $\mathcal{K}^{-1}\mathcal{J} = -4\partial_y^{-1}Q\partial_y Q(1-\partial_x^2)$ and (3.18) with $n=0$ are used. This completes the proof of the lemma in general. $\square$

Proof of Theorem 3.1. (i) We begin with the $(\mathrm{mCH})_{n+1}$ equation for $n \geq 1$, which can be written as

$$m_t = \left(\mathcal{K}\,\mathcal{J}^{-1}\right)^n(-2m_x) = \partial_x m\partial_x^{-1}m(1-\partial_x^2)^{-1}\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^{n-1}(-2m_x). \tag{3.19}$$

Therefore, by the transformation (3.3), and using (3.19), we find

$$m_t = m_\tau + m_y\int^x m_t(t,\xi)\,\mathrm{d}\xi = m_\tau + m_y m\partial_x^{-1}m(1-\partial_x^2)^{-1}\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^{n-1}(-2m_x),$$

and then

$$\begin{aligned} m_\tau + m_y m\partial_x^{-1}m(1-\partial_x^2)^{-1}\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^{n-1}&(-2m_x) \\ -\partial_x m\partial_x^{-1}m(1-\partial_x^2)^{-1}&\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^{n-1}(-2m_x) = 0, \end{aligned}$$

which yields

$$m_\tau - m^2(1-\partial_x^2)^{-1}\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^{n-1}(-2m_x) = 0.$$

Next, according to Lemma 3.2, the preceding equation reduces to

$$-\frac{\mathrm{i}}{2\,Q^2}\left[Q_\tau + \frac{\mathrm{i}}{(-4)^n}\left(\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\right)^{n-1}(1-\partial_x^2)^{-1}\left(\frac{Q_y}{Q^3}\right)\right] = 0$$

under the Liouville correspondence (3.4). Formula (3.11) allows us to conclude that $Q(\tau,y)$ satisfies

$$\left(1-\frac{Q_y}{4Q^3}\partial_y + \frac{1}{4Q^2}\partial_y^2\right)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1}Q_\tau + \frac{\mathrm{i}}{(-4)^n}\left(\frac{Q_y}{Q^3}\right) = 0. \tag{3.20}$$

On the other hand, for the $(\mathrm{mKdV})_{-n}$ equation (3.8), a nonzero solution $Q(\tau,y)$ satisfies

$$\left(\frac{1}{4Q}\partial_y + \partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1}Q_\tau - \frac{\bar{C}_{-n}}{Q} = 0.$$

Differentiating with respect to y and multiplying the resulting equation by $1/Q$ yields

$$\left(1+\frac{1}{4Q}\partial_y\frac{1}{Q}\partial_y\right)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1}Q_\tau + \bar{C}_{-n}\frac{Q_y}{Q^3} = 0. \tag{3.21}$$

Comparing (3.20) and (3.21) immediately reveals that if $m(t,x)$ is a solution of the $(\mathrm{mCH})_{n+1}$ equation for $n \geq 1$, then the corresponding $Q(\tau,y)$ satisfies the $(\mathrm{mKdV})_{-n}$ equation (3.8), with $\bar{C}_{-n} = \mathrm{i}/(-4)^n$.

For the remaining case $n=0$, the substitution of (3.3) into the $(\mathrm{mCH})_1$ equation

$$m_t + 2m_x = 0$$

yields

$$m_t = m_\tau + m_y\int^x(-2m_\xi)\,\mathrm{d}\xi = m_\tau - 2m_y m = -2m_x = -2m_y m,$$

so $m_\tau = 0$. Then (3.4) gives rise to $Q_\tau = 0$, which means that if $m(t,x)$ is a solution of the $(\mathrm{mCH})_1$ equation, then the corresponding $Q(\tau,y)$ solves the $(\mathrm{mKdV})_0$ equation.

Conversely, if $Q(\tau,y)$ is a solution of the $(\mathrm{mKdV})_{-n}$ equation for $n \geq 0$, since the transformations (3.3) and (3.4) are the bijections, tracing the previous steps backwards suffices to verify the reverse argument is also true. Part (i) is thereby proved.

(ii) Now, assume that m is the solution of a rescaled version of the $(\mathrm{mCH})_{-n}$ equation (3.10) for $n \geq 1$, then, subject to the transformation (3.3),

$$m_t = m_\tau + m_y \int^x m_t(t,\xi)\,\mathrm{d}\xi = m_\tau + \frac{(-1)^{n+1}}{2^{2n-1}} m_y \int^x \left(\mathcal{J}\,\mathcal{K}^{-1}\right)^{n-1} \mathcal{J}\frac{1}{m^2}\,\mathrm{d}\xi$$
$$= m_\tau - \frac{(-1)^{n+1}}{2^{2n-1}} m_y (1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2}.$$

Thus,

$$0 = m_\tau - \frac{(-1)^{n+1}}{2^{2n-1}} m_y (1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2} + \frac{(-1)^{n+1}}{2^{2n-1}}\partial_x (1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2}$$
$$= m_\tau - \frac{(-1)^{n+1}}{2^{2n-1}} (m_y - m\partial_y)(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2}.$$

Next, plugging (3.4) into the preceding equation, we find

$$\left(-\frac{\mathrm{i}}{2\,Q^2}\right)\left[Q_\tau + \frac{(-1)^{n+1}}{2^{2n-3}}(Q_y + Q\partial_y)(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1} Q^2\right] = 0,$$

which implies that $Q(\tau, y)$ solves the equation

$$Q_\tau + \frac{(-1)^{n+1}}{2^{2n-3}}\partial_y Q(1-\partial_x^2)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1} Q^2 = 0.$$

Finally, thanks to (3.17), we conclude that $Q(\tau, y)$ satisfies

$$Q_\tau + \partial_y\left(\tfrac14\partial_y + Q\partial_y^{-1}Q\right)\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n-1}(4Q_y) = Q_\tau + \left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^{n}(4Q_y) = 0,$$

which is exactly the $(\mathrm{mKdV})_{n+1}$ equation (3.7). This proves the first statement.

For $n = 0$, applying the transformations (3.3) and (3.4) to the $(\mathrm{mCH})_0$ equation $m_t = 0$ produces

$$m_t = m_\tau + C m_y = 0,$$

where C is the integration constant, and hence

$$-\frac{\mathrm{i}}{2\,Q^2}(Q_\tau + C Q_y) = 0.$$

This shows that the $(\mathrm{mCH})_0$ equation is mapped into the $(\mathrm{mKdV})_1$ equation, provided that we choose $C = 4$. In analogy with the proof of part (i), the reverse argument follows from the fact that (3.3) and (3.4) are the bijections. We thus complete the proof of Theorem 3.1 for all $l \in \mathbb{Z}$. □

3.3. **The correspondence between the mCH equation and the CH equation.** The well-known fact that the KdV equation and the mKdV equation are linked by the celebrated Miura transformation, which is a particular Bäcklund transformation [49], motivates asking whether there exists a transformation relating their respective dual counterparts. In other words, our aim is to find a transformation between the CH equation (1.4) and the mCH equation (1.2).

From the viewpoint of tri-Hamiltonian duality, the CH equation (1.4) is regarded as the dual integrable counterpart of the KdV equation (1.3). The corresponding KdV hierarchy takes the form

$$P_\tau + \left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)^n (4P_y) = 0, \qquad n = 0, 1, \ldots, \tag{3.22}$$

where

$$\bar{\mathcal{L}} = \tfrac14\partial_y^3 - \tfrac12\left(P\partial_y + \partial_y P\right) \qquad \text{and} \qquad \bar{\mathcal{D}} = \partial_y,$$

are the compatible bi-Hamiltonian operators. It is well-known that the KdV equation (1.3) is related to the mKdV equation (1.1) via the Miura transformation

$$\mathcal{B}(P,Q) \equiv P + Q^2 - \mathrm{i}\,Q_y = 0. \tag{3.23}$$

Furthermore, Fokas and Fuchssteiner [20] proved that all the members of the KdV hierarchy (3.22) admit the same Miura transformation. More precisely, $\mathcal{B}(P,Q)=0$ defines, for each integer $n\geq 0$, a Bäcklund transformation between (3.22) and (3.7). The key to prove this claim is the fact that the corresponding recursion operators $\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}$ and $\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}$ are related through (3.23). Indeed, set

$$T\equiv \mathcal{B}_Q^{-1}\mathcal{B}_P=\left(2\,Q-\mathrm{i}\,\partial_y\right)^{-1}, \tag{3.24}$$

where, for any "test function" ϕ

$$\mathcal{B}_P\phi=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\mathcal{B}(P+\epsilon\phi,Q)=\phi \quad \text{and} \quad \mathcal{B}_Q\phi=\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\mathcal{B}(P,Q+\epsilon\phi)=\left(2\,Q-\mathrm{i}\,\partial_y\right)\phi$$

are the Fréchet derivatives of P,Q, respectively, [42]. Then, assuming (3.23),

$$\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}=T\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)T^{-1}. \tag{3.25}$$

In view of this relation, we claim that the first negative flow of the mKdV hierarchy can be mapped into the first negative flow of the KdV hierarchy by the Miura transformation (3.23). Indeed, we have the following result.

Proposition 3.1. *Assume that Q satisfies the equation*

$$\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)\,Q_\tau=0. \tag{3.26}$$

Then $P=-Q^2+\mathrm{i}\,Q_y$ satisfies

$$\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)\,P_\tau=0. \tag{3.27}$$

Proof. Using (3.23) and (3.24), we obtain

$$P_\tau=-\left(2\,Q-\mathrm{i}\,\partial_y\right)Q_\tau=-T^{-1}Q_\tau. \tag{3.28}$$

This, together with (3.25), yields

$$\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)\,P_\tau=-\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)T^{-1}Q_\tau=-T^{-1}\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)Q_\tau.$$

Therefore, if $\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\,Q_\tau=0$, then

$$\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)\,P_\tau=-\left(2\,Q-\mathrm{i}\,\partial_y\right)0=0,$$

proving the proposition. □

Remark 3.2. *In general, applying* (3.25) *successively implies that, for each integer $n\geq 1$,* (3.23) *maps the equation $\left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\right)^n Q_\tau=0$ to $\left(\bar{\mathcal{L}}\,\bar{\mathcal{D}}^{-1}\right)^n P_\tau=0$.*

Remark 3.3. *Note that the Bäcklund transformation* (3.23) *does not define a bijection between the respective negative hierarchies. Indeed, for example, suppose $P=-Q^2+\mathrm{i}\,Q_y$ satisfies* (3.27)*. Using* (3.25) *and* (3.28)*, we find*

$$\left(2\,Q-\mathrm{i}\,\partial_y\right)\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\,Q_\tau=0,$$

which, clearly, is not equivalent to equation (3.26) .

Using Proposition 3.1, we are able to construct a transformation from the mCH equation (1.2) to the CH equation (1.4). First, it was shown in [22, 29, 37] that the following Liouville transformation

$$P(y)=\frac{1}{\rho(x)}\left(\frac{1}{4}-\frac{\left(\rho(x)^{-1/4}\right)_{xx}}{\rho(x)^{-1/4}}\right), \qquad \text{where} \qquad y=\int^x\sqrt{\rho(\xi)}\,\mathrm{d}\xi, \quad \rho=v-v_{xx}, \tag{3.29}$$

relating the respective isospectral problems for the CH hierarchy and the KdV hierarchy, gives rise to the one-to-one correspondence between the CH equation (1.4) and the first negative flow (3.27). On the other hand, from Theorem 3.1, $m(t,x)$ satisfies the mCH equation (1.2) if and only if

$$Q(\tau,y)=\frac{\mathrm{i}}{2m(t,x)},\qquad \text{where}\qquad y=\int^x m(t,\xi)\,\mathrm{d}\xi,\quad \tau=t,\tag{3.30}$$

is the solution of equation (3.26). We deduce that the composite transformation including (3.23), (3.29), and (3.30) defines a map from the mCH equation (1.2) to the CH equation (1.4), albeit not univalent.

Proposition 3.2. *Assume $m(t,x)$ is the solution of the mCH equation* (1.2)*. Then, $\rho(t,x)$ satisfies the CH equation* (1.4)*, where $\rho(t,x)$ is determined by the relation* (3.29) *with $P(\tau,y)=-Q^2(\tau,y)+\mathrm{i}\,Q_y(\tau,y)$ and $Q(\tau,y)$ defined by* (3.30).

4. Hamiltonian conservation laws of the mCH equation

4.1. **The correspondence between the Hamiltonian conservation laws of the mCH and mKdV equations.** The Magri scheme enables one to recursively construct an infinite hierarchy of Hamiltonian conservation laws of any bi-Hamiltonian system. In particular, for the mCH equation (1.2), at the n-th stage we determine the Hamiltonian conservation laws $\mathcal{H}_n$ satisfying the recursive formula

$$\mathcal{K}\,\frac{\delta\mathcal{H}_{n-1}}{\delta m}=\mathcal{J}\,\frac{\delta\mathcal{H}_n}{\delta m},\qquad n\in\mathbb{Z},\tag{4.1}$$

where $\mathcal{K}$ and $\mathcal{J}$ are the two compatible Hamiltonian operators (2.2) admitted by the mCH equation. On the other hand, the recursive formula

$$\bar{\mathcal{K}}\,\frac{\delta\bar{\mathcal{H}}_{n-1}}{\delta Q}=\bar{\mathcal{J}}\,\frac{\delta\bar{\mathcal{H}}_n}{\delta Q},\qquad n\in\mathbb{Z}\tag{4.2}$$

will formally provide an infinite collection of Hamiltonian conservation laws for the mKdV equation (1.1), using the Hamiltonian operators $\bar{\mathcal{K}}$ and $\bar{\mathcal{J}}$ defined in (2.8) .

In this subsection we investigate the effect of the transformations (3.3) and (3.4) on the two hierarchies of Hamiltonian conservation laws.

Lemma 4.1. *Let $\{\mathcal{H}_n\}$ and $\{\bar{\mathcal{H}}_n\}$ be the hierarchies of conserved functionals determined by the recursive formulae* (4.1) *and* (4.2)*, respectively. Then their corresponding variational derivatives satisfy the relation*

$$\frac{\delta\mathcal{H}_{-n}}{\delta m}=(-1)^{n-1}2^{2n-1}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_n}{\delta Q},\qquad 0\neq n\in\mathbb{Z}.\tag{4.3}$$

Proof. We first prove (4.3) for $n\geq 1$ by induction. Since $\delta\mathcal{H}_{-1}/\delta m=-1/m^2$ and $\delta\bar{\mathcal{H}}_1/\delta Q=4Q$, it's easy to see (4.3) holds for $n=1$. Assume now (4.3) holds for $n=k$ with $k\geq 1$, say

$$\frac{\delta\mathcal{H}_{-k}}{\delta m}=(-1)^{k-1}2^{2k-1}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_k}{\delta Q}.$$

Then, for $n=k+1$, thanks to (4.1) and (4.2), we find

$$\begin{aligned}\frac{\delta\mathcal{H}_{-(k+1)}}{\delta m}&=\mathcal{K}^{-1}\mathcal{J}\frac{\delta\mathcal{H}_{-k}}{\delta m}=(-1)^{k-1}2^{2k-1}\mathcal{K}^{-1}\mathcal{J}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_k}{\delta Q}\\&=(-1)^k2^{2k+1}\partial_y^{-1}Q\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1}\frac{1}{Q}\partial_y\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_{k+1}}{\delta Q}=(-1)^k2^{2k+1}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta\bar{\mathcal{H}}_{k+1}}{\delta Q},\end{aligned}$$

where we have made use of the formula (3.16). This verifies (4.3) for $n\geq 1$, completing the first step.

Next, note that, in the case of $n = -1$, formula (4.3) is equivalent to

$$\frac{\delta \bar{\mathcal{H}}_{-1}}{\delta Q} = 8\,\bar{\mathcal{J}}^{-1}\frac{1}{Q}\bar{\mathcal{J}}\frac{\delta \mathcal{H}_1}{\delta m}. \tag{4.4}$$

To prove (4.4), if we use the fact that $\bar{\mathcal{H}}_{-1}$ is a Casimir functional of the Hamiltonian operator $\bar{\mathcal{K}}$, we only need to verify that

$$h(m) = 8\,\bar{\mathcal{J}}^{-1}\frac{1}{Q}\bar{\mathcal{J}}\frac{\delta \mathcal{H}_1}{\delta m}$$

is the variational derivative of the Casimir functional admitted by $\bar{\mathcal{K}}$. In fact, since

$$\frac{\delta \mathcal{H}_1}{\delta m} = 2u = 2\left(1-\partial_x^2\right)^{-1} m,$$

one has

$$h(m) = -32\,\mathrm{i}\,\partial_x^{-1} m \partial_x \left(1-\partial_x^2\right)^{-1} m.$$

Moreover, by (3.16),

$$\bar{\mathcal{K}} = \tfrac{1}{4}\partial_x \mathcal{K}^{-1}\mathcal{J}\partial_x^{-1}\frac{1}{m}\partial_x.$$

Hence

$$\bar{\mathcal{K}} h(m) = 8\,\mathrm{i}\,\frac{1}{m}\partial_x\frac{1}{m}\partial_x^{-1}\left(1-\partial_x^2\right)\partial_x\left(1-\partial_x^2\right)^{-1} m = 8\,\mathrm{i}\,\frac{1}{m}\partial_x\, 1 = 0,$$

which proves the claim, and shows (4.3) holds in the case of $n = -1$.

Finally, to prove (4.3) holds for all $n \le -1$, we proceed by induction on n, so assume that (4.3) holds for $n = k$. Then for $n = k-1$, in view of (4.1) and (4.2) and using (3.16) again, we arrive at

$$\begin{aligned}
\frac{\delta \mathcal{H}_{-(k-1)}}{\delta m} &= \mathcal{J}^{-1}\mathcal{K}\frac{\delta \mathcal{H}_{-k}}{\delta m} = (-1)^{k-1}2^{2k-1}\mathcal{J}^{-1}\mathcal{K}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta \bar{\mathcal{H}}_k}{\delta Q}\\
&= (-1)^{k-2}2^{2k-3}\partial_y^{-1}Q\bar{\mathcal{J}}\,\bar{\mathcal{K}}^{-1}\frac{1}{Q}\partial_y\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\frac{\delta \bar{\mathcal{H}}_{k-1}}{\delta Q} = (-1)^{k-2}2^{2k-3}\bar{\mathcal{J}}^{-1}Q\bar{\mathcal{J}}\frac{\delta \bar{H}_{k-1}}{\delta Q},
\end{aligned}$$

which completes the induction step, and thus establishes (4.3) for $n \le -1$. This verifies (4.3) holds for all $0 \ne n \in \mathbb{Z}$, proving the lemma. □

In addition, in order to establish the correspondence between the conserved functionals admitted by the mCH and the mKdV equations, we require a formula for the change of the variational derivative under (3.3) and (3.4).

Lemma 4.2. *Let $m(t,x)$ and $Q(\tau,y)$ be related by the transformations* (3.3) *and* (3.4)*. If $\mathcal{H}(m) = \bar{\mathcal{H}}(Q)$, then*

$$\frac{\delta \mathcal{H}}{\delta m} = \frac{1}{Q}\left(\tfrac{1}{4}\bar{\mathcal{J}}^2 - \bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\right)\frac{\delta \bar{\mathcal{H}}}{\delta Q}, \tag{4.5}$$

where $\bar{\mathcal{J}}$ and $\bar{\mathcal{K}}$ are the Hamiltonian operators given by (2.8).

Proof. Referring back to (3.3) and denoting

$$Q(\tau,y) = F[m(t,x)] \equiv \frac{\mathrm{i}}{2m(t,x)},$$

we arrive at

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} y(m+\epsilon v) = \partial_x^{-1}v = -2\,\mathrm{i}\,\partial_y^{-1}Qv.$$

Then, on the one hand,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} F[m+\epsilon v] &= Q_y \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} y(m+\epsilon v) + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}^{y\,fixed} F[m+\epsilon v] \\ &= -2\,\mathrm{i}\, Q_y \partial_y^{-1} Q v + \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}^{y\,fixed} F[m+\epsilon v].\end{aligned}$$

On the other hand, by the definition of the Fréchet derivative

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} F[m+\epsilon v] = \mathcal{D}_{F[m]}(v) = -\frac{\mathrm{i}}{2m^2} v = 2\,\mathrm{i}\, Q^2 v.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}^{y\,fixed} F[m+\epsilon v] = 2\,\mathrm{i}\,(Q^2 + Q_y \partial_y^{-1} Q) v.$$

Furthermore, by the assumption, we find

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{H}(m+\epsilon v) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \bar{\mathcal{H}}\left(F[m+\epsilon v]\right),$$

where we use the usual definition

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{H}(m+\epsilon v) = \int \frac{\delta \mathcal{H}}{\delta m} \cdot v \, \mathrm{d}x \tag{4.6}$$

of the variational derivative. Since the Hamiltonian operators $\bar{\mathcal{J}}$ and $\bar{\mathcal{K}}$ are skew-adjoint, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \bar{\mathcal{H}}\left(F[m+\epsilon v]\right) &= \int \frac{\delta \bar{\mathcal{H}}}{\delta Q} \cdot \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}^{y\,fixed} F[m+\epsilon v]\,\mathrm{d}y = 2\,\mathrm{i} \int \frac{\delta \bar{\mathcal{H}}}{\delta Q} \cdot (Q^2 + Q_y \partial_y^{-1} Q) v \,\mathrm{d}y \\ &= 2\,\mathrm{i} \int \left(\bar{\mathcal{K}}\,\bar{\mathcal{J}}^{-1} - \tfrac{1}{4}\bar{\mathcal{J}}^2\right)^* \frac{\delta \bar{\mathcal{H}}}{\delta Q} \cdot v\,\mathrm{d}y = \int \frac{1}{Q}\left(\tfrac{1}{4}\bar{\mathcal{J}}^2 - \bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\right) \frac{\delta \bar{\mathcal{H}}}{\delta Q} \cdot v\,\mathrm{d}x,\end{aligned}$$

which, on comparison with (4.6) verifies (4.5), proving the lemma. □

Finally, under the hypothesis of Lemma 4.2, we define the functional

$$\mathcal{G}_n(Q) \equiv \mathcal{H}_{-n}(m),$$

for each $0 \neq n \in \mathbb{Z}$. Thanks to Lemma 4.2,

$$\frac{\delta \mathcal{H}_{-n}}{\delta m} = \frac{1}{Q}\left(\tfrac{1}{4}\bar{\mathcal{J}}^2 - \bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\right) \frac{\delta \mathcal{G}_n}{\delta Q}.$$

while according to Lemma 4.1,

$$\frac{\delta \mathcal{H}_{-n}}{\delta m} = (-1)^{n-1} 2^{2n-1} \bar{\mathcal{J}}^{-1} Q \bar{\mathcal{J}} \frac{\delta \bar{\mathcal{H}}_n}{\delta Q}.$$

It follows that

$$\begin{aligned}(-1)^{n-1} 2^{2n-1} \bar{\mathcal{J}}^{-1} Q \bar{\mathcal{J}} \frac{\delta \bar{\mathcal{H}}_n}{\delta Q} &= \frac{1}{Q}\left(\tfrac{1}{4}\bar{\mathcal{J}}^2 - \bar{\mathcal{J}}^{-1}\bar{\mathcal{K}}\right) \frac{\delta \mathcal{G}_n}{\delta Q} \\ &= \frac{1}{Q}\left(\tfrac{1}{4}\partial_y^2 - \partial_y^{-1}\left(\tfrac{1}{4}\partial_y^3 + \partial_y Q \partial_y^{-1} Q \partial_y\right)\right) \frac{\delta \mathcal{G}_n}{\delta Q} = -\partial_y^{-1} Q \partial_y \frac{\delta \mathcal{G}_n}{\delta Q} = -\bar{\mathcal{J}}^{-1} Q \bar{\mathcal{J}} \frac{\delta \mathcal{G}_n}{\delta Q},\end{aligned}$$

which yields

$$\frac{\delta \mathcal{G}_n}{\delta Q} = (-1)^n 2^{2n-1} \frac{\delta \bar{\mathcal{H}}_n}{\delta Q},$$

from which

$$\mathcal{G}_n(Q) = (-1)^n 2^{2n-1} \bar{\mathcal{H}}_n(Q) \tag{4.7}$$

follows. Consequently, we conclude that formula (4.7), combined with our hypothesis, produces the correspondence between the sequences of the Hamiltonian conservation laws admitted by the mCH and the mKdV equations. Thus we have now proved Theorem 2.2.

4.2. **Hamiltonian conservation laws of the mCH equation.** In this subsection, we study the Hamiltonian conservation laws (2.22) of the mCH equation based on its bi-Hamiltonian representation. We focus our attention on the sequence of conserved functionals $\mathcal{H}_{-n}$, $n \geq 1$, in the negative direction of the hierarchy. Hereafter, for simplicity, we denote the variational derivative of the functional $\mathcal{H}_{-n}$ by $V_{-n} = \delta\mathcal{H}_{-n}/\delta m$. Hence, starting from

$$V_{-1}[m] = \frac{\delta\mathcal{H}_{-1}[m]}{\delta m} = -\frac{1}{m^2},$$

and using the recursive construction (4.1), the sequence of variational derivatives satisfies, successively,

$$V_{-(n+1)} = \mathcal{K}^{-1}\mathcal{J}\,V_{-n}, \qquad n = 1, 2, \ldots. \tag{4.8}$$

Our aim is to identify the local nature and homogeneity of $\mathcal{H}_{-n}$. The main result is presented in Theorem 2.3, whose proof relies on two propositions.

First, the following lemma is a corollary of Theorem 4.7 in [42]; see also [30].

Lemma 4.3. *If a local functional with the corresponding differential function $P[m]$ satisfies*

$$\int P[m]\,\mathrm{d}x = 0, \qquad m \in X,$$

then there exists a local differential function $R[m]$, unique up to addition of a constant, such that $P[m] = D_x R[m]$ is its total $x-$derivative.

The first proposition is concerned with the local character of the V_{-n}'s.

Proposition 4.1. *All variational derivatives V_{-n} $(n \geq 1)$ in the sequence (4.8) are differential functions of m.*

Proof. We use an inductive argument. First of all, in view of the explicit forms of operators $\mathcal{K}$ and $\mathcal{J}$ in (2.2), the equality (4.8) can be written as

$$\partial_x V_{-(n+1)} = \frac{1}{m}\partial_x\frac{1}{m}\left(1-\partial_x^2\right)V_{-n}. \tag{4.9}$$

In terms of the inverse operator $\mathcal{K}^{-1}$ along with (4.8), we deduce from (4.9) that the sequence V_{-n}'s satisfy the following equality

$$\partial_x V_{-(n+1)} = \frac{1}{m}\partial_x\frac{1}{m}\left(1-\partial_x^2\right)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}V_{-1} = -\frac{1}{m}\partial_x\frac{1}{m}\left(1-\partial_x^2\right)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2}, \tag{4.10}$$

for $n \geq 1$. In particular,

$$V_{-2} = -\frac{3}{4m^4} + 5\frac{m_x^2}{m^6} - 2\frac{m_{xx}}{m^5}, \tag{4.11}$$

is a differential function of m.

Now suppose, by induction, that V_{-n} is a differential function of m. Define the local differential function

$$X_{-n} = \frac{1}{m}\partial_x\frac{1}{m}\left(1-\partial_x^2\right)V_{-n} = -\frac{1}{m}\partial_x\frac{1}{m}\left(1-\partial_x^2\right)\left(\mathcal{K}^{-1}\mathcal{J}\right)^{n-1}\frac{1}{m^2}, \tag{4.12}$$

so that

$$X_{-n} = D_x V_{-(n+1)}. \tag{4.13}$$

We claim that $\int X_{-n}\,\mathrm{d}x = 0$. Lemma 4.3 then implies that $V_{-(n+1)}$ is a differential function, which establishes the induction step and thus proves the proposition.

To verify the claim, first note that, since both the operators $\mathcal{J}$ and $\mathcal{K}$ are skew-adjoint,

$$\left[\left(\mathcal{K}^{-1}\mathcal{J}\right)^n\right]^* = \left(\mathcal{J}\,\mathcal{K}^{-1}\right)^n, \qquad n = 1, 2, \ldots.$$

Applying integration by parts,

$$\begin{aligned}\int X_{-n}\,\mathrm{d}x &= -\int \frac{1}{m}\partial_x\frac{1}{m}\big(1-\partial_x^2\big)\big(\mathcal{K}^{-1}\mathcal{J}\big)^{n-1}\Big(\frac{1}{m^2}\Big)\,\mathrm{d}x\\ &= -\int \frac{m_x}{m^3}\big(1-\partial_x^2\big)\big(\mathcal{K}^{-1}\mathcal{J}\big)^{n-1}\Big(\frac{1}{m^2}\Big)\,\mathrm{d}x\\ &= -\int \frac{1}{m^2}\big(\mathcal{J}\,\mathcal{K}^{-1}\big)^{n-1}\big(1-\partial_x^2\big)\Big(\frac{m_x}{m^3}\Big)\,\mathrm{d}x\\ &= -\frac{1}{2}\int \big(\mathcal{J}\,\mathcal{K}^{-1}\big)^{n-1}\mathcal{J}\Big(\frac{1}{m^2}\Big)\cdot\frac{1}{m^2}\,\mathrm{d}x\\ &= -\frac{1}{2}\int \big(1-\partial_x^2\big)\big(\mathcal{K}^{-1}\mathcal{J}\big)^{n-1}\Big(\frac{1}{m^2}\Big)\cdot\partial_x\Big(\frac{1}{m^2}\Big)\,\mathrm{d}x = -\int X_{-n}\,\mathrm{d}x.\end{aligned}$$

This immediately implies that $\int X_{-n}\,\mathrm{d}x = 0$, as claimed. □

The homogeneity of the dual Hamiltonian operator $\mathcal{K}$ implies that the variational derivatives of Hamiltonian conservation laws $\mathcal{H}_{-n}$ $(n \geq 1)$ constructed by (4.8) are all homogeneous under scaling transformation $m \mapsto \sigma m$.

Proposition 4.2. *The differential functions $V_{-n}[m] = \delta\mathcal{H}_{-n}[m]/\delta m$, $n \geq 1$, satisfy*

$$V_{-n}[\sigma m] = \sigma^{-2n}V_{-n}[m], \qquad 0 \neq \sigma \in \mathbb{R}. \tag{4.14}$$

Proof. Clearly (4.14) holds when $n = 1$. Assume that (4.14) holds for $n = j$. Then for $n = j+1$, formula (4.8) implies that

$$\begin{aligned}V_{-(j+1)}[\sigma m] &= \mathcal{K}^{-1}(\sigma m)\mathcal{J}(\sigma m)\,V_{-j}[\sigma m]\\ &= \partial_x^{-1}\frac{1}{\sigma m}\partial_x\frac{1}{\sigma m}\left(1-\partial_x^2\right)\left(\sigma^{-2j}V_{-j}[m]\right) = \sigma^{-2(j+1)}V_{-(j+1)}[m].\end{aligned}$$

Therefore, the proposition follows by induction. □

In addition, it is necessary to show that their Fréchet derivatives, $\mathcal{D}_{V_{-n}}$, are self-adjoint operators. Since the Hamiltonian operator $\mathcal{K}$ is non-degenerate, we have a slight variant of a lemma given in [42].

Lemma 4.4. *Suppose $\mathcal{K}$ and $\mathcal{J}$ are the two compatible Hamiltonian operators for the mCH equation defined by* (2.2)*. Assume that there are three differential functions $P[m]$, $Q[m]$ and $R[m]$ satisfying*

$$\mathcal{K}P = \mathcal{J}Q \quad \textit{and} \quad \mathcal{K}R = \mathcal{J}P.$$

If $P[m] = \delta\mathcal{P}[m]/\delta m$ and $Q[m] = \delta\mathcal{Q}[m]/\delta m$ are variational derivatives of local functionals $\mathcal{P}[m]$ and $\mathcal{Q}[m]$, respectively, then the Fréchet derivative $\mathcal{D}_{R[m]}$ is self-adjoint and hence $R[m] = \delta\mathcal{R}[m]/\delta m$ is also a variational derivative for some local functional $\mathcal{R}[m]$.

Since the sequence of differential functions V_{-n} satisfies

$$\mathcal{K}V_{-n} = \mathcal{J}V_{-n+1}, \qquad \mathcal{K}V_{-(n+1)} = \mathcal{J}V_{-n},$$

Lemma 4.4 implies that, if

$$V_{-n+1} = \frac{\delta\mathcal{H}_{-n+1}}{\delta m} \qquad \text{and} \qquad V_{-n} = \frac{\delta\mathcal{H}_{-n}}{\delta m}, \qquad \text{then} \qquad V_{-(n+1)} = \frac{\delta\mathcal{H}_{-(n+1)}}{\delta m}$$

for some local functional $\mathcal{H}_{-(n+1)}[m]$. Since we already know the local functional $\mathcal{H}_{-1}[m] = \int (1/m)\,\mathrm{d}x$ has $V_{-1}[m] = -1/m^2$ as its variational derivative, and the local functional $\mathcal{H}_{-2}[m] = \int \big(1/(4m^3) + m_x^2/m^5\big)\,\mathrm{d}x$ has $V_{-2}[m]$ given by (4.11) as its variational derivative, we are able to recursively construct all higher-order local functionals.

Proof of Theorem 2.3. Since the Fréchet derivative of $V_{-n}[m]$ is self-adjoint: $\mathcal{D}^*_{V_{-n}} = \mathcal{D}_{V_{-n}}$, the *homotopy formula*, [42, (5.123)]:

$$\mathcal{H}_{-n}[m] = \int_0^1 \int m\, V_{-n}[\lambda\, m]\, \mathrm{d}x\, \mathrm{d}\lambda$$

defines a local functional $\mathcal{H}_{-n}[m]$ which admits the corresponding variational derivative $\delta\mathcal{H}_{-n}/\delta m = V_{-n}[m]$ for $n \geq 1$. Furthermore, using the homogeneity of V_{-n} given in (4.14),

$$\mathcal{H}_{-n}[m] = \left(\int_0^1 \lambda^{-2n}\, \mathrm{d}\lambda\right)\left(\int m\, V_{-n}[m]\, \mathrm{d}x\right) = \frac{1}{1-2n}\int m\, V_{-n}[m]\, \mathrm{d}x.$$

Finally, the homogeneity of $\mathcal{H}_{-n}[m]$ directly follows from that of the variational derivatives. In fact, it follows from above and (4.14) that

$$\mathcal{H}_{-n}[\sigma m] = \frac{1}{1-2n}\int \sigma m V_{-n}[\sigma m]\ \mathrm{d}x = \frac{\sigma^{1-2n}}{1-2n}\int m V_{-n}[m]\ \mathrm{d}x = \sigma^{1-2n}\mathcal{H}_{-n}[m],$$

which completes the proof. □

Remark 4.1. *The preceding proof implies the explicit formulae*

$$\mathcal{H}_{-n}[m] = \frac{1}{1-2n}\int m\,\frac{\delta\mathcal{H}_{-n}}{\delta m}\,\mathrm{d}x, \qquad n = 1, 2, \ldots,$$

for the conservation laws, whose variational derivatives are constructed recursively by

$$\frac{\delta\mathcal{H}_{-(n+1)}}{\delta m} = \mathcal{K}^{-1}\mathcal{J}\,\frac{\delta\mathcal{H}_{-n}}{\delta m}, \qquad n = 1, 2, \ldots,$$

starting from $\delta\mathcal{H}_{-1}/\delta m = -1/m^2$.

As for the positive hierarchy, $\mathcal{H}_n$ for $n \geq 0$ in (2.22), we begin with the sequence of their variational derivatives $V_n = \delta\mathcal{H}_n/\delta m$, which satisfy the recursion formulae (4.1). Then (4.1) yields for $n \geq 0$,

$$V_{n+1} = \mathcal{J}^{-1}\mathcal{K}\, V_n = (1-\partial_x^2)^{-1} m\,\partial_x^{-1} m\,\partial_x \left(\mathcal{J}^{-1}\mathcal{K}\right)^n V_0. \tag{4.15}$$

Now, consider $F_n = \mathcal{K}\big(\mathcal{J}^{-1}\mathcal{K}\big)^n V_0,\ n \geq 0$, we claim that

$$\int F_n(m)\ \mathrm{d}x = 0$$

holds for each $n \geq 0$. In fact, since $V_0 = 1$,

$$\begin{aligned}\int F_n(m)\ \mathrm{d}x &= \int \mathcal{K}\big(\mathcal{J}^{-1}\mathcal{K}\big)^n(1)\ \mathrm{d}x = \int 1\cdot\left(\mathcal{K}\,\mathcal{J}^{-1}\right)^n\mathcal{K}\,(1)\ \mathrm{d}x \\ &= -\int \mathcal{K}\big(\mathcal{J}^{-1}\mathcal{K}\big)^n(1)\ \mathrm{d}x = -\int F_n(m)\ \mathrm{d}x,\end{aligned}$$

which immediately implies the preceding claim. Since we cannot show that F_n are differential functions in m, except for the special case that $F_0 = -2m_x$, Lemma 4.3 cannot be applied to verify that F_n are of the form of the total $x-$derivatives. So we cannot make the same claim that the corresponding variational derivatives are differential functions of m, neither can we obtain a constructive scheme and algorithm for computing the sequences of conservation laws $\mathcal{H}_n$ in analogy with Theorem 2.3. Nevertheless, no matter whether they are local or nonlocal, the variational derivatives V_n are still homogeneous functions. In analogy with Proposition 4.2, we have the following result.

Proposition 4.3. *Each V_n in the hierarchy of the variational derivatives determined by the recursion formulae* (4.15) *satisfies* $V_n(\sigma m) = \sigma^{2n-1}V_n(m),\ n = 1, 2, \ldots$.

5. Concluding remarks

In this paper, an explicit correspondence between the integrable mKdV hierarchy and its dual integrable mCH hierarchy is set up through a Liouville transformation between the isospectral problems of the two hierarchies. In addition, we show that the Liouville transformation also relates their respective recursion operators and Hamiltonian conservation laws. Finally, we have constructed an implicit transformation that maps the mCH equation directly to the CH equation.

It is worth noting that the mKdV hierarchy studied in this paper is the focusing type. We can show that the Liouville correspondence also persists in the defocusing case. Indeed, for the defocusing mKdV equation

$$Q_\tau + Q_{yyy} - 6\,Q^2\,Q_y = 0,$$

the corresponding isospectral problem is

$$\mathbf{\Phi}_y = \begin{pmatrix} -\,\mathrm{i}\,\mu & Q \\ Q & \mathrm{i}\,\mu \end{pmatrix} \mathbf{\Phi}.$$

While the corresponding Liouville transformation analogous to (2.12) takes the form

$$Q(y) = \frac{1}{2m(x)}, \qquad y = \int^x m(\xi)\,\mathrm{d}\xi, \qquad m = u - u_{xx}.$$

The corresponding results to the defocusing case can be obtained and proved similarly.

In [44], several dual integrable systems have been discovered by using the tri-Hamitonian duality approach. It is anticipated that a large number of dual integrable systems can be generated from other soliton hierarchies, such as the general AKNS systems, the three wave interaction equations, and others. It is well-known that the KdV, mKdV and Schrödinger hierarchies belong to the subclasses of AKNS hierarchy. The Liouville transformation relating the KdV hierarchy and its dual hierarchy has been derived in [29, 37]. It is of great interest to develop the approaches in this paper and [29, 37] to study the general AKNS system and other integrable systems.

For the AKNS system

$$\begin{cases} q_t = \mathrm{i}\,\left(q_{xx} - q^2 r\right), \\ r_t = \mathrm{i}\,\left(-r_{xx} + r^2 q\right), \end{cases}$$

its dual system can be derived by the tri-Hamitonian duality approach, and takes the form

$$\begin{cases} u_t = \mathrm{i}\,(r + \mathrm{i}\,r_x)\,r\,q, \\ v_t = -\,\mathrm{i}\,(q - \mathrm{i}\,q_x)\,r\,q, \end{cases} \tag{5.1}$$

where $u = r + \mathrm{i}\,r_x$ and $v = q - \mathrm{i}\,q_x$. A natural question arises: what is the Liouville transformation of the AKNS hierarchy and its dual integrable hierarchy? This will be one of the future projects in our continuing study of these remarkable integrable hierarchies.

Acknowledgements. The authors thank the referees for valuable comments and suggestions. J. Kang acknowledges the support and hospitality during her stay at the University of Minnesota, where this work was completed. The work of J. Kang is supported by NSF-China grant 11471260 and the Foundation of Shannxi Education Committee-12JK0850. The work of X.C. Liu is supported by NSF-China grant 11401471 and Ph.D. Programs Foundation of Ministry of Education of China-20136101120017 and the grant under Shannxi Province 2013JQ1001. The work of P.J. Olver is partially supported by NSF grant DMS-1108894. The work of C.Z. Qu is supported by the NSF-China grant-11471174 and NSF of Ningbo grant-2014A610018.

References

[1] R. Beals, D.H. Sattinger, and J. Szmigielski, Acoustic scattering and the extended Korteweg de Vries hierarchy, *Adv. Math.*, **140** (1998), 190-206.

[2] R. Beals, D.H. Sattinger, and J. Szmigielski, Multipeakons and the classical moment problem, *Adv. Math.*, **154** (2000), 229-257.

[3] P.M. Bies, P. Gorka, and E. Reyes, The dual modified Korteweg-de Vries-Fokas-Qiao equation: geometry and local analysis, *J. Math. Phys.*, **53** (2012), 073710.

[4] R. Camassa and D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661-1664.

[5] R. Camassa, D.D. Holm, and J. Hyman, An new integrable shallow water equation, *Adv. Appl. Mech.*, **31** (1994), 1-33.

[6] C.W. Cao and X.G. Geng, A nonconfocal generator of involute systems and three associated soliton hierarchies, *J. Math. Phys.*, **32** (1991), 2323-2328.

[7] K.S. Chou and C.Z. Qu, Integrable equations arising from motions of plane curves I, *Physica D*, **162** (2002), 9-33.

[8] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), 321-362.

[9] A. Constantin, On the scattering problem for the Camassa-Holm equation, *Proc. Roy. Soc. London Ser. A*, **457** (2001), 953-970.

[10] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229-243.

[11] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa*, **26** (1998), 303-328.

[12] A. Constantin, V. Gerdjikov, and R. Ivanov, Inverse scattering transform for the Camassa-Holm equation, *Inverse problems*, **22** (2006), 2197-2207.

[13] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Rational Mech. Anal.*, **192** (2009), 165–186.

[14] A. Constantin and H.P. McKean, A shallow water equation on the circle, *Commun. Pure Appl. Math.*, **52** (1999), 949-982.

[15] A. Constantin and W. Strauss, Stability of peakons, *Commun. Pure Appl. Math.*, **53** (2000), 603-610.

[16] K. El Dika and L. Molinet, Stability of multipeakons, *Ann. Inst. Henri Poincaré Anal. Nonlinéaire*, **18** (2009), 1517-1532.

[17] M. Fisher and J. Schiff, The Camassa Holm equation: conserved quantities and the initial value problem, *Phys. Lett. A*, **259** (1999), 371-376.

[18] A.S. Fokas, The Korteweg-de Vries equation and beyond, *Acta Appl. Math.*, **39** (1995), 295-305.

[19] A.S. Fokas, On a class of physically important integrable equations, *Physica D*, **87** (1995), 145-150.

[20] A.S. Fokas and B. Fuchssteiner, Bäcklund transformations for hereditary symmetries, *Nonlinear Anal.*, **5** (1981), 423-432.

[21] A.S. Fokas, P.J. Olver, and P. Rosenau, A plethora of integrable bi-Hamiltonian equations, in: *Algebraic aspects of integrable systems*, Progr. Nonlinear Differential Equations Appl., vol. 26, Birkhäuser Boston, Boston , MA, 1997, pp. 93-101.

[22] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa-Holm equation, *Physica D*, **95** (1996), 229-243.

[23] B. Fuchssteiner and A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D*, **4** (1981/1982), 47-66.

[24] G.L. Gui, Y. Liu, P.J. Olver, and C.Z. Qu, Wave-breaking and peakons for a modified Camassa-Holm equation, *Commun. Math. Phys.*, **319** (2013), 731-759.

[25] R. Hernandez-Heredero and E. Reyes, Geometric integrability of the Camassa-Holm equation II, *Int. Math. Res. Not.*, **2012** (2012), 3089-3125.

[26] R.S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, **455** (2002), 63-82.

[27] R.S. Johnson, On solutions of the Camassa-Holm equation, *Proc. Roy. Soc. London Ser. A*, **459** (2003), 1687-1708.

[28] S. Kouranbaeva, The Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.*, **40** (1999), 857-868.

[29] J. Lenells, The correspondence between KdV and Camassa-Holm, *Int. Math. Res. Not.*, **71** (2004), 3797-3811.

[30] J. Lenells, Conservation laws of the Camassa-Holm equation, *J. Phys. A: Math. Gen.*, **38** (2005), 869-880.

[31] Y.A. Li and P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Diff. Eq.*, **162** (2000), 27-63.

[32] Y.A. Li, P.J. Olver, and P. Rosenau, Non-analytic solutions of nonlinear wave models, in: *Nonlinear Theory of Generalized Functions*, M. Grosser, G. Höormann, M. Kunzinger, and M. Oberguggenberger, eds., Research Notes in Mathematics, vol. 401, Chapman and Hall/CRC, New York, 1999, pp. 129-145.

[33] X.C. Liu, Y. Liu, P.J. Olver, and C.Z. Qu, Orbital Stability of peakons for a generalization of the modified Camassa-Holm equation, *Nonlinearity*, **22** (2014), 2297-2319.

[34] X.C. Liu, Y. Liu, and C.Z. Qu, Orbital stability of the train of peakons for an integrable modified Camassa-Holm equation, *Adv. Math.*, **255** (2014), 1-37.

[35] Y. Liu, P.J. Olver, C.Z. Qu, and S.H. Zhang, On the blow-up of solutions to the integrable modified Camassa-Holm equation, *Anal. Appl.* (Singap.), **12** (2014), 355-368.

[36] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.*, **19** (1978), 1156-1162.

[37] H.P. McKean, The Liouville correspondence between the Korteweg-de Vries and the Camassa-Holm hierarchies, *Commun. Pure Appl. Math.*, **56** (2003), 998-1015.

[38] R. Milson, Liouville transformation and exactly solvable Schrödinger equations, *Int. J. Theo. Phys.*, **37** (1998), 1735-1752.

[39] A.C. Newell, Solitons in Mathematics and Physics, in: *CBMS-NSF Regional Conference Series in Applied Mathematics*, **48**, SIAM, Philadelphia, (1985).

[40] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.

[41] P.J. Olver, Evolution equations possessing infinitely many symmetries, *J. Math. Phys.*, **18** (1977), 1212-1215.

[42] P.J. Olver, Applications of Lie Groups to Differential Equations. Second edition, *Graduate Texts in Mathematics*, **107**, Springer-Verlag, New York, 1993.

[43] P.J. Olver, Nonlocal symmetries and ghosts, in: New Trends in Integrability and Partial Solvability, A.B. Shabat et. al., eds., Kluwer Acad. Publ., Dordrecht, Netherlands, 2004, pp. 199-215.

[44] P.J. Olver and P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, *Phys. Rev. E*, **53** (1996), 1900-1906.

[45] P.J. Olver, J. Sanders, and J.P. Wang, Ghost symmetries, *J. Nonlinear Math. Phys.*, **9**, Suppl. 1 (2002), 164-172.

[46] Z. Qiao, A new integrable equation with cuspons and W/M-shape-peaks solitons, *J. Math. Phys.*, **47** (2006), 112701, 9pp.

[47] C.Z. Qu, X.C. Liu, and Y. Liu, Stability of peakons for an integrable modified Camassa-Holm equation with cubic nonlinearity, *Commun. Math. Phys.*, **322** (2013), 967-997.

[48] E. Reyes, Geometric integrability of the Camassa-Holm equation, *Lett. Math. Phys.*, **59** (2002), 117-131.

[49] C. Rogers and W.K. Schief, Bäcklund and Darboux Transformations, Cambridge University Press, Cambridge, 2002.

[50] P. Rosenau, On solitons, compactons, and Lagrange maps, *Phys. Lett. A*, **211** (1996), 265-275.
[51] J. Schiff, Zero curvature formulations of dual hierarchies, *J. Math. Phys.*, **37** (1996), 1928-1938.
[52] J. Schiff, The Camassa-Holm equation: a loop group approach, *Physica D*, **121** (1998), 24-43.

Jing Kang
Center for Nonlinear Studies and School of Mathematics, Northwest University, Xi'an 710069, P. R. China
E-mail address: jingkang@nwu.edu.cn

Xiaochuan Liu
Center for Nonlinear Studies and School of Mathematics, Northwest University, Xi'an 710069, P. R. China
E-mail address: liuxc@nwu.edu.cn

Peter J. Olver
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
E-mail address: olver@umn.edu

Changzheng Qu
Department of Mathematics, Ningbo University, Ningbo 315211, P. R. China
E-mail address: quchangzheng@nbu.edu.cn