

# NUMERICAL SIMULATION OF NONLINEAR DISPERSIVE QUANTIZATION

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ABSTRACT. When posed on a periodic domain in one space variable, linear dispersive evolution equations with integral polynomial dispersion relations exhibit strikingly different behaviors depending upon whether the time is rational or irrational relative to the length of the interval, thus producing the Talbot effect of dispersive quantization and fractalization. The goal here is to show that these remarkable phenomena extend to nonlinear dispersive evolution equations. We will present numerical simulations, based on operator splitting methods, of the nonlinear Schrödinger and Korteweg–deVries equations with step function initial data and periodic boundary conditions. For the integrable nonlinear Schrödinger equation, our observations have been rigorously confirmed in a recent paper of Erdoğan and Tzirakis, [10].

## 1. INTRODUCTION

In the early 1990’s, Michael Berry, [1, 2], discovered that the time evolution of rough initial data on periodic domains through the free space linear Schrödinger equation exhibits radically different behavior depending upon whether the elapsed time is a rational or irrational multiple of the length of the space interval. Specifically, given a step function as initial conditions, one finds that, at rational times, the solution is piecewise constant, but discontinuous, whereas at irrational times it is a continuous but nowhere differentiable fractal-like functions. More generally, when starting with more general initial data, the solution profile at rational times is a linear combination of finitely many translates of the initial data, which explains the appearance of piecewise constant profiles obtained when starting with a step function. Berry named this striking phenomenon the *Talbot effect*, after an interesting optical experiment first performed by the inventor of the photographic negative, [27]. Berry’s discovery was then rigorously analyzed and further investigated in [18, 23, 24, 25, 28].

In [22, 23], it was shown that the same Talbot effect of dispersive quantization and fractalization appears in general periodic linear dispersive equations whose dispersion relation is a multiple of a polynomial with integer coefficients (an “integral polynomial”), the prototypical example being the linearized Korteweg–deVries equation. Subsequently, it was numerically observed, [6], that the effect persists for more general dispersion relations which are asymptotically polynomial:  $\omega(k) \sim ck^n$  for large wave numbers  $k \gg 0$ , where  $c \in \mathbb{R}$  and  $2 \leq n \in \mathbb{N}$ . However,

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equations having other large wave dispersive asymptotics exhibit a wide variety of fascinating and as yet poorly understood behaviors, including large scale oscillations with gradually accumulating waviness, dispersive oscillations leading to a slightly fractal wave form superimposed over a slowly oscillating ocean, slowly varying traveling waves, oscillatory waves that interact and eventually become fractal, and fully fractal/quantized behavior. As yet, apart from the integral polynomially dispersive case, all these results are based on numerical observations, and, despite being simple linear partial differential equations, rigorous statements and proofs appear to be very difficult. The paper [6] also displayed some preliminary numerical computations that strongly indicate that the Talbot effect of dispersive quantization and fractalization persists into the nonlinear regime — for both integral and non-integrable evolution equations whose linear part has an integral polynomial dispersion relation.

The goal of the present study is to continue our explorations of the effect of periodicity on rough initial data for nonlinear evolution equations in the context of two important examples: the nonlinear Schrödinger (nLS) and Korteweg–deVries (KdV) equations, possessing, respectively, elementary second and third order monomial dispersion. Our basic numerical tool is the operator splitting method, [13], which serves to highlight the interplay between the behaviors induced by the linear and nonlinear parts of the equation. Earlier rigorous results concerning the operator splitting method for the Korteweg–deVries, generalized Korteweg–deVries, and nonlinear Schrödinger equations can be found in [14, 15, 17, 20]. We also refer the reader to [16] and the references therein for a discussion of alternative numerical schemes and convergence thereof for  $L^2$  initial data on the real line. See also [11] for additional results comparing the behavior of linear and nonlinear dispersive equations.

Since a preliminary version of this paper appeared online, Erdoğan and Tzirakis, [10], have now proved the Talbot effect for the integrable nonlinear Schrödinger equation, showing that at rational times the solution is quantized, while at irrational times it is a continuous, nowhere differentiable function with fractal profile, thus confirming our numerical experiments. Rigorously establishing such observed effects in the third order Korteweg–deVries equation, as well as nonlinear Schrödinger equations with more general nonlinearities remains open problems.

## 2. OPERATOR SPLITTING

Let us first summarize the basic ideas behind the operator splitting method for approximating the solutions to nonlinear evolution equations. Details and proofs can be found in various references, for example [13, 15].

Formally, consider the initial value problem for a dynamical differential equation:

$$u_t = K[u], \quad u(0) = u_0, \quad (2.1)$$

where  $K$  is a differential operator in the spatial variable with no explicit time dependence. We use  $\Phi_K(t)$  to denote the *flow* induced by the initial value problem, whereby  $u(t) = \Phi_K(t)u_0$  is the solution to (2.1), which is defined on some time

interval  $t \in [0, T]$ . The time-independence of the right hand side and uniqueness of solutions to the initial value problem implies the flow condition  $\Phi_K(t + s) = \Phi_K(t) \Phi_K(s)$  for any  $t, s \geq 0$  such that the indicated flow operators are defined.

Suppose that we can write  $K = L + N$ , where  $L$  and  $N$  are “simpler” differential operators, meaning that their corresponding flow operators  $\Phi_L(t), \Phi_N(t)$  can be determined explicitly, or are easily approximated by a numerical scheme. In our applications,  $L$  is a linear differential operator, while  $N$  is nonlinear. The basic idea of *operator splitting*, [13], is to approximate the flow of their sum  $K = L + N$  by repeated application of the individual simpler flows over a sequence of small time steps. For simplicity, we adopt a uniform time step  $0 < \Delta t \ll 1$ , and seek to approximate the solution  $u(t_n)$  at the successive times  $t_n = n \Delta t$  for  $n = 0, 1, 2, \dots$

The simplest version is the *Godunov splitting* scheme, which iteratively defines the approximations

$$u_\Delta(t_{n+1}) = \Phi_L(\Delta t) \Phi_N(\Delta t) u_\Delta(t_n), \quad n = 0, 1, 2, \dots, \quad u_\Delta(0) = u_0. \quad (2.2)$$

In other words,

$$u_\Delta(t_n) \simeq (\Phi_L(\Delta t) \Phi_N(\Delta t))^n u_0 \quad (2.3)$$

Specification of a suitable function space norm allows one to prove that, under appropriate hypotheses on the operators,  $K, L, N$  and the initial data, the Godunov approximations converge to the true solution:

$$\|u_\Delta(t_n) - u(t_n)\| \longrightarrow 0 \quad \text{as} \quad \Delta t \longrightarrow 0. \quad (2.4)$$

A slightly more sophisticated method is the *Strang splitting* scheme, which replaces the iterative step in (2.2) by

$$u_\Delta(t_{n+1}) = \Phi_N\left(\frac{1}{2} \Delta t\right) \Phi_L(\Delta t) \Phi_N\left(\frac{1}{2} \Delta t\right) u_\Delta(t_n). \quad (2.5)$$

The Strang scheme typically enjoys a higher rate of convergence. However, to date, we have found not much difference between the two schemes in our numerical experiments, and most of our calculations are based on the Godunov splitting scheme.

### 3. DISPERSIVE QUANTIZATION IN SCHRÖDINGER EQUATIONS

We first consider the free space linear Schrödinger equation

$$u_t = i u_{xx}, \quad (3.1)$$

where  $i = \sqrt{-1}$ . (Throughout the paper, we use subscript notation for derivatives.) We are interested in the behavior of rough solutions on a periodic domain, which we take to be  $0 < x < 2\pi$ . This is epitomized by considering the step function initial data

$$u(0, x) = f(x) = \begin{cases} -1, & 0 < x < \pi, \\ 1, & \pi < x < 2\pi, \end{cases} \quad (3.2)$$

in analogy with the Riemann problem for first order hyperbolic systems, [26, 30].

It is straightforward to write an explicit Fourier series solution to the initial value problem (3.1), (3.2), which takes the form of a number-theoretic Gauss

sum, [29]. Michael Berry, [1, 2], discovered the Talbot phenomenon of dispersive quantization and fractalization of this solution. Briefly, when the time is rational (relative to  $\pi$ ), so  $t/\pi \in \mathbb{Q}$ , the real part, complex part, and norm of the solution  $u(t, x)$  are discontinuous and piecewise constant, whereas when the time is irrational,  $t/\pi \notin \mathbb{Q}$ , the graphs of real part, complex part and norm of the solution are continuous but nowhere differentiable fractal-like functions. Additional details can be found in [18, 23, 24, 25].

Our goal here is to investigate to what extent the dispersive quantization and fractalization phenomena persist into the nonlinear regime. We consider nonlinear Schrödinger equations of the form

$$u_t = i(u_{xx} + |u|^p u), \quad (3.3)$$

subject to periodic boundary conditions on  $[0, 2\pi]$ , and use the same step function (3.2) as initial data. In (3.3), the power is assumed to satisfy  $p \geq 1$ , with the particular case  $p = 2$  being an integrable, soliton equation, of importance in fiber optics, [8, 30]. In this paper, numerical experiments for the integrable case  $p = 2$  as well as the non-integrable cases  $p = 1$  and 3 will be displayed. As we will see, similar phenomena are observed in all three cases, and hence the qualitative features are not dependent upon the underlying integrability of the evolution equation.

We will numerically approximate the solutions to the nonlinear Schrödinger equation (3.3) by use of operator splitting. We decompose its right hand side  $K[u] = L[u] + N[u]$  as the sum of its linear and nonlinear components:

$$L[u] = i u_{xx}, \quad N[u] = i |u|^p u.$$

The resulting approximations to the solution have been previously analyzed in [15, 20]. The flows of the linear dispersive operator  $L$  and the nonlinear operator  $N$  are easily determined, and so we need to investigate their interaction in order to understand the behavior of solutions to the full nonlinear initial value problem.

**3.1. The Linear Schrödinger Equation.** We begin by plotting some representative graphs of the solution to the periodic initial value problem (3.1), (3.2) for the linear Schrödinger equation. These were obtained by using the Fast Fourier Transform (FFT) based on 2048 space nodes. Each row in Figure 3.1 displays the real part, the complex part, and the norm of the solution to the initial value problem at the indicated time. The first three times  $t = .3, .31$ , and  $.314$  are irrational (relative to  $\pi$ ) and increasingly close to the final rational time  $t = .1\pi$ , at which time the solution can be rigorously proved to be piecewise constant, [22]. A small but noticeable Gibbs effect appears in the graphs due to the FFT approximation. On the other hand, at each irrational time, the solution is a continuous, but non-differentiable fractal-like function, [23]. Between times  $t = .31$  and  $.314$ , the sudden appearance of a noticeable quantization effect, albeit still modulated by fractal behavior, is striking.

We remark that the periodic linear Schrödinger equation preserves the  $L^2$  norm of the solution.

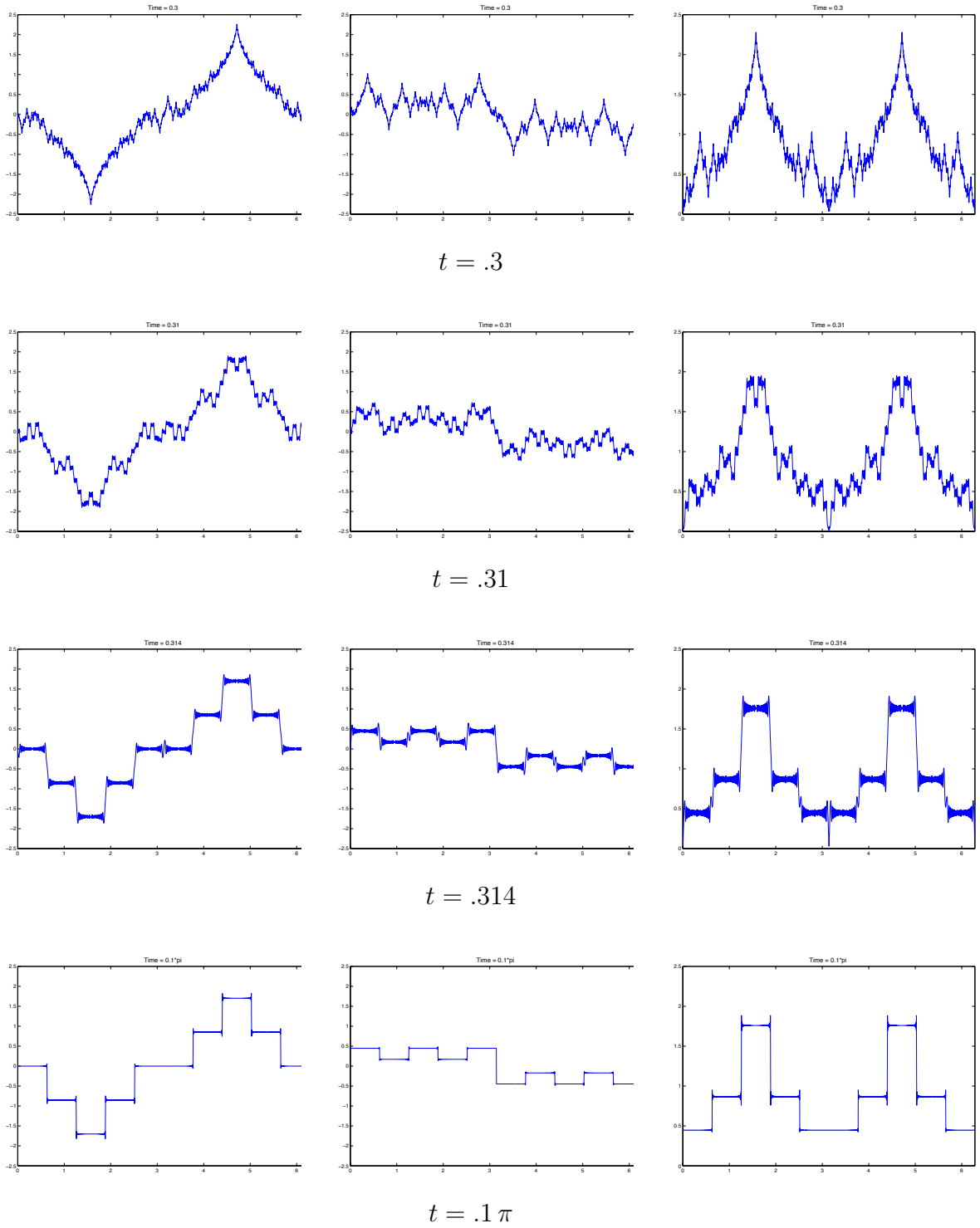


FIGURE 3.1. The Solution to the Periodic Linear Schrödinger Equation.

**Lemma 1.** *If the initial data  $u(0, x)$  is in  $L^2$ , then the  $L^2$  norm of the solution to the linear Schrödinger equation (3.1) on a periodic domain is constant in time.*

*Proof.* When  $u$  is sufficiently smooth, we have

$$\begin{aligned} \frac{d}{dt} \int_0^{2\pi} |u|^2 dx &= \int_0^{2\pi} [u_t \bar{u} + u \bar{u}_t] dx \\ &= i \int_0^{2\pi} [u_{xx} \bar{u} - u \bar{u}_{xx}] dx = i \int_0^{2\pi} (u_x \bar{u} - u \bar{u}_x)_x dx = 0, \end{aligned} \quad (3.4)$$

owing to the periodicity of our problem.

When the initial data  $u(0, x) = f(x)$  is not smooth, we can smooth it using mollification, and then apply the preceding argument. Specifically, let  $\phi(x)$  be a standard mollifier, e.g. a Gaussian, and let  $u_\epsilon(0, x)$  be the mollified initial data obtained by convolution of  $u(0, x)$  and  $\phi_\epsilon(x) = \epsilon^{-1} \phi(x/\epsilon)$ , so that

$$\lim_{\epsilon \rightarrow 0} \|u(0, x) - u_\epsilon(0, x)\|_{L^2} = 0. \quad (3.5)$$

By the preceding calculation, the  $L^2$  norm of the mollified solution  $u_\epsilon(t, x)$  is preserved. Moreover, by (3.5), the  $L^2$  norm of the mollified solution converges to the  $L^2$  norm of  $u(t, x)$ , which completes the proof.  $\square$

In particular, if the  $L^2$  norm of the initial data of (3.1) is small, so is the  $L^2$  norm of the solution, proving well-posedness.

**3.2. The Nonlinear Schrödinger Equation.** Splitting off the nonlinear part of (3.3) leads to the evolution equation

$$u_t = N[u] = i |u|^p u, \quad (3.6)$$

Because no  $x$  derivatives are involved, (3.6) is effectively an ordinary differential equation in which  $x$  serves as a parameter. Observe that

$$\frac{d}{dt} |u|^2 = u_t \bar{u} + u \bar{u}_t = i |u|^p u \bar{u} - i |u|^p u \bar{u} = 0, \quad (3.7)$$

and hence, for each fixed  $x$ , the modulus  $|u(t, x)|$  is constant as  $t$  evolves. This implies that the solution to (3.6) is explicitly given by

$$u(t, x) = f(x) e^{it|f(x)|^p} \quad \text{where} \quad u(0, x) = f(x) \quad (3.8)$$

is the initial condition. Moreover, combining this observation with Lemma 1, we see that the  $L^2$  norm is preserved under the full nonlinear Schrödinger flow (3.3).

**3.3. Numerical Results.** Figures 3.2, 3.3, 3.4, display the results of our numerical integration of the initial value problem (3.2), for the powers  $p = 1, 2,$  and  $3,$  respectively, in the nonlinear Schrödinger equation (3.3).

In all cases, we use Godunov splitting to approximate the solution to the periodic initial value problem with step function initial data. Although the Strang splitting method has higher rate of convergence, all the numerical results that it

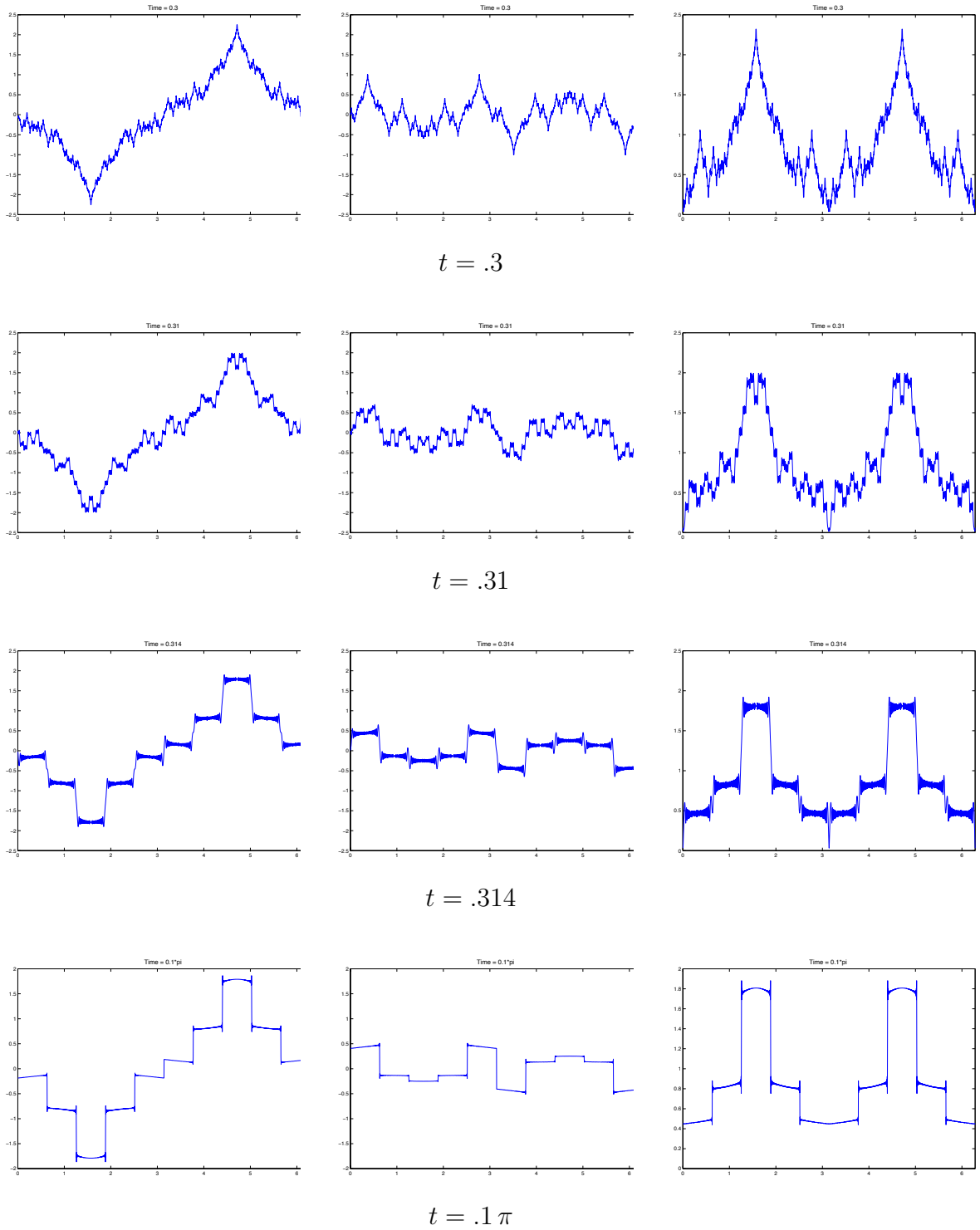


FIGURE 3.2. The Solution to the Periodic Nonlinear Schrödinger Equation,  $p = 1$ .

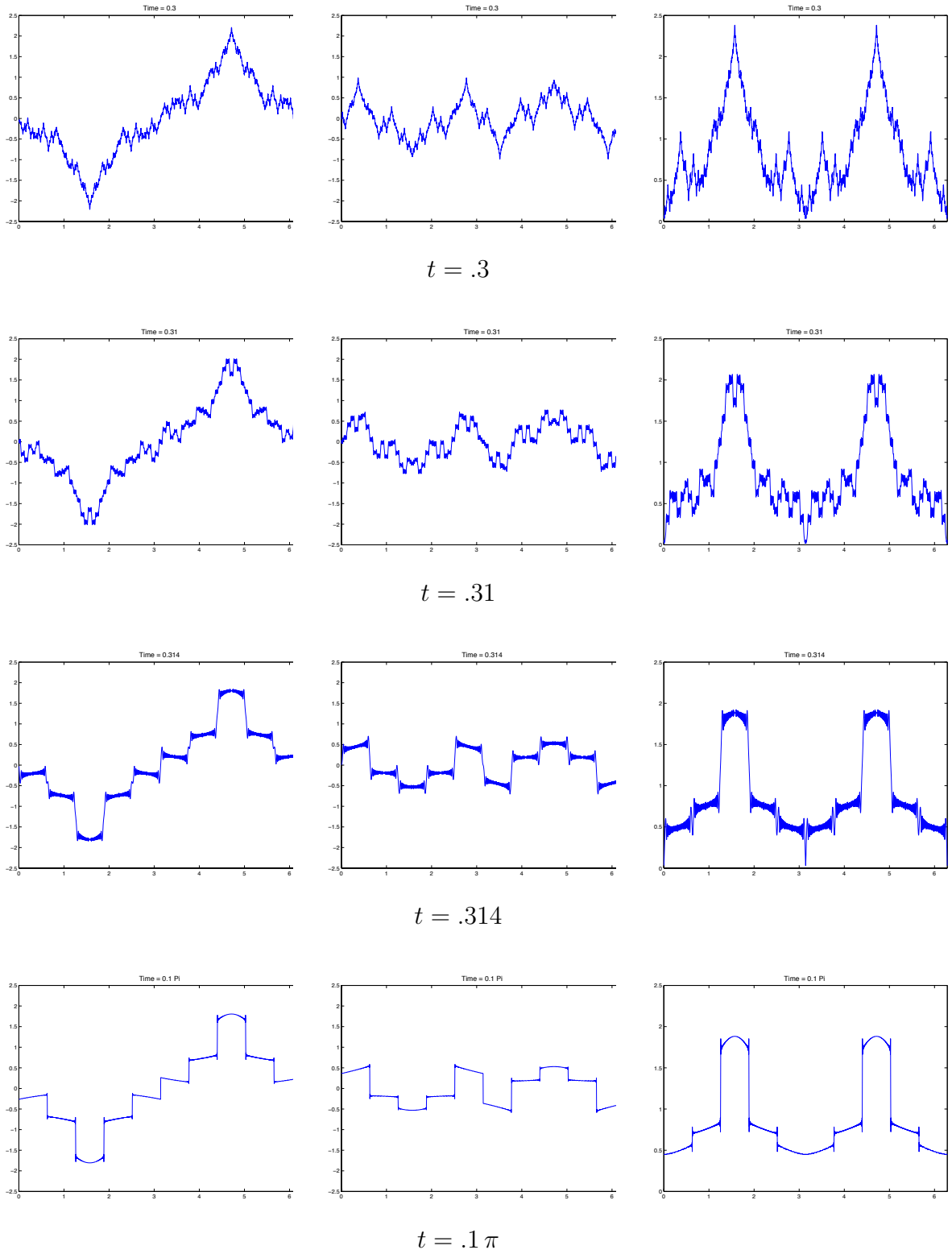


FIGURE 3.3. The Solution to the Periodic Nonlinear Schrödinger Equation,  $p = 2$ .



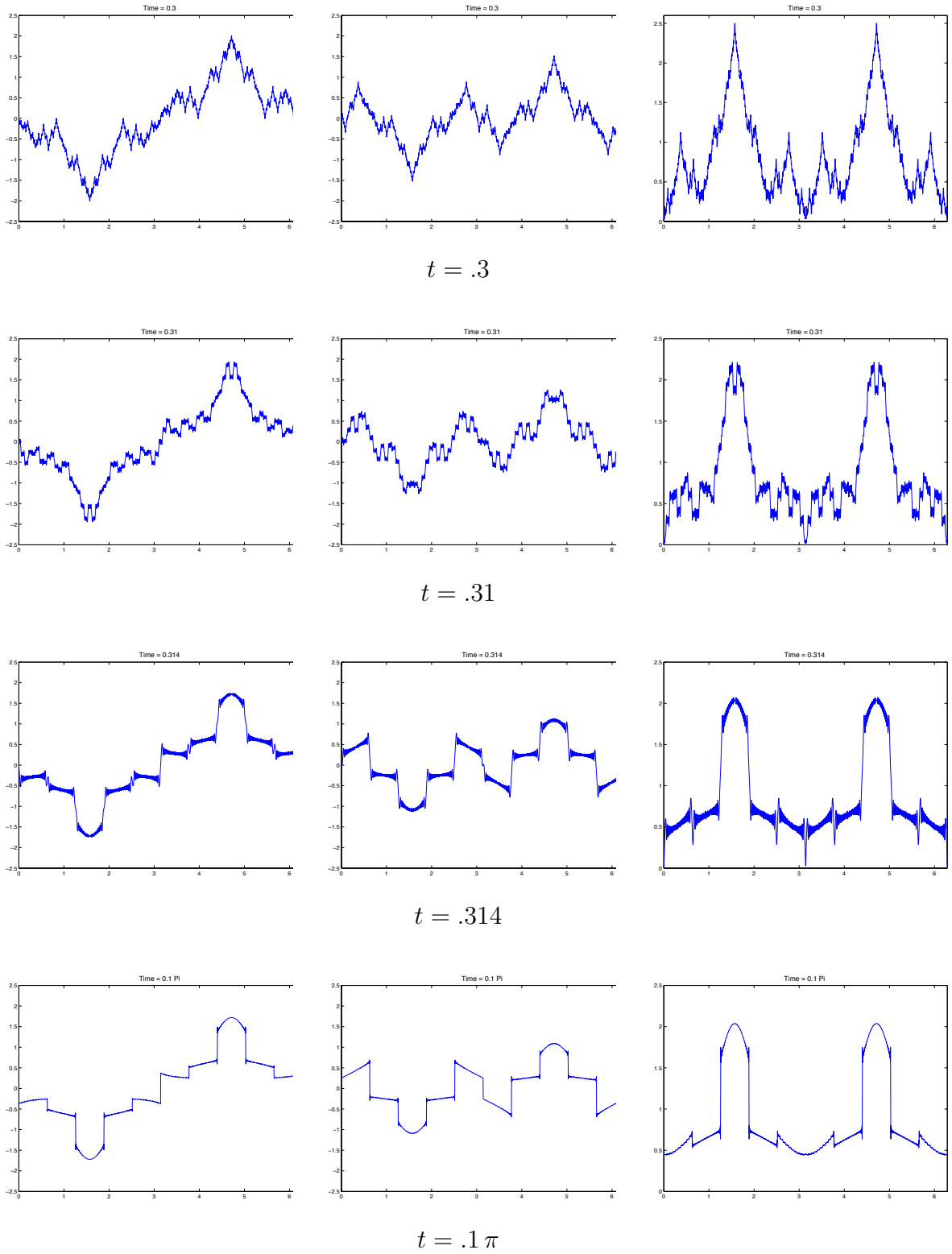


FIGURE 3.4. The Solution to the Periodic Nonlinear Schrödinger Equation,  $p = 3$ .

generated turned out to be extremely similar to those obtained via the simpler Godunov scheme, and so we concentrate on the latter approximations. The nonlinear flow is computed exactly using (3.8), while the linear flow is computed through use of the FFT based on  $n = 2048$  space nodes. The time step is  $\Delta t = .00005\pi \approx .00015708$  and, as before, we display our numerical approximations at the irrational times  $t = .3, .31, .314$  and the rational time  $t = .1\pi$ . To be certain that a subtle rational/irrational dichotomy does not affect the numerical computations, we also ran some simulations with irrational<sup>1</sup> time steps  $\Delta t = .00016$  and  $.0002$ , and, as expected, did not observe any appreciable differences among the computed solutions.

All three sets of plots are fairly similar to those from the linear evolution in Figure 3.1, although closer inspection will reveal small differences. The most noticeable is that, at the rational time  $t = .1\pi$ , the quantized solution is no longer piecewise constant, and its shape exhibits the effect of the nonlinearity.

To analyze the numerical solution, we consider the phenomenon caused by the linear and nonlinear components separately. When time is a rational multiple of  $\pi$ , the linear evolution (3.1) will generate a piecewise constant solution profile. Because the nonlinear equation (3.6) is effectively an ordinary differential equation, with explicit solution formula (3.8), its effect is simply to shift the solution graph vertically by an amount depending upon the initial height, and hence preserves the piecewise constant nature of the solution. On the other hand, when the time is an irrational multiple of  $\pi$ , the linear term will generate a continuous but nowhere differential profile. The subsequent vertical movement caused by the nonlinear term will be uniform, which means that, on a small time interval, the resulting graph will be very close to nowhere differentiable.

The presumed convergence of the operator splitting scheme then implies that the solution of the nonlinear Schrödinger equation will exhibit the same quantization/fractalization Talbot phenomenon, an observation that is confirmed by recent rigorous results of Erdoğan and Tzirakis, [10], for the integrable case  $p = 2$ . While their methods do not as yet apply to when  $p > 2$ , our numerical experiments indicate that, as long as there is no blow-up, the Talbot phenomenon in fact appears for all exponents  $p \geq 1$  (at least). In particular, at rational times, the numerical solution profile exhibits a finite number of jump discontinuities, between which it is no longer constant, but appears to assume, roughly, a parabolic shape, although we do not know the precise analytic forms of these rational solution profiles. Further, as the power  $p$  increases, the parabolic profile becomes increasingly curved.

#### 4. DISPERSIVE QUANTIZATION FOR THE KORTEWEG–DEVRIES EQUATION

Our second example is the Korteweg–deVries (KdV) equation, [8], a justly famous integrable evolution equation modeling shallow water waves, plasma waves,

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<sup>1</sup>Of course, a digital computer can only work with rational (indeed, dyadic) numbers. However, the robustness of the results coming from our computations shows that this is also not an issue.

etc. We will take the KdV equation in the normalized form

$$u_t + u_{xxx} - uu_x = 0, \tag{4.1}$$

and consider the periodic initial-boundary value problem on the interval  $[0, 2\pi]$ . For the *linearized KdV equation*

$$u_t + u_{xxx} = 0, \tag{4.2}$$

(also known as the *Airy partial differential equation*), the solution to the periodic initial value problem can be easily expressed in Fourier series form. In particular, the dispersive quantization and fractalization phenomenon for the step function initial data were established in [22, 23].

**4.1. Numerical Solutions.** The authors' previous paper, [6], presented the results of numerical experiments on the periodic problem with the step function (3.2) as initial data, using operator splitting to numerically approximate the solution. It was found that the effects of dispersive quantization and fractalization persist into the nonlinear regime. In Figures 4.1 and 4.2 we display some representative graphs of the numerical solution at, respectively, some irrational times, which exhibit a fractal, non-differentiable profile, and at some rational times, whose profiles appear to be quantized and (almost) smoothly varying between the jumps. It is not clear whether the small oscillations appearing between the jumps are due to numerical error or the persistence of a small fractal contribution.

**4.2. Operator Splitting.** In order to apply operator splitting to the KdV flow, we decompose  $K[u] = -u_{xxx} + uu_x$  into the sum of its linear part  $L[u] = -u_{xxx}$ , leading to the linearized KdV equation (4.2), and its nonlinear part  $N[u] = uu_x$ , whose corresponding evolution equations is the *nonlinear transport equation*

$$u_t = uu_x, \tag{4.3}$$

also known as the *Riemann equation* or *inviscid Burgers' equation*, [30]. Unfortunately, we have similarly been unable to rigorously establish convergence of the operator splitting method for the periodic KdV equation with rough initial data.

Let us present what is known about this problem and some discussion of how one might proceed. First of all, convergence for sufficiently smooth initial data has been established in [14].

**Theorem 2.** *Fix  $T > 0$ . Given initial data  $u_0 \in H^s$  with  $s \geq 5$ , let  $u$  be the exact solution and  $u_\Delta$  the numerical solution of the periodic Korteweg–de Vries equation (4.1) obtained through Godunov splitting based on the linear and nonlinear parts. Then, for sufficiently small  $\Delta t \ll 1$ ,*

$$\|u_\Delta(t) - u(t)\|_{H^{s-3}} \leq C\Delta t, \quad 0 \leq t \leq T, \tag{4.4}$$

where  $C$  depends only on  $s, T$ , and the initial data  $u_0$ .

The difficulty that is preventing us from extending this result to rough data, such as the step function, is fairly subtle. The key stumbling block is that one can control the  $L^2$  behavior of the linear KdV flow (4.2), and the  $L^1$  behavior of the nonlinear transport flow (4.3). More specifically, the difference between two

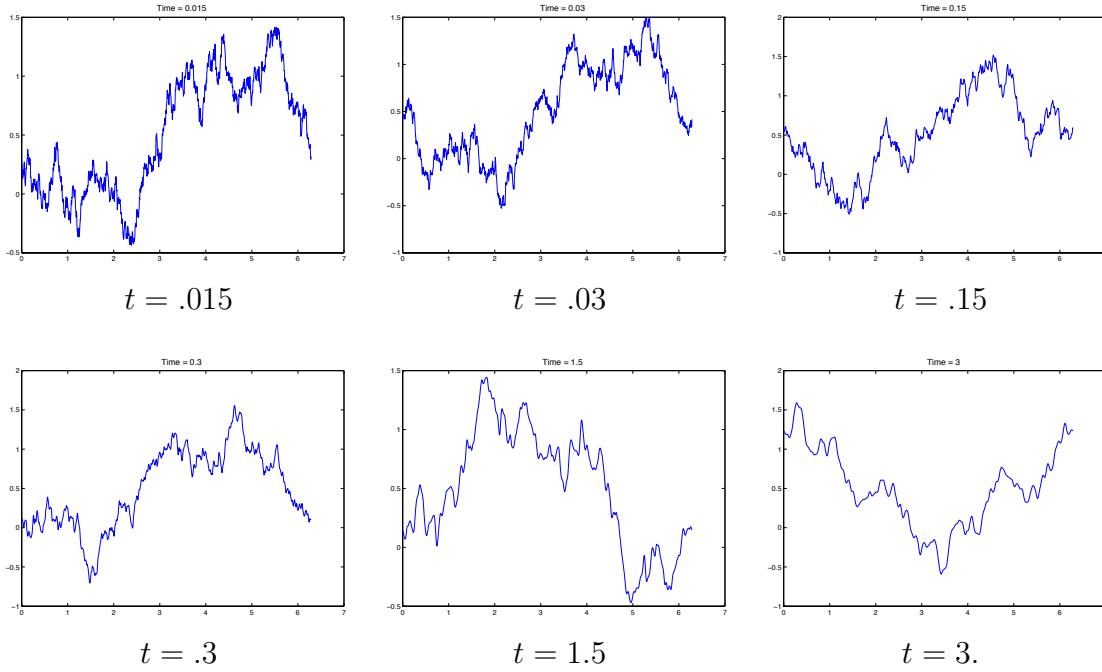


FIGURE 4.1. Solution to the KdV equation at irrational times.

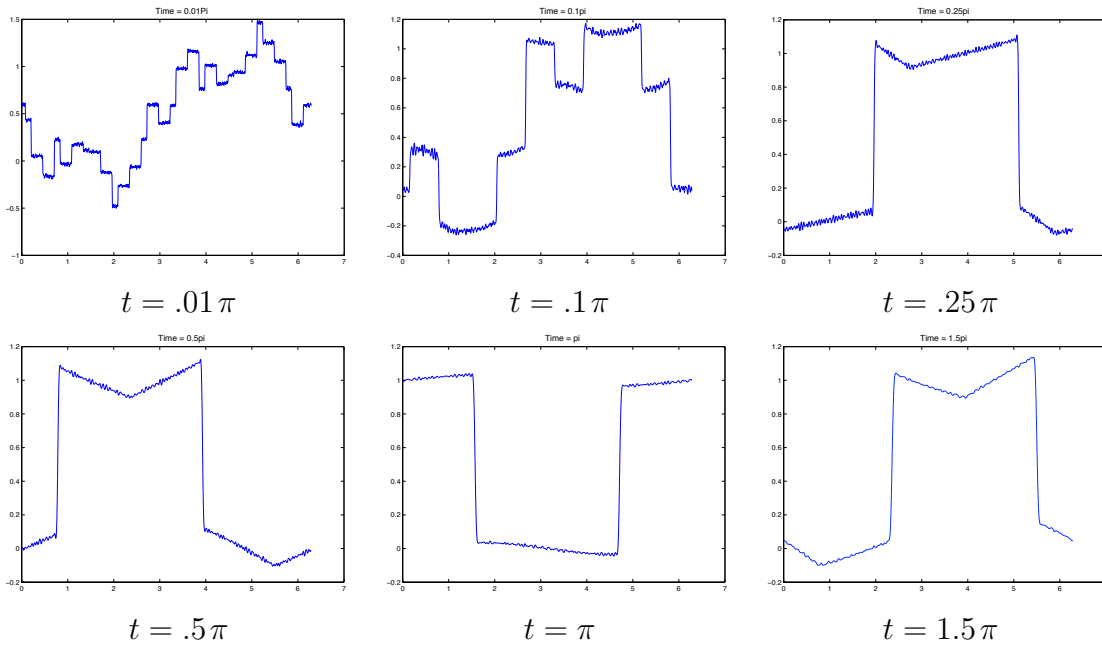


FIGURE 4.2. Solution to the KdV equation at rational times.

solutions to the nonlinear component will behave well in  $L^1$  whereas the difference between two solutions of the linear flow behaves well in  $L^2$ . However, there is not an evident method that will allow us to interpolate between these two different Banach spaces.

Nevertheless, the following theorem concerning convergence of the operator splitting scheme to weak solutions has been proved in [15]. Moreover, Zhou, [31], proves that weak solutions to the KdV equation are uniquely determined by their initial data.

**Theorem 3.** *Suppose that  $u_0 \in L^2$ . Suppose that the sequence of numerical solutions  $\{u_\Delta\}$  generated by the operator splitting method converges strongly in  $L^2([0, 2\pi] \times [0, T])$  to the solution  $u$  as  $\Delta t \rightarrow 0^+$ , Then  $u \in L^\infty([0, T]; L^2)$  is a weak solution of the KdV equation that has  $u_0$  as its initial data.*

**4.3. Well-posedness.** Next, recall that the well-posedness of the periodic Korteweg–deVries equation in  $L^2$  has been established by Bourgain, [5].

**Theorem 4.** *The periodic KdV equation is well-posed in  $L^2$ .*

On the other hand, the well-posedness of the periodic linearized KdV equation in  $L^2$  is easily established, using the conservation of  $L^2$  norm.

**Theorem 5.** *Given initial data  $u(0) = u_0$  and  $v(0) = v_0$ , let  $u_L(t) = \Phi_L(t)u_0$  and  $v_L(t) = \Phi_L(t)v_0$  be the corresponding periodic solutions of the linearized KdV equation (4.2). Then*

$$\|u_L(t) - v_L(t)\|_{L^2} \leq \|u_0 - v_0\|_{L^2}.$$

Now, if we know that our solution  $u_L(t)$  has bounded  $L^\infty$  norm, then the estimate in Theorem 5 implies that the flow is  $L^1$  contractive. Moreover, Oskolkov, [23], has established a global boundedness result for suitable initial data.

**Theorem 6.** *If the initial data has bounded BV norm, then the resulting solution to the periodic linearized KdV equation is uniformly bounded in  $L^\infty$ , the bound depending on the BV and  $L^\infty$  norms of the initial data.*

Furthermore, Oskolkov proves that when time is irrational (relative to  $\pi$ ), the solution of the linearized KdV is nowhere differentiable, and has unbounded BV norm. The difficulty is that the  $L^\infty$  bound in Oskolkov’s result depends upon the BV norm of the initial data. It is hard to apply this theorem in our case since we do not have good control of the BV norm for the nonlinear flow.

On the other hand, consider the nonlinear transport equation (4.3), whose flow is denoted by  $\Phi_N(t)$ . Of course, the formation of shock waves implies that  $\Phi_N(t)$  does not preserve smoothness or continuity of the initial data. In [14], the authors state that the flow is both  $L^1$  contractive and uniformly bounded.

**Theorem 7.** *The solution operator to the nonlinear transport equation (4.3) is  $L^1$  contractive. In other words, given solutions  $u_N(t) = \Phi_N(t)u_0$  and  $v_N(t) = \Phi_N(t)v_0$ , we have*

$$\|u_N(t) - v_N(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}.$$

Moreover, the solutions remain uniformly bounded in  $L^\infty$  norm:

$$\|u_N(t)\|_\infty \leq \|u_0\|_\infty.$$

However, when attempting to interpolate these estimates between  $L^1$  and  $L^2$ , at each step in the iterative procedure we will introduce a factor that we are unable to control as the time step goes to 0 and the number of iterations goes to infinity. For example, although we can bound the nonlinear flow, the known BV norm bound is not easy to apply in the context of Oskolkov’s Theorem 6.

Results on the decay of solutions to conservation laws, [7], imply the decay of the total variation of entropy solutions of nonlinear transport equation:

**Theorem 8.** *Suppose that the initial data to the nonlinear transport equation on a periodic domain satisfies  $u_0 \in L^\infty([0, 2\pi])$ . Then the total variation (TV) of the resulting entropy solution  $u_N(t) = \Phi_N(t)u_0$  over the interval will satisfy the decay estimate:  $TV[u_N(t)] < c/t$ , where  $c$  is a constant that is independent of the initial data.*

However, as the time step  $\Delta t \rightarrow 0$ , we lose the control of the constant which will appear when we interpolate between  $L^1$  and  $L^2$ . At this point, we do not have any sharper estimate for the BV norm of the nonlinear flow, since, when the time step is irrational, the linear flow will generate the initial data for the nonlinear flow whose BV norm is unbounded.

Motivated by Theorem 5, we can try some other ways to show the compactness of the sequence of numerical solutions in  $L^2$ . A standard approach is based on the Kolmogorov–Riesz  $L^p$  compactness theorem, [12].

**Definition 9.** Let  $1 \leq p < \infty$ . A subset  $F$  of  $L^p(\mathbb{R}^n)$  is *totally bounded* if and only if

- (1)  $F$  is bounded
- (2) for every  $\epsilon > 0$ , there is an  $R > 0$  such that

$$\int_{|x|>R} |f(x)|^p dx < \epsilon^p, \quad \text{for every } f \in F. \tag{4.5}$$

- (3) for every  $\epsilon > 0$  there is a  $\rho > 0$  such that

$$\int_{\mathbb{R}^n} |f(x+y) - f(x)|^p dx < \epsilon^p, \quad \text{for every } f \in F \text{ and } y \in \mathbb{R}^n \text{ such that } |y| < \rho. \tag{4.6}$$

Our goal is to prove compactness in  $L^2$ . In our case, condition 2 is redundant. Condition 1 is clear since boundness in  $L^2$  is straightforward: The solution of nonlinear part is stable in  $L^2$  and the  $L^2$  norm of the solution of linear part is preserved. So, no matter many iterations we take, the family of numerical solutions will be uniformly bounded in  $L^2$ . Thus, the remaining task is to verify condition 3. As we can see, when we iterate, the linear flow will behave well in  $L^2$  but we are unable to rigorously control the nonlinear flow as before. On the

other hand, our numerical experiments indicate that the numerical solutions are bounded in  $L^\infty$ , which will effectively make condition 3 immediate. Motivated by this fact, we can try to prove the uniform boundedness in  $L^\infty$  of the solution to the nonlinear part. However, the problem is that we do not have a Maximum Principle, and so cannot uniformly control the  $L^\infty$  norm for the linear flow when we perform the iterative step. Thus, we are left tantalizingly short of our goal to prove convergence of the operator splitting scheme for the KdV evolution of periodic rough initial data. We hope to report further on the convergence for nonlinear dispersive equations in a subsequent publication.

**4.4. Final Remarks.** Although we are as yet unable to prove the convergence of the numerical scheme, we can intuitively understand the behavior of the nonlinear KdV from the behavior of its linearization. The well-known studies of the “zero dispersion limit” initiated by Lax and Levermore, [19], are complemented by the following recent result of Erdoğan and Tzirakis, [9] that compares solutions to the linear and nonlinear equations.

**Theorem 10.** *Consider the Korteweg-de Vries (KdV) equation with periodic boundary conditions. For  $H^s$  initial data,  $s > -\frac{1}{2}$ , and for any  $r < \min(3s + 1, s + 1)$ , the difference between the nonlinear and linear evolutions is in  $H^r$  for all times, with at most polynomially growing  $H^r$  norm.*

This result implies that we can regard the nonlinear KdV flow as a perturbation of the linearized KdV flow. Therefore, the observed persistence of the fractalization and quantization behaviors is to be expected.

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