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9/7/2011

1.1 Examples of fluid flows

Let us consider a few examples of phenomena studied in fluid mechanics.

Example 1

We consider a ball of radius R moving in a fluid of density ρ at speed U. The problem is to determine the force F on the body caused by the "resistance of the fluid". (Often the force is called the "drag force".) We will soon see that the problem is somewhat undetermined, we need to know more about the fluid[1] to determine F precisely. In fact, even the definition of F itself is not as straightforward as it might seem: the force F can depend on time (even when U does not), so we really have in mind some "average force". For now let us disregard these complications. We will discuss them later. The problem of finding a formula for F was already considered by Newton in *Principia Mathematica* (published in 1687). He gives a formula

$$F = c\rho R^2 U^2 \,, \tag{1.1}$$

where c is constant. Modulo some adjustments (to be discussed later) the formula works surprisingly well, especially given our incomplete description of the fluid. At the same time, there are many subtle points related to it and the modifications which make it more precise.

Example 2

Consider a flow of fluid of density ρ in a pipe of radius R. We define the average flow velocity U in the pipe

$$U = \frac{\text{volume of fluid which passed through the pipe in time } T}{\pi R^2 \ (= \text{area of the section of the pipe}) \ \ T} \,. \tag{1.2}$$

Let the pressure at the beginning of the pipe be P_1 and the pressure at the end of the pipe be P_2, and let L be the length of the pipe. We let

$$P' = \frac{P_1 - P_2}{L} \tag{1.3}$$

be the drop of pressure per unit length. Can we determine U in terms of P', R and ρ? There are similar complications as in the previous problem: we should know more about the fluid, the quantities U, P' should be considered as

[1] For example, one should distinguish compressible and incompressible fluids and consider the viscosity of the fluid.

some time averages, etc. Nevertheless, one has the so-called Darcy-Weischbach formula[2]

$$U = c\sqrt{\frac{P'R}{\rho}}\,, \tag{1.4}$$

where c is a constant. Similar to Newton's formula (1.1), this formula needs some adjustments in certain regimes, but given the incompleteness of the data and the complicated nature of the flow inside a pipe[3], it works surprisingly well.

Example 3
Most of you probably saw the following experiment. Create a jet of air flowing upwards (against the direction of gravity), for example by blowing into a straw. Place a ping-pong ball in the jet. The ping-pong ball will be "floating in the air", staying roughly at the center of the jet at a certain distance upwards from the origin of the jet, perhaps with some oscillations. The configuration is quite stable. Even if we tilt the jet slightly away from the vertical direction, the ball will still float, it will not fall down. It is easy to understand that the jet should produce some force in the upward direction, due to the drag force from Example 1. But how can we explain that the configuration is stable? For example, if instead of a jet of air we will have a jet of sand in vacuum, we do not expect that the configuration would be stable.

The formulae (1.1),(1.4), and the stability effect in Example 3 are not easy to understand from the "first principles", mostly because they are related to the phenomena of turbulence.

1.2 Dimensional Analysis

One can approach problems in fluid mechanics from several angles. In the next few lectures we will derive the basic equations of fluid mechanics, and in principle the phenomena in the examples above should be possible to explain by solving the equations (with relevant boundary conditions). However, in many cases this turns out to be unrealistic, even if we have large computers at our disposal. The behavior of fluids in the regimes relevant for the above examples is simply too complicated for us being able to track everything which is going on in the fluid.

When finding relevant solutions of the full equations describing a problem is unrealistic, it is often useful to adopt a *phenomenological approach*. We do not try to derive our formulae "from the first principles", but from "rougher" considerations[4]. A good example is provided by *dimensional analysis*, which we will now illustrate in the context of equation (1.1).

[2]Perhaps known already in 1770s to A. Chezy.

[3]Here we have in mind the regime of turbulent flows, which includes most pipe flows we encounter "in practice".

[4]Often the "rough considerations" can be quite robust, and survive a paradigm shift concerning the first principles. This may be the case even with the Fluid Mechanics. We do not really know whether the standard equations of fluid mechanics (which we will soon derive) lead (in some approximation) to formulae (1.1) in very turbulent regimes, even though it is

Assume that in the situation of Example 1 we can express the force F in terms of ρ, R, U. In other words, we assume that

$$F = \phi(\rho, R, U) \tag{1.5}$$

for some function $\phi\colon \mathbf{R}^3 \to \mathbf{R}$. Seemingly there is not much we can say about ϕ at this level of generality, but we can invoke an important principle: equation(1.5) should be independent of the choice of basic units. Let us look in more detail at what we mean by this. When we say that the radius of the ball is R, we implicitly assume that we have chosen a unit of length and that this unit of length fits $R-$times into the radius of the ball. So R is not really just a number, it also implicitly includes some choice of the unit of length. To emphasize this we can write

$$R = rL\,, \tag{1.6}$$

where L is our unit of length and r is a "pure number". We apply similar considerations to other quantities involved in (1.5). When the unit of length is fixed, we have to fix a unit of time to give a meaning to U. Assuming the unit of time is T, we can write (in dimension 3)

$$U = u\,\frac{L}{T}\,, \tag{1.7}$$

where u is again a pure number. To include density and force, we also need a unit of mass. Let us call it M. We can then write

$$\rho = \varrho\,\frac{M}{L^3} \tag{1.8}$$

and

$$F = f\,\frac{ML}{T^2}\,, \tag{1.9}$$

where ϱ and f are pure numbers. We see that in our situation we need to specify three independent units to express formula (1.5). We chose L, T, M, and the real physical quantities ρ, R, U, F are then expressed in terms of the pure numbers ϱ, r, u, f. Formula (1.5) then means that, with our choice of units L, T, M, we have

$$f = \phi(\varrho, r, u)\,. \tag{1.10}$$

So far we have nothing surprising, we have just gone in some detail over what we really mean by (1.5). The key point is the following principle:

(P) Relations between physical quantities should be independent of the choice of the basic units.

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more or less universally assumed. In fact, we do not even know if the standard equations have good solutions in such regimes. In the unlikely case that the standard equations would not have good solutions, or would not lead to (1.1), we would have to admit that the equations are inadequate and may need some adjustment, which would certainly be a paradigm shift. Formula (1.1) would survive such a shift. Similar considerations apply to formula (1.4).

In our situation this means the following. We have chosen the basic units to be L, T, M. However, this choice is quite arbitrary. Somebody else looking at this problem might choose the basic units of length, time and mass differently, say as L', T', M', and instead of the "pure numbers" ϱ, r, u, f, they will work with the "pure numbers" ϱ', r', u', f', defined analogously. The above postulate says that these two quadruples of pure numbers should both satisfy the same equation:

$$f = \phi(\varrho, r, u), \qquad \text{and} \qquad f' = \phi(\varrho', r', u'). \tag{1.11}$$

It is clear that this requirement puts a very strong restriction on the function ϕ. To see what it is explicitly, let us write

$$L = \lambda L', \quad T = \tau T', \quad M = \mu M', \tag{1.12}$$

where $\lambda, \tau, \mu > 0$. Then

$$\varrho' = \frac{\mu}{\lambda^3}\,\varrho, \quad r' = \lambda\, r, \quad u' = \frac{\lambda}{\tau}\,u, \quad f' = \frac{\mu\lambda}{\tau^2}\,f\,, \tag{1.13}$$

and (1.11) gives

$$\phi(\frac{\mu}{\lambda^3}\ \varrho, \lambda\, r,\ \frac{\lambda}{\tau}\,u) = \frac{\mu\lambda}{\tau^2}\phi(\varrho, r, u)\,, \quad \mu, \lambda, \tau > 0. \tag{1.14}$$

Setting

$$\frac{\mu}{\lambda^3} = \alpha, \quad \lambda = \beta, \quad \frac{\lambda}{\tau} = \gamma, \tag{1.15}$$

we can re-write (1.14) as

$$\phi(\,\alpha\varrho\,,\,\beta r\,,\,\gamma u\,) = \alpha\beta^2\gamma^2\phi(\varrho, r, u)\,, \quad \alpha, \beta, \gamma > 0, \tag{1.16}$$

which means that

$$\phi(\varrho, r, u) = \varrho\, r^2 u^2 \phi(1, 1, 1) = c\varrho\, r^2 u^2, \quad \varrho,\, r, u > 0\,, \tag{1.17}$$

with $c = \phi(1, 1, 1)$. This gives (1.1).

Exercise

Use dimensional analysis to derive formula (1.4). [5]

*Remark** [6]

The above considerations can be generalized as follows: assume that we have some physical quantities $X_1, \ldots, X_n$ which depend on m basic units $A_1, \ldots, A_m$. Let us think of $X_1, \ldots, X_n$ as numbers representing the quantities for a given choice of units. Then a new choice of units

$$A_j \to A'_j, \quad A_j = \lambda_j A'_j, \quad \lambda_j > 0, \quad j = 1, \ldots, m \tag{1.18}$$

[5] Note that the pressure is defined as force per unit area.

[6] In general, statements with * are optional and not essential for following the course.

can be though of as a scaling transformation

$$X_j \to X_j' = \lambda_1^{\alpha_{j1}} \lambda_2^{\alpha_{j2}} \dots \lambda_m^{\alpha_{jm}} X_j, \qquad j = 1, \dots, n\,, \tag{1.19}$$

on the $n-$tuple $X_1, \dots, X_n$, where the exponents α_{jk} are given by the relations between X_j and the basic units. Letting $X = (X_1, \dots, X_n)$, and $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{R}_+^m$, we can think of (1.19) as the action of the multiplicative group $\mathbf{R}_+^m$ on $\mathbf{R}^n$, and write it schematically as

$$X \to \lambda \cdot X = X'\,. \tag{1.20}$$

According to postulate (P), a physically meaningful relation

$$\phi(X) = \phi(X_1, \dots, X_n) = 0 \tag{1.21}$$

between the quantities should be invariant under the above group action:

$$\phi(X) = 0 \Longleftrightarrow \phi(\lambda \cdot X) = 0, \qquad \lambda \in \mathbf{R}_+^m\,. \tag{1.22}$$

Heuristically it is clear that this should put restrictions on ϕ. Under some non-degeneracy assumptions one can show that in the situation above one can define $l = n - m$ quantities $\pi_1 = \pi_1(X), \dots, \pi_l = \pi_l(X)$ which are "dimensionless", in the sense that $\lambda \cdot \pi_j = \pi_j, \quad j = 1, \dots, l$, and a function $\psi = \psi(\pi_1, \dots, \pi_l)$ such that

$$\phi(X_1, \dots, X_n) = 0 \quad \text{ is equivalent to } \quad \psi(\pi_1, \dots, \pi_l) = 0. \tag{1.23}$$

Variants of this statement are known as the $\pi-$theorem, formulated in general terms by E. Buckingham[7]. At this stage we need not to go into the details of this statement in its full generality. We will soon see some additional elementary examples.

[7]Buckingham, E. (1914) ,"On physically similar systems; illustrations of the use of dimensional equations". Physical Review 4 (4), 345-376. See also the Wikipedia entry for "Buckingham $\pi-$ theorem".

2

9/9/2011

Last time we talked about some phenomenological formulae, based on approaches where we do not try to describe the fluid motion in detail. Today we start describing a more fundamental approach based on imagining the fluid as a continuum which is deformed by smooth transformations. (An even more fundamental approach would be to start from the atomic picture of the fluid, but we will not pursue this direction here, we will not go beyond the continuum description.)

We will first be dealing with *kinematics*, which can be thought of as pure description of motion, without the consideration of the causes of motion, such as the forces. It essentially sets up a system for keeping track of the motion.

2.1 Eulerian and Lagrangian description of fluid motion

We assume that our fluid is contained in a domain[8] $\Omega \subset \mathbf{R}^3$. The domain can be unbounded, such as $\Omega = \mathbf{R}^3$ (all space) or $\Omega = \{x \in \mathbf{R}^3,\, x_3 > 0\}$ (half-space), or bounded, such as $\Omega = B_r = \{x,\, |x| < r\}$. The motion of the fluid in Ω is described by a velocity field $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ in Ω. The value of u at the point $x \in \Omega$ and time t is the velocity at which the "fluid particle at x" moves at time t. Of course, the notion of the fluid particle at x is somewhat problematic from the atomistic point of view[9], but here we are dealing with a continuum model. The question of justifying the continuum model from the atomistic point of view is of great importance, but we will not pursue it here. We will take the continuum model for granted.
The velocity field u will be usually assumed to be smooth (in both x and t), and well–defined up to the boundary $\partial\Omega$, which is assumed to be smooth. In fact, most of the time we will assume that u is smooth in $\overline{\Omega} \times [t_1, t_2]$, where $[t_1, t_2]$ is a closed time interval relevant for our considerations.
If we assume that no fluid passes through the boundary of the domain Ω and Ω is stationary, then we have the restriction

$$u(x,t)\, n(x) = 0, \qquad x \in \partial\Omega\,, \tag{2.1}$$

where $n(x) = (n_1(x), n_2(x), n_3(x))$ is the unit normal to the boundary. In fact, in viscous fluids[10] a good boundary condition for stationary "containers" Ω is

$$u(x,t) = 0\,, \qquad x \in \partial\Omega\,. \tag{2.2}$$

[8] By definition, a domain is a connected open set.

[9] For example, we implicitly assume that at each point of space there is some fluid particle.

[10] The precise meaning of viscosity will be defined later, but for now we can say that these are the fluids with some internal friction. All fluids one encounters in "everyday life" are viscous, even though the viscosity can be very small.

If the boundary $\partial\Omega$ is not stationary, the equations (2.1) and (2.2) have to be changed to

$$u(x,t)\, n(x,t) = v(x,t)\, n(x,t), \qquad x \in \partial\Omega_t \tag{2.3}$$

and

$$u(x,t) = v(x,t), \qquad x \in \partial\Omega_t \tag{2.4}$$

respectively, where $\partial\Omega_t$ indicates that the domain may be time-dependent (but does not have to be so), and $v(x,t)$ is the velocity of the point $x \in \partial\Omega_t$ at time t.

From the velocity field $u(x,t)$ we can reconstruct the trajectory of each particle by solving the ODE

$$\dot{x} = \frac{dx}{dt} = u(x,t), \qquad x(0) = \alpha \tag{2.5}$$

Here α is the position of the particle at time $t = 0$. The position of the particle α at time t will be denoted by $\phi^t(\alpha)$. Sometimes we will also use the notation $\alpha = x_0$, or even $\alpha = x$, when there is no danger of misunderstanding[11]. Assuming that Ω is stationary, for each $t \in \mathbf{R}$ the mapping $\alpha \to \phi^t(\alpha)$ is a diffeomorphism of Ω and, in fact, if $u(x,t)$ is smooth up to the boundary, it is a diffeomorphism of $\overline{\Omega}$, the closure of Ω. We note that ϕ^t maps the interior of Ω into itself and the boundary $\partial\Omega$ into itself. In particular, in this model the fluid particles from the interior of Ω never reach the boundary $\partial\Omega$, and the fluid particles from $\partial\Omega$ never leave $\partial\Omega$.
The particle trajectories are given by

$$t \to \phi^t(\alpha)\,. \tag{2.6}$$

The trajectory (2.6) passes through α at $t = 0$.

The description of the fluid motion in terms of the velocity field $u(x,t)$ was introduced by L.Euler around 1757, and is called the *Eulerian description*. The description in terms of the diffeomorhisms ϕ^t is called the *Lagrangian description*. The two descriptions are related through equation (2.5), which can be also written as

$$\dot{\phi}^t(\alpha) = u(\phi^t(\alpha), t)\,, \tag{2.7}$$

where, as above, $\dot{\phi}^t(\alpha) = \frac{d}{dt}\phi^t(\alpha)$.

In this description of fluid motion we still in principle track what is going on with each "fluid particle" in the continuum model. In comparison with the atomistic model, the simplification should be that the field $u(x,t)$ is smooth. We imagine that behind each "fluid particle" in the continuum model there is some averaging process involving a very large number of atoms of fluid, and we do not have to track the motions of these individual atoms. That should be

[11] Of course, one can write x_0 in place of α in (2.5), but writing x instead of α in (2.5) would not be good notation.

the source of a tremendous reduction in the number of "degrees of freedom" which we need to track. This will happen if the distances over which $u(x,t)$ and some of its derivatives change significantly are much longer than atomistic scales, which indeed seems to be the case. Still, in turbulent flows the field $u(x,t)$ changes very rapidly in both x and t, and for many common flows (such as the flow of air around a car going at 60 mph) it is so complicated that the full description of the field $u(x,t)$ is challenging even for the largest computers[12]

2.2 Acceleration

From Newton's formula force=mass $\times$ acceleration it is clear that acceleration of the fluid particles will play an important role. The formula for acceleration is straightforward in the Lagrangian description:

The acceleration along the particle trajectory $t \to \phi^t(\alpha)$ is simply

$$\ddot{\phi}^t(\alpha) = \frac{d^2}{dt^2}\phi^t(\alpha). \tag{2.8}$$

Denoting the acceleration of a particle at point x at time t by $a(x,t)$, we can write (2.8) as

$$a(\phi^t(\alpha),t) = \ddot{\phi}^t(\alpha)\,. \tag{2.9}$$

In the Eulerian description we can calculate the acceleration by taking the time derivative of equation (2.5). We obtain

$$\ddot{x}(t) = \frac{d^2}{dt^2}x(t) = \frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{\partial u}{\partial x_j}(x(t),t)\dot{x}_j(t)\ , \tag{2.10}$$

where we have used summation convention of summing over repeated indices. For example the expression $y_j z_j$ means $\sum_{j=1}^3 y_j z_j$. Expressing $\dot{x}(t)$ from (2.5), denoting $u_t = \frac{\partial u}{\partial t}$ and using the notation $u\nabla = u_j \frac{\partial}{\partial x_j}$, we can write

$$\ddot{x} = u_t + u\nabla u\,, \tag{2.11}$$

where the expression on the right-hand side is evaluated at the point $x(t)$ and time t, or

$$a(x,t) = u_t(x,t) + u(x,t)\nabla u(x,t)\,. \tag{2.12}$$

2.3 Free particles

We can use the formula for the acceleration to write down the equation of a fluid consisting of "free particles", with no interaction between them. Such fluids of course do not exists in the physical world, it is only a mathematical idealization,

[12]In practice this is addressed by additional averaging. The images of calculated fields of flows around cars which you have undoubtedly seen are in reality not the snapshots of the whole field $u(x,t)$, but rather of some of its averages. The calculation of $u(x,t)$ itself remains often beyond our possibilities.

but if we think in terms of a very fine dust of very small density set in a slow motion, and observing it during a time interval when no dust particles collide, then we can get a good picture.

The Lagrangian description is simple: the particles do not interact, and therefor each particle moves at a constant speed:

$$\phi^t(\alpha) = \alpha + tv(\alpha)\,, \tag{2.13}$$

where we labeled the particles by their position at time $t = 0$ (the label α is the same as the position at time $t = 0$), and $v(\alpha)$ is the velocity of the particle with the label α. As an exercise, you can prove that when the vector field $\alpha \to v(\alpha)$ is smooth and vanishes outside large ball, then there exist a time interval (t_1, t_2) with $t_1 < 0 < t_2$ such that (2.13) defines a diffeomorphism of $\mathbf{R}^3$ for $t \in (t_1, t_2)$.

In the Eulerian description, the vanishing of the acceleration $a(x, t)$ gives

$$u_t + u\nabla u = 0\ . \tag{2.14}$$

This is the *Burgers equation*. It is a non-linear equation, but because of the free-particle interpretation and the explicit solution in the Lagrangian description, we can understand the solutions quite well. If $u(x, 0 = v(x)$ with $v(x)$ smooth and vanishing outside a bounded set, then $u(x, t)$ is given by

$$u(x + v(x)t, t) = v(x) \tag{2.15}$$

in some time interval $t \in (t_1, t_2)$, where $t_1 < 0 < t_2$ are such that $x \to x + tv(x)$ is a diffeomorphism of $\mathbf{R}^3$ for $t \in (t_1, t_2)$. When $x' + \bar{t}\, v(x') = x'' + \bar{t}\, v(x'') = y$ for some $x' \neq x''$, then the smooth solution $u(x, t)$ cannot be continued up to time $\bar{t}$, as (2.15) gives two conflicting requirement for the value of $u(y, \bar{t})$, corresponding to the situation that two particles with different velocities reach the point y at time $\bar{t}$. We have here an example of a situation where a solution of a locally-in-time well-posed nonlinear evolution equation can break down after some time.[13]

2.4 Equation $f_t + u\nabla f = 0$.

Equation (2.14) is closely related to the linear transport equation

$$f_t + u(x, t)\nabla f = 0\,, \tag{2.16}$$

where the vector field $u(x, t)$ is considered as known and $f = f(x, t)$ is the unknown. The equation just says that the function f is constant along the trajectories of the vector field $u(x, t)$. If $\phi^t(\alpha)$ represent the Lagrangian description of the trajectories given by the vector field $u(x, t)$, then the solutions $f(x, t)$ of (2.16) satisfy

$$f(\phi^t(\alpha), t) = f(\alpha, 0)\,, \tag{2.17}$$

[13]The questions if the solution can in some meaningful sense be continued beyond the singularity has been studied in detail, see for example the book "Shock waves and reaction diffusion equations" by J. Smaller or a more recent book on hyperbolic conservation laws by C. Dafermos.

which can be used to determine f if the initial condition $f(x,0)$ is known. For example, of $\mathcal{O}_t$ is a region "moving with the fluid", that is $\mathcal{O}_t = \phi^t(\mathcal{O}_0)$, then the function $f(x,t) = \chi_{\mathcal{O}_t}(x)$ (the characteristic function of $\mathcal{O}_t$) should in some sense satisfy (2.16). (Some caution is needed as $\chi_{\mathcal{O}_t}$ is not classically differentiable. We will not go into details at this point. It is however clear that (2.17) is satisfied.)

2.5 The equation of continuity

The density $\rho(x,t)$ of the fluid is defined by the requirement

$$\int_{\mathcal{O}} \rho(x,t)\,dx = \text{mass of the fluid contained in } \mathcal{O} \text{ at time } t. \tag{2.18}$$

If the Lagrangian trajectories are known, then $\rho(x,t)$ is determined by $\rho(x,0)$ (and *vice versa*) from the formula

$$\rho(\phi^t(\alpha),t)\,\det\nabla\phi^t(\alpha) = \rho(\alpha,0) \tag{2.19}$$

which follows from definition (2.18) of ρ and the substitution formula

$$\int_{\mathcal{O}} \rho(\phi^t(\alpha),t)\det\nabla\phi^t(\alpha)\,d\alpha = \int_{\phi^t(\mathcal{O})} \rho(x,t)\,dx\,. \tag{2.20}$$

Formula (2.19) should be compared with formula (2.17). The Eulerian description counterpart of (2.19) is the *continuity equation*

$$\rho_t + \operatorname{div}(\rho u) = 0\,, \tag{2.21}$$

where

$$\operatorname{div} u = \frac{\partial u_j}{\partial x_j}, \tag{2.22}$$

with the summation convention understood. We will derive this equation directly in the Eulerian coordinates, but it can be also obtained directly from (2.19) by differentiating in t.[14]

[14] It is a good exercise to do this calculation. If you have not differentiated determinats before, it may look complicated, but it is worth learning how to handle this calculation. Do it first at $t = 0$. We remark that the differentiation of (2.19) gives (2.21) in the form $\rho_t + u\nabla\rho + \rho\operatorname{div} u = 0$, which is equivalent to (2.21) for sufficiently regular functions.

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9/12/2011

We continue to discuss the equation of continuity. The Eulerian description version (2.21) can be easily derived directly in the Eulerian description, without relying on (2.19). Let us consider an arbitrary smooth domain $\mathcal{O} \subset \Omega$, and let us consider the quantity

$$\frac{d}{dt} \int_{\mathcal{O}} \rho(x,t)\, dx = \int_{\mathcal{O}} \rho_t(x,t)\, dx\,. \tag{3.1}$$

(We assume that $\rho(x,t)$ is sufficiently regular.) This quantity represents the instantaneous rate of increase of mass of the fluid in $\mathcal{O}$. Any change in mass of the fluid in $\mathcal{O}$ can only be a result of a flux of the fluid through the boundary $\partial\mathcal{O}$ of the region $\mathcal{O}$. The instantaneous rate of flow of mass through $\partial\mathcal{O}$ is clearly

$$-\int_{\partial\mathcal{O}} \rho u\, n\, dx \;= -\int_{\partial\mathcal{O}} \rho(x,t) u(x,t) n(x)\, dx \tag{3.2}$$

where $n = n(x)$ is the outward unit normal to the boundary, $u\, n$ is the scalar product of u and n (which is a function of x, so we can also write $u(x,t)\, n(x) = u_j(x,t) n_j(x)$), and dx denotes the natural surface measure on $\partial\mathcal{O}$. The expressions (3.1) and (3.2) have to be equal:

$$\int_{\mathcal{O}} \rho_t(x,t)\, dx \;= -\int_{\partial\mathcal{O}} \rho(x,t) u(x,t) n(x)\, dx\,. \tag{3.3}$$

Recalling the Gauss formula

$$\int_{\partial\mathcal{O}} v(x) n(x)\, dx = \int_{\mathcal{O}} \operatorname{div} v(x)\, dx\ , \tag{3.4}$$

which is valid for any sufficiently regular vector field v, we see that we can rewrite (3.3) as

$$\int_{\mathcal{O}} \rho_t(x,t)\, dx + \int_{\mathcal{O}} \operatorname{div}(\rho u)(x,t)\, dx = 0\,. \tag{3.5}$$

Since the domain $\mathcal{O}$ was arbitrary, we see that

$$\rho_t + \operatorname{div}(\rho u) = 0\,. \tag{3.6}$$

The equation of continuity is "dual" to the transport equation $f_t + u\nabla f = 0$. [15] If $f = f(x,t)$ is a function compactly supported in Ω for each t satisfying the

[15] In general, a (formal) L^2- dual L^* of an operator L (in our case $L = \frac{\partial}{\partial t} + u(x,t)\nabla$ is defined by

$$\int\int L\varphi\overline{\psi}\, dx\, dt = \int\int \varphi\, \overline{L^*\psi}\, dx\, dt\,, \tag{3.7}$$

where $\varphi(x,t), \psi(x,t)$ are smooth, compactly supported functions. (The formula is written for complex-valued functions, with $\overline{\psi}$ denoting the complex conjugate.)

transport equation and if $\rho = \rho(x,t)$ satisfies the continuity equation (for the same vector field $u(x,t)$), then an easy calculation shows (see also (3.10) below)

$$\frac{d}{dt}\int_\Omega \rho f\, dx = 0\,, \tag{3.8}$$

where we use the shorthand notation $\int_\Omega \rho f\, dx = \int_\Omega \rho(x,t) f(x,t)\, dx$.

To illustrate the meaning of (3.8), we can consider the following special case. Let $\mathcal{O}$ be a (smooth) domain, and let $\mathcal{O}_t$ describe its motion with the flow. In other words, using the Lagrangian description (2.7), we have $\mathcal{O}_t = \phi^t(\mathcal{O})$. Let

$$f(x,t) = \chi_{\mathcal{O}_t}(x)\,. \tag{3.9}$$

Then we can think of $f(x,t)$ as a solution of the transport equation (2.16) (or (2.17)). Strictly speaking, the function f is not smooth and therefore one must be somewhat careful in interpreting the equation. This is a relatively minor technical detail which nevertheless must be taken care of in a rigorous treatment, but we will ignore it for now. For this choice of f, equation (3.8) says that the mass of fluid in $\mathcal{O}_t$, given by $\int_{\mathcal{O}_t} \rho(x,t)\, dx$ does not change with t, which is of course to be expected from the definition: the "fluid particles" in $\mathcal{O}_t$ are always the same.

If $\eta(x,t)$ is any sufficiently regular function (not necessarily satisfying the continuity equation), and $f(x,t)$ is compactly supported in Ω (for each t) and satisfies the transport equation (2.16), then at any time t we have

$$\frac{d}{dt}\int_\Omega \eta f\, dx = \int_\Omega (\eta_t f + \eta f_t)\, dx = \int_\Omega (\eta_t f - \eta u \nabla f)\, dx = \int_\Omega (\eta_t + \operatorname{div}(\eta u)) f\, dx\,. \tag{3.10}$$

Taking $f(x,t) = \chi_{\mathcal{O}_t}(x)$ as above, we obtain

$$\frac{d}{dt}\int_{\mathcal{O}_t} \eta(x,t)\, dx = \int_{\mathcal{O}_t} (\eta_t + \operatorname{div}(\eta u))\, dx = \int_{\mathcal{O}_t} \eta_t + \int_{\partial\mathcal{O}_t} \eta u\, n\, dx\ , \tag{3.11}$$

which is a useful formula for following quantities associated with "moving volumes".

Example.

Let us consider the motion of free particles described by the Burgers equation (2.14). We assume for simplicity that $\Omega = \mathbf{R}^3$ and that u is smooth vanishes outside a bounded set. (In particular, we consider the motion only during a times interval (t_1, t_2) when there are no collisions between the particles.) Let $\rho(x,t)$ be the density of the particles. The total kinetic energy of the particles is given by

$$\int_\Omega \rho(x,t)\frac{|u(x,t)|^2}{2}\, dx\,. \tag{3.12}$$

We expect that it will not change with time. This is obvious in the Lagrangian description (2.13). You can check by a direct calculation in the Eulerian variable that the time derivative of (3.12) vanishes. It can be also deduced from (3.8), since the function $f(x,t) = |u(x,t)|^2/2$ satisfies the transport equation. In fact, we can replace $|u|^2/2$ by $b(u) = b(u(x,t))$, where b is any function on $\mathbf{R}^3$. For the free particles the conservation of energy (and the more general quantities given by $b(u)$) can be localized to the "moving volumes": if $f(x,t)$ is a solution of the transport equation, then

$$\frac{d}{dt}\int \rho(x,t)b(u(x,t))f(x,t)\,dx = 0\,. \tag{3.13}$$

This can again be seen from (3.8), since the function $b(u(x,t))f(x,t)$ satisfies the transport equation. Taking $b(u) = u$ (the reader can check that there is no problem with vector-valued $b(u)$), we obtain the conservation of momemtum

$$\frac{d}{dt}\int \rho(x,t)u(x,t)f(x,t)\,dx = 0\,. \tag{3.14}$$

Of course, these properties of the motion of the free particles are not very deep and are to be expected, but going through these calculation is a good exercise in checking our formulae and in making sure that they express what we expect.

3.1 Vorticity

What makes the motion of fluids interesting is that they are easily deformed. The deformation of course arises when not all points move in the same way. We will look at this effect more closely in a small neighborhood of a fluid particle trajectory $y(t)$. We assume $\dot{y}(t) = u(y(t),t)$. Let us look at the motion of particles close to $y(t)$. For that purpose it is useful to follow the deformations in a system of coordinates moving with the point $y(t)$ (at speed $v(t) = \dot{y}(t) = u(y(t),t)$). We will denote these coordinates by $\tilde{x}$. We have, for each t,

$$\tilde{x} = x - y(t) \tag{3.15}$$

The velocity field in the coordinates $\tilde{x}$ is

$$\tilde{u}(\tilde{x},t) = u(\tilde{x}+y(t),t) - v(t) = u(\tilde{x}+y(t),t) - u(y(t),t)\,, \tag{3.16}$$

and the particle trajectories in the $\tilde{x}$ coordinates are given by

$$\frac{d}{dt}\tilde{x}(t) = \tilde{u}(\tilde{x},t)\,. \tag{3.17}$$

By the construction, we have $\tilde{u}(0,t) = 0$. Dropping the tildes, we see that we can reduce the study of the deformations of the fluid in a neighborhood of a given fluid particle (moving with the fluid) to the study of the deformations generated by

$$\dot{x} = u(x,t)\,, \tag{3.18}$$

where the field $u(x,t)$ now satisfies $u(0,t) = 0$ and we are interested in trajectories close to 0. In the first approximation it is reasonable to look at the linearization:

$$\dot{x} = \nabla u(0,t)x\,, \tag{3.19}$$

which we will write as

$$\dot{x} = A(t)x\,, \tag{3.20}$$

where $A(t)$ is the 3×3 matrix $\nabla u(0,t)$. We will now consider the special case when $A(t) = A$ is a constant matrix. We are therefore dealing with the simple constant-coefficient linear system

$$\dot{x} = Ax\,. \tag{3.21}$$

The solutions of the system are given by

$$x(t) = e^{tA}x_0\,, \tag{3.22}$$

where $x_0 = x(0)$. We wish to understand how the map $x \to e^{tA}x$ deforms the space. We will first consider two special cases:

Case I: A is anti-symmetric ($a_{ij} = -a_{ji}$).
Case II: A is symmetric ($a_{ij} = a_{ji}$).

This decomposition is relevant for our considerations concerning deformations since the anti-symmetric matrices generate the tangent space to the special orthogonal group $SO(3)$ at the identity, and the symmetric matrices generate a normal space to $SO(3)$ at the identity.

In suitable orthogonal coordinates, every anti-symmetric matrix looks like

$$A = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

In this case the map $x \to e^{tA}x$ represents the rotation by angle at about the x_3-axis:

$$e^{tA} = \begin{pmatrix} \cos at & -\sin at & 0 \\ \sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.24}$$

The fluid moves as a rigid body and is not deformed.

In suitable orthogonal coordinates, every symmetric matrix looks like

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{3.25}$$

In this case the map $x \to e^{tA}$ represents stretching or compression along the coordinate axis, depending on the sign of λ_j.

$$e^{tA} = \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix}. \tag{3.26}$$

When $A \neq 0$, this represents a real deformation of the fluid.

The general matrix A can be written as $A = A_{\text{sym}} + A_{\text{asym}}$ and we can think of $e^{tA}x$ as a result of applying succesively small rotations $e^{\tau A_{\text{asym}}}$ and small "pure deformations" $e^{\tau A_{\text{sym}}}$, as in the Trotter formula

$$e^{t(A+B)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}A}\, e^{\frac{t}{n}B} \right)^n. \tag{3.27}$$

We see that it is the symmetric part of A which is responsible for the real deformation, whereas the anti-symmetric part only rotates the fluid as a rigid body.

The case when $A = A(t)$ is similar: if $A(t)$ is anti-symmetric for each t, then the solution of (3.21) represents a rigid rotation. The symmetric part of $A(t)$ will be responsible for the deformations. This shows that when looking at the deformations of the fluid, the decomposition of $\nabla u(x,t)$ into the symmetric and anti-symmetric part should play an important role.

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Last time we looked at the matrix $\nabla u(x,t)$ (where ∇, as always, means ∇_x), and we saw that its symmetric part and and its anti-symmetric part each has a definite geometric meaning related to respectively the deformation or the rotation of infinitesimal volumes of fluid. In dimension $n = 3$ the anti-symmetric matrices can be identified with vectors by the formula

$$Ax = a \times x\,, \tag{4.1}$$

where $\times$ denotes the cross product. We can also write

$$a = (a_1, a_2, a_3) \longleftrightarrow A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}, \tag{4.2}$$

or

$$A_{ik} = \epsilon_{ijk} a_j \quad \text{and} \quad a_j = \frac{1}{2}\epsilon_{ijk} A_{ik}\,, \tag{4.3}$$

where ϵ_{ijk} is the Levi-Civita symbol, defined as $\epsilon_{ijk} = 1$ when ijk is an even permutation of 123, $\epsilon_{ijk} = -1$ when ijk is an odd permutation of 123, and $\epsilon_{ijk} = 0$ otherwise.[16]

In view of the above identification of the anti-symmetric matrices and vectors in dimension $n = 3$, we can identify the anti-symmetric part of the matrix ∇u with the vector

$$\omega = \operatorname{curl} u = \nabla \times u, \qquad \text{or} \qquad (\operatorname{curl} u)_i = \epsilon_{ijk}\partial_j u_k = \epsilon_{jik} u_{j,k}\,. \tag{4.4}$$

The vector field ω is called *vorticity*, and plays an important role in the study of fluid motion.

Note that the normalization is such that for $v(x) = a \times x$ we have $\operatorname{curl} v = 2a$. The vector $\operatorname{curl} u$ taken at a point x represents the axis of rotation for the infinitesimal rotation generated by the anti-symmetric part of $\nabla u(x)$, the rate of rotation being $\frac{1}{2}|\operatorname{curl} u|$.

The reader can check the following formulae[17]:

[16] In this notation the cross product is $(a \times x)_i = \epsilon_{ijk} a_j x_k$. Note that it is important to verify that these formulae work the same in any orthogonal coordinate system with the right orientation. This amounts to verifying that ϵ_{ijk} is a (pseudo-)tensor, which means that $q_{il} q_{jm} q_{kn} \epsilon_{lmn} = \epsilon_{ijk}$ for any matrix $\{q_{ij}\} \in SO(3)$. This is an example of the geometric counter-part of the principle we used in the first lecture that the physical formulae should be independent of the choice of units. In a similar way, formulae with a geometric meaning should be independent of the choice of coordinates.

[17] For verifying some of them the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ is useful.

$$\begin{aligned}
\operatorname{curl} \nabla f &= 0\,, && (4.5)\\
\operatorname{div} \operatorname{curl} u &= 0\,, && (4.6)\\
\operatorname{curl} \operatorname{curl} u &= -\Delta u + \nabla \operatorname{div} u\,, && (4.7)\\
\int_{\mathbf{R}^3} \left(|\operatorname{curl} u|^2 + |\operatorname{div} u|^2 \right)\, dx &= \int_{\mathbf{R}^3} |\nabla u|^2\, dx\,, \quad \nabla u \in L^2(\mathbf{R}^3). && (4.8)
\end{aligned}$$

We can see from (4.8) or (4.7) that a vector field in $\mathbf{R}^3$ vanishing near ∞ can be reconstructed from $\omega = \operatorname{curl} u$ and $\operatorname{div} u$.

For incompressible fluids we have $\operatorname{div} u = 0$ and hence the equations

$$\operatorname{curl} u = \omega, \qquad \operatorname{div} u = 0\ . \tag{4.9}$$

These equations are the same as the equations for static magnetic fields generated by steady electric currents. If we denote the magnetic field by B and the current by J, then the equations are

$$\operatorname{curl} B = \mu_0 J, \qquad \operatorname{div} B = 0\ , \tag{4.10}$$

where μ_0 is the magnetic constant, which can be taken $\mu_0 = 1$ in a suitable system of units. Therefore if you have seen pictures of magnetic fields around electrical wires, it can help you to imagine what u is if you know ω in (4.9).

The symmetric part of the matric ∇u will also play an important role. It is known as the *deformation tensor*, and in the literature it can be denoted by D, or by S, or by e_{ij}. For example, in the last notation we write

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\ . \tag{4.11}$$

You can verify that

$$\int_{\mathbf{R}^3} e_{ij} e_{ij}\, dx = \frac{1}{2} \int_{\mathbf{R}^3} \left(|\nabla u|^2 + |\operatorname{div} u|^2 \right)\, dx\,, \qquad \nabla u \in L^2(\mathbf{R}^3)\,. \tag{4.12}$$

Therefore in $\mathbf{R}^3$ the tensor e_{ij} also determines u uniquely if we assume suitable decay at ∞.[18]

4.1 The Cauchy Stress Tensor

We now begin the study of the dynamics and the relevant forces acting in the fluid. In a continuum the forces are usually described by "fields". The simplest

[18] In fact, if $e_{ij} = 0$, then $u(x) = Ax + b$ for some anti-symmetric matrix A. Intuitively this should be clear: if $e_{ij} = 0$ then there is no real deformation, and hence u is a velocity field of a rigid motion. A rigorous proof requires some thought, if you have not seen it before. It is a good (non-trivial) exercise.

example is a vector field $f(x,t)$, which represents a force density: the total force due to the field $f(x,t)$ on any subdomain $\mathcal{O} \subset \Omega$ (at time t) is

$$\int_{\mathcal{O}} f(x,t)\, dx\,. \tag{4.13}$$

For example, the force due to gravity is decribed by the field

$$f(x,t) = -\rho(x,t)g \tag{4.14}$$

where ρ is the density of the fluid, and g is the vector of the acceleration due to gravity.[19]

In addition to the "external" forces such as (4.14), the description of which is more or less obvious, we have to describe the macroscopic effect of the forces due to the interaction of the atoms of the of the fluid[20] we are considering. The basic assumption here is that the range of these forces is very small, and in the continuum description we have in mind here it can be taken to be zero. That means that if we think of a (smooth) domain $\mathcal{O} \subset \Omega$, then the net result of the interatomic forces acting on $\mathcal{O}$ can be thought of as a result of a force density acting only on the boundary of $\partial\mathcal{O}$ of $\mathcal{O}$. We postulate that the $i-$th component of the net force with which the complement of $\mathcal{O}$ acts on $\mathcal{O}$ through $\partial\mathcal{O}$ is given by

$$\int_{\partial\mathcal{O}} \tau_{ij}(x) n_j(x)\, dx\,, \tag{4.15}$$

where $\tau_{ij} = \tau_{ij}(x)$ is a two-tensor[21] ($\sim$ matrix field) in Ω and $n = (n_1, n_2, n_3)$ is the outward unit normal to $\partial\mathcal{O}$. If a time-dependent situation is considered, the tensor τ can also depend on t, of course. The tensor τ is called the *Cauchy stress tensor*.

Strictly speaking, the above reasoning concerning the nature of the interatomic forces justifies immediately only the assumption that the force is given by

$$\int_{\partial\mathcal{O}} \phi(x, n(x))\, dx \tag{4.16}$$

where $\phi(x,n)$ is a vector-valued function of $x \in \Omega$ and $n \in \mathbf{R}^3, |n| = 1$ with suitable transformation properties so that the expression of the force is independent of the choice coordinate system. It may not be instantaneously obvious that the expression $\phi_i(x,n)$ should be linear in the vector n. However, starting from the expression (4.16) one can actually derive that – under some natural assumptions – the dependence on n must be linear as in (4.15). We will not

[19] We wrote g as being independent of x, t, which is adequate for scales much smaller than the planetary scales. However, in general we can have $g = g(x,t)$.

[20] or elastic body

[21] We are always working in orthogonal frames and therefore we do not have to distinguish between covariant and contravariant tensors.

pursue this here, but we recommend the reader to think about this - it is a good exercise.[22]

We will now derive the conditions under which a continuum with Cauchy stress tensor τ and a force density f is in equilibrium. We recall that in the simpler situation of a rigid body on which we act by forces $F^{(1)}, \dots, F^{(m)}$ at the points $x^{(1)}, \dots, x^{(m)}$ respectively the equilibrium conditions are

a) the total force is zero

$$F^{(1)} + \cdots + F^{(m)} = 0\,, \tag{4.17}$$

and

b) the total moment of force is zero

$$F^{(1)} \times x^{(1)} + \cdots + F^{(m)} \times x^{(m)} = 0\ . \tag{4.18}$$

Note that once (4.17) is satisfied, then (4.18) is independent of the choice of the coordinate system.

In the situation with the continuum we are considering, the analogue of (4.17) is

$$\int_{\mathcal{O}} f\,dx + \int_{\partial\mathcal{O}} \tau \cdot n = 0, \qquad \text{for each smooth } \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega\,, \tag{4.19}$$

and the analogue of (4.18) is

$$\int_{\mathcal{O}} f \times x\,dx + \int_{\partial\mathcal{O}} (\tau \cdot n) \times x\,dx = 0, \qquad \text{for each smooth } \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega\,, \tag{4.20}$$

where we used an index-free notation for simplicity and keep in mind that f, τ, n depend on x. Using the notation

$$(\operatorname{div}\tau)_i = \frac{\partial \tau_{ij}}{\partial x_j} \tag{4.21}$$

we know from the Gauss theorem that

$$\int_{\partial\mathcal{O}} \tau \cdot n = \int_{\mathcal{O}} \operatorname{div}\tau \tag{4.22}$$

and hence (4.19) gives

$$\operatorname{div}\tau + f = 0\,. \tag{4.23}$$

In a similar way, we can write the surface integral in (4.20) as

$$\int_{\partial\mathcal{O}} \epsilon_{ijk}\tau_{jl}n_l x_k\,dx = \int_{\mathcal{O}} \epsilon_{ijk}\frac{\partial}{\partial x_l}\left(\tau_{jl}x_k\right) = \int_{\mathcal{O}} \epsilon_{ijk}\left(\frac{\partial \tau_{jl}}{\partial x_l}x_k + \tau_{jk}\right)\,. \tag{4.24}$$

[22] Perhaps moderately difficult if you are new to Continuum Mechanics.

Using (4.21) and substituting $-f_j$ for $\frac{\partial \tau_{jl}}{\partial x_l}$, we see that (4.20) gives

$$\int_{\mathcal{O}} \epsilon_{ijk}\tau_{jk} = 0\,, \qquad \text{for each smooth } \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega\,. \tag{4.25}$$

This clearly means τ is symmetric, i. e.

$$\tau_{ij} = \tau_{ji} \quad \text{in } \Omega\,. \tag{4.26}$$

Hence the conditions for equilibrium are given by (4.23) and (4.26).

We have worked under the assumption that the continuum is at equilibrium, which may at first seem somewhat restrictive. The key point here the classical d'Alembert's principle: even if a system is in motion, it can be considered as being in equilibrium if we take into account the inertial forces due to acceleration.

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5.1 Cauchy stress tensor in fluids

By definition, an *ideal fluid* is a continuum in which the Cauchy stress tensor always has the form

$$\tau_{ij} = -p\,\delta_{ij}\,, \tag{5.1}$$

where p can be a function of x and t. We emphasize that this is assumed to be correct even when the fluid is in motion. We will see later that in most "real fluids" there is an additional stress which arises when the fluid moves, the so called *viscous stress*. This will be discussed in some detail later, but we can state even at this point that this additional stress arises only when the fluid moves. For a fluid which is at rest, the Cauchy stress is always given by (5.1). The function p is called the pressure, and its physical dimension is force per unit area, or $ML^{-1}T^{-2}$ in terms of the units of mass, length and time.

The above definition expresses the observation that the fluids cannot resist any "shear stresses"[23] (unlike, say, elastic bodies). In the ideal fluids this property is extended to the dynamical regime.[24]

In the real fluids the pressure is related to the other thermodynamical quantities, the main ones being (in addition to p) the density ρ, and the temperature T, [25] and we typically assume relations such as $p = p(\rho, T)$.[26] Sometimes the models disregarding T and assuming only $p = p(\rho)$ [27], which is often considered for compressible fluids, can work quite well.

For many fluids it is reasonable to make the assumption that they are incompressible.[28] In this case the Lagrangian mappings ϕ^t we considered in lecture 2 satisfy $\det \nabla \phi^t = 1$ in Ω for each t, and the velocity field $u(x,t)$ satisfies the constraint

$$\operatorname{div} u = 0 \quad \text{in } \Omega. \tag{5.2}$$

[23] A typical situation when a shear stress arises is the following: glue a cube of a material to a surface and then act on it by a force parallel to the surface. See also the Wikipedia entry for Shear Stress.

[24] This produces some unrealistic effects, as we will see, but the model still gives a good picture for many phenomena.

[25] In more complicated fluids one also has to consider additional quantities such as concentration of various components, for example.

[26] These relations are often written in a form which does not satisfy the principle (P) from Lecture 1. For example, one often writes (for simplified models) $p = p(\rho)$, whereas what one really means is $\frac{p}{p_0} = \phi(\frac{\rho}{\rho_0})$ where p_0 and ρ_0 are some normalized values of the pressure and density respectively, which may depend on the choice of units.

[27] see the previous footnote

[28] Incompressible fluids which are *homogeneous* (= the same at each point) have constant density, i. e. $\rho(x,t) = \rho_0 = \text{const.}$ in Ω. One can also consider incompressible fluids with non-constant density, such as a mixture of water and, say, ethanol, with concentration which depends on x. In these notes we will mostly concentrate on the former situation with $\rho = \text{const.}$

In this model the pressure depends only on the inertial forces (and the external forces, if present and are not div-free), and it is simply whatever is necessary so that the equations of motion keep the constraint. The situation is similar to what we have when a particle moves while being restricted to a ideally rigid circle. In that case the forces on the particle which are perpendicular to the circle have always exactly the right size to keep the particle on the circle. The pressure in incompressible fluids is similar. It arises as a result of the incompressibility constraint and is unrelated to other thermodynamical quantities. (The density ρ will not be a thermodynamical quantity either.) In fact, in the incompressible model the pressure is typically not even uniquely defined, it is only defined for each time up to a constant. We can change $p(x,t)$ to $p(x,t) + f(t)$ without any effect on anything else. When dealing with issues arising in this context, it is useful keep in mind that all real fluids are compressible, and that the incompressible model is really an idealized limiting case of the situation when the dependence of p on ρ is very "steep". It is a similar idealization as the concept of a rigid body.

One drawback of the incompressible model is that it becomes unrealistic when the pressure becomes negative. While typical real fluids can withstand large positive pressures without much compression, the situation is different for negative pressures. For example, at low pressure water begins to boil even at the room temperature, and does not "resists" the negative pressures in a way which is predicted by the incompressibility condition. In many situations the pressure is of the form

$$p = p_{\text{atm}} + p' \,, \tag{5.3}$$

where p_{atm} is the atmospheric pressure and p' is not negative enough for the above effect to become significant. In those cases the incompressible model works well. In other cases the additional thermodynamical effects of the low pressure have to be taken in account, such as in the phenomenon of cavitation, which can become a serious concern in pumps, turbines, propellers and other devices which create flows where the pressure becomes low.

The incompressible model has other drawbacks which we may mention later as we get into details of certain phenomena, but overall it works very well in many situations and is widely used. It is as with other mathematical models: they work well in the situations for which they were designed, but one should know their limitations.

5.2 Hydrostatics

Hydrostatics deals with fluids which are not moving and – unlike hydrodynamics (dealing with the moving fluids) – it is quite simple. Assuming the density of the forces is $f(x) = (f_1(x), f_2(x), f_3(x))$ and the stress tensor is $\tau_{ij} = -p(x)\delta_{ij}$, the equilibrium equations (4.23) become

$$\nabla p = f \,. \tag{5.4}$$

This means that the non-moving fluid can be at equilibrium only when the (volume) forces acting on it are potential forces (i. e. arise as a gradient of a function). For incompressible fluids the situation is particularly simple. In that case the pressure is uncoupled from the other quantities describing the state of the fluid, and if $f = \nabla\phi$, the solution of (5.4) is trivial: we simply set

$$p = \phi + \text{const.}, \tag{5.5}$$

where the constant can be arbitrary. (It can be fixed by demanding that the pressure has a specific value at some point in the fluid, for example.)

Let us consider some simple examples

Example 1 (Incompressible fluid in a constant field of gravity.)

Assuming that the fluid occupied a domain Ω and $f = -\rho_0 g e_3$ (with $e_3 = (0, 0, 1)$), the solution of (5.4) clearly is

$$p = -\rho_0 g x_3 + \text{const.}\,, \tag{5.6}$$

confirming the well-known elementary formula. If a body $\mathcal{O}$ is submerged into the fluid, the resulting force due to the pressure on the body will be

$$F = \int_{\partial\mathcal{O}} -p(x)n(x)\,dx = |\mathcal{O}|\,\rho_0 g\,e_3\,, \tag{5.7}$$

where $|\mathcal{O}|$ denotes the volume of $\mathcal{O}$. This is the Archimedes' principle, which can also be seen without calculation, by replacing the immersed body $\mathcal{O}$ with the fluid and using the fact that the fluid can be at rest when we do that.

Example 2 (Incompressible fluid in a closed container undergoing acceleration.)

Let us assume we have an ideally rigid container Ω which is closed and completely filled with incompressible fluid of constant density ρ_0. Assume the boundary of the container moves as

$$x \to x + b(t)\,, \tag{5.8}$$

where $b(t) = (b_1(t), b_2(t), b_3(t))$ is any (sufficiently regular) function of time.[29] How will the fluid move? The answer is simple: the motion of the fluid will be given again by(5.8). In other words, the container and the fluid in it will move together as a rigid body. There will be no mixing of the fluid. This is easily seen if we view the situation from the coordinate system of the container. In that system the container is stationary and the fluid is subject to inertial forces of the form

$$f = -\ddot{b}(t)\rho_0\,. \tag{5.9}$$

[29] You can imagine that we shake the container, but only by translational motions, with no rotations.

Equation (5.4) can be easily solved (in the coordinate system moving with the container) for each t by letting

$$p(y,t) = -y_j \ddot{b}_j(t)\rho_0 + c(t)\,, \tag{5.10}$$

where $c(t)$ is any function of t, and y are the coordinates in the system of the container. The fluid is incompressible, and therefore the pressure will have no visible effect on it.

Examples 1 and 2 can be used to explain the following experiment.[30] Assume that you are standing in bus which is going at relatively high speed and you hold in your hand a party balloon filled with helium. To be precise, you hold a string to which the balloon is attached and the balloon floats in the air, pulling slightly up on the string. If the driver steps on the breaks and you manage to stay in your position, how will the balloon move? For the purpose of this experiment it is reasonable to assume that the air is incompressible.[31]

Example 3 (Exponential atmosphere)

Let us assume the force of gravity on air is given by the force density $-\rho(x)ge_3$, where $\rho(x)$ is the air density, and g is the acceleration due to gravity, which is assumed to be constant. Let us assume that the air is at constant temperature and follows the ideal gas law which in this situation predicts

$$\frac{p}{p_0} = \frac{\rho}{\rho_0}\,, \tag{5.11}$$

where p_0, ρ_0 are respectively the pressure and the density at a reference position, which we will take to be the surface $x_3 = 0$. Assume the air is not moving. How will p and ρ depend on x_3?

We see from equation (5.4) that p can only depend on x_3. Hence, using (5.11), we see that $p = p(x_3), \rho = \rho(x_3)$ and

$$\frac{p_0}{\rho_0}\rho' = -\rho\, g\,. \tag{5.12}$$

Elementary integration now gives

$$p(x_3) = p_0\, e^{-\frac{\rho_0}{p_0} g x_3}\,. \tag{5.13}$$

5.3 The equations of motion

Let us now consider the equations of motion for ideal fluids, for which we already know all the forces acting in it. By d'Alembert's principle, the equation motion

[30] I thank to David V., a high-school student in Prague for bringing this experiment to my attention.

[31] If you think about this situation using Examples 1 and 2 above, you will see that the balloon will move in backwards.

are obtained if we add the inertial forces to the equilibrium equations (4.23). The inertial forces are given by

$$-\rho(x,t)a(x,t)\,, \tag{5.14}$$

where $a(x,t)$ is the acceleration, which can be expressed in terms of $u(x,t)$ by (2.12). We obtain

$$\rho u_t + \rho u \nabla u + \nabla p = f\,, \tag{5.15}$$

where $f = f(x,t)$ describes "external forces". For incompressible fluids we have $\operatorname{div} u = 0$. If ρ is constant, $\rho = \rho_0$, then we get a closed system of equations

$$\begin{aligned} \rho_0 u_t + \rho_0 u \nabla u + \nabla p &= f(x,t)\ , & (5.16)\\ \operatorname{div} u &= 0 & (5.17) \end{aligned}$$

for the unknown functions $u(x,t), p(x,t)$. These are the *incompressible Euler's equations*, derived by L. Euler around 1757.

When the fluid is compressible, we can add the equation of continuity (3.6) to the equation (5.15), but we still need one more equation to close the system, as now both $p(x,t)$ and $\rho(x,t)$ are unknown. The simplest way to close the system is to assume a thermodynamical relation $p = p(\rho)$. This way we get the so-called isentropic Euler's equations

$$\begin{aligned} \rho u_t + \rho u \nabla u + \nabla p &= f\ , & (5.18)\\ \rho_t + \operatorname{div}(\rho u) &= 0\,, & (5.19)\\ p &= p(\rho)\,. & (5.20) \end{aligned}$$

This is a good model in many cases, although it sometimes oversimplifies temperature effects.[32] We also remark that – as we already mentioned in a previous footnote (page 21) – writing $p = p(\rho)$ does not conform to the principle (P) from lecture 1, and one should really write $p = p_0\phi(\frac{\rho}{\rho_0})$. We will follow the custom and write $p = p(\rho)$, even though that one should really write $p = p_0\phi(\frac{\rho}{\rho_0})$ to have the principle (P). Similar practice is sometimes referred to as saying that we assume that the equations are already "non-dimensionalized".

To get a feeling for the system (5.18)-(5.20), we will derive its linearization about the trivial solution $\rho = \rho_0, p = p_0 = p(\rho_0), u = 0$ representing a fluid at constant density at rest, with no forces acting on it. We assume that the trivial solution is perturbed to a smooth solution close to it as

$$\begin{aligned} \rho_0 &\to \rho = \rho_0 + \varepsilon\eta + O(\varepsilon^2), & (5.21)\\ p_0 &\to p = p_0 + p'(\rho_0)\varepsilon\eta + O(\varepsilon^2), & (5.22)\\ u = 0 &\to u = \varepsilon v + O(\varepsilon^2)\,, & (5.23)\\ f = 0 &\to \varepsilon f\,. & (5.24) \end{aligned}$$

[32] The temperature is not present in the above equations, so that we really assume that either the temperature is constant (one extreme case) or that the processes are "locally adiabatic", with no heat exchange between the fluid particles (the other extreme case).

where $O(\varepsilon^2)$ represents terms of order ε^2. An easy calculation shows that in the limit $\varepsilon \to 0$ we get the following equations

$$\rho_0 v_t + p'(\rho_0)\nabla\eta = f \tag{5.25}$$
$$\eta_t + \rho_0 \operatorname{div} v = 0\,. \tag{5.26}$$

Taking the divergence of the first equation and subtracting from it the time derivative of the second equation we obtain

$$-\eta_{tt} + c^2\Delta\eta = \operatorname{div} f \tag{5.27}$$

with

$$c = \sqrt{p'(\rho_0)}\,. \tag{5.28}$$

This is the wave equation for waves with speed of propagation c. Once we know η (from solving (5.27)), we can calculate v from (5.25). The linearized equations are a good approximation for propagation of sound in a regime when the non-linear effects of the full equations can be neglected, which is practically any sound which we can comfortably listen to.[33] At this point this should be taken as an empirical statement. A rigorous mathematical investigation of the relation between the solutions of the linearized equations and system (5.18)–(5.20) is non-trivial.[34]

[33]The sound level in the usual units of dB is defined as $20\log_{10}(\frac{q}{q_{\text{ref}}})$, where q measures the size of the oscillations of the pressure and $q_{\text{ref}} = 2\cdot 10^{-5}$ Pa in the SI unit system. From this you can see that the amplitude of the pressure oscillations in usual sounds is small. The velocity v of the particles of air will be of the order $\frac{q}{\rho_0 c}$, which is also very small (as opposed to the speed of propagation c).

[34]Moreover, the "real equations" may not be exactly (5.18)– (5.20), as the derivation of this model already involved some idealized assumptions about the behavior of air.

6

9/19/2011

6.1 Helmholtz decomposition

Let us now look at the linearization of the incompressible model (5.16)–(5.17) at $u = 0$. We assume that an incompressible fluid of constant density ρ_0 in a container Ω is at rest and we apply infinitesimally small forces $f(x,t)$ to it. The domain Ω is assumed to be smooth and bounded for simplicity. We also consider the boundary condition

$$u(x,t)n(x) = 0 \qquad \text{at } \partial\Omega\,. \tag{6.1}$$

The linearization of (5.16)–(5.17) is

$$\begin{aligned} \rho_0 v_t + \nabla p &= f(x,t) && (6.2)\\ \operatorname{div} v &= 0\,, && (6.3) \end{aligned}$$

together with the boundary condition $v(x,t)n(x) = 0$. If the initial velocity $u(x,0)$ vanishes, then at time $t = 0$ this equation (with $v = u$) is valid exactly for (5.16)–(5.17), as the term $u\nabla u$ vanishes at $t = 0$ in that case. We look at (6.2) for a fixed time, say $t = 0$. The right-hand side of (6.2) is a general ("sufficiently regular") vector field in Ω. Each term on the left-hand side is special: the vector field $g = \rho_0 v_t$ is div-free and satisfies the boundary condition $g\,n = 0$ at $\partial\Omega$. The vector field ∇p is a gradient field. Only the field g will be responsible for accelerating the fluid. The field ∇p has no visible effect on the fluid, due to the assumption of incompressibility. In some sense, the fluid will decompose the general force f for us into two special forces: a gradient force ∇p and a div-free force g satisfying the boundary condition $g\,n = 0$. This decomposition is known as the *Helmholtz decomposition*.

The situation is quite similar to the following finite-dimensional scenario. Assume we have a plane $\Sigma \subset \mathbf{R}^3$ and a particle with coordinates x which is constrained to the plane. The constraint is assumed to be ideally rigid. Within the plane Σ the particle can move freely. Assume the particle is at rest and let us act on it by a force F which can be any vector in $\mathbf{R}^3$. We can decompose the force as

$$F = F^{\perp} + F^{||}\,, \tag{6.4}$$

where $F^{\perp}$ is perpendicular to Σ and $F^{||}$ is parallel to Σ. The force $F^{\perp}$ has no effect on the particle, it is exactly countered by the forces responsible for the constraint. The force $F^{||}$ will cause the particle to accelerate in the plane Σ, with the acceleration $a^{||}$ satisfying $ma^{||} = F^{||}$, where m is the mass of the particle. In the context of (6.2), the role of F is played by f, the role of $F^{\perp}$ is played by ∇p, and the role of $F^{||}$ is played by $\rho_0 v_t$.

The above analogy is even more complete: we can introduce a scalar product of the vector fields in Ω as

$$(f, g) = \int_\Omega fg\, dx = \int_\Omega f_i(x) g_i(x)\, dx. \tag{6.5}$$

Let us now take a smooth div-free vector field g satisfying $g\, n = 0$ at $\partial\Omega$, and a gradient field $\nabla\varphi$, where φ is any smooth function. We have

$$(g, \nabla\varphi) = \int_\Omega g\nabla\varphi = \int_{\partial\Omega} (g\, n)\varphi - \int_\Omega (\operatorname{div} g)\varphi = 0\,. \tag{6.6}$$

In other words if we let

$$X = \{g\colon \Omega \to \mathbf{R}^3,\ g \text{ is smooth, } \operatorname{div} g = 0 \text{ and } g\, n = 0 \text{ at } \partial\Omega\} \tag{6.7}$$

and

$$Y = \{\nabla\varphi,\, \varphi\colon \Omega \to \mathbf{R} \text{ is a smooth}\}\,, \tag{6.8}$$

then these two linear spaces are perpendicular to each other with respect to the scalar product (6.5). Let us denote, as usual, by $L^2(\Omega, \mathbf{R}^3)$ the Hilbert space of functions which is obtained by taking the completion of the smooth vector fields in Ω with respect to the norm $||f|| = \sqrt{(f, f)}$. Let $\overline{X}$ be the closure of X is $L^2(\Omega, \mathbf{R}^3)$, and let $\overline{Y}$ be the closure of Y in $L^2(\Omega, \mathbf{R}^3)$. Then clearly $\overline{X}$ is perpendicular to $\overline{Y}$, by the continuity of the scalar product. In fact, we have the following result:

Theorem 1 (Helmholtz decomposition)

$$L^2(\Omega, \mathbf{R}^3) = \overline{X} \oplus \overline{Y}\ . \tag{6.9}$$

In addition with the easy statement that $\overline{X}$ is perpendicular to $\overline{Y}$, this contains the less obvious claim that every $f \in L^2(\Omega, \mathbf{R}^3)$ can be written as a sum $f = g + h$, with $g \in \overline{X}$ and $h \in \overline{Y}$, exactly as suggested by the behavior of incompressible fluids in (6.2). This is in complete analogy with the obvious decomposition $\mathbf{R}^3 = \Sigma \oplus \Sigma^\perp$ in the example with F above.

We will not go into the proof of the theorem,[35] but we will mention a method for calculating the decomposition. Assume for simplicity that f is smooth. The domain Ω is also assumed to be smooth. Assuming $f = g + \nabla\varphi$ and taking div, we obtain $\Delta\varphi = \operatorname{div} f$, with the boundary condition $n\nabla\varphi = nf$ at $\partial\Omega$. We solve this problem for φ and then set $g = f - \nabla\varphi$.

The problem of finding the decomposition $f = g + \nabla\varphi$ can be also formulated as a variational problem: *Minimize the functional* $\varphi \to \int_\Omega |f - \nabla\varphi|^2\, dx$ *over the space of all functions* φ *with* $\nabla\varphi \in L^2(\Omega)$.

[35] The proof follows from standard considerations in the theory of the Laplace equation.

This is again similar to the example with F above, where one can find the decomposition (6.4) by minimizing $H \to |F - H|^2$ over $H \in \Sigma^\perp$.

All the above remains true, in one form or another, for unbounded domains, although one has to adjust the exact definitions of the spaces. The definitions above are not satisfactory for unbounded domains since they do not guarantee that all the integrals converge.

6.2 Transport of vector fields

We start with some definitions from elementary differential geometry, which will be very useful for the description of some laws of fluid motion.

Let $v = v(x)$ be a vector field in Ω and let $\phi\colon \Omega \to \Omega$ be a diffeomorphism. The *push forward* $\phi_* v$ of v by ϕ is defined by

$$\phi_* v(x) = \nabla\phi(\phi^{-1}(x))\, v(\phi^{-1}(x))\,, \tag{6.10}$$

or, equivalently,

$$\phi_* v(\phi(x)) = \nabla\phi(x) v(x)\,. \tag{6.11}$$

The meaning of the definition can be illustrated by considering the one-parameter group of diffeomorphisms ψ^t generated by the vector field v. Recall that $\psi^t(\alpha)$ is the solution of $\dot{x} = v(x)$ with $x(0) = \alpha$. Since $v(x)$ is independent of t, we have $\psi^t \circ \psi^s = \psi^{t+s}$. The notation $\psi^t = \exp tv$ is often used in this situation. (Warning: the Lagrangian maps ϕ^t we discussed in lecture 2 do not form a one-parameter group of diffeomorphisms in general, since the generating vector field $u(x,t)$ may depend on t.)

If ψ^t is a one-parameter group of diffeomophisms and ϕ is a given diffeomorphism, then $\phi\circ\psi^t\circ\phi^{-1}$ is clearly also a one parameter family of diffeomorphisms. As an execise, you can check the formula

$$\exp\left(t\phi_* v\right) = \phi \circ \psi^t \circ \phi^{-1}\,, \tag{6.12}$$

which gives a good idea about the geometric meaning of $\phi_* v$.

Another way to look at the situation is to consider ϕ as a change of coordinates. If $x = (x^1, \dots, x^n)$ are the original coordinates and $y = \phi(x)$ are the new coordinates, then the coordinates v^i of a vector field in the old coordinates will transform as to

$$w^i = v^j \frac{\partial y^i}{\partial x^j}, \tag{6.13}$$

where the value of w^i is taken at y and the values of the quantities on the right-hand side are taken at x. This is the same as (6.11). We have used the convention of writing contra-variant vectors with upper indices, which one should do once working with general coordinates.

Consider now two vector fields u, v. The *Lie bracket* $w = [u, v]$ of the two vector fields is defined by

$$w_i = u_j v_{i,j} - v_j u_{i,j} \,. \tag{6.14}$$

In the orthodox notation we should really write the vector fields with upper indices and $w^i = u_j \frac{\partial v^i}{\partial x^j} - v_j \frac{\partial u^i}{\partial x^j}$, but for our purposes here it is acceptable to use the notation in (6.14). An important property of the Lie bracket, which can be verified by direct calculation, is

$$\phi_* [u, v] = [\phi_* u, \phi_* v] \,. \tag{6.15}$$

One way to look at this formula is that it shows that the Lie bracket is a "geometric object", independent of the choice of any general coordinates given by $y = \phi(x)$.

The Lie bracket measures how far the fields u, v are from "commuting" in the sense the the derivatives $u\nabla$ and $v\nabla$ commute, or in the sense the one parameter groups $\theta^s = \exp su$ and $\psi^t = \exp tv$ commute. As an exercise, you can check the following formula[36]

$$\psi^{-t} \circ \theta^{-s} \circ \psi^t \circ \theta^s(x) = x + st[u, v](x) + O(|s|^3 + |t|^3) \,, \tag{6.17}$$

which also gives a good idea about the nature of $[u, v]$ (and also implies (6.15)). In fact, it can be shown that if $[u, v] = 0$, then the left-hand side of (6.17) vanishes, and the one-parameter groups θ^s, ψ^t commute (so that we have a commutative two-parameter group of diffeomorphisms).[37]

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[36] It may require some patience. The most direct way to prove the formula is the following. We note that the Taylor expansion gives

$$\theta^s(x) = x + su(x) + \frac{s^2}{2} u\nabla u(x) + O(|s|^3) \,, \qquad \psi^t(x) = x + tv(x) + \frac{t^2}{2} v\nabla v(x) + O(|t|^3) \,, \tag{6.16}$$

where we used formula (2.12) for the second time derivative. This means

$$\begin{aligned} \psi^t \circ \theta^s(x) \quad = \quad & x + su(x) + \frac{s^2}{2} u\nabla u(x) + O(|s|^3) + \\ & + tv(x + su(x) + \frac{s^2}{2} u\nabla u(x) + O(|s|^3)) + \\ & + \frac{t^2}{2} (v\nabla v)(x + su(x) + \frac{s^2}{2} u\nabla u(x) + O(|s|^3)) + O(|s|^3 + |t|^3) \,. \end{aligned}$$

In this expression we can drop a number of terms which only contribute $O(|s|^3 + |t|^3)$ and obtain

$$\begin{aligned} \psi^t \circ \theta^s(x) \quad &= \quad x + su(x) + \frac{s^2}{2} u\nabla u(x) + tv(x + su(x)) + \frac{t^2}{2} v\nabla v(x) + O(|s|^3 + |t|^3) \\ &= \quad x + su(x) + \frac{s^2}{2} u\nabla u(x) + tv(x) + stv\nabla u(x) + \frac{t^2}{2} v\nabla v(x) + O(|s|^3 + |t|^3) \,. \end{aligned}$$

Continuing this procedure by two more steps, we obtain the result.

[37] The proof is not hard but we will not pursue it at this point. We may return to this topic later.

Consider a fluid flow with velocity field $u(x,t)$ in some domain Ω, and assume its Lagrangian description is given by $\phi^t(\alpha)$, see lecture 2, section 2.1. Let $\overline{v} = \overline{v}(x)$ be a fixed vector field (independent of t). Consider the time-dependent vector field

$$v(x,t) = \phi^t_* \overline{v}(x)\,. \tag{6.18}$$

Lemma 1

With the notation introduced above, the vector field $v(x,t)$ given by (6.18) satisfies the equation

$$v_t + [u,v] = 0\,. \tag{6.19}$$

Vice versa, if a smooth vector field v satisfies (6.19) and $v(x,0) = \overline{v}(x)$, then $v(x,t)$ is given by (6.18).

Proof

The first statement can be verified by taking the time derivative of (6.18). To prove the second statement, assume that $v(x,t)$ satisfies (6.19) with the initial condition $v(x,t) = \overline{v}(x)$. In the class of the smooth vector fields where we are working, equation (6.19) is of the form $v_t + u\nabla v + A(x,t)v = 0$ for a suitable matrix $A(x,t)$. The solution of this equation reduces to integration of ODEs along the characteristics (given by the fluid particle trajectories $t \to \phi^t(\alpha)$) and hence the solution is unique. At the same time, formula (6.18) already provides a solution. Due to the uniqueness, our solution $v(x,t)$ has to coincide with the one provided by the formula.

The equation (6.19) can be considered as a vector field analogue of the transport equation (2.16) and formula (6.17) can be considered as a vector field analogue of (2.17).

7

9/21/2011

7.1 Evolution of vorticity and the Helmholtz law

Let us consider the momentum part of Euler's equations (compressible or incompressible) in the form

$$u_t + u\nabla u + \frac{\nabla p}{\rho} = f(x,t)\,. \tag{7.1}$$

In the compressible case we assume $p = p(\rho)$, in the incompressible case we assume $\rho = \rho_0 = \text{const}$.
We can write

$$(u\nabla u)_i = u_j u_{i,j} = u_j(u_{i,j} - u_{j,i}) + u_j u_{j,i}\,. \tag{7.2}$$

By inspecting the definitions in lecture 4 (see e. g. (4.4)), and denoting $\omega = \operatorname{curl} u$ as usual, we see that (7.2) is the same as

$$u\nabla u = \omega \times u + \nabla\,\frac{|u|^2}{2}\,. \tag{7.3}$$

We also note that in the compressible case we can write

$$\frac{\nabla p}{\rho} = \frac{p'(\rho)}{\rho}\nabla\rho = \nabla P \tag{7.4}$$

where $P(x,t) = \tilde{P}(\rho(x,t))$, with $\tilde{P}$ being a primitive of the function $\rho \to \frac{p'(\rho)}{\rho}$. Hence we can write

$$u_t + \omega \times u + \nabla\left(\frac{|u|^2}{2} + P\right) = f\ , \qquad \text{compressible case} \tag{7.5}$$

and

$$u_t + \omega \times u + \nabla\left(\frac{|u|^2}{2} + \frac{p}{\rho_0}\right) = f\ , \qquad \text{incompressible case.} \tag{7.6}$$

We now recall the standard formula

$$\operatorname{curl}(a \times b) = -a\nabla b + b\nabla a + a\operatorname{div} b - b\operatorname{div} a\,, \tag{7.7}$$

which can be easily checked from the definitions in lecture 4. For example, you can use (4.4) together with

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{il}\delta_{km} - \delta_{jm}\delta_{kl} \tag{7.8}$$

to obtain (7.7). Formula (7.7) can also be written in terms of the Lie bracket

$$\operatorname{curl}(a \times b) = [b,a] + a\operatorname{div} b - b\operatorname{div} a\,. \tag{7.9}$$

Let us now assume that the force $f(x,t)$ is *potential*, in the sense that

$$f(x,t) = \nabla\phi(x,t) \tag{7.10}$$

for a suitable scalar function $\phi(x,t)$. [38] We can take curl of (7.5) and use (7.9) (keeping in mind that $\operatorname{div}\omega = 0$) to obtain

$$\omega_t + [u,\omega] + \omega \operatorname{div} u = 0\,, \qquad \text{compressible case.} \tag{7.11}$$

In the same way we obtain from (7.6) and the constraint $\operatorname{div} u = 0$

$$\omega_t + [u,\omega] = 0\,, \qquad \text{incompressible case.} \tag{7.12}$$

The last equation is familiar to us from Lemma 1 in lecture 6. From the lemma we can therefore obtain the following fundamental result.

Theorem (Helmholtz's vorticity law)[39]

If $\omega(x,t)$ is the vorticity of an incompressible fluid of constant density moving in a potential force field, ϕ^t is the Lagrangian description of the motion, and $\omega_0(x) = \omega(x,0)$ is the vorticity at time $t = 0$, then

$$\omega(x,t) = \phi^t_*\omega_0\,(x)\,. \tag{7.13}$$

In other words, the vorticity "moves with the fluid".

This result has a number of consequences which we will discuss in some detail as we proceed. At the moment let us see what is the consequence of (7.11) for the compressible fluids. Recalling the equation of continuity $\rho_t + \operatorname{div}(\rho u) = 0$, it is easy to check that (7.11) implies

$$\left(\frac{\omega}{\rho}\right)_t + [u, \left(\frac{\omega}{\rho}\right)] = 0\,. \tag{7.14}$$

We see that in the compressible case the vector field $\frac{\omega}{\rho}$ moves with the flow (as long as $\rho > 0$ and the flow remains sufficiently regular, of course).

*Remark**

The above calculations are based on traditional vector analysis formulae, such as (7.7) and (7.3). There are "more geometric" formulae one can work with. For example, let $\tilde{u}$ be the one-form $u_i\,dx_i$ (as opposed to the vector field $u_i\frac{\partial}{\partial x_i}$). We can identify $u\nabla u$ with $L_u\tilde{u} - d(\frac{|u|^2}{2})$, where L_u is the Lie derivative in the

[38] We have seen in lecture 6 that such forces have no effect on the motion of an incompressible fluid, as they are completely resisted by the incompressibility constraint.

[39] The original 1858 paper of H. Helmholtz is in the Journal für die reine und angewandte Mathematik, vol. 55, pp. 25-55. An English translation, "On integrals of the hydrodynamical equations which express vortex motion", was published in Philosophical Magazine, vol. 33, pp. 485-512 (1867).

direction of u, which acts on one-forms by $(L_u a)_i = u^j \frac{\partial a_i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} a_j$. Euler's equation (with potential forces) can be written as

$$\tilde{u}_t + L_u \tilde{u} + d\pi = 0\,, \tag{7.15}$$

where π is a suitable function. Taking the exterior derivative d of the equation and using $d\,L_u = L_u\,d$ one obtains

$$(d\tilde{u})_t + L_u(d\tilde{u}) = 0\,, \tag{7.16}$$

which says that the differential two-form $d\tilde{u}$ is transported with the flow. In dimension three, once a volume element is given, we can identify two-forms and vector fields, and this way we arrive at the same results about the vorticity transportation as obtained above.

8

9/23/2011

8.1 Potential flows of ideal fluid

We saw last time that in ideal incompressible fluid (with only potential forces acting on it) the vorticity "moves with the flow", according to formula (7.13). Note that the law is completely "local", in that for its validity for a given fluid particle it is only important that the force $f(x,t)$ is potential in any small neighborhood of the fluid particle. Even when when the boundaries of the domain occupied by the fluid change with time, the law is still valid. In particular the law implies that if the vorticity of the initial velocity field $u(x,0)$ vanishes, it will vanish for all times unless there is some non-potential forcing term $f(x,t)$ in the equation (7.1).[40] For simply connected domains the condition $\operatorname{curl} u = 0$ is equivalent to

$$u(x,t) = \nabla h(x,t) = \nabla_x h(x,t) \tag{8.1}$$

and by the Helholtz law, if this is true at any time, it will be always true (unless non-potential volume forces act on the fluid. We will see later that the condition that the domain be simply connected can be in fact easily removed[41], and the condition (8.1) is preserved in time in without any constraints on the topology of the domain (as long as the forcing term $f(x,t)$ acting on the fluid is potential, of course). This can be also seen in other ways, for example by checking that a) formula (8.1) provides a solution and b) the solutions are unique.

The potential flows (8.1) (combined with the fact that they are preserved by the time evolution) offer some good insights into the nature of the ideal incompressible fluids. We will see shortly that in some cases the behavior of this model does not correspond to what we see in the real (almost) incompressible fluids, such as water.

We will consider potential flows (8.1) in a time-dependent domain $\Omega_t \subset \mathbf{R}^3$. We start by the obvious observation that the condition

$$\operatorname{div} u = 0 \tag{8.2}$$

implies

$$\Delta h(x,t) = \Delta_x h(x,t) = 0 \qquad \text{in } \Omega_t \tag{8.3}$$

for each time t. In other words, the function $h(x,t)$ has to be harmonic in x for each time t. The boundary condition at the (possibly moving) boundary $\partial\Omega_t$ is

$$u(x,t)n(x,t) = v(x,t)n(x,t)\,, \quad x \in \partial\Omega_t\,, \tag{8.4}$$

[40] Note that the law is much more "local" than this statement: for general flows, if a "fluid particle" has zero vorticity at some time t_1 and $\operatorname{curl} f(x,t) = 0$ at the particle for any time, then the particle will always have zero vorticity as it moves.

[41] e. g. by Kelvin's reformulation of the Helholtz law, which we will discussed soon.

where $n(x,t)$ is the normal to the boundary $\partial\Omega_t$ at time t, and $v(x,t)$ is the (prescribed) velocity of the boundary. (We assume that the motion of the boundary is given.) In terms of $h(x,t)$ this means

$$\frac{\partial h}{\partial n} = v(x,t)n(x,t)\,, \quad x \in \partial\Omega_t\,. \tag{8.5}$$

Equation (8.3) together with the boundary condition (8.4) represent the well-known Neumann boundary-value problem for the Laplace equation in the domain Ω_t. This problem has to be solved for each time. The time only plays a role of a passive parameter. We know from the theory of the Laplace equation that the Neumann problem is uniquely solvable (modulo constants) under natural assumptions.[42] Let us assume that $h(x,t)$ is a solution.

We claim that $u(x,t) = \nabla h(x,t)$ satisfies the Euler equation (5.16) with

$$\frac{p}{\rho_0} = -h_t - \frac{|\nabla h|^2}{2} \tag{8.6}$$

and $f = 0$. [43] This is immediately seen from the form (7.6) of the equation: in our case $\omega = 0$ and $u_t = \nabla h_t$, and (7.6) becomes

$$\nabla\left(h_t + \frac{|\nabla h|^2}{2} + \frac{p}{\rho_0}\right) = 0\,, \tag{8.7}$$

which is obviously satisfied if p is given by (8.6). As usual in incompressible fluid, we can change the pressure by an arbitrary function of time without affecting the motion.

We can reach the following conclusions concerning the potential solutions of the form $u(x,t) = \nabla h(x,t)$:

1. The solutions are uniquely determined by the motion of the boundary. If $v(x,t)n(x,t) = 0$ then $u(x,t) = 0$.

2. The response to the motion of the boundary is instantaneous. If the boundary starts moving in the sense that $v(x,t)n(x,t) \neq 0$ at some point, the fluid almost everywhere in Ω_t starts moving.[44] If the motion of the boundary stops, the fluid will also immediately stop moving everywhere.

3. If we move the boundary a certain way during a time interval $(0, t_1)$ and then we retrace the same motion of the boundary backwards during a time interval

[42] For example, for bounded Ω_t the right-hand side in (8.5) must satisfy $\int_{\partial\Omega_t} v(x,t)n(x,t) = 0$, which is obviously true in our situation.

[43] In fact, as we have Δu=0, the fields $u(x,t) = \nabla h(x,t)$ and p given by (8.6) even satisfy the Navier-Stokes equations describing the simplest viscous fluids (which we will discuss soon). However, (8.5) is usually not a good boundary condition for the viscous fluids.

[44] If a harmonic function h is non-constant, then it is analytic and its gradient ∇h can only vanish on a small set. The set is closed in Ω_t and has to have measure zero, for example. In fact, it typically contains only isolated points, although in some degenerate cases it can be larger.

(t_1, t_2), at time $t = t_2$ each fluid particle will be exactly where it was at time $t = 0$.

To summarize, we can say that "the degrees of freedom" of an ideal incompressible fluid with no initial vorticity are only at the boundary of the fluid. The degrees of freedom of the interior of the fluid can only be "activated" if we act on the fluid by non-potential forces. In the absence of these and in the absence of vorticity at time $t = 0$, the degrees of freedom in the interior of the fluid remain "locked" and the motion of the fluid is completely determined by the motion of the boundaries. If the motion of the boundaries is given, there is no dynamics, in that the motion is completely determined only by the constraint of incompressibility. (Strictly speaking, we proved this only for simply-connected domains, but it remains true in general.)

These conclusions are useful for example in the study of the motion of the rigid bodies in ideal fluids (a topic which we may return to later). In particular, they show that the (unknown) motion of finitely many rigid bodies moving in an ideal fluid with no initial vorticity is described by a finite-dimensional dynamical system, which may seem surprising at first.

8.2 d'Alembert's paradox

Let us consider a bounded body $\mathcal{O}$ with smooth boundary submerged in an ideal incompressible fluid which we imagine fills the whole space R^3. (For simplicity you can assume that $\mathbf{R}^3 \backslash \mathcal{O}$ is simply connected, but it is not necessary.) Assume at time $t = 0$ both the fluid and the body are at rest. We then start acting by a force on the body, so that it will start moving. (There is no force acting on the fluid other than the one caused by the motion of the body.[45]) Assume that a regime is achieved such that $\mathcal{O}$ moves at a constant speed, say, v. (We can consider the motion of $\mathcal{O}$ as given, applying whatever forces necessary to the body to achieve this.) If there have been no non-potential forces in the fluid, the flow will be potential:

$$\tilde{u}(x,t) = \nabla \tilde{h}(x,t). \tag{8.8}$$

Let us view the situation after the constant speed v of the body is achieved in the coordinate frame moving with the body. In this frame the body is stationary and the velocity field of the fluid is $u = \tilde{u} - v$, and is independent of time. The flow is still potential:

$$u(x) = \nabla h(x)\,. \tag{8.9}$$

[45] One can consider potential forces, such as gravity, and you can check that they will not change any of the considerations below, except possibly for the effects due to the Archimedes principle we considered in lecture 5. To eliminate these, one can assume that the density of $\mathcal{O}$ is the same as the fluid.

Let $U = -v$. The function h satisfies

$$\begin{aligned} \Delta h(x) &= 0 \qquad \text{in } \mathbf{R}^3 \setminus \mathcal{O}\,, && (8.10)\\ \frac{\partial h}{\partial n} &= 0 \qquad \text{at } \partial\mathcal{O}\,, && (8.11)\\ \lim_{x\to\infty} \nabla h(x) &\to U\,. && (8.12) \end{aligned}$$

The pressure is (up to a constant)

$$p = -\rho_0 \frac{|u|^2}{2} = -\rho_0 \frac{|\nabla h|^2}{2}\,. \tag{8.13}$$

Let us calculate the force $F = (F_1, F_2, F_3)$ on the body due to the fluid. It is given by

$$F = \int_{\partial\mathcal{O}} -p\, n\, dx\,, \tag{8.14}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to $\mathcal{O}$. To evaluate the integral (8.14), let us consider the tensor

$$T_{ij} = \rho_0 u_i u_j + p\delta_{ij} = \rho_0 h_i h_j - \rho_0 \frac{|\nabla h|^2}{2}\delta_{ij}\,, \tag{8.15}$$

where we use the notation $h_i = \frac{\partial h}{\partial x_i}$. The tensor T_{ij} describes locally the transport and transfer of momentum in the fluid. We note that

$$T_{ij,j} = \frac{\partial T_{ij}}{\partial x_j} = 0\,, \tag{8.16}$$

due to the Euler equation. One can also check that, regardless of Euler's equation,

$$\left(h_i h_j - \frac{|\nabla h|^2}{2}\delta_{ij} \right)_{,j} = 0 \tag{8.17}$$

for any harmonic function h.[46] For large radii R let us set

$$\Omega_R = B_R \setminus \mathcal{O}\,. \tag{8.18}$$

Due to the boundary condition $u\, n = 0$ at $\partial\mathcal{O}$ and (8.16), we have

$$F_i = \int_{\partial B_R} T_{ij} n_j\,. \tag{8.19}$$

We will evaluate this integral in the limit $R \to \infty$. Let us consider the expansion of h at ∞

$$h(x) = Ux + c + \frac{a_0}{|x|} + a_j \frac{\partial}{\partial x_j}\frac{1}{|x|} + O(R^{-3})\,. \tag{8.20}$$

[46] The tensor in (8.17) is known as the energy-momentum tensor of the harmonic function h and is known to be div-free. You can either check it directly, or you can verify that the condition $\frac{d}{d\varepsilon}|_{\varepsilon=0} \int |\nabla_x h(x+\varepsilon\xi(x))|^2\, dx = 0$ for each compactly supported smooth vector field ξ leads to (8.17).

Then

$$\nabla h(x) = U + a_0 \frac{x}{|x|^3} + O(R^{-3})\,. \tag{8.21}$$

Substituting this expression in (8.19) and letting $R \to \infty$, we obtain

$$F = 0\,. \tag{8.22}$$

This means that when the body does not accelerate, there is no drag or lift on the body in an ideal incompressible fluid in the situation we have considered. [47] This result was discovered in 1752 by d'Alembert, and is often called d'Alembert paradox.[48] It had been puzzling for some time because the inner friction in air and water is very small, so in some sense both air and water are quite close to ideal fluids. A good explanation was given around 1904 by L. Prandtl, and his explanation is almost universally accepted today, although its rigorous mathematical proof remains far out of reach. We will discuss it later when we introduce viscosity.

[47] Among other things, it means that airplanes could not fly in an ideal incompressible fluid.

[48] Compressibility does not save the situation, at least for subsonic flows, when the speeds do not exceed the speed of sound in the fluid. Nevertheless, it should be noted that a compressible fluid always has "its own degrees of freedom", even when no vorticity is present. Potential flows are not "slaved" to the motion of the boundaries as in the incompressible case. For super-sonic flows one has new effects and drag due to shock waves is possible even for potential flows. We will not go into a discussion of these topics here. In general, the effects due to compressibility, while sometimes not important for practical purposes (such as in flows around cars), are mathematically significant in that – from the point of view of the PDE theory – they can change the nature of the equations. For example, it is known that compressible flows of ideal compressible fluids typically develop singularities, in a way somewhat similar to what we have seen for free particles in lecture 2. On the other hand, the question about the existence of singularities in incompressible flows is open and is generally considered difficult.

9

9/26/2011

9.1 Kelvin's circulation theorem

Let us consider an ideal fluid in a domain Ω, either incompressible with constant density $\rho = \rho_0$ or compressible with $p = p(\rho)$. The domain can depend on time, so we can also write $\Omega = \Omega_t$ if needed. Let $u(x,t)$ be the velocity field of the fluid and let ϕ^t be the corresponding Lagrangian maps, as in lecture 2. Let us consider a time-dependent closed curve γ^t in Ω which "moves with the fluid", i. e.

$$\gamma^t = \phi^t(\overline{\gamma})\,, \tag{9.1}$$

where $\overline{\gamma}$ is some closed curve in Ω at time $t = 0$. We will assume that $\overline{\gamma}$ is a smooth curve with no self-intersections and is given by a map $s \in [0,1] \to \overline{\gamma}(s) \in \Omega$, with $\overline{\gamma}(0) = \overline{\gamma}(1)$. It is natural to parametrize the curves γ^t by s as follows

$$\gamma^t(s) = \gamma(s,t) = \phi^t(\overline{\gamma}(s))\,. \tag{9.2}$$

We will consider the circulation of the field $u(x,t)$ along γ^t, given by the curve integral

$$\oint_{\gamma^t} u(x,t)\,dx = \int_0^1 u_i(\gamma(s,t),t)\gamma_i'(s,t)\,ds\,, \tag{9.3}$$

where $\gamma'(s,t) = \frac{\partial}{\partial s}\gamma(s,t)$.
In 1869, motivated by the Helmholtz theorem (published in English in 1867), Kelvin proved the following result:

Theorem (Kelvin)
In the situation above, assume that the force field f in the incompressible Euler equations (5.16)–(5.17) or the compressible barotropic Euler equations (5.18)–(5.20) is such that $\frac{f}{\rho}$ is potential. Then

$$\frac{d}{dt}\oint_{\gamma^t} u(x,t)\,dx = 0\,. \tag{9.4}$$

Remark
We will see that the Helmoltz law can be obtained by applying this result to the special case when $\overline{\gamma}$ are suitable infinitesimally small circles. In that sense, the Helmholtz law is a localized version of Kelvin's theorem.

Proof of the theorem
The proof is by straightforward calculation. We have

$$\frac{d}{dt}\oint_{\gamma^t} u(x,t)\,dx = \int_0^1 \frac{d}{dt}\left[u_i(\gamma(s,t),t)\gamma_i'(s,t)\right]\,ds\,. \tag{9.5}$$

We calculate, using Euler's equation

$$\frac{d}{dt}\left[u_i(\gamma(s,t),t)\gamma_i'(s,t)\right] = (u_t + u\nabla u)\gamma' + u\frac{\partial^2}{\partial s \partial t}\gamma \tag{9.6}$$

$$= -\nabla F \gamma' + \frac{\partial}{\partial s}\frac{|u|^2}{2}\,, \tag{9.7}$$

where the values of the functions are taken at $(\gamma(s,t),t)$, and $F(x,t)$ is a suitable function corresponding to $-\frac{p}{\rho_0} +$ potential of $\frac{f}{\rho_0}$ in the compressible case and to $-P +$ potential of $\frac{f}{\rho}$ in the incompressible case, with P as in (7.4). Since

$$\oint_{\gamma^t} \nabla F(x,t)\,dx = 0, \qquad \int_0^1 \frac{\partial}{\partial s}\frac{|u(\gamma(s,t),t)|^2}{2}\,ds = 0\,, \tag{9.8}$$

the proof is finished.

The curve γ^t moving with the flow can be considered as a limiting case of a divergence-free vector field moving with the flow.[49] Let us look at this analogy in more detail, first at some fixed time, so that we drop t from the notation. Let be γ is any closed curve. Let us think of it as an electric wire with an electric current J passing through it. The idea of an infinitely thin wire with a steady electric current is of course an idealization. In reality the wire has some finite thickness, say, $\varepsilon > 0$, and the current in it is a vector field $J(x)$ which can be defined for $x \in \mathbf{R}^3$, with $J(x) = 0$ outside the wire. The current will satisfy $\operatorname{div} J = 0$ in $\mathbf{R}^3$. We can even imagine that $J(x)$ depends smoothly on x, even though the derivatives within the wire will be large, of order $\frac{1}{\varepsilon}$. (This may not be the case for real wires, but that is not our concern at the moment.) In this sense we can think about a closed curve as of a limiting case of a div-free vector field.[50] In this picture the curve integral

$$\oint_\gamma u(x)\,dx \tag{9.10}$$

is replaced by

$$\int_{\mathbf{R}^3} u(x)J(x)\,dx = \int_{\mathbf{R}^3} u_i(x)J_i(x)\,dx\,. \tag{9.11}$$

We have the following variant of Kelvin's theorem, which – as we will see – is suitable for generalizations.

[49] This idea is generalized in many directions in Geometric Measure Theory, where it is formalized in the notion of *current*. See for example the book "Geometric Measure Theory" by H. Federer.

[50] Starting from a closed curve $\gamma(s)$ parametrized by $s \in [0,1]$, with $\gamma(0) = \gamma(1)$, one can define such a field as follows: let $\varphi(x)$ be a smooth function in $\mathbf{R}^3$ supported in the unit ball with $\int_{\mathbf{R}^3} \varphi = 1$. Set $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$, so that φ_ε is the usual mollifying function. Now define

$$v_\varepsilon(x) = \int_0^1 \varphi_\varepsilon(x - \gamma(s))\gamma'(s)\,ds\,. \tag{9.9}$$

You can check that v_ε is supported in an $\varepsilon-$ neighborhood of the curve, $\operatorname{div} v_\varepsilon = 0$, and it has other expected properties.

Kelvin's Theorem, version 2

Let u be as in Kelvin's theorem, and let $v(x,t)$ be a divergence-free field compactly supported in Ω_t and satisfying

$$v_t + [u,v] + v \operatorname{div} u = 0\,. \tag{9.12}$$

Then

$$\frac{d}{dt}\int_{\Omega_t} u(x,t)v(x,t)\,dx \;=\; 0\,. \tag{9.13}$$

Remark

1. It is easy to check that the equation (9.12) preserves the condition $\operatorname{div} v = 0$ and the compact support of v. For incompressible flows (when $\operatorname{div} u \;=\; 0$) the field v is just transported with the flow (in analogy with the transport of the curve γ^t above). If the flow is compressible, the pure transport equation $v_t + [u,v] \;=\; 0$ does not preserve $\operatorname{div} v \;=\; 0$ and the term $v \operatorname{div} u$ restores the preservation of this condition, so that the analogy with the transport of the curve γ^t is kept.

2. Unlike the Helmholtz law or the first formulation of Kelvin's theorem, version 2 of Kelvin's theorem is suitable for a generalization to viscous fluids, as we will see later.

Proof

We write

$$\frac{d}{dt}\int_{\Omega_t} u(x,t)v(x,t)\,dx = \int_{\Omega_t} u_t v + u v_t\,dx\,, \tag{9.14}$$

express u_t and v_t from the equations, and integrate by parts to see the terms in last integral in (9.14) either vanish or cancel each other, somewhat similarly to (9.6)–(9.8).

We will now derive Helmholtz's law from Kelvin's theorem. For that we recall the Stokes formula. Let $\Sigma \subset \mathbf{R}^3$ be a smoothly embedded two-dimensional disc (with no self-intersections) and let $\gamma = \partial\Sigma$ be its boundary, which is assumed to be a smooth curve without self-intersections. We choose a unit normal $n(x)$ to Σ and an orientation of γ so that the orientation of γ with respect to n is positive.[51] If u is a smooth vector field defined in a neighborhood of the closure of Σ then

$$\oint_\gamma u\,dx = \int_\Sigma n \operatorname{curl} u\,dx\,, \tag{9.15}$$

[51] Roughly speaking, if we stand on Σ in the direction of $n(x)$, then we see the orientation of γ as counter-clockwise.

where the integral over Σ is with respect to the usual surface measure.[52]
Assume now that we have diffeomorphism from another surface $\tilde{\Sigma}$ to Σ,

$$\phi\colon \tilde{\Sigma} \to \Sigma\,. \tag{9.17}$$

In fact, for our purposes here we can assume that ϕ is defined in a neighborhood of the closure of $\tilde{\Sigma}$, and is an orientation-preserving diffeomorphism of this neighborhood and a neighborhood of a closure of Σ. The boundary of $\tilde{\Sigma}$ is $\tilde{\gamma}$, the normal to $\tilde{\Sigma}$ is $\tilde{n}$, and they are oriented so that ϕ preserves the orientation. If f is any smooth vector field defined in a neighborhood of the closure of Σ, then we have the "change of variables" formula

$$\int_{\Sigma} n(y)f(y)\,dy = \int_{\tilde{\Sigma}} [\operatorname{Cof}\nabla\phi(x)n(x)]f(\phi(x))\,dx = \int_{\tilde{\Sigma}} n(x)[\operatorname{Adj}\nabla\phi(x)f(\phi(x))]\,dx\,, \tag{9.18}$$

where, as usual, the matrix $\operatorname{Cof} A$ is the transpose of $A^{-1}\det A = \operatorname{Adj} A$.[53] Let us now apply (9.18) to the Lagrangian map $\phi = \phi^t$ and $f(y) = \omega(y,t)$. We obtain

$$\int_{\Sigma} n(y)\omega(y,t)\,dy = \int_{\tilde{\Sigma}} n(x)[\operatorname{Adj}\nabla\phi^t(x)\omega(\phi^t(x),t)]\,dx\,. \tag{9.19}$$

At the same time, from Kelvin's theorem and (9.15) we see that

$$\int_{\Sigma} n(y)\omega(y,t)\,dy = \int_{\tilde{\Sigma}} n(x)\omega(x,0)\,dx\,. \tag{9.20}$$

Therefore

$$\int_{\tilde{\Sigma}} n(x)[\operatorname{Adj}\nabla\phi^t(x)\omega(\phi^t(x),t)]\,dx = \int_{\tilde{\Sigma}} n(x)\omega(x,0)\,dx\,. \tag{9.21}$$

This argument is valid for any Σ as above, and hence $\tilde{\Sigma}$ can be any disc in Ω at time $t = 0$. We see that

$$\operatorname{Adj}\nabla\phi^t(x)\omega(\phi^t(x),t) = \omega(x,0)\,, \tag{9.22}$$

which, taking into account (2.19), is easily seen to be equivalent to the Helmholtz law that the vector field $\frac{\omega}{\rho}$ moves with the flow.

[52] This is of course more elegantly formulated in terms of differential forms: identifying u with the one-form $u_i\,dx^i$, we can write (9.15) as

$$\int_{\gamma} u = \int_{\Sigma} du\,, \tag{9.16}$$

where du is the exterior derivative of u.

[53] In term of differential forms, this formula just says $\int_{\tilde{\Sigma}} \phi^*\beta = \int_{\Sigma}\beta$ for any two-form β, where $\phi^*\beta$ is the pull-back of β. The form (9.18) of this formula can be easily inferred from the identity $(Aa \times Ab) = (\operatorname{Cof} A)(a \times b)$ which is true for $a, b \in \mathbf{R}^3$ and any 3×3 matrix A.

10

9/28/2011

10.1 The Biot-Savart law

Today we will mostly have in mind incompressible fluid in the whole space $\mathbf{R}^3$.[54] Let us imagine a flow with a smooth velocity field $u(x,t)$ which "decays to zero as $x \to \infty$." As we shall see, one has to be somewhat careful about the exact rate of decay, since in incompressible fluids certain disturbances can propagate with infinite speed and this can immediately create some action at infinity even when the initial velocity field is compactly supported. This effect is relatively harmless and mostly of technical interest in the mathematical investigations of the equations, but it is useful to keep it in mind. As before, we let

$$\omega = \operatorname{curl} u \tag{10.1}$$

be the vorticity. As the flow is incompressible, we have

$$\operatorname{div} u = 0\,. \tag{10.2}$$

The equation for ω is (see lecture 7)

$$\omega_t + [u, \omega] = 0\,. \tag{10.3}$$

It is often useful to view ω in (10.3) as the primary unknown quantity, and u as a quantity expressed in terms of ω, by a suitable operator $\omega \to u = U(\omega)$, and think of (10.3) as

$$\omega_t + [U(\omega), \omega] = 0\,. \tag{10.4}$$

We will discuss the operator $\omega \to u$ in some detail. Given ω the equations for u are

$$\begin{aligned} \operatorname{curl} u &= \omega \qquad \text{in } \mathbf{R}^3\,, && (10.5)\\ \operatorname{div} u &= 0 \qquad \text{in } \mathbf{R}^3\,, && (10.6)\\ u &\to 0 \qquad \text{at } \infty. && (10.7) \end{aligned}$$

the For simplicity we can assume that ω is smooth and compactly supported. The equations determine u uniquely: if both $\operatorname{curl} u$ and $\operatorname{div} u$ vanish, then $u = \nabla h$ for some harmonic function h and from (10.7) and the Liouville theorem we see that $u = 0$ in that case.[55] We also note that when ω solves (10.3) for

[54] Generalizations to domains with boundaries are possible, and at some point these will also be of interest to us, but today we consider only the case of the whole space.

[55] If $\mathbf{R}^3$ is replaced by a bounded domain Ω then the condition (10.7) should be replaced by the condition $u\,n = 0$ at $\partial\Omega$. If Ω is simply connected, then u is again uniquely determined from Ω. However, for general domains this may no longer be the case, and the system might have non-trivial solutions. For smooth bounded domains the space of solutions will be finite-dimensional, with the dimension coinciding with the first Betti number of the domain (by the Hodge theory). It is clear from (7.6) that these solutions will also satisfy the steady Euler equation, with $p = \rho_0 \frac{|u|^2}{2}$.

some smooth bounded velocity field u and $\omega(x,0)$ is compactly supported, than $\omega(x,t)$ will also be compactly supported. Therefore the condition that ω be compactly supported is kept during the evolution, if a good solution exists. In the equations (10.5)–(10.8) the time only plays a role of a passive parameter, and hence we can temporarily drop t from our notation. Under our assumptions the solution of(10.5)–(10.7) is straightforward. Taking curl of (10.5) and using the formula $\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div}$, we obtain

$$-\Delta u = \operatorname{curl} \omega \tag{10.8}$$

and we can express u in terms of the Newton potential

$$G(x) = \frac{1}{4\pi} \frac{1}{|x|} \tag{10.9}$$

as

$$u(x) = \int_{\mathbf{R}^3} G(x-y) \operatorname{curl} \omega(y)\, dy\,. \tag{10.10}$$

Integrating by parts and letting

$$K(x) = -\nabla G(x) \tag{10.11}$$

we can also write

$$u(x) = \int_{\mathbf{R}^3} K(x-y) \times \omega(y)\, dx\,. \tag{10.12}$$

This is the Biot-Savart law.

Sometimes it is useful to write the solution u in terms of a vector potential A as

$$u = \operatorname{curl} A \tag{10.13}$$

with

$$A(x) = \int_{\mathbf{R}^3} G(x-y)\omega(y)\, dy\,. \tag{10.14}$$

As an easy exercise, you can check that formulae (10.13) and (10.14) are equivalent to (10.12) (under our assumptions).

We note that the formulae make sense for any vector field ω, but produce the solution of (10.5)–(10.7) only when $\operatorname{div}\omega = 0$, which is a necessary condition for the solvability of (10.5)–(10.7), due to the identity $\operatorname{div}\operatorname{curl} = 0$. For a general ω, not necessarily div-free, and u given by the formulae above we obtain

$$\operatorname{curl} u = \operatorname{curl}\operatorname{curl} A = -\Delta A + \nabla \operatorname{div} A = \omega + \nabla(G * \operatorname{div}\omega)\,, \tag{10.15}$$

where we use the usual notation for convolution, $f*g\,(x) = \int_{\mathbf{R}^3} f(x-y)g(y)\, dy\,.$ When $\operatorname{div}\omega = 0$, we obtain $\operatorname{curl} u = \omega$. We note that for general ω the expression $\omega + \nabla G * \operatorname{div}\omega$ is exactly the div-free part of the Helmholtz decomposition of ω which we discussed in lecture 6.

If ω is supported in a ball B_R, then outside B_R the field u will be potential, as $\operatorname{curl} u = 0$ in $\mathcal{O}_R = \mathbf{R}^3 \setminus B_R$ and every closed curve in $\mathcal{O}_R$ bounds a surface. In other words, we will have $u = \nabla h$ in $\mathcal{O}_R$, where h is a harmonic function in $\mathcal{O}_R$. That correspond to the fact that if the vorticity is compactly supported, then the motion of the fluid near infinity will be potential - the fluid near infinity does not have its independent degrees of freedom and only passively reacts to what is going on in the regions where the vorticity is active. Therefore, in some sense, the field u at near infinity is not of primary interest, it is not where the action is. Nevertheless, it is still interesting to look at some of its features as it is related to some quantities which are conserved by Euler's equations.

We first consider the rate of decay of u to zero. Its expansion at ∞ can be obtained in various ways. The most direct one is to use in (10.14) the Taylor expansion

$$G(x-y) = G(x) - G_i(x)y_i + \frac{1}{2}G_{ij}(x)y_iy_j + \dots \tag{10.16}$$

For y is B_R and $|x| > 2R$ the series is easily seen to converge.[56] Then

$$A_k(x) = G(x)a_k + G_i(x)a_{ik} + G_{ij}(x)a_{ijk} + \dots\,, \tag{10.19}$$

where

$$a_k = \int_{\mathbf{R}^3} \omega_k\, dy, \quad a_{ik} = \int_{\mathbf{R}^3} y_i\omega_k(y)\, dy, \quad a_{ijk} = \frac{1}{2}\int_{\mathbf{R}^3} y_iy_j\omega_k(y)\, dy, \quad \dots \tag{10.20}$$

The condition $\operatorname{div}\omega = 0$ has important consequences for these coefficients. Roughly speaking, the symmetry properties of the coefficients are the same as if they were coming from a current defined by a closed loop, as discussed in lecture 9. If ω would correspond to a closed curve $\gamma(s)$ (with $\gamma(0) = \gamma(1)$), we would have

$$a_k = \int_0^1 \gamma_k'(s)\, ds, \qquad a_{ij} = \int_0^1 \gamma_i(s)\gamma_j'(s)\, ds, \qquad a_{ijk} = \int_0^1 \gamma_i\gamma_j\gamma_k'\, ds, \quad \dots \tag{10.21}$$

In this case we see easily the following relations

$$a_k = 0, \qquad a_{ij} = -a_{ji}, \qquad a_{ijk} = a_{jik},\ a_{ijk} + a_{kij} + a_{jki} = 0\,, \quad \dots \tag{10.22}$$

The relations (10.22) are valid also for the coefficients (10.20), for any compactly supported ω with $\operatorname{div}\omega = 0$. For example, the condition $a_k = 0$ follows from

$$0 = -\int_{\mathbf{R}^3} (\operatorname{div}\omega) f\, dy = \int_{\mathbf{R}^3} (\omega\nabla f)\, dy \tag{10.23}$$

[56] In fact, as noticed by Legendre around 1782, one can write this expansion in a more subtle way using

$$\frac{1}{|x-y|} = \frac{1}{|x|}\,\frac{1}{\sqrt{1 - 2\frac{xy}{|x|^2} + \frac{|y|^2}{|x|^2}}} \tag{10.17}$$

and the Taylor expansion (or the Newton formula) for

$$(1-z)^{-\frac{1}{2}} \tag{10.18}$$

but we will not need this form of the expansion.

by taking $f = x_k$. For the other conditions we use

$$0 = -\int_{\mathbf{R}^3} (\operatorname{div}\omega) f_1 f_2 \dots f_m \, dy = \int_{\mathbf{R}^3} \omega \nabla(f_1 f_2 \dots f_k) \, dy \tag{10.24}$$

and use various coordinate function x_{k_j} for f_j .[57]

We see that the first term on the right-hand side in (10.19) vanishes and we conclude that we will have

$$u(x) = O(|x|^{-3}) \,, \qquad x \to \infty. \tag{10.25}$$

In fact, it is not difficult to calculate the leading term of u at ∞ directly. One has

$$u(x) = \nabla(M \nabla G(x)) + O(|x|^{-4}) \,, \qquad x \to \infty \,, \tag{10.26}$$

with

$$M = \frac{1}{2} \int_{\mathbf{R}^3} y \times \omega(y) \, dy \,. \tag{10.27}$$

In magnetostatics this vector would be called the *magnetic moment*. The pictures of the field $\nabla(M \nabla G(x))$ for $M = e_3$ can be found in many textbooks of electromagnetism.

With some more work one can get

$$u(x) = \nabla(M \nabla G(x)) + \nabla(\operatorname{Tr}(C \, \nabla^2 G(x))) + O(|x|^{-5}) \,, \quad x \to \infty \tag{10.28}$$

where C is the symmetric part of the matrix

$$c_{ij} = \frac{1}{3} \int_{\mathbf{R}^3} y_i \, (y \times \omega(y))_j \, dy \,. \tag{10.29}$$

The reason why we look at these expansions in some detail is that (12.27) and (10.29) are related to quantities which are conserved in the evolution by Euler's equations, as we will soon see.

*Remark**

Some of the above calculation can be also done in the following way, which is related to formula (10.15). Let us assume

$$\omega = \operatorname{curl} b \tag{10.30}$$

for a compactly supported field b, not necessarily divergence free. Such representation is possible for any compactly supported ω with $\operatorname{div}\omega = 0$. For

[57] Another way to see (10.21) is from the Fourier transform. If v is a div-free field in R^n which is integrable, then its Fourier transform $\hat{v}$ is continuous and satisfies $\xi_j \hat{v}_j = 0$ pointwise. It is not hard to see that in these circumstances we have $\hat{v}(0) = 0$, which gives the condition $a_k = 0$ above.

example, we can smoothly extend the function h discussed in the paragraph before (10.16) from $\mathcal{O}_R$ to $\mathbf{R}^3$ and set $b = u - \nabla h$. Different (and more geometric) constructions are possible. For the potential A given by (10.14) we can write

$$A = G * \omega = G * (\operatorname{curl} b) = \operatorname{curl}(G * b)\,. \tag{10.31}$$

Then

$$u = \operatorname{curl} A = \operatorname{curl}\operatorname{curl}(G*b) = -\Delta(G*b)+\nabla\operatorname{div}(G*b) = b+\nabla(G*\operatorname{div} b) = b+\nabla(G_i*b_i)\,. \tag{10.32}$$

We note that we can write

$$\nabla(G_i * b_i) = \int_{\mathbf{R}^3} \nabla(b(y)\nabla_x G(x-y))\,dy\,, \tag{10.33}$$

and we see that the field near ∞ (outside the support of b), is a linear combination of the various shifts of the elementary fields $\nabla(M\nabla G)$. Formulae (10.26) and (10.28) can be also derived from this last formula (and probably with less work, although some calculation is still needed to relate b to M and C defined respectively by (12.27) and (10.29)).

10.2 The pressure near ∞

Let us consider the incompressible Euler equation

$$u_t + u\nabla u + \frac{\nabla p}{\rho_0} = 0, \qquad \operatorname{div} u = 0 \tag{10.34}$$

in the whole space $\mathbf{R}^3$. Let us take $\rho_0 = 1$ in what follows to simplify notation. Let us assume that at time $t = 0$ the velocity field $u(x, 0)$ is compactly supported and let us calculate $u_t(x, t)$ near ∞, where clearly $u_t(x, 0) = -\nabla p(x, 0)$. We need to calculate p. The equation for p is obtained by taking div of the first equation in (10.34). This gives

$$-\Delta p = \operatorname{div}(u\nabla u) = \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j)\,, \tag{10.35}$$

where we used that $\operatorname{div} u = 0$. Using again the expansion (10.16), we obtain for large x

$$p(x) = G(x)a + G_i(x)a_i + G_{ij}a_{ij} + \dots \tag{10.36}$$

with

$$a = \int_{\mathbf{R}^3} f(y)\,dy, \qquad a_i = \int_{\mathbf{R}^3} y_i f(y)\,dy, \qquad a_{ij} = \int_{\mathbf{R}^3} y_i y_j f(y)\,dy\,, \tag{10.37}$$

where

$$f = \frac{\partial^2}{\partial y_i \partial y_j}(u_i u_j)\,. \tag{10.38}$$

Integrating by parts, one sees that $a = 0$ and $a_i = 0$ and typically (unless u is very special) $a_{ij} \neq 0$ at least for some i, j. Therefore we expect

$$p \sim O(|x|^{-3}), \qquad \nabla p \sim O(|x|^{-4}) \,, \tag{10.39}$$

and hence at time $t = 0$

$$u_t \sim O(|x|^{-4}) \,. \tag{10.40}$$

For later times the support of u will no longer be compact, but one can expect that the term $u\nabla u$ will decay quite fast to zero as $x \to \infty$, so that the above considerations are still valid at the level of the main term $O(|x|^{-4})$. Therefore we still expect that $u_t \sim O(|x|^{-4})$ (and generically not better) as long as a smooth solution exists. In fact, the decay of $u\nabla u$ is sufficiently fast even with assuming only that the vorticity is compact to reach the same conclusion. Therefore when, say, the initial vorticity is compact, we expect that the equation can change the field $u(x, t)$ for x near ∞ only at the orders $O(|x|^{-4})$ or higher. Therefore the leading term $\nabla(M\nabla G)$ should be conserved by the evolution. This leads us to the consideration of the conservation laws for the equations, a topic which we will address next time.

11

9/30/2011

11.1 The classical conservation laws for a particle system

In an classical isolated system of particles of finitely many particles (possibly interacting with one another) in the whole space $\mathbf{R}^3$ the following quantities are known to be conserved, under some natural assumption of frame invariance, see the remark about the symmetries below. The number of particles is denoted by m.

1. The total momentum of the system

$$P = \sum p^{(i)} = \sum m^{(i)} v^{(i)} \,, \tag{11.1}$$

   where $m^{(i)}$ is the mass of the $i-$the particle, $v^{(i)}$ is its velocity and $p^{(i)} = m^{(i)} v^{(i)}$ (no summation) is its momentum.

2. The total angular momentum of the system

$$L = \sum x^{(i)} \times p^{(i)} \,, \tag{11.2}$$

   where $x^{(i)}$ is the position of the $i-$th particle.

3. The total energy of the system

$$E = \sum \frac{1}{2} m^{(i)} |v^{(i)}|^2 + V(x^{(1)}, x^{(2)}, \dots x^{(m)}) \,, \tag{11.3}$$

   where $V(x^{(1)}, x^{(2)}, \dots x^{(m)})$ represents the potential energy of the m particles when the $i-$th particle has position $x^{(i)}$. (This function has to satisfy some natural assumptions to obtain the conservation laws, see below.) For example, when the interaction between the particles is only due to the newtonian gravity, we have

$$V = \sum_{\substack{i,j=1 \\ i \neq j}}^{m} - \frac{\kappa \, m^{(i)} m^{(j)}}{|x^{(i)} - x^{(j)}|} \,, \tag{11.4}$$

   where κ is the gravitational constant.

We will se later that these conservation laws are closely related to some natural assumptions about the symmetries of the system.[58] The conservation of the

[58] It is easy to come up with mathematical examples of V for which the conservation of momentum and angular momentum fail. However, these examples would violate the symmetry properties which we usually implicitly assume when talking about an isolated system of particles.

momentum is related to the translational symmetry (if $x^{(i)}(t)$ is a solution and a is a fixed vector in $\mathbf{R}^3$, then $x^{(i)}(t)+a$ is also a solution). The conservation of the angular momentum is related to the symmetry of the system under rotations (if $x^{(i)}$(t) is a solution, and $Q \in SO(3)$, then $Qx^{(i)}(t)$ is also a solution). Finally the conservation of energy is related to the translation invariance in t (if $x^{(i)}(t)$ is a solution and $t_0 \in \mathbf{R}$, then $x^{(i)}(t-t_0)$ is again a solution).

11.2 Analogy between fluids and a system of particles

A fluid filling all space can be considered as continuum version of finite particle system in which the interaction between the particles generates the pressure. The incompressible fluid can be thought of as a limit example when the potential energy between the particles is finite only for volume-preserving deformations.[59]

Let us think of a situation that the fluid is at rest until time t_1, then between times t_1 and $t_2 > t_1$ a force with a smooth compactly supported volume density $f(x,t)$ acts on the fluid, and after time t_2 the fluid is "left alone", and evolves by the Euler equation with zero right-hand side. In what follows we will assume that all solutions involved in our considerations are smooth. This is known to be the case for a short time period after t_2, but it is open whether long-time smooth solutions exists for incompressible fluids, and it is known that for compressible fluids singularities can form in finite time, somewhat similarly to what we saw for the free particles in lecture 2, although the situation is not as straightforward and proofs can be quite harder. In the consideration below we will always assume that the solutions we are dealing with are sufficiently regular.

Let us have a look at the analogies of the conservation laws corresponding the conservation of momentum, angular momentum, and energy for the finite particle systems.

11.3 Conservation of momentum in fluids

The natural analogue of the momentum (11.1) is

$$\int_{\mathbf{R}^3} \rho(x,t)u(x,t)\,dx \tag{11.5}$$

[59] It should however be borne in mind that this limit procedure may be non-trivial to justify mathematically. In fact, it is not completely trivial even in finite dimensions, when for a finite system of particles we want to impose a constraint, such as that distances between the particles are fixed and therefore they form a rigid body. The heuristics of "almost imposing" this constraint by, say, rods with high but finite stiffness is good for many purposes. However, we have to keep in mind that when considering the dynamical behavior of the system, the rods with high finite stiffness will lead to high-frequency oscillations, in which some energy can possibly accumulate. The situation with the incompressible fluids is similar. If we think of them as a limit of compressible fluids, we have to keep in mind that the degrees of freedom responsible for the compressibility can undergo fast oscillations (known as *acoustic waves* in this context, which make the limit procedure to incompressible fluids mathematically non-trivial.

in the compressible case and

$$\int_{\mathbf{R}^3} \rho_0 u(x,t)\, dx \tag{11.6}$$

in the incompressible case. (For simplicity we will only consider the homogeneous incompressible fluids for which the density is constant.) We expect these quantities to be constant after time t_2, when there are no longer any exterior forces acting on the fluid. This is true, as we shall see shortly, although one has to be somewhat careful in the incompressible case, as based on the last lecture (formula (10.26) we can expect that the integral in (11.6) may not be convergent.

In fact, consideration about momentum conservation can be "localized" and taken as a basis for another derivation of the equations of motions (= Euler's equations) in a way which is different from our derivation in lecture 5, in the sense that it postulates the local momentum conservation and derives the equations of motion from this postulate. This way we of course do not prove the momentum conservation from the equations, but rather we derive the equations from the momentum conservation. This approach is important and therefore we look at it in some detail.

Consider a compressible or incompressible fluid, and let us look at the tensor (depending on x, t)

$$T_{ij} = \rho u_i u_j + \delta_{ij} p\,, \tag{11.7}$$

which we already considered in lecture 8. The part $\delta_{ij} p$ is minus the Cauchy stress tensor in the fluid, and given a (non-moving) domain $\mathcal{O}$ in the region occupied by the fluid, it represents the forces acting on $\mathcal{O}$ by the fluid outside of $\mathcal{O}$, this time with a sign such that the forces can be thought of as causing a loss of momentum of $\mathcal{O}$. The term

$$\int_{\partial\mathcal{O}} p\, n\; dx \tag{11.8}$$

represents the rate of loss of momentum of $\mathcal{O}$ due to these forces. Another way momentum in $\mathcal{O}$ can be lost is that particles carrying momentum move out of $\mathcal{O}$. The rate of this loss is easily seen to be given by the integral

$$\int_{\partial\mathcal{O}} (u\, n) \rho\, u\; dx\,. \tag{11.9}$$

The integral

$$\int_{\partial\mathcal{O}} T_{ij} n_j \tag{11.10}$$

represents the sum of (11.8) and (11.9), and therefore it represents the total rate of loss of the momentum in $\mathcal{O}$. On the other hand, the integral

$$\frac{\partial}{\partial t} \int_{\partial\mathcal{O}} \rho u\, dx \tag{11.11}$$

represents the gain of momentum in $\mathcal{O}$. In the absence of other forces, the gain (11.11) and the loss (11.10) must balance each other and we have

$$\frac{\partial}{\partial t}\int_{\mathcal{O}} \rho u + \int_{\partial\mathcal{O}} Tn = 0\,. \tag{11.12}$$

Integrating by parts in the second term on the left-hand side of (11.12) and using the fact that $\mathcal{O}$ can be chosen arbitrarily (as long as it is smooth), we obtain

$$(\rho u)_t + \operatorname{div} T = 0\,, \tag{11.13}$$

where, as usual,

$$(\operatorname{div} T)_i = \frac{\partial T_{ij}}{\partial x_j}\,. \tag{11.14}$$

If volume forces $f(x,t)$ are present, similar considerations give

$$(\rho u)_t + \operatorname{div} T = f(x,t)\,. \tag{11.15}$$

Vice versa, if (11.13) is satisfied, we have the local momentum conservation in the sense of (11.12). Therefore we see that the local momentum conservation is really equivalent to (11.13). For incompressible fluids it is easy to check that (11.15) is the same as (5.16). Therefore (5.16) implies the local conservation of momentum in the sense above.

For compressible fluids the equation (11.15) is of a slightly different form than (5.18). However, it is easy to see that

$$(\rho u)_t + \operatorname{div} T - \rho u_t - \rho u\nabla u - \nabla p = u(\rho_t + \operatorname{div}(\rho u)) \tag{11.16}$$

and the last term vanishes whenever the equation of continuity is satisfied, which is always the case of fluid motion. So we see that the local conservation of momentum is equivalent to the equation of motion (5.18) also in the compressible case (assuming the equation of continuity is satisfied).

The conservation of the total momentum is obtained from (11.13) by integrating over x. This presents no problem in the compressible case, where in the situation which we described in our thought experiment in section 11.2 the velocity field will be compactly supported (which we are now claiming without a proof [60]). In this case the integration of (11.15) immediately gives the conservation of (11.5).

In the compressible case the velocity field $u(x,t)$ will typically not be integrable. One can define the non-convergent integral $\int_{\mathbf{R}^3} \rho_0 u(x,t)\,dx$ in various ways and obtain the conservation law. One way to do it is to use vorticity. Assume that $v(x)$ is a smooth compactly supported vector field (not necessarily div-free). Then one has

$$\int_{\mathbf{R}^3} x \times \operatorname{curl} v(x)\,dx = 2\int_{\mathbf{R}^3} v(x)\,dx\,. \tag{11.17}$$

[60] The proof is not hard but we will not pursue it at this point. However, we saw the finite speed of propagation of disturbances for the linearized equations in lecture 5.

Let us therefore look at the evolution of the quantity

$$\int_{\mathbf{R}^3} x \times \omega(x,t)\, dx \tag{11.18}$$

in the situation described in our thought experiment in section 11.2. For simplicity we replace f by $\rho_0 f$ in what follows. Taking curl of the Euler equation (5.16) and using (5.17) we obtain, similarly as in lecture 7

$$\omega_t + [u,\omega] = \operatorname{curl} f\,. \tag{11.19}$$

Hence

$$\frac{\partial}{\partial t}\int_{\mathbf{R}^3} x\times\omega\, dx + \int_{\mathbf{R}^3} x\times[u,\omega]\, dx = \int_{\mathbf{R}^3} x\times\operatorname{curl} f\, dx = 2\int_{\mathbf{R}^3} f(x,t)\, dx\,. \tag{11.20}$$

As an exercise, you can check that, when ω is compactly supported (which is our case here), then

$$\int_{\mathbf{R}^3} x \times [u,\omega]\, dx = 0\,. \tag{11.21}$$

Therefore we obtain

$$\frac{\partial}{\partial t}\ \frac{1}{2}\int_{\mathbf{R}^3} x\times\omega\, dx = \int_{\mathbf{R}^3} f(x,t)\, dx\,. \tag{11.22}$$

This shows that, in the thought experiment described above, the quantity

$$\frac{1}{2}\int_{\mathbf{R}^3} x\times\omega\, dx \tag{11.23}$$

will be conserved after time t_2, and its value (after t_2) will be

$$\int_{t_1}^{t_2}\int_{\mathbf{R}^3} f(x,t)\, dx\, dt \tag{11.24}$$

which can be thought of as the total impulse applied to the fluid. Therefore the quantity (11.23) should represent the total momentum of the fluid.[61] We note that it is the same quantity as that giving the leading order term for u at large distances, see (10.26), (12.27). Therefore the leading order term of the large distance behavior is a conserved quantity, and depends only on the total impulse given to the fluid (assuming the fluid was initially at rest). We emphasize that this assumes the incompressibility of the fluid and that the density is constant. We also remark that due to the identities (10.22) the conservation of (11.23) is equivalent to the conservation of $\int_{\mathbf{R}^3} x_j\omega_k(x)\, dx$ for $j,k = 1,2,3$.

[61] In fact, as an exercise you can prove the following: consider a smooth radial compactly supported function φ with $\varphi = 1$ in a neighborhood of the origin, let ω be compactly supported div-free vector field, and let u be obtained from ω by the Biot-Savart law (10.12). Then

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^3} u\varphi_\varepsilon\, dx = \frac{1}{2}\int_{\mathbf{R}^3} x\times\omega(x)\, dx\,. \tag{11.25}$$

We see that in our situation the integral $\int_{\mathbf{R}^3} u\, dx$, though not absolutely convergent, can be defined for example in this way.

11.4 Conservation of the angular momentum in fluids.

The angular momentum is

$$\int_{\mathbf{R}^3} x \times \rho u \, dx \, . \tag{11.26}$$

The meaning of this integral is clear when u is compactly supported, as will be the case for the compressible fluids in our thought experiment above. In this case its conservation can be proved by a direct calculation, which in the end relies on the fact the tensor T_{ij} defined by (11.7) is symmetric.

In the incompressible case the integral (11.26) is not convergent. We can however replace it by

$$\frac{1}{3} \int_{\mathbf{R}^3} x \times (x \times \omega) \, dx \tag{11.27}$$

which can be shown to be a conserved quantity, with a calculation similar to (11.20), and the identity

$$\int_{\mathbf{R}^3} x \times (x \times \operatorname{curl} v(x)) \, dx = 3 \int_{\mathbf{R}^3} x \times v(x) \, dx \tag{11.28}$$

playing the role of (11.17). Due to identities (10.22) the vector (11.27) is proportional to $\int_{\mathbf{R}^3} |x|^2 \omega \, dx$ or to $\int_{\mathbf{R}^3} x(x\omega) \, dx$. From (10.28) we see that the second term in the asymptotics of u does not contain information about these quantities.

12

10/3/2011

12.1 Conservation of energy in fluids

Let us first consider the incompressible case. In this case the only form of energy of the fluid we have to deal with (in the model we consider) is the kinetic energy.[62] The kinetic energy is given by

$$E = \int_{\mathbf{R}^3} \frac{1}{2}\,\rho_0\,|u(x,t)|^2\,dx\,. \tag{12.1}$$

We note that when $u = O(|x|^{-3})$ as $x \to \infty$ (as we expect in our thought experiment from section 11.2), then the integral is clearly convergent. To see that E is conserved, we consider the identity

$$\partial_t \left(\rho_0 \frac{|u|^2}{2}\right) + \operatorname{div}\left[u\left(\rho_0\frac{|u|^2}{2} + p\right)\right] = f(x,t)\,u\ , \tag{12.2}$$

which can be obtained by multiplying the Euler equation (5.16) by u. This identity shows what is happening locally with the density of kinetic energy $\rho_0\frac{|u|^2}{2}$. Assuming that $f = 0$ (no outside forces) and integrating (12.2) over a fixed volume $\mathcal{O}$, we obtain

$$\partial_t \int_{\mathcal{O}} \rho_0 \frac{|u|^2}{2} = \int_{\partial\mathcal{O}} \left(-(u\,n)\rho_0\frac{|u|^2}{2} - (u\,n)p\right) dx\,. \tag{12.3}$$

The first term on the right expresses the rate of change of the energy in $\mathcal{O}$ due to particles carrying energy arriving and leaving $\mathcal{O}$, whereas the second term represents the rate of change change of the energy in $\mathcal{O}$ due to forces generated by the pressure. (Recall that in the ideal fluids these are the only forces due to the interaction between the fluid particles.) In our thought experiment in section 11.2 we expect $u = O(|x|^{-3})$ and $\nabla p = O(|x|^{-3})$ [63]. This means that (12.2) can be integrated over $\mathbf{R}^3$, and when $f = 0$ we obtain that E is constant, as expected.

We can also check what happens with the energy in a volume "moving with the fluid", i. e. $\mathcal{O}^t = \phi^t(\mathcal{O})$, with the usual meaning of ϕ^t, see lecture 2. Using (3.11) and (12.2) with $f = 0$, we obtain

$$\partial_t \int_{\mathcal{O}_t} \rho_0 \frac{|u|^2}{2}\,dx = \int_{\partial\mathcal{O}_t} -(u\,n)p\;dx\,. \tag{12.4}$$

[62] This is similar for finite systems of particles when the interaction energy V in (11.3) is replaced by rigid constraints. For example if distances between the particles are fixed and the particles form an ideal rigid body, there is no interaction energy, and the kinetic energy is conserved (assuming there are no outside forces acting on the body, of course). On the other hand, in the case when the particles are connected by rods of great but still finite stiffness, one has to consider the potential energy related to the deformations of the rods to obtain the energy conservation.

[63] when $f \neq 0$; for $f = 0$ we expect $p = O(|x|^{-4})$ as discussed in lecture 10.

This is exactly what we should expect, since this time there is no transport of the fluid particles through the boundary of $\mathcal{O}_t$.

Let us now turn to the compressible case. In this case we also have to include the potential energy related to the volume changes of the fluid. We assume a relation $p = p(\rho)$ between the density and the pressure.[64] We also emphasize that in what follows we assume that the solutions are sufficiently regular. It is known that the compressible Euler equations can generate singularities from smooth initial data, and for some singularities the conservation of energy can be violated once a singularity is reached.
The kinetic energy is naturally given by

$$\int_{\mathbf{R}^3} \rho \frac{|u|^2}{2}\, dx\,. \tag{12.5}$$

The potential density should be given by

$$F(\rho(x,t))\, dx\,, \tag{12.6}$$

where F expresses the energy due to compression of a unit volume of the fluid of density ρ. To calculate F in terms of the function $p(\rho)$, we expand the unit volume of fluid to the whole space and calculate how much energy this will generate. If during the expansion the original unit volume expanded to volume V, the density will be $\frac{\rho}{V}$ and the pressure will be $p(\frac{\rho}{V})$. The work done by an infinitesimal expansion $V \to V + dV$ is $p(\frac{\rho}{V})\, dV$. Hence

$$F(\rho) = \int_1^\infty p(\frac{\rho}{V})\, dV \;= \int_0^1 p(s\rho)\frac{ds}{s^2} = \int_0^\rho \frac{\rho\, p(\sigma)}{\sigma^2}\, d\sigma\,. \tag{12.7}$$

Note that this assumes that the function $\frac{p(\rho)}{\rho^2}$ is integrable. [65] From (12.7) one can see that

$$F'(\rho) = \int_0^\rho \frac{p'(\sigma)}{\sigma}\, d\sigma\,, \tag{12.8}$$

so that F' can be identified (modulo a constant) with the function P in (7.4). One can also see from (12.7) that

$$\rho F' - F = p\,. \tag{12.9}$$

The energy conservation can be seen from the identity

$$\left(\rho\frac{|u|^2}{2} + F(\rho)\right)_t + \operatorname{div}\left[\, u\, \left(\rho\frac{|u|^2}{2} + \rho F'(\rho)\right)\right] = f(x,t)u\,, \tag{12.10}$$

[64] In other words, the fluid is assumed to be barotropic.

[65] The case $p(\rho) = c\rho$ is sometimes also considered. In this case one can define F from equation (12.9) below, and there is an interpretation to fit this case into the picture considered above via a suitable re-calibration procedure. We will not go into the details.

which can be obtained by multiplying the Euler equation (5.18) by u and suitably re-grouping the terms, taking into account the equation of continuity (5.19). Identity (12.10) can also be re-written, using (12.9), as

$$\left(\rho\frac{|u|^2}{2}+F(\rho)\right)_t+\operatorname{div}\left[u\left(\rho\frac{|u|^2}{2}+F(\rho)+p\right)\right]=f(x,t)u\,, \tag{12.11}$$

and we see that the situation is in fact quite similar to the incompressible case: the term $u(\rho\frac{|u|^2}{2}+F(\rho))$ expresses the transport of energy due to motion of the fluid and the term $u\,p$ expresses the changes of energy due to the forces associated with the pressure. When $f=0$ and we consider a domain $\mathcal{O}_t$ which moves with the fluid, we obtain, in a similar way as in the incompressible case

$$\partial_t\int_{\mathcal{O}_t}\left(\rho\frac{|u|^2}{2}+F(\rho)\right)\,dx=\int_{\partial\mathcal{O}_t}-(u\,n)\,p\;dx\,, \tag{12.12}$$

which again agrees with the assumption that the only forces by which the fluid particle interact with each other are due to the pressure.

12.2 Conservation of helicity

Consider again the situation described in 11.2. The quantity

$$\int_{\mathbf{R}^3}u\omega\,dx=\int_{\mathbf{R}^3}u_i\omega_i\,dx \tag{12.13}$$

is called *helicity*. It is a conserved quantity, which can be seen easily from our second version of Kelvin's circulation theorem, see (9.13). We can use (9.13) with $v=\omega$, and we see that (12.13) is a conserved quantity. The helicity is a topological quantity in that is does not change if change ω to $\tilde{\omega}=\psi_*\omega$ for a diffeomorphism ψ (assumed to approach the identity at ∞), and also change u to $\tilde{u}$, calculated from $\tilde{\omega}$ using the Biot-Savart law. Helicity is related to the mutual entanglement of the integral lines of the vorticity field. We will not study this quantity in the near future, but we list it here for completeness.

12.3 First observations about the incompressible fluid motion

Let us consider an incompressible fluid in a bounded domain. We can think about a closed large rigid box Ω completely filled with an incompressible fluid.[66] We assume that at time $t=0$ the vorticity field is $\overline{\omega}$. The velocity field is determined from the equations

$$\operatorname{curl}u=\omega,\quad \operatorname{div}u=0,\quad \text{in }\Omega\text{ and }\quad u\,n=0\text{ at }\partial\Omega\,, \tag{12.14}$$

[66] Our considerations will be quite rough, so we do not have to pay attention to issues like the possible lack of smoothness of the boundary at the edges of the box. In any case, we can consider that the edges are smoothed out.

where n is the unit normal to the boundary as usual. (So far we have only studied this system in $\mathbf{R}^3$ in lecture 10, but the situation in Ω is not significantly different for our purposes in connection with the situation we will consider.)

When considering the fluid in the box Ω rather than in $\mathbf{R}^3$, we will not have the conservation of the total momentum or the total angular momentum.[67] These global conservation laws do not seem to be important for the local study of the fluid in the areas of the high vorticity, as the components of the densities $x \times \omega$ and $x \times (x \times \omega)$ can change sign and therefore the value of their integrals do not put significant restrictions on the vorticity field. Therefore we will not lose much in a situation when these conservation laws are not valid.

In Ω we consider a circle

$$a(s) = (r \cos s, r \sin s, 0)\,, \qquad s \in [0, 2\pi)\,. \tag{12.15}$$

(We assume that Ω is sufficiently large so that all our constructions will fit into it.) In addition to the circle, let us also consider a second circle

$$b(s) = (0, r + r \cos s, -r \sin s)\,, \qquad s \in [0, 2\pi)\,. \tag{12.16}$$

Note that the two circles are "linked" as in a way similar to two links in a chain. Let us now imagine that a current of unit size passes through circle a in the direction of the derivative $a'(s)$. We now smooth the current to a smooth div-free vector field A supported in a tube of radius ε with center a, similar to what we discussed in lecture 9 in connection with formulae (9.10), (9.11). We assume that the integral lines of the field A are again circles. The support of the field A is a solid torus (= a doughnut shape), with the curve b passing through its opening. We now take a smooth volume-preserving diffeomorphism ψ_0 of Ω and deform the whole configuration quite a bit, so that we create some "waves" on the torus, both in the x_3 direction, and the directions perpendicular to it. (This deformation is done to destroy the symmetries of the field A, so that the evolution becomes more chaotic. The field A itself represents a nice "vortex ring" and its evolution can be quite regular, at least until it hits the boundaries of Ω several times – we will discuss the motion of such vortex rings later. For now we can say that under the Euler evolution the undeformed vorticity field A will move up the x_3 axis, perhaps with some secondary "inner motions" in it, and as long as it stays away from $\partial\Omega$.) Let us assume that $r \sim 20$ and $\varepsilon \sim 1$ and let us denote the deformed field A by $\overline{\omega}$. The motion starting with the vorticity field $\overline{\omega}$ will not be simple, and it may look quite chaotic. Strictly speaking, we do not know for how long the equations of motion can be solved, it is conceivable that the solution will develop a singularity in finite time.[68] Let

[67] the angular momentum is conserved when Ω is a ball, though.

[68] The opinions as to whether or not solutions of incompressible Euler's equation can develop a singularity do not seem to be uniform. In the 1980s and 1990s several authors reported evidence for singularities in numerical simulations, but in some cases the interpretation of these calculations had to be re-evaluated after a higher resolution calculation did not support the conclusion, and in other cases there is still no uniform agreement as to what the calculations say. The difficulties in the numerics are significant. In fact, the 3d incompressible Euler equation may be one of the hardest equations to solve numerically with some reliability.

us nevertheless assume that a smooth solution exists during the relatively long time interval we are interested in.

We note that for any closed curve Γ which can be deformed to the curve b without crossing the support of $\overline{\omega}$ we will have

$$\oint_\Gamma \overline{u}\, dx = 1\,. \tag{12.17}$$

This follows from the formula (9.15) applied to a surface Σ with $\partial\Sigma = \Gamma - b$, such that $\overline{\omega}$ vanishes on Σ.

Assume that at time $t = 0$ we dye the support of $\overline{\omega}$ dark red. We know from the Helmholtz theorem that at a later time t the support of the vorticity field $\omega(t)$ will be exactly where the dye is at t. How will the dye be distributed after a sufficiently long time? We know that the volume of the red region has to be constant and the density of the dye at each point is also constant. However, can the motion deform the red region into some fine structures in such a way that from a distance all the fluid will look light red and only if we look closer we will see that the seemingly uniform light red color is actually caused by a distribution of some fine structures obtained by a complicated deformation of the original support of $\overline{\omega}$? We do not really know. There appears to be no obvious obstacle to such a scenario, although the conservation of energy does put some constraints on the possible fine structures, see the comments about "folding" following (12.27) below.

We can make some interesting conclusions from the Helmholtz and the Kelvin laws. For example, let us take the curve $\overline{\gamma}(s) = \psi_0(a(s))$, the deformation of the original circle a by the diffeomorphism ψ_0 above. Let us assume that $|A| = 1$ on the circle a. Let $\bar{l}$ be the length of the curve $\overline{\gamma}$. We have

$$\bar{l} = \int_0^{2\pi} |\overline{\gamma}'(s)|\, ds\,. \tag{12.18}$$

We note that

$$\overline{\omega}(\overline{\gamma}(s)) = \nabla\psi_0(a(s))\frac{a'(s)}{2\pi r} \tag{12.19}$$

We imagine that the derivatives of ψ_0 are of order unity, so that

$$\bar{l} \sim 2\pi r\ , \tag{12.20}$$

where by $\sim$ we have in mind an equality modulo a multiplicative factor relatively close to 1, such as 2 or $\frac{1}{2}$. It is quite plausible that after some time the length l^t of the curve

$$\gamma^t = \phi^t(\overline{\gamma})\,, \tag{12.21}$$

where ϕ^t is the usual Lagrangian map (see lecture 2) is much larger than $\bar{l}$. We have

$$l^t = \int_0^{2\pi} |(\gamma^t)'(s)|\, ds = \int_0^{2\pi} |\nabla\phi^t(\overline{\gamma}(s))\overline{\gamma}'(s)|\, ds\,. \tag{12.22}$$

By Helmholtz's law we have

$$\nabla\phi^t(\overline{\gamma}(s))\frac{\overline{\gamma}'(s)}{2\pi r} = \omega(\phi^t(\overline{\gamma}(s)),t) \tag{12.23}$$

and therefore

$$\text{the average size of } \omega(\phi^t(\overline{\gamma}(s)),t) \text{ (with respect to } s) \quad \sim \frac{l^t}{\overline{l}}\,. \tag{12.24}$$

We see that there is a potential for a significant growth of ω during the evolution.

Let us now fix some time t and consider any curve C which can be obtained by a continuous deformation of the curve $\phi^t(\Gamma)$ (with the same meaning of Γ as above) outside of the support of $\omega(x,t)$. By Kelvin's theorem we have

$$\oint_C u(x,t)\,dx = \oint_\Gamma \overline{u}(x)\,dx = 1\,. \tag{12.25}$$

Denoting by $|C|$ the length of the curve C, we see that

$$\text{the average velocity } |u(x,t)| \text{ along the curve } C \text{ will be at least } \tfrac{1}{|C|}\,. \tag{12.26}$$

If $|C|$ can be taken small, which means that the (deformed) torus $\phi^t(K)$, where K being the support of $\overline{\omega}$, is "pinched" in at least one place, than the velocity u will be quite large near the pinched area of the deformed torus. This illustrates the effects of the "vorticity stretching" which will discuss some more next time.

In all these considerations we must keep in mind the energy conservation. Basically, we can summarize what we know about the vorticity field ω at time t in the following.

1. $\omega = \phi_*\overline{\omega}$ for some volume preserving diffeomorphism ϕ. (Of course, $\phi = \phi^t$, our Lagrangian map from lecture 2.)
2. Energy conservation. If u is the velocity field generated by ω, and $\overline{u}$ is the velocity field generated by $\overline{\omega}$, then

$$\int_\Omega |u|^2\,dx = \int_\Omega |\overline{u}|^2\,dx\,. \tag{12.27}$$

The energy conservation (12.27) does put constraints on ϕ. For example, let us assume that ω will keep the form of a tube around a curve, except that the curve will become much longer and the tube will become much thinner (to preserve the volume). "Most" long thin tubes we would draw without energy considerations will have too much energy. To conserve energy, one possibility is that sections of our long thin tube will "fold", such as when a curve makes a sharp 180 degrees turn and runs back to an existing section in the opposite direction for a while. This way we can make long thin tubes and conserve energy at the same time. Of course, if some scenario is consistent with 1 and 2 above, it does not mean that it can be realized by some actual solutions of the equations, but some signs of the folding mentioned above are seen in the real solutions.

13

10/5/2011

13.1 Vorticity stretching

Last time we saw the potential for the growth of the vorticity field $\omega(x,t)$ during the evolution by Euler's equations. The possible stretching effect is already apparent in the Helmholtz law

$$\omega(\phi^t(x),t) = \nabla\phi^t(x)\omega(x,0)\,. \tag{13.1}$$

The matrix $A = \nabla\phi^t(x,t)$ satisfies $\det A = 1$, and, as for any matrix with positive determinant, we can write it as

$$A = QB \tag{13.2}$$

where Q is a rotation ($Q \in SO(3)$) and B is a symmetric positive-definite matrix. In a suitable coordinate frame we have

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{13.3}$$

where

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0, \qquad \lambda_1\lambda_2\lambda_3 = 1\,. \tag{13.4}$$

It is very plausible that after a relatively long evolution, the largest eigenvalue λ_1 will be typically quite large, and therefore the ω_1 component (in the frame we are considering) will be significantly magnified by (13.1). Therefore, unless the first eigenvector e_1 of B (which depends on (x,t)) stays practically perpendicular to $\omega(x,0)$, the vorticity will be stretched. The reversibility of the equation implies that there will be solutions where the stretching does not happen at least for some periods of time: if we take a solution which stretches the vorticity and run in backward, we do not get expect stretching effect in the backward solution. However such solutions without the stretching seem to be quite exceptional, somewhat similarly to the solutions of the equations of motion for molecules of a mixture of two gases which "unmix" the two gases. Such solutions must exist (we just run backward the solutions which mix), but we can see them only for very carefully prepared initial data. For "generic" data the molecules of the two gases do not "unmix". In a similar way, we expect that for the "generic smooth data", the solutions of the Euler equations will stretch the vorticity. Statements in this direction are notoriously hard to prove rigorously and for Euler's equation there are no rigorous results of such form, although numerical simulations and observations of fluids seem to support the discussed scenarios.

The potential for the stretching effect can also be seen directly from the vorticity equation (7.12). Let us write the equation in the form

$$\omega_t + u\nabla\omega = \omega\nabla u\,. \tag{13.5}$$

The term on the left is the "material derivative" of the vorticity: we follow a fixed particle of the fluid, observe its vorticity as a time-dependent vector, and take the time derivative. Often the notation

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\nabla \tag{13.6}$$

is used, so that we can write (13.5) as

$$\frac{D\omega}{Dt} = \omega\nabla u\,. \tag{13.7}$$

Let us now look at the term $\omega\nabla u$. Let us write (at a given point (x,t))

$$\nabla u = S + A\,, \tag{13.8}$$

where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the symmetric part of ∇u and $A_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ is the anti-symmetric part of ∇u. By the definition of ω, we have $A\,y = \frac{1}{2}\,\omega \times y$ and therefore $A\omega = 0$. Hence we can write

$$\omega\nabla u = (\nabla u)\omega = (S + A)\omega = S\omega\,. \tag{13.9}$$

The matrix S is sometimes called the deformation matrix. (In some texts it is denoted by D, and it can be also denoted by e_{ij}.) Equation (13.7) can now be written as

$$\frac{D\omega}{Dt} = S\omega\,, \tag{13.10}$$

where $S = S(t)$ is the deformation matrix at the position of the moving particle we are following. For the vorticity vector $\omega(t)$ of the moving particle this is just the ODE

$$\frac{d}{dt}\omega(t) = S(t)\omega(t)\,. \tag{13.11}$$

For any given particle this equation is not "closed", as $S(t)$ can depend on the whole field $\omega(x,t)$, and not just on the single vector $\omega(t)$. If we consider $S(t)$ as given, then of of course (13.11) gives the evolution of $\omega(t)$. The matrix $S(t)$ is symmetric, and its trace vanishes, due to the incompressibility condition $\operatorname{div} u = 0$. Therefore, for any given time t we have, in a suitable coordinate frame (depending on t)

$$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{13.12}$$

where

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\,. \tag{13.13}$$

Assuming the coordinate frame is chosen so that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3\,, \tag{13.14}$$

we can now repeat the considerations following (13.3) at the "differential level". After some time we expect λ_1 to be generically quite large, and therefore $\omega(t)$ will be stretched in the e_1 direction (which may depend on time, of course).

For the special case when the velocity field $u(x,t)$ is linear in x, the solutions of (13.11) provide examples of exact solutions of Euler's equations. (They may not be very physical when considered in all space $\mathbf{R}^3$, due to their linear growth as $x \to \infty$, but locally they are of interest and illustrate some aspects of the vorticity stretching.) Assuming

$$u(x,t) = S(t)x + \frac{1}{2}\,\omega(t) \times x \tag{13.15}$$

the full vorticity equation (13.5) becomes exactly (13.11). Hence the field (13.15) will solve the Euler equations (for a suitable pressure) if and only if (13.11) is satisfied. The time-dependent matrix $S(t)$ can be arbitrarily chosen, and then (13.11) can be solved for ω, after choosing an initial condition $\omega(0) = \omega_0$. For example, we can choose $S(t)$ to be a constant diagonal matrix of the form (13.12), and obtain

$$\omega(t) = \begin{pmatrix} e^{\lambda_1 t}\omega_{01} \\ e^{\lambda_2 t}\omega_{02} \\ e^{\lambda_3 t}\omega_{03} \end{pmatrix}. \tag{13.16}$$

Equation (13.11) can also be used for some heuristics concerning the possible blow-up scenarios for the solutions of Euler's equations. If we know the vorticity field $\omega(x,t)$, we can calculate the velocity field $u(x,t)$ from the Biot-Savart law and then calculate the matrix S at all points. From (4.8) and (4.12) we see that

$$\int_{\mathbf{R}^3} 2|S|^2\,dx = \int_{\mathbf{R}^3} |\omega|^2\,dx\,. \tag{13.17}$$

So if we compare the sizes of S and ω in the space L^2, we can say

$$||\omega|| \sim ||S|| \quad \text{in } L^2\,. \tag{13.18}$$

We can now speculate about a situation which would most likely lead to a finite-time blow-up for an initially smooth solution in the following terms:

1. It might perhaps be possible that for some solutions we not only have $||\omega|| \sim ||S||$ in L^2, but also $|S(t)| \sim |\omega(t)|$ at some moving particle.

2. In addition, it might perhaps be even possible that at this particle the direction of the eigenvector of $S(t)$ corresponding to the largest eigenvalue and the direction of $\omega(t)$ are always aligned to the degree that not only $|S(t)\omega(t)| \sim |\omega(t)|^2$, but in fact

$$\frac{d}{dt}|\omega(t)| \sim |\omega(t)|^2\,. \tag{13.19}$$

The solution of the equation

$$\dot{y} = cy^2, \qquad y(0) = y_0 \tag{13.20}$$

is

$$y(t) = \frac{y_0}{1 - cty_0} \tag{13.21}$$

and blows up at time $T = \frac{1}{cy_0}$ with blow-up rate $\sim \frac{1}{T-t}$.
Therefore, if it is possible for some solutions to really achieve the necessary alignments, these solution will blow-up at some finite time T with the blow-up rate

$$\sup_x |\omega(x,t)| \sim \frac{1}{T-t}\,. \tag{13.22}$$

In fact, to have the blow-up, the alignments do not have to be so perfect, it is enough to achieve

$$\frac{d}{dt}|\omega(t)| \geq c|\omega(t)|^{1+\varepsilon} \tag{13.23}$$

for some $\varepsilon > 0$. The solution of

$$\frac{d}{dt}y = cy^{1+\varepsilon}\,, \qquad y(0) = y_0 \tag{13.24}$$

is

$$y(t) = \frac{y_0}{(1 - \varepsilon c y_0^{\varepsilon} t)^{\frac{1}{\varepsilon}}} \tag{13.25}$$

and it blows up at time $T = \frac{1}{\varepsilon c y_0^{\varepsilon}}$ at rate $\sim \frac{1}{(T-t)^{\frac{1}{\varepsilon}}}$. Note that the smaller the ε is, the longer it takes the equations to blow up, and the higher the power in the blow-up rate is. Therefore we can say that the slower the blow-up is, the higher the power in the blow-up rate is. This may look counter-intuitive at first, but it actually makes a lot of sense: in a slower blow-up a solution has to build up the size more gradually, and therefore it has to be large longer before it finally reaches ∞.

The above considerations suggest that the lowest power in the and possible blow-up rate of $\sup_x |\omega(x,t)|$ in Euler's equations is at least

$$\sim \frac{1}{T-t}\,. \tag{13.26}$$

The stretching mechanism we described is not strong enough to produce faster blow-up rates (which - as explained above - would have a lower power) such as

$$\sim \frac{1}{(T-t)^{1-\delta}} \tag{13.27}$$

for any $\delta > 0$. There is a rigorous results, due to Beale-Kato-Majda[69] which confirms this expectation. Let us formulate it in the context of a known existence result about the solutions of Euler's equations.

[69] J. T. Beale, T. Kato and A. J. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys., 94(1984), 61-66.

Let ω_0 be a smooth, compactly supported div-free field. Consider the initial value problem

$$\begin{aligned} \omega_t + [u, \omega] &= 0 \qquad \text{in } \mathbf{R}^3 \times [0, T)\,. && (13.28) \\ \omega(x, 0) &= \omega_0(x), \qquad x \in \mathbf{R}^3\,, && (13.29) \end{aligned}$$

where at each time the velocity field u is obtained from $\omega(x,t)$ by the Biot-Savart law (10.12) and we require that at each time the vorticity field maintains its fast decay at ∞. (In fact, we can demand that ω stays compactly supported in x.)
We have the following result:

Theorem

In the situation as above we have the following results

1. A smooth solution[70] *(with fast decay of ω as $x \to \infty$) of problem (13.28), (13.29), if it exists, is unique.*

2. There always exists some positive time $T > 0$ such that a smooth solution with fast decay of ω at ∞ exists in $\mathbf{R}^3 \times [0, T)$. (If the support of ω_0 is compact, then the support of $\omega(x,t)$ will be compact for each $t \in [0, T)$.)

3. (Beale-Kato-Majda criterion)
If a smooth solution in $\mathbf{R}^3 \times [0, T)$ with fast decaying vorticity cannot be continued beyond T, then

$$\int_0^T \sup_x |\omega(x,t)|\, dt = +\infty\,. \qquad (13.30)$$

At this point we will not go into the proof of these results. The interested readers can consult the book "Vorticity and Incompressible Flow" by A. Majda and A. Bertozzi for details.

Note that the criterion (13.30) is in agreement with our heuristic expectations concerning the possible blow-up rates, see (13.26), (13.27).

[70] The full smoothness can be relaxed to "sufficient regularity". The problem of finding optimal low-regularity classes of solutions in which uniqueness remains valid is difficult.

14

10/7/2011

14.1 Two-dimensional incompressible flows

So far we have considered the fluids in three space dimensions. It is also useful to consider two-dimensional flows. In reality it is quite hard to realize purely two dimensional flows in practice, but there are some situations where fluid flows are nearly two-dimensional, and therefore the study of the idealized situation where the flow is exactly two-dimensional is useful. Also, from the purely mathematical point of view, it is interesting to see what happens in dimension 2. The situation is quite simpler than for the 3d flows, especially for the incompressible flow on which we will focus our attention. At the same time, there are still various non-trivial phenomena and many open problems.

We will consider a domain $\Omega \subset \mathbf{R}^2$ with smooth boundary and divergence-free vector fields u in Ω satisfying the boundary condition

$$u\, n = 0 \qquad \text{at } \partial\Omega, \tag{14.1}$$

where n is the outward unit normal. The whole picture can be of course imbedded in three dimensions by considering the domain

$$\tilde{\Omega} = \Omega \times \mathbf{R} \tag{14.2}$$

with the normal $\tilde{n} = (n_1, n_2, 0)$ and the vector field

$$\tilde{u}(x) = \begin{pmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \\ 0 \end{pmatrix}. \tag{14.3}$$

The vorticity of $\tilde{u}$ is

$$\tilde{\omega} = \operatorname{curl} \tilde{u} = \begin{pmatrix} 0 \\ 0 \\ u_{2,1} - u_{1,2} \end{pmatrix}, \tag{14.4}$$

where, as usual, $u_{i,j}$ is used to denote the partial derivative $\frac{\partial u_i}{\partial x_j}$. It is therefore natural to define the vorticity in two dimensions as the scalar

$$\omega = u_{2,1} - u_{1,2}\,. \tag{14.5}$$

As in dimension three, the condition $\operatorname{curl} u = 0$ is necessary and sufficient for the field u to be *locally* expressible as a gradient of a scalar function, $u = \nabla\varphi$. The condition

$$\operatorname{div} u = u_{1,1} + u_{2,2} = 0 \tag{14.6}$$

can be written as

$$\operatorname{curl} \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} = 0\,, \tag{14.7}$$

and hence we can *locally* write

$$u = \begin{pmatrix} -\psi_{,2} \\ \psi_{,1} \end{pmatrix}\,. \tag{14.8}$$

Under our assumptions and with the boundary condition $u\,n = 0$ this is possible even globally[71]. To see this, we recall that a necessary and sufficient condition for the existence of a function ψ with

$$\nabla\psi = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix} \tag{14.9}$$

is that

$$\oint_\gamma u_2\,dx_1 - u_1\,dx_2 \;=\; 0 \tag{14.10}$$

for every closed smooth curve $\gamma \subset \Omega$. In what follows we will assume that Ω is a smooth domain with finitely many boundary components. If $\operatorname{div} u = 0$, the condition (14.10) will be satisfied when

$$\oint_{\Gamma_j} u_2\,dx_1 - u_1\,dx_2 = 0 \tag{14.11}$$

for every bounded connected component Γ_j of $\partial\Omega$. This translates to

$$\int_{\Gamma_j} u\,n = 0\,, \tag{14.12}$$

which is automatically satisfied when $u\,n = 0$.

We see that *with the boundary condition (14.1) we have the representation (14.8) even globally.*[72]

The function ψ is called the *stream function*. It can be represented as

$$\psi(x) = \oint_{\gamma_{a,x}} u_2\,dx_1 - u_1\,dx_2 = \int_{\gamma_{a,x}} u\,n\,dx \tag{14.13}$$

where a is some given point in Ω, $\gamma_{a,x}$ is a curve in Ω joining a and x, and n is a vector perpendicular to $\gamma_{a,x}$ oriented so that the orientation of the pair of vectors $\gamma'_{a,x}(s), n$ (with n taken at $\gamma_{a,x}(s)$) is positive. The conditions $\operatorname{div} u = 0$ in Ω and $u\,n = 0$ at $\partial\Omega$ guarantee that the definition is independent of a particular choice of $\gamma_{a,x}$.

[71] under reasonable assumptions on Ω, see below

[72] Under our assumptions that Ω is smooth with finitely many boundary components. This assumption can be still further relaxed, but this is not important for our purposes here.

The stream function can also be understood in terms of the vector potential of the field $\tilde{u}$. Letting

$$\tilde{A} = \begin{pmatrix} 0 \\ 0 \\ -\psi(x_1, x_2) \end{pmatrix}, \tag{14.14}$$

we have

$$\tilde{u} = \operatorname{curl} \tilde{A}\,. \tag{14.15}$$

Remark
Let X^h be the space of the smooth vector fields of the form (14.3) and let X^v be the space of the smooth vector fields of the form (14.14). It is easy to see that the operation $v \to \operatorname{curl} v$ maps X^h into X^v and vice versa.

We can easily check that

$$\omega = \operatorname{curl} u = \Delta\psi\,. \tag{14.16}$$

The representation (14.8) is often written as

$$u = \nabla^{\perp}\psi \tag{14.17}$$

or

$$\omega = J\nabla\psi \tag{14.18}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{14.19}$$

We note that the integral lines of u [73] are the connected components of the level sets

$$\{\psi = c\}\,, \tag{14.20}$$

at least when we avoid various degenerate situations by assuming that $\nabla\psi \neq 0$ on $\{\psi = c\}$. (By Sard's theorem this assumption is satisfied for almost every c when u is smooth.) If we apply these observation to the case $\Omega = \mathbf{R}^2$ and a compactly supported u (with $\operatorname{div} u = 0$), we see that almost all integral lines of u are smooth closed curves and, in some sense, the vector field u is a "continuous linear combination" of currents in closed loops which we discussed in lecture 9, just before (9.10). In dimension three such simple picture cannot be true, but some of the properties the simple fields obtained as continuous sums of loops (such as the relations (10.22)) survive the passage to general div-free fields.

The Euler equations (5.16),(5.17) remain the same in dimension two, and the same is true about Kelvin's circulation theorem (lecture 9). However, there is a significant simplification in the vorticity equation and the Helmholtz law.

Let us first look at the vorticity equation for the natural 3d extensions $\tilde{u}$ and $\tilde{\omega}$ defined by (14.3), (14.4). In lecture 13 we discussed the stretching term $\omega\nabla u$

[73] also called the streamlines in our context

for the 3d flows. It is easy to see that for our special fields $\tilde{u}$ and $\tilde{\omega}$ of the form (14.3) and (14.4) we have

$$\tilde{\omega}\nabla\tilde{u} = 0\,. \tag{14.21}$$

This means that there is no vorticity stretching. In the 2d notation, the equation for the scalar $\omega = \operatorname{curl} u$ is

$$\omega_t + u\nabla\omega = 0\,, \tag{14.22}$$

which is the transport equation (2.16) we discussed in lecture 2. This means that the scalar ω is transported with the flow given by $u(x,t)$. In terms of the Lagrangian maps ϕ^t introduced in lecture 2, we have

$$\omega(\phi^t(x),t) = \omega(x,0)\,. \tag{14.23}$$

As simple as this looks, it still leads to non-trivial behavior of solutions which is not completely understood, even though the problems are at a different level than in dimension three, where we cannot answer even the basic question about the existence of the solutions. In dimension two we know that the solutions exist (this is not surprising if we combine (13.30) and (14.23)), but there are many open problems about their behavior over long time intervals.

The equation (14.22) is of course non-linear since u depends on ω. For example, in all space $\Omega = \mathbf{R}^2$ we can write down explicitly the 2d analogue of the Biot-Savart law we discussed in lecture 10 in dimension three. In the 2d situation the problem

$$\operatorname{curl} u = \omega, \quad \operatorname{div} u = 0 \quad \text{in } \mathbf{R}^2, \qquad u(x) \to 0 \text{ as } x \to \infty\,, \tag{14.24}$$

can be solved in a way similar to (10.14), by using equation (14.16). Letting

$$G(x) = \frac{1}{2\pi}\log|x|\,, \tag{14.25}$$

we can write

$$\psi = G * \omega \tag{14.26}$$

and

$$u = \nabla^\perp\psi = K * \omega, \qquad K = \nabla^\perp G\,. \tag{14.27}$$

This is the 2d Biot-Savart law.

We introduce the notation

$$\{\psi,\omega\} = -\psi_{,2}\omega_{,1} + \psi_{,1}\omega_{,2} = \det(\nabla\omega,\nabla\psi)\,. \tag{14.28}$$

(The bracket $\{\psi,\omega\}$ is a special case of the so-called Poisson bracket.) The incompressible Euler equation can then be written as

$$\omega_t + \{\psi,\omega\} = 0\,. \tag{14.29}$$

The boundary condition $u\,n = 0$ at $\partial\Omega$ is equivalent to the condition that ψ be constant on each connected component Γ_j of the boundary $\partial\Omega$ (with the constant depending on the component). In other words

$$\psi|_{\Gamma_j} = c_j\,. \tag{14.30}$$

Under some special conditions the constants c_j can be preserved under the Euler's equations (in the absence of forces), but in general this may not be the case. The natural constants of motion in this context are

$$\gamma_j = \int_{\Gamma_j} \frac{\partial\psi}{\partial n}$$

which can be identified with

$$\int_{\Gamma_j} u_i dx_i\,.$$

From (14.29) and the boundary condition (14.30) one can also see that the equation has many steady states. The steady states are determined by

$$\{\psi, \omega\} = 0\,, \tag{14.31}$$

which says that ω is locally constant along the level sets of ω. In a neighborhood of a point $a \in \Omega$ with $\nabla\psi(a) \neq 0$ this means that $\omega = F_a(\psi)$ for some function F_a. If $\omega = F(\psi)$ everywhere in Ω for some function F, and ψ satisfies (14.30), then ψ gives a steady-state solution of (14.29). There are many such solutions, as the function F can be chosen. For example, any eigenfunction of the laplacian

$$\Delta\psi = -\lambda\psi, \qquad \psi|_{\partial\Omega} = 0 \tag{14.32}$$

gives a steady state solution of Euler's equation. (We emphasize that we assume that the dimension is two.)

14.2 Zhukovski's Theorem

There is a classical result concerning the lift force produced by ideal incompressible flows around a wing which we should mention in connection with 2d flows. Let us consider a connected compact set $K \subset R^2$ and assume that its complement $\Omega = \mathbf{R}^2 \setminus K$ is also connected and has smooth boundary.[74] We can think of the set K as a wing profile. Let us consider steady flows u in Ω which satisfy the following conditions:

1. The vector field u is smooth in Ω (up to the boundary).
2. At the boundary $\partial\Omega$ we have $u\,n = 0$, where n the unit normal of $\partial\Omega$.
3. $\lim_{x\to\infty} u(x) = U$, where $U \in \mathbf{R}^2$ is a given vector.

[74] The assumptions imply that the situation is a deformation of the case when K is a disc.

4. $\operatorname{curl} u = 0, \quad \operatorname{div} u = 0$ in Ω.

We wish to calculate the force F on the wing profile K produced by the flow. We saw in lecture 8 [75] that in dimension $n = 3$ and in the case when Ω is simply connected, our assumptions imply that $F = 0$. The same will be true, by the same proof, in any dimension if we know that $u = \nabla h$ for some smooth function h. The assumptions 1-4 above do not imply that $u = \nabla h$, however. For example, when K is a disc and $u = \nabla^\perp \log|x|$ it is clear that there is no function h in Ω such that $u = \nabla h$. (This is of course due to the fact that Ω is not simply connected, in any domain $\Omega_1 \subset \Omega$ which is simply connected, we do have $u = \nabla h_1$ for some function h_1 defined in Ω_1.)
We aim to show that under the assumptions 1-4 above a flow u which is not of the form $u = \nabla h$ in Ω can generate a "lift force" on the profile K.

Let us consider a curve $\gamma(s) = R\cos s + R\sin s$ where R is sufficiently large so that $\gamma \subset \Omega$. We note that the only obstacle to the field u being a gradient flow in Ω is a possible non-zero value the curve integral

$$\Gamma = \oint_\gamma u\, dx. \tag{14.33}$$

We can write $u = \nabla h$ in Ω if and only if $\Gamma = 0$.

If $\Gamma \neq 0$, we can consider the field

$$\tilde{u} = u - \frac{\Gamma}{2\pi}\nabla^\perp \log|x|\,. \tag{14.34}$$

Clearly

$$\oint_\gamma \tilde{u}\, dx = 0 \tag{14.35}$$

and since $\operatorname{curl} \nabla^\perp \log|x| = 0$, we see that

$$\tilde{u} = \nabla \tilde{h} \tag{14.36}$$

for some function $\tilde{h}$ in Ω. The function $\tilde{h}$ will satisfy $\Delta\tilde{h} =$ in Ω. By (14.34) we have

$$u = \tilde{u} + \frac{\Gamma}{2\pi}\nabla^\perp \log|x| = \nabla\tilde{h} + \frac{\Gamma}{2\pi}\nabla^\perp \log|x|\ , \tag{14.37}$$

and hence for some $a \in \mathbf{R}$ we can write

$$u = U + a\nabla \log|x| + \frac{\Gamma}{2\pi}\nabla^\perp \log|x| + O(|x|^{-2}) = U + \frac{ax}{|x|^2} + \frac{\Gamma\, x^\perp}{2\pi|x|^2} + O(|x|^{-2}), \quad x \to \infty\,. \tag{14.38}$$

Repeating the calculation from lecture 8, we can substitute the expression (14.38) into formula (8.19) and let $R \to \infty$ to obtain

$$F = -\Gamma J U \tag{14.39}$$

[75] See the section on d'Alembert's paradox

where J is rotation by $\pi/2$, i. e.

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{14.40}$$

The expression (14.39) for the force is known as Zhukovski theorem or Zhukovski formula. [76] Note that the assumption $n = 2$ is crucial. The formula gives some idea about the origin of the lift force, but the 3d picture is more complicated. Even in the 2d picture, mathematically it is not quite obvious how a circulation $\Gamma \neq 0$ can be established. In view of Kelvin's circulation theorem, viscosity has to play an important rôle.[77]

[76] After N. E. Zhukovski who derived it around 1906.

[77] The 3d picture of the flow around wings was anticipated already in early 1890s by F. W. Lanchester and is more complicated. You can see some of the typical pictures if you type "trailing vortices" into a search engine. A mathematical analysis of these flows has been achieved only at the level of certain approximations.

Homework Assignment 1 due October 26

Do one or both of the following problems:

Problem 1

Consider a ball of incompressible fluid of constant density ρ and radius R in the otherwise empty space $\mathbf{R}^3$. The fluid is at rest, its velocity field vanishes at each point. Assuming that gravity acts according to Newton's law, [78] calculate the pressure p due to gravity in the fluid at the center of the ball in two ways:

1. Assume that $p = f(\kappa, \rho, R)$, where κ is the gravitational constant, show that the requirement that f be independent of a specific choice of units determines f uniquely up to a multiplicative constant. In other words, use dimensional analysis. (Hint: Use the same methodology which was described in lecture 1.)

2. Use the equations of hydrostatics (lecture 5) to calculate the pressure explicitly.

Problem 2

A rigid ball of radius R and density ρ is suspended (by a long thread of negligible thickness) in the middle of a very large tank of an ideal incompressible fluid of density $\rho_0 < \rho$. Until time $t = 0$ everything is at rest. At time $t = 0$ the ball is released from the suspension and starts descending towards the bottom of the tank (due to the gravitational force). Assuming that the tank is all of $\mathbf{R^3}$, find the formula for the velocity of the ball at time $t > 0$.

(Hint: Use lecture 8 and the following formula: let $U = (U_1, U_2, U_3)$ be a vector in $\mathbf{R}^3$. Then the function $h(x) = (Ux)(1 + \frac{R^3}{2|x|^3})$ describes a potential flow around a ball of radius R with velocity U at ∞.)

[78] This means that a masses two point masses m_1 and m_2 which are at distance r from each attract one another by a force of size $\frac{\kappa m_1 m_2}{r^2}$, where κ is the gravitational constant.

15

10/17/2011

15.1 Point Vortices

Let us first consider a very simple class of solutions of the 2d incompressible Euler equation in all space $\mathbf{R}^2$. We will consider the vorticity form (14.22) of the equation we derived in lecture 14,

$$\omega_t + u\nabla\omega = 0\,. \tag{15.1}$$

where u is given by (14.27). Let $r = |x|$, and let us consider any smooth compactly supported function ω which is radially symmetric, i. e.

$$\omega = \omega(r)\,. \tag{15.2}$$

We let

$$\gamma = \int_{\mathbf{R}^2} \omega\, dx\ . \tag{15.3}$$

It is easy to see that ω will be a steady-state solution of (15.1): the stream function ψ given by (14.27) will clearly also be radially symmetric, $\psi = \psi(r)$, and the field $u = \nabla^\perp\psi$ is tangential to the circles $\{x, |x| = r\}$. As an exercise in the theory of the Laplace equation, you can check that if the support of ω is contained in $\{x, |x| \le R\}$, than for $|x| > R$ the stream function ψ is given exactly by

$$\psi(x) = \frac{\gamma}{2\pi}\log|x|\,. \tag{15.4}$$

The pressure p will be also radially symmetric and will satisfy the equation

$$p_{,r} = \frac{|u|^2}{r}\,. \tag{15.5}$$

Note that the pressure is an increasing function of r. It means that if it is fixed at ∞, its value at $x = 0$ can get low. This family of solutions is easy to understand: the force field generated by the centrifugal force acting on the fluid particles is a gradient field and can be balanced by a suitably chosen pressure. We can consider a family of such solutions with a fixed γ, where the support of ω shrinks to one point. For example, we can set for $\varepsilon > 0$

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon^2}\overline{\omega}(\frac{x}{\varepsilon})\,, \tag{15.6}$$

and let $\varepsilon \to 0$. Note that the corresponding stream functions $\overline{\psi}_\varepsilon$ will converge locally uniformly in $\mathbf{R}^2 \setminus \{0\}$ (together with all their derivatives) to the stream function (15.4). The limit solution can be thought of as

$$\omega = \gamma\delta, \qquad \psi = \frac{\gamma}{2\pi}\log|x|\,, \tag{15.7}$$

where δ is the Dirac function. This solution represents an idealized situation where all the vorticity is concentrated at just one point. Note that the velocity field

$$u = \nabla^\perp \psi \tag{15.8}$$

is not locally square integrable near $x = 0$. This means that the kinetic energy of the field u is infinite in any neighborhood of 0. [79]

It is worth recalling that the following quantities are conserved for the 2d incompressible Euler equation:

$$\frac{1}{2}\rho_0 \int_{\mathbf{R}^2} |u(x,t)|^2 \, dx \qquad \text{(kinetic energy)} \tag{15.9}$$

$$\int_{\mathbf{R}^3} x\omega(x,t) \, dx \tag{15.10}$$

This is the 2d version of the integral $\int_{\mathbf{R}^3} x \times \omega(x,t) \, dx$ and is related to the (possibly not absolutely convergent) integral $\rho_0 \int_{\mathbf{R}^2} u(x,t) \, dx$ which should define the total momentum of the fluid. The conservation of $\int_{\mathbf{R}^2} x\omega(x,t) \, dx$ means that the "center of mass" of the measure $\omega(x,t) \, dx$ is preserved during the evolution.

$$\int_{\mathbf{R}^2} |x|^2 \omega(x,t) \, dx \tag{15.11}$$

This is the 2d version of the integral $\int_{\mathbf{R}^3} x \times (x \times \omega(x,t)) \, dx$ and is related to the (possibly not absolutely convergent) integral $\rho_0 \int_{\mathbf{R}^3} (x \times u(x,t)) \, dx$ which should define the total angular momentum of the fluid.

$$\int_{\mathbf{R}^2} \omega(x,t) \, dx \qquad \text{total vorticity} \tag{15.12}$$

In dimension 3 the analogue is not $\int_{\mathbf{R}^3} \omega \, dx$ (which vanishes for any div-free field ω with sufficiently fast decay at ∞, see lecture 11), but rather the integral $\oint_C u \, dx = \int_\Sigma \omega \, n$, where C is a curve around a vortex, Σ is a surface bounded by the curve, and n is a suitably oriented normal to the surface Σ.

These conservation laws can be used to get some idea about the stability of the radial solutions we have been considering in the case when $\omega \geq 0$. If we assume a vortex solution with a radial non-negative vorticity $\omega = \omega(r)$ which is concentrated near the origin and satisfies $\int_{\mathbf{R}^2} \omega = \gamma$ is slightly perturbed away from a radially symmetric profile to a new function $\tilde{\omega} \geq 0$ we see from the above conservation laws that "most of the mass" $\tilde{\omega}$ will stay close to the origin in the evolution and therefore the solutions with a radial, non-negative, and compactly supported vorticity should be relatively stable under perturbations. At this moment we leave this statement at a heuristic level, we will not go into precise definitions.

[79] We will not address here the question in what sense (if any) the solution satisfies the Euler equation across the origin. The usual weak formulation of the steady incompressible Euler equation for a div-free vector field u in $\mathbf{R}^2$ is that $\int_{\mathbf{R}^2} u_i u_j \varphi_{i,j} = 0$ for each smooth div-free, compactly supported vector field φ. However, in this definition needs that u be locally square integrable, so it does not apply directly to the singular solution we are considering here.

Let us now consider two vortices, one given by a vorticity function $\omega^{(1)}$ which is supported near a point $x^{(1)}$ and radial with respect to it, and one given by $\omega^{(2)}$ which is supported near $x^{(2)}$ and radial with respect to it. We let

$$\gamma^{(j)} = \int_{\mathbf{R}^2} \omega^{(j)} \, dx \,, \qquad j = 1, 2 \,, \tag{15.13}$$

and

$$\omega = \omega^{(1)} + \omega^{(2)} \,. \tag{15.14}$$

We also let $u^{(j)}$ be the velocity field generated by $\omega^{(j)}$ and

$$u = u^{(1)} + u^{(2)} \ . \tag{15.15}$$

We now consider the evolution of this vorticity field under Euler's equation. Let us calculate the time derivative ω_t (at the moment of time when the evolution is started) from the equation of motion (15.1). We have

$$\omega_t = -u\nabla\omega = -(u^{(1)} + u^{(2)})\nabla(\omega^{(1)} + \omega^{(2)}) = -u^{(1)}\nabla\omega^{(2)} - u^{(2)}\nabla\omega^{(1)} \,, \tag{15.16}$$

as the terms $u^{(j)}\nabla\omega^{(j)}, \ j = 1, 2$ vanish (at the moment of time when the evolution started). We see that the vortex (1) is moved by the velocity field generated from vortex (2) and vice versa. At later times the quantity $u^{(j)}\nabla\omega^{(j)} = 0$ may not be exactly zero, but we can think of the limiting situation when each of the vortices is supported at one point. In this case we have

$$u^{(1)}(x^{(2)}) = \frac{\gamma_1}{2\pi}\nabla^{\perp}_{x^{(2)}} \log|x^{(1)} - x^{(2)}| \,, \qquad u^{(2)}(x^{(1)}) = \frac{\gamma_2}{2\pi}\nabla^{\perp}_{x^{(1)}} \log|x^{(1)} - x^{(2)}| \ . \tag{15.17}$$

Therefore we expect that the point vortices will move according to the system of ODEs

$$\frac{d}{dt}x^{(1)} = \frac{\gamma_2}{2\pi}\nabla^{\perp}_{x^{(1)}} \log|x^{(1)} - x^{(2)}| \,, \tag{15.18}$$

$$\frac{d}{dt}x^{(2)} = \frac{\gamma_1}{2\pi}\nabla^{\perp}_{x^{(2)}} \log|x^{(1)} - x^{(2)}| \,. \tag{15.19}$$

It is natural to expect that even when the initial vorticity is not concentrated at the two points but it is smooth and slighly "smeared" around these points (while staying close to them), the evolution will still be similar to the case described by the ODEs (15.18), (15.19) for some time, which will become larger as we focus the support more and more to the two points. This has been proved rigorously by C. Marchioro and M. Pulvirenti (for any number of vortices, under natural assumptions) , but the proof is not easy.[80] The system of ODEs we get for any finite number m of vortices of strengths $\gamma_1, \ldots, \gamma_m$ with trajectories $x^{(1)}(t), \ldots, x^{(m)}(t)$ is an obvious generalization of (15.18), (15.19):

$$\frac{d}{dt}x^{(j)}(t) = \sum_{k \neq j} u^{(k)}(x^{(j)}) \,, \tag{15.20}$$

[80] See the paper Marchioro, C., Pulvirenti, M., Vortices and localization in Euler flows. Comm. Math. Phys. 154 (1993), no. 1, 49-61.

where

$$u^{(k)}(x) = \frac{\gamma_k}{2\pi} \nabla_x^{\perp} \log |x - x^{(k)}| \,. \tag{15.21}$$

As an exercise, you can solve the system (15.18), (15.19) explicitly in the case when $\gamma_1 = \gamma_2$ and in the case when $\gamma_1 = -\gamma_2$ respectively. In the former case the vortices move along a circle, whereas in the latter case they are translated at a constant speed in the direction perpendicular to the segment joining them.

The conservation laws (15.9)– (15.12) give conserved quantities also for the point vortex motion. The corresponding quantities are

$$\sum_{j \neq k} \frac{\gamma_j \gamma_k}{2\pi} \log |x^{(j)} - x^{(k)}| \,, \qquad \begin{array}{l}\text{(energy, not including the infinite} \\ \text{“self-energy” of the vortices)}\end{array} \quad . \tag{15.22}$$

$$\sum_k \gamma_k x^{(k)} \,, \tag{15.23}$$

$$\sum_k \gamma_k |x^{(k)}|^2 \,, \tag{15.24}$$

$$\sum_k \gamma_k \,, \qquad \text{(conserved trivially)} \,. \tag{15.25}$$

These conservation laws are particularly useful when all γ_k have the same sign. In that case one can see from the conservation laws that during the evolution the vortices will always stay in some bounded region and will never collide. This is no longer the case when γ_k can have different signs. The conservation laws are of course also helpful for integrating the equations. The equations are always integrable[81] when the number of vortices does not exceed three.[82] Systems of four or more vortices may no longer be integrable.[83]

Point vortices can also be considered a bounded domain Ω. In that case the stream function is given by

$$\psi(x) = \sum_k \gamma_k G(x, x^{(k)}) \tag{15.26}$$

where G is Green's function of the Laplacian in Ω, with a suitable boundary condition. In simply connected domains one can take $G(x, y) = 0$ when $x \in \partial\Omega$.

In some cases one can use symmetries to replace Ω by the whole space at the cost of introducing auxiliary vortices in the complement of Ω. (This is sometimes called the method of images, and you probably saw it in the theory of the Laplace equation.) For example, if we have one vortex $x = (x_1, x_2)$ in the half-space $\Omega = \{x,\, x_2 > 0\}$, we can add a vortex of the opposite sign at the point $x^* = (x_1, -x_2)$ and calculate the motion of the pair x, x^* in $\mathbf{R}^2$. The resulting motion of x will be a translation at a constant speed parallel to the boundary.

[81] In the sense of Hamiltonian systems - we will discuss these notions later.
[82] This is a classical result going back to the 19th century.
[83] Results in this direction are not so old.

16

11/19/2011

16.1 Vortex filaments

The point vortices we considered last time can also be thought of in the 3d picture. Let us first consider a smooth 2d solution given by a smooth compactly supported radially symmetric function $\overline{\omega} = \overline{\omega}(r)$ with $r = \sqrt{x_1^2 + x_2^2}$, similar to (15.2). It is natural to think of this solution also in the 3d picture, in which the vorticity field $\omega(x)$ is given by

$$\omega(x) = \begin{pmatrix} 0 \\ 0 \\ \overline{\omega}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \overline{\omega}(r) \end{pmatrix} . \tag{16.1}$$

In the 3d picture we can calculate the velocity field $u = (u_1, u_2, u_3)$ [84] from the Biot-Savart law (10.12), i. e.

$$u(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{y - x}{|y - x|} \times \omega(y)\, dy\,. \tag{16.2}$$

This formula should of course reproduce the 2d formulae from the last lecture (such as (15.4)) if we write $u = (u_1(x_1, x_2), u_2(x_1, x_2), 0)$. [85] The 3d solu-

[84] Our notation is somewhat loose in that we do not systematically distinguish between the "row vectors" such as $u = (u_1, u_2, u_3)$ and the "column vectors", such as $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$. Sometimes it is useful to make a distinction between the two notations (e. g. when co-variant and contra-variant vectors need to be carefully distinguished), but in our situation here the distiction will not play an important role, and we sometimes use both notations intergangeably.

[85] This is related to the "method of descent" which is sometimes used to calculate lower-dimensional fundamental solutions from the higher-dimensional ones. In the case here we are really dealing with the method of descent applied to the Laplace equation. Its 3d fundamental solution is $G_3(x) = \frac{1}{4\pi|x|}$ and it can be used to calculate its 2d fundamental solution by considering the potential of a uniform distribution of a charge along the x_3 axis. The integral for the corresponding potential

$$u(x_1, x_2) = \int_{-\infty}^{\infty} \frac{dx_3}{4\pi\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

is divergent, but this can be easily fixed. For example, we note that the integrals for the partial derivative

$$u_{x_1}(x_1, x_2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{-x_1 dx_3}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}}$$

is convergent, and similarly for $u_{x_2}(x_1, x_2)$. Alternatively, one can calculate

$$\int_{-L}^{L} \frac{dx_3}{4\pi\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{1}{2\pi} \log \frac{1}{\sqrt{x_1^2 + x_2^2}} + \log 2L + o(1), \qquad L \to \infty$$

and subtract the constant $\log(2L)$ as $L \to \infty$ to recover the 2d fundamental solution $G_2(x_1, x_2) = \frac{1}{2\pi} \log \frac{1}{\sqrt{x_1^2 + x_2^2}}$.

tion $u(x)$, sometimes referred to as a columnar vortex is quite interesting, in spite of its apparent simplicity. For example, just as we looked at the linearization of the compressible Euler equations around the trivial solution $(0, \rho_0)$ in lecture 5 and obtained the wave equation (5.27), we can linearize the incompressible Euler equation about a smooth columnar vortex solution $u(x)$ above, and obtain a linear equation for the perturbations. At this point we will not pursue this calculation (made first by Kelvin in 1880 [86]), but we can mention that the linear equation for the perturbation is more complicated than the wave equation and shows the possibility of various types of "vibrations" of the vortex and "wave-like" disturbances propagating away from the vibrating vortex, at various speeds. Some of the complexity of the behavior of solutions 3d incompressible Euler is revealed already at this linearized level. In fact, some important natural questions about this linearized system remain open to this date. We may revisit this topic later, but today we will look at different solution obtained by a more serious modification of the columnar vortex.

Let us denote

$$\Gamma = \int \overline{\omega}(x_1, x_2)\, dx_1\, dx_2\,, \tag{16.3}$$

which can be thought of as the strength of the vortex. If we shrink the support of $\overline{\omega}$ to one point (e. g. as in (15.6)), we get a limiting solution which can be though of as solution in which the vorticity is concentrated in a line, which in our case here coincides with the x_3-axis. In analogy with the picture of an electric wire we discussed in lecture 9, we can imagine that all the "current" represented by ω passes through an infinitely thin wire represented by the line.

What happens when we change the line in this picture to a closed curve? Let us consider a closed smooth curve

$$\gamma\colon [0, L] \to \mathbf{R}^3, \qquad \gamma(0) = \gamma(L)\,. \tag{16.4}$$

We assume that the curve has no self-intersections and that it is parametrized by length, i. e. $|\gamma'(s)| = 1$. Let us now think of a "vorticity current" of strength Γ passing through the curve γ. (We can think in terms of formula (9.9) in lecture 9 and the limit case $\varepsilon \to 0$.) The velocity field generated by the "vorticity current" of strength Γ in the curve γ is given by the Biot-Savart law:

$$u(x) = \int_0^L K(x - \gamma(s)) \times \gamma'(s)\, ds = \frac{1}{4\pi} \int_0^L \frac{\gamma(s) - x}{|\gamma(s) - x|^3} \times \gamma'(s)\, ds\,. \tag{16.5}$$

In analogy with the 2d point vortices we discussed in the last lecture, it is natural to study the velocity field $u(x)$ near the curve γ. Let us consider a point x which is close to the curve γ, at distance $r > 0$ from in, say, with r small. It is not hard to see from the expression (16.5) that for $r \to 0$ the magnitude $|u(x)|$ of the velocity field at x grows as $\sim \frac{1}{r}$, similarly to what we have in the case of a rectilinear vortex. In the case of the rectilinear vortex the velocity field does not

[86] Lord Kelvin, Vibration of a columnar vortex, Philos. Mag. 10: 155, 1880.

move the vortex. The fluid circulates around it at high speed, but the line of the vortex itself is not moved. In some sense, the velocity at the line itself vanishes, and the rectilinear vortex is at rest. How does this picture change when the line of the vortex is bent? The precise calculation is not quite easy, due to the fact that it is more or less clear that, similarly to the rectilinear case, the trajectories of the velocity field circle around the vortex curve and "at the leading order" do not seem to move it. The difficulty of course is that the velocity field $u(x)$ does not really have a well-defined limiting value as x approaches the curve γ. Strictly speaking, for $x \in \gamma$ the value $u(x)$ is not well-defined.

The correct solution of this problem consist in regularising the curve into a smooth vector field ω (e. g. by using formula (9.9)) calculating the velocity field across the support of ω, and isolating the part of the velocity field which "moves ω". Such calculations were done still in the 1860s by Kelvin, although a fully rigorous justification was achieved only much later.[87]

Here we do only a rough (and not quite rigorous) calculation, which nevertheless gives some idea about what one can expect. We will attempt to calculate the value $u(x)$ for $x \in \gamma$. For this purpose we will assume that the surve γ is parametrized by a length parameter $s \in [-L/2, L/2)$, with

$$\gamma(0) = 0, \quad \gamma'(0) = e_1, \quad \gamma''(0) = \kappa e_2\,, \tag{16.6}$$

where κ denotes the curvature of the curve at $x = 0$, and $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$. These assumptions can be made without loss of generality, as the general case can be reduced to the one above by a suitable choice coordinates. [88] The Biot-Savart law (16.5) gives the following formal expression for $u(0)$

$$u(0) = \frac{\Gamma}{4\pi} \int_{-L/2}^{L/2} \frac{\gamma(s) \times \gamma'(s)}{|\gamma(s)|^3}\, ds\,. \tag{16.7}$$

Let us take some small (but fixed) number $l > 0$ and split the integral (16.7) as

$$\int_{-L/2}^{L/2} \ldots ds = \int_{|s|>l} \ldots ds + \int_{-l}^{l} \ldots ds\,. \tag{16.8}$$

It is clear that the contribution from the integral $\int_{|s|>l} \ldots ds$ (which is clearly well-defined) is a bounded vector. To calculate the integral $\int_{-l}^{l} \ldots ds$ with a precision up to a well-defined bounded vector, it is clearly enough to consider the first two non-zero terms of the Taylor expansion of γ as $s = 0$,

$$\gamma(s) \sim s e_1 + \frac{1}{2}\kappa s^2 e_2 \tag{16.9}$$

[87] See, for example, Fraenkel, L.E., On Steady Vortex Rings of Small Cross Section in an Ideal Fluid, Proc. of the Royal Society of London, Series A, Vol. 316, No. 1524 (1970), 29-62.

[88] We recall that when we parametrize a curve by length, then $|\gamma'(s)|^2 = \gamma'(s)\,\gamma'(s) = 1$, and this implies that $\gamma'(s)\,\gamma''(s) = 0$.

With these approximations we can write

$$u(0) = \frac{\Gamma}{4\pi} \int_{-l}^{l} \frac{(se_1 + \frac{1}{2}\kappa s^2 e_2) \times (e_1 + \kappa s e_2)}{|s|^3} \quad + \quad \Gamma \text{ [a well-defined bounded vector]} \tag{16.10}$$

We have

$$(se_1 + \frac{1}{2}\kappa s^2 e_2) \times (e_1 + \kappa s e_2) = \frac{1}{2}\kappa s^2 e_3 \,, \tag{16.11}$$

and therefore

$$u(0) = \frac{\Gamma}{4\pi} \kappa e_3 \int_{-l}^{l} \frac{ds}{2|s|} \quad + \quad \Gamma \text{ [a well-defined bounded vector]} \,. \tag{16.12}$$

The integral in (16.12) is of course divergent (and equal $+\infty$), suggesting that an infinitely thin vortex filament with some curvature will try move at an infinite speed, which means that the motion of such a filament is not really well-defined. [89] We therefore have to replace the infinitely thin filament by a filament of a small but finite radius which we will denote by $\varepsilon > 0$. We can then write, with some approximation

$$u(0) = \frac{\Gamma}{4\pi} \kappa e_3 \int_{10\varepsilon}^{l} \frac{ds}{s} + \Gamma \text{ [a well-defined bounded vector]} \,, \tag{16.13}$$

as the contribution from the integration $\int_{-10\varepsilon}^{10\varepsilon} \dots ds$ is replaced by

$$\int_{|y| \le 10\varepsilon} \frac{y \times \omega_\varepsilon(y)}{|y|^3} \, dy \,, \tag{16.14}$$

where ω_ε is the vector field obtained by a regularization of the current thought γ, e. g. by formula (9.9). Now the integrant in (16.14) can still be of size ε^{-4} (as one can see from (15.6), for example) and we integrate over an area of volume ε^3, so there is still potential for divergence. However, we note that the ε^{-1} part of the integral vanishes, due to cancelations between the contributions from y and $-y$. So our final conclusion is that

$$u(0) = \frac{\Gamma}{4\pi} \kappa e_3 \log \frac{l}{\varepsilon} \quad + \quad \Gamma \text{ [a well-defined bounded vector]} \,. \tag{16.15}$$

We see that the velocity of the filament will be large for small ε, as we already expect. The [well-defined bounded vector] from (16.15) is not easy to determine exactly, but we can remove this term by considering the limiting regime of weak filaments. A weak filament is a filament with a small total flux of vorticity Γ. If we choose $\Gamma = \Gamma_\varepsilon$ so that

$$\frac{\Gamma}{4\pi} \log \frac{l}{\varepsilon} = 1 \tag{16.16}$$

[89] Note however that the divergence is quite mild. The Biot-Savart kernel $K(x)$ is -2-homogeneous, and if it were $-2+\varepsilon$ homogeneous for some $\varepsilon > 0$, the integral we would obtain in place of (16.12) would be convergent.

we see that the second term on the right-hand side of (16.15) converges to 0 as $\varepsilon \to 0$, and in the limit $\varepsilon \to 0$ we can then write

$$u(0) = \kappa e_3 \,. \tag{16.17}$$

Now in the coordinate choice given by (16.6) the vector e_3 coincides with the binormal[90] b of the curve γ at the point $x = 0$. The above calculation applies to every point of the curve and therefore we see that the weak vortex filaments normalized by (16.16) should move by

$$u(x) = \kappa(x) b(x), \qquad x \in \gamma \,, \tag{16.18}$$

where $\kappa(x)$ is the curvature of γ at x and $b(x)$ is the binormal vector of γ at x. It turns out that equation (16.18) (which was during the last 100 years or so derived independently several times by various authors,[91] and is called binormal curvature flow) has remarkable properties. It turns out to be completely integrable and equivalent to the cubic non-linear Schrödinger equation.[92] Our derivation above was not very rigorous. One can do more careful calculations, but it remains an open problem if the 3d Euler equation has solutions which would "shadow" the solutions of (16.18). (This can only be viable as long as the solutions of (16.18) exist as embedded curves.) We should emphasize that this can be possible only in the limit of weak vortex filaments, the behavior of vortex filaments of a finite strength is not really compatible with (16.18), due to effects such as vorticity stretching (which we discussed in lecture 13).

[90] given by $b = [\text{tangent}] \times [\text{normal}]$

[91] See, for example,

L. S. Da Rios, Sul moto di un filetto vorticoso di forma qualunque, Rend. del Circolo Mat. di Palermo 22 (1906), 117135 and 29 (1910), 354-368.

Arms, R.J., Hamma, F.R., Localized-induction concept on a curved vortex and motion of an elliptic vortex ring, Phys. Fluids 8, 553–559, 1965

[92] See H. Hasimoto, A Soliton on a vortex filament, Journal of Fluid Mechanics, Volume 51, Issue 03, pp 477 – 485, 1972.

17

10/21/2011

17.1 Axi-symmetric solutions

We will now look at a special class of solutions where some of the calculations from the last lecture can be done in more detail.

We say that a vector field u in $\mathbf{R}^3$ is axi-symmetric if in a suitable Cartesian coordinate frame x_1, x_2, x_3 we have

$$u(Rx) = Ru(x) \tag{17.1}$$

for each $x \in \mathbf{R}^3$ and each rotation R about the x_3 axis.

If $u(x,t), p(x,t)$ is a smooth solution of Euler's equation in $\mathbf{R}^3 \times [0,T)$ with $u(x,0) = u_0(x)$, then it is easy to see that $\tilde{u}(x,t) = Ru(R^{-1}x,t), p(R^{-1}x,t)$ is a solutions of Euler's equation with $\tilde{u}(x,0) = Ru_0(R^{-1}x)$, and therefore we expect that the class of axi-symmetric vector fields is preserved under the evolution by Euler's equations.[93] Strictly speaking, to prove this rigorously we need to apply some uniqueness result about the solutions of Euler's equations, such as the Theorem in lecture 13, and therefore some assumptions are needed.[94] At this stage we will not go into technicalities concerning this point and we will take for granted that axi-symmetric initial data lead to an axi-symmetric solution.

We recall that the cylindrical coordinates r, θ, z are given by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z\,. \tag{17.2}$$

We consider the natural orthogonal frame e_r, e_θ, e_z in these coordinates, where the cartesian coordinates of the vectors e_r, e_θ and e_z are given by

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_z = (0,0,1)\,, \quad r = \sqrt{x_1^2 + x_2^2}\,. \tag{17.3}$$

In the cylindrical coordinates we will write

$$u = u^{(r)}\, e_r + u^{(\theta)}\, e_\theta + u^{(z)}\, e_z\ , \tag{17.4}$$

and – if there is no danger of confusion – also

$$u = \begin{pmatrix} u^{(r)} \\ u^{(\theta)} \\ u^{(z)} \end{pmatrix}\ , \tag{17.5}$$

[93] Here we have in mind the incompressible Euler equation, but the statement remains true also for the compressible equations, with the obvious requirement that the initial density should also be axi-symmetric.

[94] For example, we can assume that we work in the class of solutions with fast decay of vorticity as $x \to \infty$.

or

$$u = (u^{(r)}, u^{(\theta)}, u^{(z)})\,. \tag{17.6}$$

A field of the form (17.4) is axi-symmetric if the coordinates $u^{(r)}$, $u^{(\theta)}$, $u^{(z)}$ depend only on r, z and not on θ, i. e.

$$u^{(r)} = u^{(r)}(r,z), \quad u^{(\theta)} = u^{(\theta)}(r,z), \quad u^{(z)} = u^{(z)}(r,z)\,. \tag{17.7}$$

The component $u^{(\theta)}$ of u is called the swirl, and the field (17.7) with $u^{(\theta)} = 0$ are called *axi-symmetric fields with no swirl*. An axi-symmetric $u = u(x)$ has no swirl if and only if any plane containing the x_3-axis is invariant under the flow of u.

We note that for smooth axi-symmetric solutions we can apply Kelvin's Circulation Theorem (lecture 9) to see that the quantity $ru^{(\theta)}(r,z)$ is conserved along particle trajectories under the evolution by Euler's equation. In particular, the condition of "no swirl" is conserved under the evolution by Euler's equation, under quite general assumptions.

In what follows we will always assume that the components $v^{(r)}, v^{(\theta)}, v^{(z)}$ of all the vector fields depend only on r and z. We will not consider any fields with the components in the cylindrical coordinates depending on θ.

We recall some simple formulae. The equation $\omega = \operatorname{curl} u$ for a axi-symmetric vector field u with no swirl looks in the cylindrical coordinates as follows:

$$\operatorname{curl} \begin{pmatrix} u^{(r)} \\ 0 \\ u^{(z)} \end{pmatrix} = \begin{pmatrix} 0 \\ -u^{(z)}_{,r} + u^{(r)}_{,z} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega^{(\theta)} \\ 0 \end{pmatrix}\,. \tag{17.8}$$

For a vector field $A = A^{(\theta)}(r,z)\, e_\theta$ we have

$$\operatorname{curl} A = \operatorname{curl} \begin{pmatrix} 0 \\ A^{(\theta)} \\ 0 \end{pmatrix} = \begin{pmatrix} -A^{(\theta)}_{,z} \\ 0 \\ \frac{1}{r}(rA^{(\theta)})_{,r} \end{pmatrix}\,. \tag{17.9}$$

We see that the operation $v \to \operatorname{curl} v$ takes the axi-symmetric fields with no swirl to the axi-symmetric fields of the form $w^{(\theta)}(r,z)\, e_\theta$ (which can be considered as "pure swirl" axi-symmetric fields) and vice versa. It is worth noting the similarity of (17.8) with the formula (in cartesian coordinates)

$$\operatorname{curl} \begin{pmatrix} u_1(x_1,x_2) \\ u_2(x_1,x_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u_{2,1}(x_1,x_2) - u_{1,2}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega(x_1,x_2) \end{pmatrix} \tag{17.10}$$

and the similarity of (17.9) with the formula (in cartesian coordinates)

$$\operatorname{curl} \begin{pmatrix} 0 \\ 0 \\ A_3(x_1,x_2) \end{pmatrix} = \begin{pmatrix} A_{3,2}(x_1,x_2) \\ -A_{3,1}(x_1,x_2) \\ 0 \end{pmatrix}\,. \tag{17.11}$$

The equation $\operatorname{div} u = 0$ for an axi-symmetric field u with no swirl is

$$\operatorname{div} \begin{pmatrix} u^{(r)} \\ 0 \\ u^{(z)} \end{pmatrix} = \frac{1}{r}(ru^{(r)})_{,r} + u^{(z)}_{,z} \;=\; 0\,. \tag{17.12}$$

This can be written as

$$(ru^{(r)})_{,r} + (ru^{(z)})_{,z} = 0\,. \tag{17.13}$$

Just as in (14.7) and (14.8) this means that

$$ru^{(r)} = -\psi_z, \qquad ru^{(z)} = \psi_{,r}\,, \tag{17.14}$$

for a suitable function $\psi = \psi(r, z)$. In other words,

$$\begin{pmatrix} u^{(r)} \\ 0 \\ u^{(z)} \end{pmatrix} = \begin{pmatrix} -\frac{\psi_{,z}}{r} \\ 0 \\ \frac{\psi_{,r}}{r} \end{pmatrix} = \operatorname{curl} \begin{pmatrix} 0 \\ \frac{\psi}{r} \\ 0 \end{pmatrix}\,. \tag{17.15}$$

The function $\psi(r, z)$ is called the stream function (for the axi-symmetric flow with no swirl), and similarly to the 2d situation in lecture 14, the level sets $\{\psi = \text{const.}\}$ are the integral lines of the vector field u (in any of the planes $\theta = \text{const}\,.$) Combining (17.8) and (17.15) we obtain an expression for $\omega^{(\theta)}$ in terms of ψ, which can be thought of of an analogue of formula (14.16):

$$\omega^{(\theta)} \;=\; -\left(\frac{\psi_{,r}}{r}\right)_{,r} - \frac{\psi_{,zz}}{r} \;=\; -\frac{\psi_{,rr}}{r} + \frac{\psi_{,r}}{r^2} - \frac{\psi_{,zz}}{r}\,. \tag{17.16}$$

18

10/24/2011

18.1 Hill's spherical vortex

We first write the incompressible Euler equation for axi-symmetric solutions in the cylindrical coordinates. The equation of continuity $\operatorname{div} u = 0$ is

$$\frac{(ru^{(r)})_{,r}}{r} + \frac{u^{(\theta)}_{,\theta}}{r} + u^{(z)}_{,z} = 0 \tag{18.1}$$

and the equations (5.16) (when $f = 0$ and $\rho_0 = 1$) are

$$\begin{aligned} u^{(r)}_{,t} + (u\nabla)u^{(r)} - \frac{(u^{(\theta)})^2}{r} + p_{,r} &= 0 && (18.2)\\ u^{(\theta)}_{,t} + (u\nabla)u^{(\theta)} + \frac{u^{(r)}u^{(\theta)}}{r} + \frac{p_{,\theta}}{r} &= 0 && (18.3)\\ u^{(z)}_{,t} + (u\nabla)u^{(z)} + p_{,z} &= 0\,, && (18.4) \end{aligned}$$

where we use the notation

$$(u\nabla)h = u^{(r)}h_{,r} + u^{(\theta)}\frac{h_{,\theta}}{r} + u^{(z)}h_{,z} \tag{18.5}$$

for any scalar function h. As an exercise, you should analyze where the various terms in this system come from. Note for example that (18.2)– (18.4) only differs from the equations in cartesian coordinates by the term $-\frac{(u^{(\theta)})^2}{r}$ in the first equation and the term $\frac{u^{(r)}u^{(\theta)}}{r}$ in the second equation. There is also the difference that the derivative in the direction e_θ is $\frac{\partial}{r\partial\theta}$ and not just $\frac{\partial}{\partial\theta}$, which is however natural since we normalize the vector e_θ to unit length. For axi-symmetric vector fields with no swirl we have $u^{(\theta)} = 0$ and all the quantities depend only on r and z. In this case the system simplifies to

$$\begin{aligned} u^{(r)}_{,t} + u^{(r)}u^{(r)}_{,r} + u^{(z)}u^{(r)}_{,z} + p_{,r} &= 0 && (18.6)\\ u^{(z)}_{,t} + u^{(r)}u^{(z)}_{,r} + u^{(z)}u^{(z)}_{,z} + p_{,z} &= 0\,, && (18.7) \end{aligned}$$

together with the equation of continuity

$$\frac{(ru^{(r)})_{,r}}{r} + u^{(z)}_{,z} = 0\,. \tag{18.8}$$

Recalling formula (17.8), we obtain from (18.6)– (18.8)

$$\left(\frac{\omega^{(\theta)}}{r}\right)_{,t} + u^{(r)}\left(\frac{\omega^{(\theta)}}{r}\right)_{,r} + u^{(z)}\left(\frac{\omega^{(\theta)}}{r}\right)_{,z} = 0\,. \tag{18.9}$$

This means that the quantity $\frac{\omega^{(\theta)}}{r}$ moves with the flow. The situation is almost the same as in dimension two, where (the scalar) ω moves with the flow. Equation (18.9) can also be understood from the Helmholtz law that vorticity moved with the flow. The volume of the domain in $\mathbf{R}^3$ described by the coordinates (r', θ', z') with $r \leq r' \leq r + dr, \theta' \in [0, 2\pi), z \leq z' \leq z + dz$ is $2\pi r dr dz$. The vorticity field ω at each point is a multiple of e_θ, and the Lagrangian map ϕ^t preserves the volume and satisfies $\phi^t(Rx) = R\phi^t(x)$ for all rotations R about the x_3-axis. Putting these constraints together, we see that the Helmoltz law (7.13) implies that $\frac{\omega^{(\theta)}}{r}$ moves with the flow, which gives another confirmation of (18.9). Note that when a fluid particle moves from small radii to large radii in this setting, we will observe some stretching of vorticity, as discussed in lecture 13.

We will now consider solutions of (18.9) which are independent of time (steady solutions). Recalling the definition of the stream function (17.15) together with equation (17.16), we see that a steam function $\psi = \psi(r, z)$ defines a steady solution of (18.9) if and only if the quantity

$$\frac{\omega^{(\theta)}}{r} = -\frac{1}{r}\left(\frac{\psi_{,r}}{r}\right)_{,r} - \frac{\psi_{,zz}}{r^2} = -\frac{\psi_{,rr}}{r^2} + \frac{\psi_{,r}}{r^3} - \frac{\psi_{,zz}}{r^2}\,. \tag{18.10}$$

is constant along the streamlines

$$\psi = \text{const.} \tag{18.11}$$

We will use the notation

$$R = \sqrt{r^2 + z^2}\,. \tag{18.12}$$

We now consider a ball B_a of radius a centered at the origin. Let

$$\Omega_a = \mathbf{R}^3 \setminus \overline{B}_a\,, \tag{18.13}$$

where $\overline{B}_a$ denotes the closure of B_a.

Let us first consider a potential flow in Ω_a with velocity tangent to $\partial\Omega_a$ and $\partial\Omega_a$ and $U = (0, 0, -V)$ with $V > 0$ as $x \to \infty$. By Problem 2 of homework assignment 1 we know that the potential h of this flow is

$$h(x) = (Ux)\left(1 + \frac{a^3}{2R^3}\right) = (Ux)\left(1 + \frac{a^3}{2(r^2 + z^2)^{\frac{3}{2}}}\right)\,. \tag{18.14}$$

the velocity field in Ω_a is given by

$$u = \nabla h\,. \tag{18.15}$$

Clearly the flow is axi-symmetric with no swirl in Ω_a and therefore it must be given by a stream function $\psi = \psi(r, z)$, see (17.15). As an exercise you can calculate the stream function, starting from (18.14) and (18.15). The result is

$$\psi = -\frac{1}{2}Vr^2(1 - \frac{a^3}{R^3})\,. \tag{18.16}$$

Note that $\psi < 0$ in Ω_a with the exception of the x_3-axis (where $\psi = 0$) and $\psi = 0$ at the boundary of Ω_a. You can also check that formula (18.10) gives $\omega^{(\theta)} = 0$ in Ω_a, as it should be the case.

To the steady flow (18.16) in Ω_a we now try to match a steady flow in B_a, with the same velocity at ∂B_a as that of (18.16). As we have seen, in a steady flow the quantity (18.10) must be constant along the streamlines $\psi = \text{const.}$ and the simplest way to achieve that is to have (18.10) constant everywhere in B_a. In other words, we would like to have

$$-\psi_{,rr} + \frac{\psi_{,r}}{r} - \psi_{,zz} = Cr^2 \qquad \text{in } B_a\,, \tag{18.17}$$

for some $C \in \mathbf{R}$. To be able to match the solution to the solution (18.16) in Ω_a, the boundary ∂B_a should be a streamline (in the r, z-plane), and therefore we complement (18.17) by the boundary condition

$$\psi|_{\partial B_a} = 0\,. \tag{18.18}$$

It turns out the solution to (18.17), (18.18) can be written down explicitly:

$$\psi(r,z) = Ar^2(1 - \frac{R^2}{a^2})\,, \qquad A = \frac{Ca^2}{10}\,. \tag{18.19}$$

The functions (18.19) and (18.16) both vanish at ∂B_a. Can we choose the constant A so that the gradients of the functions on ∂B_a coincide? From the specific form of the functions we see that a necessary and sufficient condition for this is

$$\frac{d}{dR}|_{R=a}\ A(1 - \frac{R^2}{a^2}) = \frac{d}{dR}|_{R=a}\ -\frac{1}{2}V(1 - \frac{a^3}{R^3})\,. \tag{18.20}$$

A straightforward calculation now gives

$$A = \frac{3}{4}V\,. \tag{18.21}$$

Hence the function

$$\psi(r,z) = \begin{cases} -\frac{1}{2}Vr^2(1 - \frac{a^3}{R^3})\,, & R \geq a \\ \\ \frac{3}{4}Vr^2(1 - \frac{R^2}{a^2})\,, & R < a\,, \end{cases} \tag{18.22}$$

represents a steady solution with a continuous velocity field. The vorticity is discontinuous across the boundary ∂B_a. This solution is called Hill's vortex.[95] If we change coordinates so that the fluid at $x \to \infty$ is at rest, we see that the solution represents a spherical axi-symmetric vortex (with no swirl) moving through an otherwise vorticity-free fluid at constant speed V. The vortex carries with it the fluid within the sphere, the fluid outside the sphere does not travel with the vortex, its motion is the same as if the vortex was replaced by a sphere.

[95] Hill, M. J. M., Phil. Trans. Roy. Soc. London, A, Vol. 185, p. 213, 1894.

Hill's vortex can be thought of as an limiting case of a "vortex ring" solution. A typical vortex ring solution is also axi-symmetric with no swirl and consists of a vorticity area of a shape of a solid torus traveling along the x_3-axis. Existence of such solutions has been proved rigorously.[96] One extreme case of such solutions would be when the vorticity is supported at a circle obtained by rotating a point about the x_3 axis, but – as we have already seen in lecture 16 – such a circle with a non-zero "vorticity current" would have to move at infinite speed, and hence does not really exists as a solution. However, there are solutions "nearby", when the circle is modified to have a non-zero thickness, see the paper by Fraenkel mentioned in a footnote in lecture 16. If we imagine deforming a solid torus into a ball from which a very thin tube about the x_3-axis was removed, and then taking the radius of the removed tube to zero, we can think of Hill's vortex as such a limiting case. Solutions in a neighborhood of Hill's vortex have been also studied.[97]

[96] Fraenkel, L. E., Berger, M. S., A global theory of steady vortex rings in an ideal fluid, Acta Math. 132 (1974), 13-51.

[97] Amick, C. J., Turner, R. E. L., A global branch of steady vortex rings. J. Reine Angew. Math. 384 (1988), 1-23.

19

10/26/2011

19.1 Vortex rings - some calculations

We now revisit the calculations we did in lecture 16 in the special case of vortex rings. We consider a vorticity field $\omega(x)$ in $\mathbf{R}^3$ which is expressed in the cylindrical coordinates as

$$\omega = \omega^{(\theta)}(r,z)e_\theta = \omega^{(\theta)}(r,z) \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} . \tag{19.1}$$

We assume that in the (r,z) coordinate domain $r \geq 0, z \in \mathbf{R}$ the support of the function $\omega^{(\theta)}(r,z)$ is located in a small disc $\mathcal{O}_\varepsilon$ of radius ε centered at (r_0,z_0) with $r_0 > 0$. Such a vorticity field represents a vortex ring of radius approximately r_0, thickness 2ε, and the strength of the "vortex current"

$$\Gamma = \int \omega^{(\theta)}(r,z)\, dr\, dz \ .^{98} \tag{19.2}$$

From formula (16.15) we expect that the evolution by Euler's equation will result in the ring moving up along the x_3-axis at speed which is given, modulo a bounded error, by $\frac{\Gamma}{4\pi r_0}\log\frac{l}{\varepsilon}$, where l is some fixed length (roughly of order r_0).[99] However, the derivation of (16.15) was really rigorous. We will now indicate how one can carry out a more careful calculation. Similar calculations are classical, going back to a 1867 note by Kelvin, with later contributions by many other authors, see for example the 1969 paper by Fraenkel cited in a footnote in lecture 16.

Let u be the velocity field given by ω by the Biot-Savart law (10.12) and let A be the vector potential of u, given by (10.14). We know from lecture 17 (see (17.9) and (17.15)) that for axi-symmetric fields u with no swirl the potential A is related to the axi-symmetric stream function ψ by

$$A = \begin{pmatrix} 0 \\ A^{(\theta)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\psi}{r} \\ 0 \end{pmatrix} . \tag{19.3}$$

To calculate A, it is enough to evaluate $A^{(\theta)}$ at the points x with cylindrical coordinates $(r,0,z)$. We will use formula (10.14), i. e. (in cartesian coordinates)

$$A(x) = \int_{\mathbf{R}^3} \frac{\omega(y)}{4\pi|x-y|}\, dy\,. \tag{19.4}$$

[98] Note that we also have $\Gamma = \oint_C u^{(r)}\, dr + u^{(z)}\, dz$ where u is the velocity field generated by ω and C is any curve encircling the support of $\omega^{(\theta)}(r,z)$ once in the clock-wise direction.

[99] In fact, the more precise calculation we will outline shows that one should take $l = 8r_0$.

Letting (r', θ', z') be the cylindrical coordinates of y, by elementary trigonometry we can write

$$|x - y| = \sqrt{r^2 - 2rr' \cos\theta' + r'^2 + (z - z')^2}\,. \tag{19.5}$$

Hence (19.4) can be written as

$$A(x) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{\omega^{(\theta)}(r,z)}{4\pi\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} \begin{pmatrix} -\sin\theta' \\ \cos\theta' \\ 0 \end{pmatrix} r' d\theta'\, dr'\, dz'\,. \tag{19.6}$$

We let

$$\mathcal{A}(r, r'; z, z') = \int_0^{2\pi} \frac{r'}{4\pi\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} \begin{pmatrix} -\sin\theta' \\ \cos\theta' \\ 0 \end{pmatrix} d\theta'\,. \tag{19.7}$$

Clearly

$$\int_0^{2\pi} \frac{r'\sin\theta'\, d\theta'}{\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}} = 0 \tag{19.8}$$

and therefore we have

$$\mathcal{A}(r, r', z, z') = \begin{pmatrix} 0 \\ \mathcal{A}^{(\theta)}(r, r', z, z') \\ 0 \end{pmatrix}\,, \tag{19.9}$$

with

$$\mathcal{A}^{(\theta)}(r, r', z, z') = \int_0^{2\pi} \frac{r'\cos\theta'\, d\theta'}{4\pi\sqrt{r^2 - 2rr'\cos\theta' + r'^2 + (z-z')^2}}\,. \tag{19.10}$$

Hence we have

$$A(x) = A^{(\theta)}(r, z) e_\theta(x)\,, \tag{19.11}$$

with

$$A^{(\theta)}(r, z) = \int_{-\infty}^{\infty}\int_0^{\infty} \mathcal{A}^{(\theta)}(r, r', z, z')\omega^{(\theta)}(r', z')\, dr'\, dz'\,. \tag{19.12}$$

The axi-symmetric stream function $\psi(r, z)$ of the field u is then, from (19.3),

$$\psi(r, z) = \int_{-\infty}^{\infty}\int_0^{\infty} r\mathcal{A}^{(\theta)}(r, r', z, z')\omega^{(\theta)}(r', z')\, dr'\, dz'\,. \tag{19.13}$$

Defining a differential operator L by

$$L\psi = -\frac{\psi_{,rr}}{r} + \frac{\psi_{,r}}{r^2} - \frac{\psi_{,zz}}{r} \tag{19.14}$$

and recalling from lecture 17 (see (17.16)) that

$$L\psi = \omega^{(\theta)}\,, \tag{19.15}$$

we see that formula (19.13) inverts the operator L (with certain natural boundary conditions given by our setup), and hence the function $r\mathcal{A}^{(\theta)}(r,r',z,z')$ is really the Green function of the operator L. We now express $\mathcal{A}^{(\theta)}(r,r',z,z')$ somewhat more explicitly. Let us set

$$\frac{r-r'}{r'} = x, \quad \frac{z-z'}{r'} = y \tag{19.16}$$

Then

$$\mathcal{A}^{(\theta)}(r,r',z,z') = \frac{1}{2\pi\sqrt{1+x}} \int_0^\pi \frac{\cos\theta'\, d\theta'}{\sqrt{2(1-\cos\theta') + \frac{x^2+y^2}{1+x}}} \tag{19.17}$$

Letting

$$F(s) = \int_0^\pi \frac{\cos\theta\, d\theta}{\sqrt{2(1-\cos\theta)+s}}\,, \tag{19.18}$$

we can write

$$\mathcal{A}^{(\theta)}(r,r',z,z') = \frac{1}{2\pi\sqrt{1+x}}\, F\left(\frac{x^2+y^2}{1+x}\right). \tag{19.19}$$

Substituting for x, y from (19.16), we see that the Green function for the operator L above is

$$r\mathcal{A}^{(\theta)}(r,r',z,z') = \frac{\sqrt{rr'}}{2\pi} F\left(\frac{(r-r')^2+(z-z')^2}{rr'}\right) \tag{19.20}$$

and the stream function $\psi(r,z)$ is given by

$$\psi(r,z) = \int_{-\infty}^{\infty}\int_0^\infty \frac{\sqrt{rr'}}{2\pi} F\left(\frac{(r-r')^2+(z-z')^2}{rr'}\right) \omega^{(\theta)}(r',z')\, dr'\, dz'\,. \tag{19.21}$$

The velocity field u will now be given by (17.15).

The function F cannot be expressed in terms of elementary functions, but it can be expressed in terms of certain classical special functions known as the complete elliptic integrals. These are defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,, \qquad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi\,, \tag{19.22}$$

and they were studied in much detail during the 19th century.[100] In the classical monograph "Hydrodynamics" by Horace Lamb (Cambridge 1932), Art. 161 you

[100] The subject of elliptic functions is an important classical area. The name "elliptic" comes from their appearance in integrals expressing the length of an arc of an ellipse. However, their "real significance" is in their connection to the cubic curves (also called "elliptic curves"). See for example the book "Elliptic Functions" by K. Chandrasekharan (1985), or "Elliptic Function" by A. Caley (1895). Much of modern research in this area is related to the number-theoretic aspects of the corresponding curves.

can find several ways to express ψ in (19.21) in terms of K and E, including a formula due to Maxwell (who studied it in connection with magnetic fields).

Here we will only derive a few important properties of F which can be obtained by simple calculations. Setting $\varphi = \frac{\theta}{2}$ and using $\cos\theta = \cos^2\varphi - \sin^2\varphi$ in (19.18),

$$F(s) = \int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2\varphi}{\sqrt{\sin^2\varphi + \sigma^2}}\, d\varphi\,, \qquad \sigma^2 = \frac{s}{4}\,, \tag{19.23}$$

or

$$F(s) = \int_0^{\frac{\pi}{2}} \frac{1 + 2\sigma^2}{\sqrt{\sin^2\varphi + \sigma^2}}\, d\varphi - 2\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\varphi + \sigma^2}\, d\varphi\,, \quad \sigma^2 = \frac{s}{4}\,. \tag{19.24}$$

The leading term in the expression (19.4) as $s \to 0$ is

$$f(\sigma) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{\sin^2\varphi + \sigma^2}}\,. \tag{19.25}$$

Clearly $f(\sigma) \to +\infty$ as $\sigma \to 0$. To obtain a more precise estimate, we write

$$f(\sigma) = \int_0^{\frac{\pi}{2}} \frac{\cos\varphi\, d\varphi}{\sqrt{\sin^2\varphi + \sigma^2}} + \int_0^{\frac{\pi}{2}} \frac{(1 - \cos\varphi)\, d\varphi}{\sqrt{\sin^2\varphi + \sigma^2}}\,. \tag{19.26}$$

The first integral in (19.26) can be written as

$$\int_0^1 \frac{dt}{\sqrt{t^2 + \sigma^2}} = \int_0^{\frac{1}{\sigma}} \frac{dt}{\sqrt{t^2 + 1}} = \log\frac{1}{\sigma} + \log(1 + \sqrt{1 + \sigma^2}) = \log\frac{1}{\sigma} + \log 2 + O(\sigma^2)\,, \quad \sigma \to 0\,. \tag{19.27}$$

The second integral in (19.26) can be approximated as

$$\int_0^{\frac{\pi}{2}} \frac{1 - \cos\varphi}{\sin\varphi} d\varphi + O(\sigma^2 \log\frac{1}{\sigma}) = \log 2 + O(\sigma^2 \log\frac{1}{\sigma})\,, \quad \sigma \to 0\,. \tag{19.28}$$

For the second integral in (19.24) we can write

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin^2 + \sigma^2}\, d\varphi = 1 + O(\sigma^2 \log\frac{1}{\sigma})\,. \tag{19.29}$$

Hence we have

$$F(s) = \frac{1}{2}\log\frac{1}{s} + \log 8 - 2 + O(s\log\frac{1}{s})\,, \quad s \to 0_+\,. \tag{19.30}$$

Moreover, the estimate for $F'(s)$ obtained by formally taking a derivative of (19.30) is also correct.[101]

[101] In fact, it is not hard to check that one has an expansion

$$F(s) = \left(\log\frac{1}{s}\right)\left(a_0 + a_1\, s + a_2 s^2 + \dots\right) + \left(b_0 + b_1\, s + b_2 s^2 + \dots\right). \tag{19.31}$$

The above calculation determines $a_0 = \frac{1}{2}$ and $b_0 = \log 8 - 2$. Similar calculations "by hand" can be done to determine higher order terms, if necessary. The expansion can also be obtained from the theory of the elliptic integrals (19.22).

With these approximations for F, one can now go back to (19.21) and use it when $\omega^{(\theta)}$ is approximately uniformly distributed of the disc $\mathcal{O}_\varepsilon$. In view of (18.9), the most natural $\omega^{(\theta)}$ considered in $\mathcal{O}_\varepsilon$ is such that $\frac{\omega^{(\theta)}}{r}$ is constant. With some additional work we get a formula for the speed of propagation of the vortex ring obtained by Kelvin in 1867:

$$V = \frac{\Gamma}{4\pi r_0}\left(\log\frac{8r_0}{\varepsilon} - \frac{1}{4}\right) + o(\varepsilon)\,, \qquad \varepsilon \to 0_+\,. \tag{19.32}$$

It should be emphasized that the solutions we have considered do not really precisely keep their shape as they move, the $o(\varepsilon)$ part of the velocity field can in general slightly change the shape of the vortex (while keeping all the conserved quantities constant, of course). One can however construct precise vortex ring solutions which solve Euler's equations and are exactly translated along the x_3 axis at constant speed, see for example the paper of Fraenkel we quoted in a footnote in lecture 16.

20

10/28/2011

20.1 Some general properties of Euler steady-states

i) Natural appearance of tori[102]

Let u be a steady-state solution[103] of the incompressible Euler equations (5.16) in $\mathbf{R}^3$ with zero forcing term ($f = 0$). We assume that the density ρ_0 of the fluid is constant. We will assume (without loss of generality) $\rho_0 = 1$. As usual, we denote by ω the vorticity field, i. e. $\omega = \operatorname{curl} u$. One useful form of the equation for the steady-state follows from (7.6):

$$\nabla \left(\frac{|u|^2}{2} + p \right) = u \times \omega \, . \tag{20.1}$$

Another useful form follows from (7.13)

$$[u, \omega] = 0 \, . \tag{20.2}$$

Both these equations can give some insights into the structure of the steady state solutions. Let us first look at (20.1). We will use the notation

$$H = \frac{|u|^2}{2} + p \, . \tag{20.3}$$

This is the "Bernoulli quantity" (which becomes $\rho_0 \frac{|u|^2}{2} + p$ if we "restore" ρ_0). As $(u \times \omega)u = 0$ and $(u \times \omega)\omega = 0$, we see from equation (20.1) that H is constant along the particle trajectories and along the vorticity lines.

Let us consider a situation when

$$H \to c_0 \text{ as } x \to \infty \tag{20.4}$$

for some constant c_0. Let J be the range of the function H. Under the assumption (20.4) the set J is a bounded interval, non-trivial when H is not identically constant. Let us consider the level sets

$$\Sigma_c = \{x \in \mathbf{R}^3, \ H(x) = c\} \, . \tag{20.5}$$

By Sard's theorem, for almost every $c \in J$ the set Σ_c has the property that

$$\nabla H(x) \neq 0, \qquad x \in \Sigma_c \, . \tag{20.6}$$

[102] See

Arnold, V., I., On the topology of three-dimensional steady flows of an ideal fluid. Prikl. Mat. Meh. 30 183–185 (Russian); translated as J. Appl. Math. Mech. 30 1966 223-226.

[103] As usual, a steady-state solution is a solution which does not depend on the time t. In other words $u = u(x), p = p(x)$.

Let us denote by $J_{\rm reg}$ the set of such c, and let us also denote $J'_{\rm reg} = J_{\rm reg} \setminus \{c_0\}$ Assuming the solution u, p is smooth, we see that for each $c \in J'_{\rm reg}$ the set Σ_c is a compact smooth oriented surface (with possibly more than one connected component). At each point $x \in \Sigma_c$ the vector fields u and ω are tangent to Σ_c and hence can be considered as vector fields on Σ_c. From (20.2) and the condition $\nabla H \neq 0$ in Σ_c we see that u and ω form a basis of the tangent space of Σ_c at each point of Σ_c. The only connected orientable compact two-dimensional manifold for which such fields can exist is the torus. Hence, under our assumptions, *each connected component of* Σ_c *is a torus.* The tori will be invariant under the flow generated by u and also under the flow along the vorticity lines given by ω. Equation (20.2) gives us another look at the situation. Let ϕ^t be the flow generated by u and let ψ^s be the flow generated by ω. The condition $[u, \omega] = 0$ means that the two flows commute: $\phi^t \psi^s = \psi^s \phi^t$, see (6.17). Let us take $\overline{x} \in \Sigma_c$ and consider the map

$$\vartheta \colon \mathbf{R}^2 \to \Sigma_c \,, \qquad \vartheta(s,t) = \psi^s \phi^t(\overline{x}) \,. \tag{20.7}$$

Clearly ϑ is locally a diffeomorphism, and therefore globally is has to be a covering of Σ_c. Moreover, the set

$$\Lambda = \{(s,t) \in \mathbf{R}^2 \,, \vartheta(s,t) = \overline{x}\} \tag{20.8}$$

is a discrete subgroup of $\mathbf{R}^2$ and

$$\Sigma_c \sim \mathbf{R}^2/\Lambda \,. \tag{20.9}$$

Since Σ_c is compact, we see that Λ has to be a lattice of rank two, and (20.9) gives another confirmation that Σ_c is a torus. Moreover, we see that the map ϑ provides coordinates on the torus in which the flow by both u and ω is linear. The reader can identify the corresponding tori in the Hill's vortex we considered in lecture 18. (Note that in the region where the vorticity vanishes the function H is constant. Therefore in the case of the Hill's vortex the above picture holds only in the ball B_a where the vorticity does not vanish.) For the axi-symmetric flows without swirl the flow on the tori will be quite special and all the integral lines of u and ω will be closed. This may no longer be the case for more general flows, such as axi-symmetric flows with swirl, where the flow of u or ω can resemble a flow on $\mathbf{R}^2/\mathbf{Z^2}$ generated by a vector with an irrational slope.

The above picture is smoothly deformed as we move c in $J'_{\rm reg}$.

(ii) A variational characterization[104]

Let us consider a smooth vorticity field ω_0 in R^3. We can assume that ω_0 is compactly supported.

[104]See Arnold, V. I., Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 1966 fasc. 1, 319-361.

This paper contains many more results. We will study some of them in more detail later.

If ω_0 evolves as $\omega(x,t)$ according to Euler's equation (7.12), with $u(x,t)$ determined by the Biot-Savart law (10.12), we know from Helmholtz's law (lecture 7) that the field $\omega(x,t)$ will stay in the set

$$\mathcal{O}_{\omega_0} = \{\phi_*\omega_0\,,\ \phi\colon \mathbf{R}^3 \to \mathbf{R}^3 \quad \text{is a volume-preserving diffeomorphism}\} \tag{20.10}$$

We could impose some conditions on the behavior of ϕ as $x \to \infty$ to make the set $\mathcal{O}_{\omega_0}$ smaller, but this will not be important for our purpose at the moment. The sets $\mathcal{O}_{\omega_0}$ can be viewed as orbits of the natural action of the volume-preserving diffeomorphism groups on the vorticity fields.

Let us now think as if the function spaces were finite-dimensional, ignoring at first the considerable complications due to infinite dimensions.[105]
Let us pretend that $\mathcal{O}_{\omega_0}$ are submanifolds of the space of all "suitable" vorticity fields. (We can consider all compactly supported smooth div-free fields, for example.)

To each (smooth, compactly supported) vorticity field ω we can associate its energy by

$$E(\omega) = \int_{\mathbf{R}^3} \frac{1}{2}|u|^2\,dx\ , \tag{20.11}$$

where u is obtained from ω by the Biot-Savart law (10.12).

We now prove the following important fact:

Proposition 1

Let ω be a compactly supported (or sufficiently fast decaying) smooth vorticity field (and hence div-free). Then ω is a steady-state solution of incompressible Euler's equation if and only if it is a formal critical point of the energy E restricted to the orbit $\mathcal{O}_{\overline{\omega}}$.

The word "formal" is used to emphasize that the orbit $\mathcal{O}_\omega$ is not really a submanifold of the space of all relevant vorticity fields.

Proof

We first identify the formal tangent space to $\mathcal{O}_\omega$ at ω. This is given by all (smooth) vector fields of the form

$$\frac{d}{dt}|_{t=0}\ \phi^t_*\omega\,, \tag{20.12}$$

where ϕ^t is a flow generated by a div-free field ξ (via $\frac{d}{dt}\phi^t(x) = \xi(\phi^t(x))$). This is the same as all (smooth) vector fields of the form

$$[\xi, \omega]\,, \tag{20.13}$$

[105] In the case of Euler's equations in dimension three these complications are particularly severe.

where ξ runs through the smooth vorticity fields with the required decay. Now the condition that ω be a formal critical point of the restriction of the energy functional E to the orbit $\mathcal{O}_\omega$ is that

$$\frac{d}{dt}|_{t-0} E(\omega + t[\xi, \omega]) = 0 \tag{20.14}$$

for each smooth div-free field ξ with sufficiently fast decay. To calculate the derivative in the last expression, it is useful to use the vector potential A of u given by (10.13), (10.14). Let us denote

$$\dot{u} = \frac{d}{dt}|_{t=0}\ u(\omega + t[\xi, \omega]), \quad \dot{A} = \frac{d}{dt}|_{t=0}\ A(\omega + t[\xi, \omega]) \quad A = A(\omega\,), \quad u = u(\omega\,). \tag{20.15}$$

where $u(\omega)$ denotes the velocity field corresponding to ω (via the Biot-Savart law), and $A(\omega)$ the the velocity vector potential corresponding to ω (via (10.14)). Then, by integration by parts,

$$\frac{d}{dt}|_{t=0} E(\omega + t[\xi, \omega]) = \int_{\mathbf{R}^3} \frac{1}{2}(\dot{A}\,\omega + A\,[\xi, \omega]) = \int_{\mathbf{R}^3} A\,[\xi, \omega]\,dx\,. \tag{20.16}$$

We next calculate

$$\int_{\mathbf{R}^3} A\,[\xi, \omega]\,dx = \int_{\mathbf{R}^3} \frac{1}{2}\,\mathrm{curl}(A \times \omega)\,\xi\,dx = \int_{\mathbf{R}^3} \frac{1}{2}(u \times \omega)\,\xi\,dx. \tag{20.17}$$

If the last integral vanishes for each smooth compactly supported div-free field ξ, it means that

$$u \times \omega = \nabla h \tag{20.18}$$

for some scalar function h and we see that u solves (20.1) for a suitably chosen p. Vice versa, if u solves (20.1), the same calculation shows that ω is a formal critical point of the functional E restricted to the orbit $\mathcal{O}_\omega$, and the proof of Proposition 1 is finished.

Proposition 1 explains at a heuristic level many features of the set of steady state solutions. A finite-dimensional model of the situation is the following: suppose we have a space with coordinates (x, y), where $x \in R^m$ and $y \in \mathbf{R}^n$ and a smooth function $E(x, y)$. Assume now that $x \to E(x, y_0)$ has a critical point at x_0, i. e. $\nabla_x E(x_0, y_0) = 0$. If the Hessian matrix $\nabla_x^2 E(x_0, y_0)$ is not singular, then for each y close to y_0 the function $x \to E(x, y)$ will have a critical point $x = x(y)$ close to x_0, which depends smoothly on y. This follows from the Implicit Function Theorem. In other words, the critical points of the restriction of E to the surfaces given by fixing the coordinate y come (locally and under some non-degeneracy assumptions) as manifolds which are labeled by y. In a similar way, the steady states of Euler's equation should come as manifolds which are labeled by the orbits $\mathcal{O}_\omega$. Due to the difficulties coming form the infinite-dimensional nature of the real picture is more complicated (especially in dimension 3), with many open problems. (Note that both the dimension and co-dimension of the formal tangent space to $\mathcal{O}_\omega$ is infinite.)

20.2 The Clebsch Variables

In 1850s R.F.A.Clebsch came up with a remarkable way to re-write the incompressible Euler's equations.[106] We will outline the main idea.

Let f, g be two smooth scalar functions of $\mathbf{R}^3$. Let us assume that the vector field $f\nabla g$ is in $L^2(\mathbf{R}^3)$, so that the Helmholtz decomposition (lecture 6) of $f\nabla g$ is well-defined. Let u be the divergence-free part of $f\nabla g$. We have

$$u = f\nabla g + \nabla h\,, \tag{20.19}$$

where

$$\Delta h = -\operatorname{div}(f\nabla g)\,, \qquad \nabla h \in L^2(\mathbf{R}^3)\,, \tag{20.20}$$

or, equivalently,

$$h = \int_{\mathbf{R}^3} G_{,i}(x-y) f(y) \nabla g(y)\, dy\,, \tag{20.21}$$

where G is the Green function (10.9) of the laplacian.

This construction gives a (quadratic) map

$$(f, g) \to u\,, \tag{20.22}$$

so that u can be considered as a function of the pair (f, g). From the elementary identity

$$f\nabla g + \nabla h = -(\nabla f)g + \nabla(fg + h)\,, \tag{20.23}$$

we see that the map $(f, g) \to u$ is anti-symmetric on the (smooth) pairs (f, g) with $f\nabla g, (\nabla f)g \in L^2(\mathbf{R}^3)$:

$$(f, g) \to u \iff (g, f) \to -u\,, \quad f\nabla g,\, (\nabla f)g \in L^2(\mathbf{R}^3)\,. \tag{20.24}$$

The map $(f, g) \to u$ is not injective. For example, $(f, f) \to 0$ and, more generally, if $(f, g) \to u$ then also $(f, g + \varphi(f)) \to u$ (under appropriate assumptions).

The key point for our purposes here is the identity

$$\omega = \operatorname{curl} u = \nabla f \times \nabla g\,. \tag{20.25}$$

From (20.25) we see that

$$\omega\nabla f = 0, \quad \omega\nabla g = 0\,. \tag{20.26}$$

This means that *the functions f, g are constant along the integral lines of ω.*

In other words, *the curves*[107] $\{x \in \mathbf{R}^3\,,\ f(x) = a,\ g(x) = b\}$ *are the vorticity lines of the field u.*

[106] Clebsch, R.F.A., Über eine allgemeine Transformation der hydrodynamischen Gleichungen" J. Reine Angew. Math. 54(1857) 293–313.

Clebsch, R.F.A., Über die Integration der hydrodynamischen Gleichungen" J. Reine Angew. Math. 56(1859) 1–10.

[107] Strictly speaking, the set $\{x \in \mathbf{R}^3\,,\ f(x) = a,\ g(x) = b\}$ may not always be a smooth curve. However, it is a smooth curve for almost every (a, b) in the range of the map $x \in \mathbf{R}^3 \to (f(x), g(x)) \in \mathbf{R}^2$, by Sard's theorem.

We can therefore consider the values $f(x), g(x)$ as labels attached to the vorticity lines of the field u, with the caveat that the labeling is not one-to-one and that it might be degenerate at some points.[108] It is a good exercise to consider the same construction in dimension $n = 2$.

What is the range of the map $(f, g) \to u$? This question is non-trivial and we will not address it at the moment. It is clear, however, that the range is "rich", and it contains many genuinely three-dimensional vector fields. At the same time, not every smooth, div-free field u with compactly supported vorticity and $u(x) \to 0$ at ∞ is in the range of the map $(f, g) \to u$.

We can now consider the functions f, g as time dependent. The constructions gives us a time-dependent velocity field u and the corresponding time-dependent vorticity field ω. How do the Euler equations for u look in terms of f, g? The answer is surprisingly simple:

Proposition 2
With the notation above, the equations

$$f_t + u\nabla f = 0\,, \tag{20.27}$$
$$g_t + u\nabla g = 0\,. \tag{20.28}$$

give the evolution of $u = f\nabla g + \nabla h$ *and* $\omega = \nabla f \times \nabla g$ *under Euler's equations.*

We see that if we merely transport the "labels" f, g of the vorticity lines by the velocity field u (obtained from (f, g) by the map above), we get exactly the evolution by Euler's equations. Therefore, in these variables the Euler equation "reduces" to the transport of two scalars (f, g) by a velocity field u they generate via the map $(f, g) \to u$.

Proof
Let ϕ^t be the Lagrangian map generated by the vector field u, as in lecture 2. Equations (20.27), (20.28) are equivalent to

$$f(\phi^t(x), t) = f(x, 0), \qquad g(\phi^t(x), t) = g(x, 0)\,. \tag{20.29}$$

The vorticity of the velocity field $\tilde{\omega}(x, t)$ given by $f(x, t), g(x, t)$ satisfies

$$\tilde{\omega}(\phi^t(x,), t) = (\nabla f)(\phi^t(x), t) \times (\nabla g)(\phi^t(x), t)\,. \tag{20.30}$$

Let

$$A = \nabla\phi^t\,. \tag{20.31}$$

Then, from (20.29),

$$(\nabla f)(\phi^t(x), t)A = \nabla f(x, 0), \quad (\nabla g)(\phi^t(x), t)A = \nabla g(x, 0)\,. \tag{20.32}$$

[108] In other words, we are not claiming that the "manifold of the vortex lines" (an object for which we do not give a precise definition at the moment) is parametrized by (f, g).

Hence

$$(\nabla f)(\phi^t(x),t) = \nabla f(x,0)A^{-1}, \quad (\nabla g)(\phi^t(x),t) = \nabla g(x,0)A^{-1}\,. \tag{20.33}$$

and, by (20.30)

$$\tilde{\omega}(\phi^t(x),t) = (\nabla f)(\phi^t(x),t)A^{-1} \times \ (\nabla g)(\phi^t(x),t)A^{-1}\,. \tag{20.34}$$

Recalling that for any 3×3 matrix M and any vectors $a, b \in \mathbf{R}^3$ we have, with obvious notation,

$$(aM \times bM) = (M^* a \times M^* b) = \mathrm{Cof}\,(M^*)(a\times b) = \mathrm{Adj}\,(M)(a\times b)\,, \tag{20.35}$$

we see from (20.34) (using $\det \nabla\phi^t(x) = 1$) that

$$\tilde{\omega}(\phi^t(x,t),t) = \nabla\phi^t(x)\tilde{\omega}(x,0) = \nabla\phi^t(x)\omega(x,0)\,. \tag{20.36}$$

This says that the vorticity field $\tilde{\omega}(x,t)$ "moves with the flow" in the same way that the vorticity field given by evolution by Euler's equations, and the statement of Proposition 2 follows.

Let us define a functional

$$\mathcal{H}(f,g) = \int_{\mathbf{R}^3} \frac{1}{2}|u|^2\,dx, \tag{20.37}$$

where u is obtained from (f,g) by the map $(f,g)\to u$ above. Note that $\mathcal{H}$ is quartic in the variable (f,g). We calculate

$$\frac{d}{ds}|_{s=0}\mathcal{H}(f+s\varphi,g) = \int_{\mathbf{R}^3} u(\varphi\nabla g + \nabla\dot{h}) = \int_{\mathbf{R}^3}(u\nabla g)\varphi\,, \tag{20.38}$$

where $\dot{h}$ is the d/ds derivative of the div-free part of the Helmoltz decomposition of $(f+s\varphi)\nabla g$ at $s=0$. In a similar way

$$\frac{d}{ds}|_{s=0}\mathcal{H}(f,g+s\varphi) = \int_{\mathbf{R}^3} u(f\nabla\psi + \nabla\dot{h}) = \int_{\mathbf{R}^3} -(u\nabla f)\varphi\,. \tag{20.39}$$

We can write (20.38), (20.39) as

$$\frac{\delta\mathcal{H}}{\delta f}(f,g) = u\nabla g, \qquad \frac{\delta\mathcal{H}}{\delta g}(f,g) = -u\nabla f\,. \tag{20.40}$$

Therefore the evolution equations (20.27), (20.28) can be written as

$$f_t = \frac{\delta\mathcal{H}}{\delta g}(f,g)\,, \qquad g_t = -\frac{\delta\mathcal{H}}{\delta f}(f,g)\,, \tag{20.41}$$

resembling the Hamiltonian equations of Classical Mechanics

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{20.42}$$

Note also the connection with Proposition 1: the critical points of $\mathcal{H}$ are steady-state solutions. These connections are related to some general geometrical facts concerning the so-called "symplectic reduction" for hamiltonian systems, which is somewhat hidden in the background. We will investigate these issues in more detail later.

21

10/31/2011

21.1 Viscosity and the Reynolds number

Assume we have two parallel plates gliding along each other at a constant speed V. Assume the plates do not touch each other, but there is a very small gap d between them filled with a fluid. For concreteness let us assume the fluid is incompressible, although it is not really necessary. Consider a coordinate system in which the x_3 axis is perpendicular to the plates, the top of the first plate coincides with the plane $x_3 = 0$ and the bottom of the second plate coincides with the plane $x_3 = d$. Let the plate $x_3 = 0$ be at rest, and assume the plate $x_3 = d$ moves at speed $(V, 0, 0)$. Furthermore, let us assume that the velocity field in the fluid between the plates is

$$u(x) = \begin{pmatrix} V\,\frac{x_3}{d} \\ 0 \\ 0 \end{pmatrix} . \tag{21.1}$$

In a real situation this will be the case as long as the speed V does not exceed a certain critical value V_{crit} which, for a given fluid, is inversely proportional to d. For $V \geq V_{\text{crit}}$ the velocity field may no longer be of the form (21.1), but it will be more complicated (and possibly time-dependent).

In real fluids there is some "internal friction", and a force $(\tilde{F}, 0, 0)$ needs to be applied to the upper plate to keep it moving. Let F be the magnitude of the force per unit area of the plate, so that $F = \tilde{F}/A$, where A is the area of the plate. (The flow will be exactly of the form (21.1) only when the plates are infinite, but the effect due to the finiteness of the plates can be neglected as long as the dimensions of the plates are much larger than d.)

In many fluids (including water) one observes the following relation (for $V < V_{\text{crit}}$)

$$F = \mu\,\frac{V}{d}\,, \tag{21.2}$$

where μ is a constant (depending on the fluid). This is the simplest possible realistic relation between V, d and F, and the fluids for which is holds are often called newtonian fluids. In the non-newtonian fluids the relation between V, d and F can be more complicated.[109] In what follows we will only consider the newtonian fluids.

The constant μ is called the *viscosity* of the fluid. The physical dimension of μ is

$$[\mu] = \frac{\text{mass}}{(\text{length})\ (\text{time})}\,. \tag{21.3}$$

[109] Examples of non-newtonian fluids include blood, ketchup, certain paints, shampoos etc.

The *kinematic viscosity* ν is defined by

$$\nu = \frac{\mu}{\rho}\,, \tag{21.4}$$

where ρ is the density of the fluid. The kinematic viscosity is often also called just viscosity. The physical dimension of ν is

$$[\nu] = \frac{(\text{length})^2}{\text{time}}\,. \tag{21.5}$$

Already at this stage, without going to PDEs, we can return to Example 1 from lecture 1, in which we considered the drag force F on a ball moving at speed U in a fluid of density ρ. In our first calculation in lecture 1, based on the dimensional analysis and leading to formula (1.1), we only used one parameter to describe the fluid, namely the density ρ. The (kinematic) viscosity ν is another parameter which describes the properties of the fluid, and we can replace the assumption (1.5) by

$$F = \phi(\rho, R, U, \nu)\,. \tag{21.6}$$

Going through the dimensional analysis similarly as in lecture 1, we obtain

$$F = \rho R^2 U^2 \phi(1, 1, 1, \frac{\nu}{RU})\,. \tag{21.7}$$

The quantity $\frac{\nu}{RU}$ is dimensionless, i. e. it is independent of the choice of the basic units. It is customary to work with its inverse, which is called the Reynolds number and is denoted by Re,

$$\text{Re} = \frac{UR}{\nu}\,. \tag{21.8}$$

The kinematic viscosity of water in the SI units is approximately 10^{-6}, and for air it is approximately 10^{-5}. Therefore the Reynolds numbers for most everyday flows are quite high.

Relation (21.7) is then usually written as

$$F = c(\text{Re})\,\frac{1}{2}\rho R^2 U^2 \tag{21.9}$$

and the coefficient $c(\text{Re})$ (which is a function of Re) is called the *drag coefficient*. The function

$$\text{Re} \to c(\text{Re}) \tag{21.10}$$

has of course been subject to detailed experimental scrutiny and its behavior is simple and non-trivial at the same time. If you do an online image search for "drag coefficient", you will see some of the typical curves.[110] Explaining these curves mathematically remains a great challenge. We will return to this issue when we introduce the PDE describing the flow.

[110] For example, you can check http://www.aerospaceweb.org/question/aerodynamics/q0231.shtml

21.2 The Navier-Stokes equation

We will now introduce the viscosity into the equations of motion. As we discussed in lecture 5, in the ideal fluids the Cauchy stress tensor is given by

$$\tau_{ij} = -p\delta_{ij}\,. \tag{21.11}$$

In fluids with viscosity the Cauchy stress tensor contains an additional term

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}\,, \tag{21.12}$$

where σ_{ij} is the *viscous stress*, due to the viscosity of the fluid. For the simple flow (21.1) in a fluid with viscosity μ given by (21.2) we expect,

$$\sigma_{13} = \mu\frac{V}{d}\,, \tag{21.13}$$

which expresses the idea that the fluid layers $\{x_3 = \text{const.}\}$ slide along each other with some friction. (By the symmetry of the stress tensor one must also have $\sigma_{31} = \mu\frac{V}{d}$, which has a somewhat less transparent interpretation.) In general, it is natural to expect that the viscous stress $\sigma_{ij}(x)$ at a point x will depend on $\nabla u(x)$. Since the anti-symmetric part of $\nabla u(x)$ is related to a rigid rotation of an infinitesimal volume of fluid at x, see lecture 4, we expect that $\sigma_{ij}(x)$ will depend only on the symmetric part of $\nabla u(x)$, the deformation tensor $e_{i,j}(x) = \frac{1}{2}(u_{i,j}(x) + u_{j,i}(x))$. We will make two assumptions about this dependence.

1. The dependence is linear, i. e. we have at each point

   $$\sigma = L(e) \tag{21.14}$$

   for some linear map L between symmetric matrices.

2. The dependence is homogeneous, i. e. the map L in (21.14) is the same at each point x.

3. The dependence is isotropic, i. e.

   $$L(Q\,e\,Q^*) = Q\,L(e)\,Q^* \tag{21.15}$$

   for each rotation $Q \in SO(3)$.

It can be shown[111] that these three conditions imply that one has

$$\sigma_{ij} = 2\mu e_{ij} + \lambda\delta_{ij}e_{kk} \tag{21.16}$$

for some constants $\mu, \lambda \in \mathbf{R}$, where e_{kk} is the trace of e_{ij} (and therefore $e_{kk} = \operatorname{div} u$).

[111] The proof is not very difficult. One can do it "by hand", but the most natural context for the proof is that of the elementary representation theory of the group $SO(3)$. In particular, the proof follows directly from the Schur's Lemma.

For the incompressible fluids one has $e_{kk} = 0$ and therefore (21.16) reduces to

$$\sigma_{ij} = 2\mu e_{ij}\,. \tag{21.17}$$

The constant μ is the same as the one in (21.2), as one can easily check.

For compressible fluids one can indeed have "two viscosities", and both μ and λ are necessary to describe the viscous effects.[112]

Even with the assumptions 1–3 above, one cannot rule out that the "constants" μ, λ is (21.16) will in fact depend on the pressure, temperature, density and perhaps also some additional quantities. However, the assumptions that μ and λ are constant seems to work quite well in practice (for newtonian fluids).

If we now return to the derivation of the equations of motion in lecture 5, and use (21.12) for the Cauchy stress tensor, with σ_{ij} given by (21.16), we obtain

$$\rho u_t + \rho u \nabla u + \nabla p - \mu \Delta u - (\mu + \lambda)\nabla \operatorname{div} u = f(x,t)\,. \tag{21.18}$$

As in lecture 5, this equation should be considered together with the equation of continuity

$$\rho_t + \operatorname{div}(\rho u) = 0\,. \tag{21.19}$$

Moreover, for compressible fluids one has to specify the dependence of p on ρ.[113] In the incompressible case when $\rho = \rho_0 = \text{const.}$ one often writes f as $\rho_0 f$ and divides the equation by ρ_0 to obtain

$$u_t + u\nabla u + \frac{\nabla p}{\rho_0} - \nu \Delta u = f(x,t)\,, \tag{21.20}$$

$$\operatorname{div} u = 0\,. \tag{21.21}$$

This is the incompressible Navier-Stokes equation, first considered in the 1820s by C. L. Navier and derived in a definitive form by G. G. Stokes in 1840s.
The equation should augmented by a boundary condition. The correct boundary condition at rigid boundaries for most flows[114] is that the velocity of the fluid coincides with the velocity of the corresponding boundary point. In particular, if the boundary does not move, then $u = 0$ at it. This should be contrasted with the condition $u\,n = 0$ at the boundary for the ideal fluids.

21.3 The drag force calculation from the Navier-Stokes equation

Let us now go back to the problem of the drag force moving though an incompressible fluid of density ρ, (kinematic) viscosity ν and speed U. What is the

[112] See e. g. "The discussion of the first and second viscosities of fluids under the leadership of L. Rosenhead, F.R.S.", Proceedings of the Royal Society of London, Vol. 226, 1954.

[113] The models based on (21.18), (21.19) and a relation $p = p(\rho)$ are not quite right from the thermodynamical point of view, although they are adequate for many purposes. To get the thermodynamics completely right, one should introduce temperature and augment the two equation by a third one which expressed (locally) the conservation of energy.

[114] including virtually all flows we encounter in everyday life

prediction from the Navier-Stokes equation? Let $\Omega = \mathbf{R}^3 \setminus B_R$ be the domain occupied by the fluid. We would like to solve the Navier-Stokes equation in Ω with the boundary conditions $u = 0$ at the boundary $\partial\Omega$ and the condition $u \to U$ as $x \to \infty$. Once we know u and p, the drag force can be determined from the formula

$$F_i = \int_{\partial\Omega} \tau_{ij} n_j \, dx \,, \tag{21.22}$$

where $\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} = -p\delta_{ij} + 2\rho\nu e_{ij}$ is the Cauchy stress tensor and n_i is the unit normal pointing in the fluid region.[115]

Calculation 1

A reasonable starting point seems to be to find a steady-state axi-symmetric solution $u = u(x)$. It seems that it is not possible to find an explicit solution in a closed form.[116] One can nevertheless calculate solutions numerically on a computer.[117] We can start with low Reynolds numbers and continue the solution to higher Reynolds numbers. The steady axi-symmetric solution we calculate and the corresponding drag force will be in very good agreement with the experimental observations for Reynolds numbers $\mathrm{Re} \leq \mathrm{Re_c}$ with $\mathrm{Re_c}$ of order 10^2. For Reynolds numbers of this magnitude the "real flow" in the experiment becomes unstable. Numerically we can continue the steady axi-symmetric solution to higher Reynolds numbers, but neither the flow pattern nor the drag force will match the experimental data. (The drag force will be too low.) The explanation is simple: the calculated solution is unstable and for high Reynolds numbers it is never observed in the experiments. The flow observed in the experiments for higher Reynolds numbers is neither axi-symmetric nor time-independent.

Calculation 2

Following what we see experimentally, in the numerical calculation we introduce perturbations away from the steady-state axi-symmetric solutions, and solve the full time-dependent Navier-Stokes equations without enforcing any symmetries. It is quite non-trivial to do such calculations on a computer, but it can be done. We note that the force given by (21.22) becomes time-dependent, and we have to determine the drag force by taking an average:

$$F_i = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} F_i(t) \, dt \,. \tag{21.23}$$

If we start increasing the Reynolds number, we will see more and more oscillations in the solution (especially in the "wake region" behind the ball), and the

[115] Formula (21.22) can still be further manipulated by integrating by parts, and one can obtain expression which may be better from various points of view (e. g. more suitable for numerical simulation), but this is not our focus at the moment.

[116] However, we will soon see that one can calculate explicitly (following G. G. Stokes) the solution of the linearized equation, i. e. the function $\frac{d}{dU}|_{U=0} \, u(x, U)$, where U is the velocity at ∞ as above.

[117] This is not a simple task, but it can be done.

velocity field $u(x,t)$ will have large gradients, both in space and time. This phenomenon is often referred to as turbulence. If we have a large super-computer (by 2011 standards) we can continue the calculation up to Reynolds numbers of order 10^4 (or perhaps even 10^5) [118] before the oscillations of the solution become too fast to follow due to computer limitations.

A flow around a car going at 60 mph corresponds to Reynolds numbers well above 10^6, and at present it is not possible to fully solve the Navier-Stokes equations for such flows.[119] In fact, mathematically we do not even know if the Navier-Stokes equations have good solutions in such regimes.
In practice these difficulties are sidestepped by replacing the full Navier-Stokes equations by various models. The reason why this works reasonably well in many cases is that the highly oscillatory solutions of the Navier-Stokes equations (also called turbulent solutions) have some universal features, which we will discuss in more detail soon. The mathematical study of this behavior based on the Navier-Stokes equations seems to be out of reach at present, but there are "phenomenological theories" which shed some light on the problem.

[118] Opinions here can differ because opinions on what is a "well-resolved calculation" are not uniform.

[119] One can still use the incompressible equations at such speeds. The compressibility of air becomes important only when the speeds become comparable with the speed of sound.

22

11/2/2011

22.1 The scaling symmetry of the Navier-Stokes equation

The Navier-Stokes equation has the expected symmetries coming from its translational and rotational invariance. You can also check how the solutions transform under a Galilean transformation

$$x' = x - Ut\,, \qquad t' = t\,, \tag{22.1}$$

where $U = (U_1, U_2, U_3) \in R^3$ is a fixed velocity vector - one gets no surprises here. One can also note the trivial transformations

$$f(x,t) \to f(x,t) + \rho_0 \nabla g(x,t), \qquad p(x,t) \to p(x,t) + g(x,t)\,. \tag{22.2}$$

The most interesting symmetry of the Navier-Stokes equation is probably the scaling symmetry defined for any $\lambda > 0$ by

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t), \quad p(x,t) \to \lambda^2 p(\lambda x, \lambda^2 t), \quad f(x,t) \to \lambda^3 f(\lambda x, \lambda^2 t)\,. \tag{22.3}$$

Our notation means the following: if (u,p,f) satisfy (21.20), (21.21) in $\Omega \times (t_1, t_2)$ (where $\Omega \subset \mathbf{R}^3$) and we define

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)\,, \tag{22.4}$$

together with

$$\Omega_\lambda = \{\frac{x}{\lambda}\,,\ x \in \Omega\}\,, \tag{22.5}$$

then $(u_\lambda, p_\lambda, f_\lambda)$ satisfy (21.20), (21.21) in $\Omega_\lambda \times (\frac{t_1}{\lambda^2}, \frac{t_2}{\lambda^2})$.
One can for example think of λ as a dimension-less number which changes the the unit of length from L to $\frac{L}{\lambda}$ and the unit of time from T to $\frac{T}{\lambda^2}$, while describing the same solution. (Note than with this scaling the unit of kinematic the viscosity $\frac{L^2}{T}$ is unchanged, as it should be the case if we do not want to change the the viscosity in the equation.)

A typical situation in which this symmetry is relevant is as follows. Assume we wish to study a flow around a submarine of length L using a scale model of length l, with $L = \lambda l$, with both the submarine and the model being operated in the same fluid. If the flow around the submarine is $u(x,t)$, the flow $\lambda u(\lambda x, \lambda^2 t)$ (with x, t measured in the same units) gives a flow around the model and vice versa. This means that if we wish to know the flow around the submarine at speed U in all detail, we should run the model at speed λU. (We can of course come to the same conclusion by looking at the Reynolds number $\mathrm{Re} = \frac{UL}{\nu}$, by dimensional analysis from the last lecture.) This suggests that if we have a 100 meter submarine with the expected speed 10 m/s, a 10 meter model would have

to go at 100 m/s to generate a flow which would be exactly similar to the flow we are interested in. It is of course hard to operate the model at those speeds, and hence we might be tempted to conclude that the 1:10 scale model is not very useful. The situation with scale models of airplanes looks similarly hopeless at first. In practice the scale models are not as useless as the above might suggest, and one can get valuable insights from them.[120] The reason is again in a certain universality of the behavior of turbulent flows, which we already mentioned at the end of the last lecture.[121]

If we are willing to change the viscosity, we can scale by

$$\begin{array}{ll} u(x,t) \to \kappa u(\lambda x, \lambda\kappa t)\,, & p(x,t) \to \kappa^2 p(\lambda x, \lambda\kappa t) \\ f(x,t) \to \lambda\kappa^2 f(\lambda x, \lambda\kappa t)\,, & \nu \to \frac{\kappa}{\lambda}\nu\,. \end{array} \tag{22.6}$$

If U is some characteristic velocity of the flow (such as some average velocity, or the velocity at ∞), L is some characteristic length (such as the distance between two distinguished points in at the boundary of the flow region) and ν is the viscosity of the fluid, the above scaling changes these quantities as

$$U \to \kappa U, \quad L \to \frac{L}{\lambda}, \quad \nu \to \frac{\kappa}{\lambda}\,\nu\,, \tag{22.7}$$

and the Reynolds number based on these quantities, defined as

$$\mathrm{Re} = \frac{UL}{\nu}\,, \tag{22.8}$$

is preserved by this scaling. This of course should be the case, as the Reynolds number is defined exactly so that it is not changed in this situation.

22.2 Flows in pipes

Let us consider a pipe of radius R and a coordinate system in which the center of the pipe coincides with the x_1-axis. The kinematic viscosity of the fluid is ν,

[120] At the same time, one has to be cautious and know which results from the scale model can be used for the big object and which results should not be used.

[121] Once the Reynolds number is sufficiently high, it may be the case that some important quantities (such as the drag coefficient, for example) become fairly independent of it. If our scale model reaches the Reynolds number at which this effect starts taking place, it may be enough to get reasonable conclusions (although we cannot really be completely sure in situations where there is not much previous experience). We have plausible heuristic arguments which give some explanation of this phenomena, but its real mathematical understanding seems to be out of reach at present. Our basic beliefs in this area are based on experimental results and – more recently – on numerical simulations, but not on the theoretical analysis of the equations. What we know from the theory does not rule out the observed behavior, but we cannot really say that we would predict this behavior from the theoretical analysis of the equations if we did not see it before in experiments or numerical simulations. Once we know the behavior experimentally / numerically, we can try to explain how it can be allowed by the equations. Much has been done in this direction, but a clear-cut explanation without any hand-waving still seems to be elusive.

its density is ρ. There is an explicit steady-state solution of the Navier-Stokes equation representing a flow in the pipe:

$$u(x) = \begin{pmatrix} 2U(1 - \frac{x_2^2+x_3^2}{R^2}) \\ 0 \\ 0 \end{pmatrix}, \qquad p(x) = -\frac{8\rho\nu U x_1}{R^2} + \text{const.} \tag{22.9}$$

Here U is taken so that the amount of fluid passing though the pipe per unit time is (area of the pipe section) $U = \pi R^2 U$. The relation between the drop of pressure per unit length of the pipe $P' = p_{,x_1}$, the velocity U and the radius R based on this solution is

$$U = \frac{P' R^2}{8\rho\nu}\,. \tag{22.10}$$

Solution (22.9) is usually called Poiseulle's solution, and (22.10) is called Poiseulle's law.

Recall that in lecture 1 we mentioned another formula for this situation, the Darcy-Weischbach formula,

$$U = c\sqrt{\frac{P'R}{\rho}}\,, \tag{22.11}$$

which is the only possible dimensionally consistent relation if we assume that $U = \phi(\rho, R, P')$. If we repeat the derivation assuming

$$U = \phi(\rho, R, P', \nu) \tag{22.12}$$

we obtain

$$U = c(\mathrm{Re}')\sqrt{\frac{P'R}{\rho}}\,, \tag{22.13}$$

where Re' is the Reynolds number based on P', R, ρ, ν, defined as

$$\mathrm{Re}' = \frac{P'^{\frac{1}{2}} R^{\frac{3}{2}}}{\rho^{\frac{1}{2}}\nu}. \tag{22.14}$$

Alternatively, we can write

$$P' = \tilde{c}(\mathrm{Re})\,\frac{\rho U^2}{R}\,, \tag{22.15}$$

with the more usual definition

$$\mathrm{Re} = \frac{UR}{\nu}\,. \tag{22.16}$$

Experimentally, the functions $c(\mathrm{Re}')$ or $\tilde{c}(\mathrm{Re})$ are not quite constant for large Reynolds number, but typically they only change slowly (once large Reynolds

numbers are reached), so that (22.11) is not bad as a first approximation.[122] There is of course a dramatic difference between (22.10) and (22.11). The explanation is that no formula is valid universally for flows observed "in practice". Relation (22.10) is valid (quite precisely) for low Reynolds numbers (of order up to 10^3), whereas (22.11) is valid (only approximately) for flows observed "in practice" at high Reynolds numbers (of order, say, at least 10^5). At the low Reynolds numbers the observed flow is exactly (22.9). On the other hand, this flow is virtually never observed at high Reynolds numbers. Instead, the observed flow is of highly oscillatory nature, with oscillations in both space and time. We are again dealing with turbulence.[123] The simple solution (22.9) is unstable at the high Reynolds numbers[124]. The transition to turbulence in this case has been the subject of many studies, starting with the classical works of O. Reynolds in 1880s[125], and continuing to this day.[126] (The classical papers of Reynolds on the pipe flows are still a very good reading today.)

The situation with the flow (21.1) from the last lecture (often called the Couette flow), with the help of which we introduced viscosity, is similar to the situation with the pipe flow. The explicit flow (21.1) is observed only until a certain critical Reynolds number (which is again of order 10^3 or so). For high Reynolds numbers the flow between the plates will be turbulent, and formula (21.2) will no longer work. This effect must be taken into account when viscosity is measured: we must be sure that we are in the regime in which the flow is really (21.1), and not some more complicated flow, in which case the measured viscosity would be higher than the "basic viscosity" which appears in the Navier-Stokes equations if we wish to model all the details of the fluid motion. Note that when d in (21.1) is small, then we can go to high velocities V (of order $\frac{10^3\nu}{d}$) before the flow becomes unstable.

22.3 The Taylor-Couette flow

In the 1920s G. I. Taylor identified a situation where the onset of instability can be studied both theoretically and experimentally. We consider a domain between two concentric cylindrical surfaces. The domain is filled with a fluid we wish to study. Instead of looking at plates sliding along each other, we

[122] The subject of the flows in pipes is of course of great practical interest and there has been a lot of investigations in this direction, including a number of phenomenological formulae refining (22.11). For an introduction based on classical lectures by L. Prandtl see the book "Applied Hydro- and Aeromechanics" by O. G. Tietjens. For more recent texts you can type "pipe flow" in Google Books, and you will see a number of books on the subject.

[123] It is worth emphasizing that typical everyday flows are turbulent, due to the low viscosity of air and water which is of the order 10^{-5} and 10^{-6} respectively, in the SI units. When we talk about "high Reynolds numbers" we are not talking about some exotic speeds.

[124] The instability is not the traditional linearized instability. It seems to appear only at the non-linear level.

[125] Reynold's works are freely available online, see the links at the Wikipedia entry for O. Reynolds.

[126] For more recent works, see for example Hof, et al. Science 10 September 2004, 1594–1598 and Busse, Science 10 September 2004, 1574–1575.

can look at the situation with the two cylinders in which the cylinders rotate. (For example, the situation when the inner cylinder rotates produces unstable behavior.) If the cylinders are of infinite extent along their common axis of symmetry, there is a simple explicit solution of the Navier-Stokes equation, which in the cylindrical coordinates has the form

$$u = u^{(\theta)}(r)\, e_\theta\,, \qquad u^{(\theta)}(r) = ar + \frac{b}{r}\,. \tag{22.17}$$

The onset of instability for this flow can be seen from the linear analysis about it.[127] The transition to more complicated flows in this situation has been investigated in some detail, and various flow patters after the loss of stability have been studied at some length.[128] As you can already expect, the mathematical analysis of the turbulent regimes again remains elusive.

[127]Taylor, G.I. (1923). "Stability of a Viscous Liquid contained between Two Rotating Cylinders". Phil. Trans. Royal Society, 289-343.

See also the book Drazin, P. G., Read, W. H., Hydrodynamic Stability.

[128]See for example the book P. Chossat, G. Iooss, The Couette-Taylor problem, Springer-Verlag, 1994.

23

11/4/2011

23.1 The Stokes flow around a sphere

Let us consider a steady flow around a sphere at low velocities. We will work with cylindrical coordinates, and we slightly change our notation as follows.

a	.....	the radius of the sphere,
B_a	.....	the ball $\{x \in \mathbf{R}^3,\ \|x\| \le a\}$,
$(0,0,U)$	.....	the velocity of the fluid at ∞,
Ω_a	.....	the domain $\mathbf{R}^3 \setminus B_a$,
r	.....	the cylindrical coordinate $r = \sqrt{x_1^2 + x_2^2}$,
z	.....	the cylindrical coordinate $z = x_3$,
R	.....	the distance from the origin $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$,
ν	.....	the kinematic viscosity of the fluid,
ρ	.....	the density of the fluid,
F	.....	the drag force,
Re	.....	the Reynolds number $\mathrm{Re} = \frac{Ua}{\nu}$,
$c_d = c_d(\mathrm{Re})$	.....	the drag coefficient defined by $F = c_d \frac{1}{2}\rho a^2 U^2$,
$u = u(x)$	.....	the velocity field in Ω_a,
$\psi = \psi(r,z)$	.....	the axi-symmetric steam function of u (when u is axi-symmetric).

(23.1)

As in lecture 21, we wish to solve the Navier-Stokes equation in Ω_a with the boundary conditions $u(x) \to (0,0,U)$ as $x \to \infty$ and $u|_{\partial\Omega_a} = 0$, and then calculate F from formula (21.22), but this time we are interested in the situation $\mathrm{Re} \to 0_+$.

It can be proved rigorously that for sufficiently small Reynolds number, say $\mathrm{Re} < \mathrm{Re_c}$ with $\mathrm{Re_c}$ small[129], there exist a unique smooth steady-state solution of the problem.[130] Therefore the drag force F is well-defined for $\mathrm{Re} < \mathrm{Re_c}$.[131]

Let us now consider the quantities a, ρ, ν as fixed. The drag force F will be then a function of velocity U (well-defined for $0 \le U < \frac{\nu \mathrm{Re_c}}{a}$).We will write

[129]The exact value is not important at this point as we are only interested in the limit $\mathrm{Re} \to 0_+$.

[130]See the book G. Galdi: An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Volume II, Chapter IX.

[131]The existence result for the *steady-state* solutions can be in fact extended to arbitrary large Reynolds numbers, see the book of Galdi quoted above. However, the uniqueness has not been proved for large Reynolds numbers, and it can be expected to fail. It is quite possible that there might be several distinct steady-state solutions for a given large U, each of them with a different drag force F. One can speculate that the solution with the highest F will be the "most stable one" (even though it can still be unstable with respect to time-dependent perturbations). (Similar situation in fact does occur for flows between rotating cylinders.) It could perhaps even be the case that for some moderate value of Re (between, say, between 30 and 300) one might have two distinct stable solutions u, each with a different drag force F. There appears to be no result which would rule out such a situation.

$F = F(U)$. We can try to formally calculate the derivative

$$F'(0) = \frac{\partial F}{\partial U}|_{U=0} \ . \tag{23.2}$$

The reason why we say "formally" is that we have not investigated how smooth the function $F(U)$ is at $U = 0$ and, in particular, whether the derivative (23.2) is well-defined. This issue is more tricky than it might seem at first, since F is defined through a solution of a nonlinear elliptic system in the infinite domain Ω_a. If Ω_a was replaced by, say,

$$\Omega_{a,b} = \{x \in \mathbf{R}^3, \ a < |x| < b\} \tag{23.3}$$

with the boundary conditions

$$u|_{\{|x|=a\}} = 0, \qquad u|_{\{|x|=b\}} = (0, 0, U)\,, \tag{23.4}$$

it would more or less clear that F is analytic in U near $U = 0$ and it would be straightforward to justify the formal calculation we are about to do. However, for the infinite domain Ω_a the situation is more complicated. In fact, it turns out that F is not a smooth function at $U = 0$, although the singularity appears only at the level of $F'''(0)$. The usual formal calculations break down already at the level $F''(0)$ – this is the so-called Whitehead paradox (1880s). If we do the formal calculation in dimension $n = 2$, it will break down already at the first step – this is the so-called Stokes paradox (1850s). The singularity in $F(U)$ for $n = 2$ appears at the level of $F''(U)$.[132]

Fortuitously, in dimension $n = 3$ the formal calculation of $F'(0)$ turns out to lead to the right result. The calculation was first carried out by G. G. Stokes in 1850s. Let us denote by $\dot{X}$ the (formal) derivative $\frac{\partial X}{\partial U}|_{U=0}$ of a quantity X at $U = 0$. From (21.22) we have

$$\dot{F} = \int_{\Omega_a} \dot{\tau}_{ij} n_j \,. \tag{23.5}$$

Here

$$\dot{\tau}_{ij} = -\delta_{ij}\dot{p} + 2\rho\nu\dot{e}_{ij}\,, \tag{23.6}$$

where $\dot{e}_{ij}$ is the symmetric part of $\nabla\dot{u}$, with $\dot{u}, \dot{p}$ solving

$$\begin{array}{rcll} -\nu\Delta\dot{u} + \frac{\nabla\dot{p}}{\rho} & = & 0 & \\ \operatorname{div}\dot{u} & = & 0 & \end{array} \qquad \text{in } \Omega_a\,, \tag{23.7}$$

together with the boundary conditions

$$\dot{u}|_{\partial\Omega_a} = 0, \qquad \dot{u}(x) \to (0, 0, 1) \text{ as } x \to \infty. \tag{23.8}$$

[132] We refer the reader interested in the details to the following papers:
John Veysey II, Nigel Goldenfeld, Simple viscous flows: From boundary layers to the renormalization group, Reviews of Modern Physics, Vol 79, July-Sept. 2007.
Proudman, I. and Pearson, J. R. A., 1957, J. Fluid. Mech. **2**, 237.

The system (23.7) for $\dot{u}, \dot{p}$ is known as the *Stokes system*, or even more precisely as the *steady Stokes system*. Stokes found an explicit solution of (23.7) with the boundary conditions (23.8) by seeking the solution as an axi-symmetric vector field with no swirl, which he expressed in terms of the axi-symmetric stream function $\dot{\psi}(r, z)$ (see lecture 17 for the definition of the axi-symmetric stream function). Eliminating the pressure from the equations by taking curl, one obtains a PDE for $\psi(r, z)$ which can be solved by a separation of variables if we seek the solution in the form

$$\dot{\psi}(r, z) = \frac{1}{2} r^2 f(R) \,. \tag{23.9}$$

After some calculations one gets

$$\dot{\psi}(r, z) = \frac{1}{2} r^2 (1 - \frac{3a}{2R} + \frac{a^3}{2R^3}) \,, \qquad \dot{p} = -\rho\nu \frac{3az}{R^3} \,. \tag{23.10}$$

Note that (23.7) implies that $\Delta\dot{p} = 0$ and the function $\dot{p}$ in (23.10) is the simplest possible non-trivial harmonic function in Ω_a which vanishes at ∞ and is not radially symmetric. Substituting the solution in (23.5), we obtain the famous result of Stokes that

$$\dot{F} = 6\pi\rho\nu a \,. \tag{23.11}$$

The more usual formulation is that in the limit of vanishing Reynolds number one has

$$F \doteq 6\pi\rho\nu a U, \qquad \mathrm{Re} << 1. \tag{23.12}$$

If one tries to calculate $F''(0)$ by the same method, one finds that the equations for $\ddot{u}, \ddot{p}$ do not have a solution. (This is the Whitehead paradox). However, the derivative $F''(0)$ still exists. The singular behavior appears at the level of $F'''(0)$ in the form of a term with $\log(\mathrm{Re})$. Similarly, if one tries to calculate $F'(0)$ by this method in dimension $n = 2$, one finds that the equations for $\dot{u}, \dot{p}$ do not have a solution. (This is the Stokes paradox). The derivative $F'(0)$ still exists (and $= 0$), and the singular behavior appears at the level of $F''(0)$, also in the form of a term with $\log(\mathrm{Re})$. We refer the reader to the paper of Proudman and Pearson quoted above.

One should also prove uniqueness for the linear problem (23.7) for $\dot{u}, \dot{p}$ and the boundary condition (23.8), augmented with the requirement that $\dot{p} \to 0$ at ∞, for example. This can be done, with the level of difficulty depending on the class in which uniqueness is sought.

The solutions of the non-linear problem for small $\mathrm{Re} << 1$ are in some sense close to the Stokes solution, although one must be somewhat careful with the definition of what "close" means, due to the unboundedness of the domain Ω_a. This family of solutions can be numerically continued from small Reynolds numbers to large Reynolds number, where they still seem to persist, although they are unstable to time-dependent perturbations. For large Reynolds numbers the drag force for these solution will be much lower than the drag force for

the time-dependent (turbulent) solutions which are observed in real flows with large Re, somewhat similarly to the situation with difference in the resistance of pipe flows between the laminar solution (22.9) and the turbulent solutions actually observed for larger Reynolds numbers. We can speculate that the unstable continuation of the Stokes flow to large Reynolds numbers[133] perhaps plays the role which the Poiseulle's flow (22.9) plays for the pipe flows.

[133] assuming a well-defined continuation exists

24

11/7/2011

During this lecture we discussed photographs of various flows from Milton Van Dyke's book "An Album of Fluid Motion" which illustrate some of the phenomena we have encountered so far in the course. In case you have not attended the lecture and are interested in the material, the book is available in the Walter Library reserve. The commentaries in the book explain very well the various phenomena in the photographs. Many of the pictures can be found online.

25

11/9/2011

25.1 The Landau Jet

Let consider an incompressible viscous fluid of constant density ρ and viscosity $\nu > 0$ and a steady-state flow of the fluid in $\mathbf{R}^3$ satisfying

$$\begin{aligned} -\nu\Delta u + u\nabla u + \tfrac{\nabla p}{\rho} &= f(x)\,, \\ \operatorname{div} u &= 0\,, \\ u &\to 0\,, \qquad x \to \infty\,. \end{aligned} \tag{25.1}$$

Let us start with a smooth compactly supported f. When we are given an f the existence of smooth solution to (25.1) is a-priori not clear, but it was proved by J. Leray in 1930s.[134] In general we do not expect that the solutions will be unique when f is large, there might be perhaps several solutions for a given (large) f, although such examples have not been constructed rigorously, as far as I know. Let $\phi(x)$ be the usual mollifier, i. e. a radial smooth non-negative function on $\mathbf{R}^3$ with support in the unit ball such that $\int_{\mathbf{R}^3} \phi = 1$, and let $\phi_\varepsilon(x) = \varepsilon^{-3}\phi(x/\varepsilon)$. We can consider the problem above with

$$f(x) = \beta\phi_\varepsilon(x)e_3\,, \tag{25.2}$$

with a parameter $\beta > 0$ and $e_3 = (0, 0, 1)$. We can now consider the limit case $\varepsilon \to 0_+$. In the limit $\varepsilon \to 0_+$ we have

$$f(x) = \beta\,\delta\,e_3\,, \tag{25.3}$$

where δ is the Dirac distribution supported at $x = 0$. We have, with some abuse of notation,

$$\lambda^3\delta(\lambda x) = \delta(x)\,, \tag{25.4}$$

and therefore, recalling the scaling symmetry (22.3) of the Navier-Stokes equation, we see that it is reasonable to expect that the equations

$$\begin{aligned} -\nu\Delta u + u\nabla u + \tfrac{\nabla p}{\rho} &= \beta\delta e_3\,, \\ \operatorname{div} u &= 0\,, \\ u &\to 0\,, \qquad x \to \infty\,. \end{aligned} \tag{25.5}$$

have a solution u with

$$\lambda u(\lambda x) = u(x)\,, \qquad \lambda > 0\,. \tag{25.6}$$

Moreover, we can ask that u be smooth in $\mathbf{R}^3 \setminus \{0\}$. It is also reasonable to expect that the solution u might be chosen to be axi-symmetric.

[134] See for example G. Galdi's book "An introduction in the Mathematical Theory of the navier-Stokes equations", Vol. 2

Now if u satisfies (25.6) then it is determined by its restriction on the sphere $\mathbf{S}^2$ and if it is moreover axi-symmetric, then it is actually determined by a (vector-valued) function of one variable only. Therefore, for such fields the system (25.5) reduces to a system of ODE. If one writes down this system in suitable coordinates, such as the polar coordinates, it looks complicated, but in 1944 L. Landau was able to find a family of explicit solutions to this system of ODEs. The calculation can be found in standard textbooks, see for example the *Fluid Mechanics* by Landau and Lifschitz (p. 82 of the second edition), or *An Introduction to Fluid Mechanics* by G. K. Batchelor (p. 206). To write down the Landau solutions, we recall that the spherical coordinates are given by

$$\begin{aligned} x_1 &= R\cos\theta\,, \\ x_2 &= R\sin\theta\cos\varphi\,, \\ x_3 &= R\sin\theta\sin\varphi\,, \end{aligned} \tag{25.7}$$

where we use $R = |x|$ to distinguish it from the cylindrical coordinate $r = \sqrt{x_1^2 + x_2^2}$.

For $A > 1$ we set

$$b = b(A) = 16\pi\left(A + \frac{1}{2}A^2\log\frac{A-1}{A+1} + \frac{4A}{3(A^2-1)}\right) \tag{25.8}$$

The function $b(A)$ is strictly decreasing for $A \in (1,\infty)$ with $\beta \to +\infty$ as $A \to 1_+$ and $b(A) \sim \frac{16\pi}{A}$ for $A \to \infty$.

Let us consider an axi-symmetric velocity field given by the axi-symmetric stream function

$$\psi = \frac{2\nu R\sin^2\theta}{A - \cos\theta}\,. \tag{25.9}$$

Landau showed that

The velocity field given by the stream function (25.9) solves (25.5) with $\beta = \nu^2 b(A)$, with the pressure given by

$$p = \rho\nu^2\frac{4(A\cos\theta - 1)}{R^2(A-\cos\theta)^2}\,. \tag{25.10}$$

It can be shown that Landau's solution (25.9), (25.10) is the only solution of (25.5) which is smooth in $\mathbf{R}^3 \setminus \{0\}$ and satisfies (25.6).[135] It is interesting to compare the Landau solution with the fundamental solution of the linear Stokes system

$$\begin{aligned} -\nu\Delta\overline{u} + \frac{\nabla\overline{p}}{\rho} &= \beta\delta e_3\,, \\ \operatorname{div}\overline{u} &= 0\,, \\ \overline{u} &\to 0\,, \qquad x \to \infty \end{aligned} \tag{25.11}$$

which is given by the stream function

$$\overline{\psi} = \frac{\beta}{\nu}\frac{1}{8\pi}R\sin^2\theta\,. \tag{25.12}$$

[135] V. Sverak, On Landau's solution of the Navier-Stokes equation, Journal of Mathematical Sciences, 2011, Volume 179, Number 1, pp. 208–228.

In the limit $A \to \infty$, which means $\beta \to 0$, we have, in some sense,

$$\psi \sim \frac{16\pi}{A}\overline{\psi}\,. \tag{25.13}$$

although one needs to be somewhat careful about the nature of the approximation. (The approximation is certainly good on compact subsets of $\mathbf{R}^3 \setminus \{0\}$.)

In principle the Landau solution should be relevant for describing the flow which we obtain by blowing in the straw, so we can try to use it to explain Example 3 from lecture 1: why is the ping-pong ball stable if we place it in the jet from the straw (assuming we the straw is pointed up)? However, formula (25.10) for the pressure shows that in the planes $\{x_3 = c > 0\}$ the maximum pressure is at the center of the jet $(x_1, x_2, c) = (0, 0, c)$ and the pressure is decaying as we move away from the center. This would predict that the ping-pong ball put in the jet will be unstable.

25.2 Reynolds stress

To understand why the ping-pong ball in Example 3 from lecture 1 is stable, we need to know that in most jets we encounter in everyday life the flow is not given by the Landau solution from the previous section, but instead the velocity field is turbulent and oscillates, similarly as in pipe flows at high Reynolds numbers. This is again not so easy to see from the equations, and at this point we take it as an experimental observation. Let us assume the the velocity field in the jet is of the form

$$u(x,t) = \overline{u}(x) + v(x,t)\,, \tag{25.14}$$

where $\overline{u}(x)$ is an average velocity field at the point x. We emphasize that we have in mind a situation where it is reasonable to expect that some average velocity field exists, such as when we blow into a straw with a constant effort and the straw is stationary. There is more than one way to define the average field $\overline{u}(x)$. The simplest one is probably

$$\overline{u}(x) = \lim_{(t_2-t_1)\to\infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} u(x,t)\,dt\,, \tag{25.15}$$

assuming the limit exists. Proving the existence of the limit is mathematically out of reach, and we will simply *assume* that the limit exists. In general, we will use the notation

$$\overline{f}(x) = \lim_{(t_2-t_1)\to\infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(x,t)\,dt\,, \tag{25.16}$$

and we assume that the limit exists whenever the notation is used. For our velocity field $u(x,t)$ we will write

$$u(x,t) = \overline{u}(x) + v(x,t)\,, \tag{25.17}$$

or simply

$$u = \overline{u} + v \ . \tag{25.18}$$

We will write the Navier-Stokes equation in the following way

$$u_{it} + \partial_j(u_j u_i) + \frac{p_{,i}}{\rho} - \nu \Delta u_i = 0 \, , \qquad \operatorname{div} u = 0 \, . \tag{25.19}$$

We take averages to obtain an equation for $\overline{u}$. Assuming that $|u(x,t)| \leq C$ for each x, t (which we again do not know how to prove rigorously in most interesting situations) and $\rho = \text{const.}$, it is not hard to pass to averages in the linear terms of the equation:

$$\overline{u_t} = 0, \quad \overline{\Delta u} = \Delta \overline{u}, \quad \overline{\left(\frac{p_{,i}}{\rho} \right)} = \frac{\overline{p}_{,i}}{\rho} \, , \quad \overline{\operatorname{div} u} = \operatorname{div} \overline{u} \, . \tag{25.20}$$

For the term $u_i u_j$ we have

$$\overline{u_i u_j} = \overline{(\overline{u}_i + v_i)(\overline{u}_j + v_j)} = \overline{u}_i \overline{u}_j + \overline{v_j v_j} \, , \tag{25.21}$$

due to the obvious fact that $\overline{v}_i = 0$.

For general solutions $u(x,t)$ there is not much we can say about $\overline{v_i v_j}$. Easy examples show that we cannot expect it will vanish.[136] We have to accept that it is a new quantity, which – in general – is not related to $\overline{u}$ in any obvious way. We introduce the notation

$$\overline{v_i v_j} = R_{ij} = R_{ij}(x) \, . \tag{25.22}$$

The tensor R_{ij} is called *Reynolds stress*. There is some vague analogy of this tensor with the viscous stress σ_{ij} we discussed in lecture 21. The origin of the viscous stress is at the molecular level: the molecules from layer of fluid moving at different speeds mix due to their chaotic thermal motion and their mixing explains the tendency for the velocities at two "neighboring layers" to approach their average. Similarly, at a much larger scale of the motion of "fluid parcels", the various parcels of fluid mingle and interact, and this tends to average out the mean velocities, creating an (imperfect) analogy of the viscous stress at this much larger scale. The effect of this on $\overline{u}$ is described by the Reynolds stress. We can already anticipate that one the actions of the Reynolds stress will be to enhance viscous-like effects. If we look at the fluid at the larger scale where the oscillations $v(x,t)$ look small, the Reynolds stress might induce – among other things – an extra "apparent viscosity", sometimes called as *turbulent viscosity*.[137] It is of course very challenging to derive this effect mathematically with any precision, but at a heuristic level it is not hard to understand that some effect of this form should take place.

[136] For example, if $f(t) = \sin t$, then $\overline{f} = 0$ and $\overline{ff} = \frac{1}{2}$.

[137] In reality this is an over-simplification. Some experts consider the notion of turbulent viscosity as problematic.

The equation for the average velocity $\overline{u}$ are simply

$$\frac{\partial}{\partial x_j}\left(\overline{u}_i\overline{u}_j + \frac{\overline{p}}{\rho}\delta_{ij} + R_{ij} - \nu e_{ij}(\overline{u})\right) = 0\,, \tag{25.23}$$

or

$$-\nu\Delta\overline{u}_i + \overline{u}_j\overline{u}_{i,j} + \frac{\overline{p}_{,i}}{\rho} + R_{ij,j} = 0\,. \tag{25.24}$$

These equations have to be complemented by

$$\operatorname{div}\overline{u} = 0\,. \tag{25.25}$$

If we could express R_{ij} in terms of $\overline{u}, \overline{p}$, we would have a closed set of equations for $\overline{u}, \overline{p}$. However, it is not clear how to express R_{ij} in terms of $\overline{u}, \overline{p}$. Any type of expression, even non-local expressions where $R_{ij}(x)$ would depend not only on $\overline{u}(x), \nabla\overline{u}(x), \nabla^2\overline{u}(x), \dots$ but also on the values of these quantities at other points would be great, as it would close the equations. In general one would obtain an integro-differential equation for $\overline{u}$. There has been a lot of effort to come up with plausible phenomenological rules of how to determine R_{ij} from $\overline{u}_i$. Some of these rules work reasonably well for certain special flows, but the goal of finding a good rule which works for general situations remains elusive.

One can attempt to use the Navier-Stokes equation to find information about R_{ij}, and average relations for $\overline{v_iv_j}$ obtained from the equation. However, due to the non-linearity of the equations, in equations for $\overline{v_iv_j}$ we will have new terms such as $\overline{v_iv_jv_k} = R_{ijk}$, and again we cannot close the equations. This can be continued to $\overline{v_iv_jv_kv_l}$ and higher order averages, but there are always new terms which do not allow us to close the equations. This is the notorious *closure problem.*

Although the introduction of the Reynolds stresses does not lead to good equations for $\overline{u}$, we can still get very valuable insights from this concept. For example, let us look at the problem with the stability of the ping-pong ball in the jet of air as described in Example 3 from lecture 1, which we also discussed in the previous section. Let $u(x,t)$ be the flow in the jet (now assumed to be oscillating). The position of the jet in the coordinate system is assumed to be the same as for the landau jet in the previous section. Let $u = \overline{u} + v$ as above. Let R_{ij} be the Reynolds stress. What can we say about R_{ij} at a point $(0,0,c)$ with $c > 0$? Based on the symmetries, we can expect (at $(0,0,c)$) the following:

$$R_{12} = 0, \quad R_{13} = 0, \quad R_{23} = 0, \quad R_{11} = R_{22}\,. \tag{25.26}$$

Let

$$\overline{v_1^2 + v_2^2 + v_3^2} = V^2\,. \tag{25.27}$$

Then

$$R_{ij} = \frac{V^2}{3}\delta_{ij} + R'_{ij}\,, \qquad R'_{kk} = 0 \quad \text{(summation understood)}\,. \tag{25.28}$$

If we had $R'_{ij} = 0$ near the axis of the jet, the equation (25.24) would reduce to

$$-\nu\Delta\overline{u}_i + \overline{u}_j\overline{u}_{i,j} + \frac{(\overline{p} + \frac{1}{3}\rho V^2)_{,i}}{\rho} = 0\,. \tag{25.29}$$

The quantity $P = \overline{p} + \frac{1}{3}\rho V^2$ would then coincide with the pressure calculated without the presence of the Reynolds stresses, and the real (average) pressure would be

$$\overline{p} = P - \frac{1}{3}\rho V^2\,. \tag{25.30}$$

This shows that the oscillating part of the solution provides a mechanism for lowering the pressure at the center of the jet, giving potentially some explanation for the stability of the ping-pong ball. Of course, in reality we do not have $R'_{ij} = 0$ and the situation is more complicated. If we start looking at what is happening at the boundary of the ping-pong ball, things get even more complicated, because at the boundary we should have $V = 0$ (assuming the ball is stationary). Additional analysis is needed to account for all these effects, but the basic idea that the oscillatory part of the flow causes a drop in the pressure is good. In real jets, if we measure the pressure at the planes $\{x_3 = c > 0\}$, we do see that the pressure is the lowest at the axis of the jet, and this is why the ping-pong ball is stable. We see from the above analysis that the stability crucially depends on the fact that the flow in the jet is turbulent. More details about turbulent jets, including plots of the Reynolds stresses (obtained from measurements or numerical simulations) can be found in the book of S. Pope "Turbulent Flows".[138] On p. 113 of the book you can find an argument that a more precise (but still approximate) formula for the pressure $\overline{p}$ should be

$$\overline{p} = p_\infty - \rho\,\overline{v_3 v_3}\,, \tag{25.31}$$

where p_∞ is limiting value of the pressure as at $(0, 0, x_3)$ with $x_3 \to \infty$.

[138] I thank to Prof. Ivan Marusic for pointing out this reference to me, and for a discussion concerning experimental observations and their interpretations. It should be emphasized that any possible flaws in the above descriptions and heuristics are solely due to the author of these notes.

Homework assignment 2
due November 28

Do one or more of the following three problems:

Problem 1

Assume that u is a smooth divergence-free velocity field in $\mathbf{R}^3$ such that

$$\operatorname{curl} u(x) = \lambda(x)u(x)$$

in $\mathbf{R}^3$ for some function $\lambda(x)$. Such fields are called *Beltrami fields*.

1. Show that any Beltrami field is a steady-state solution of the incompressible Euler's equation (for a suitable pressure $p(x)$).

2. For a given $\xi \in \mathbf{R}^3$ and a given $\lambda \in \mathbf{R}$ find all solution of the system $\operatorname{curl} u = \lambda u, \quad \operatorname{div} u = 0$ of the form $v(x) = \hat{v}e^{i\xi x}$, where $\hat{v} \in \mathbf{C}^3$. Show that if $u(x)$ is a linear combination of solutions of this form for different ξ but the same λ, then the real part of $u(x)$ solves the steady Euler equation. Also show that the so-called ABC flows, given by $u(x) = \begin{pmatrix} A\sin x_3 + C\cos x_2 \\ B\sin x_1 + A\cos x_3 \\ C\sin x_2 + B\cos x_1 \end{pmatrix}$, can be obtained in this way.

3.* (Optional) Decide if there are non-trivial Beltrami fields in $\mathbf{R}^3$ which are compactly supported.

Problem 2

Let us consider a large axi-symmetric cylindrical container C filled with perfect incompressible fluid of density ρ which rotates about its axis of symmetry with angular velocity Ω in an otherwise empty space. Assume that the gravitational forces can be neglected.[139]

1. Show that in the coordinate frame of the cylinder in which the axis of symmetry passes through the origin the equations of motion for the fluid are

$$\begin{aligned} u_t + u\nabla u + 2\Omega \times u + \tfrac{1}{\rho}\nabla(p - \tfrac{1}{2}\rho|\Omega \times x|^2) &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \qquad \text{in } C \tag{25.32}$$

together with the boundary condition $u\,n = 0$ at the boundary ∂C of C.

2. Linearize the equations about the trivial solution $u = 0$ and, assuming that $C = \mathbf{R}^3$, find the solutions of the form

$$v(x,t) = \hat{v}e^{i\xi x - i\omega t}$$

of the linearized system, where $\hat{v} \in \mathbf{C}^3$. (The real part of these solutions can be though of as representing "waves" in the incompressible rotating fluid. Note

[139] Since the fluid is incompressible, the effect of the gravitational forces would not change the motion of the fluid, it would only change the pressure.

that the relation between ξ, ω and $\hat{v}$, also called the *dispersion relation*, is non-trivial.)

Problem 3

Consider two large plates in the planes $x_3 = 0$ and $x_3 = d > 0$ respectively, with d small. Assume that the space the between plates is filled with incompressible fluid of density ρ and kinematic viscosity ν. Assume that the upper plate oscillates in its own plane along the x_1-axis with speed $v(t) = V \cos \omega t$. Find what the motion of the fluid will be after a sufficiently long time, and find the forces per unit area acting on both plates (after a sufficiently long time).

26

11/11/2011

26.1 Reynolds stress in a channel flow

Let $L > 0$ and let

$$\Omega = \{x \in \mathbf{R}^3,\, -L < x_2 < L\} \tag{26.1}$$

Let us think of Ω as a limiting case of a "channel"

$$\Omega_{L,L'} = \{\in \mathbf{R}^3\,, -L < x_2 < L\,,\, -L' < x_3 < L'\}, \tag{26.2}$$

with $L' \to \infty$. (In this situation we usually think of x_1 as measuring the length, x_2 measuring the height, and x_3 measuring the width.) We consider a flow $u(x,t)$ in the channel such that the mean flow (defined by (25.15), which we again *assume* to be a good definition) is of the form

$$\overline{u}(x) = \begin{pmatrix} \overline{u}_1(x_2) \\ 0 \\ 0 \end{pmatrix}. \tag{26.3}$$

The boundary condition is that $u(x,t) = 0$ at $\partial\Omega$, and hence also $\overline{u}_1(-L) = \overline{u}_1(L) = 0$. We wish to obtain as much information about the Reynolds stresses as possible, based on the Navier-Stokes Equation and some plausible symmetry assumptions. As in the last lecture we assume

$$u(x,t) = \overline{u}(x) + v(x,t)\,, \qquad \text{with } \overline{v} = 0\,, \tag{26.4}$$

and we define the Reynolds stresses

$$R_{ij}(x) = \overline{v_i v_j}(x)\,. \tag{26.5}$$

We will assume the following identities, based on the symmetries of the situation:

$$\begin{aligned}
\overline{u}_1(x_2) &= \overline{u}_1(-x_2)\,, \\
R_{ij} &= R_{ij}(x_2)\,, \\
R_{13} &= 0\,, \\
R_{23} &= 0\,, \\
R_{ii}(-x_2) &= R_{ii}(x_2), \quad i = 1,2,3 \text{ (no summation)}\,, \\
R_{12}(-x_2) &= -R_{12}(x_2) \\
\overline{p} &= \overline{p}(x_1, x_2)\,, \\
\overline{p}(x_1, -x_2) &= \overline{p}(x_1, x_2)\,.
\end{aligned} \tag{26.6}$$

We also recall that, by definition, $R_{ij} = R_{ji}$. Therefore the Reynolds stress tensor in this case involves four functions of one variable: $R_{11}(x_2), R_{22}(x_2), R_{33}(x_2)$ and $R_{12}(x_2)$. In general we do not expect expect that the solution $u(x,t)$ will

have such symmetries, once we get into the regimes where the simple flow given by the explicit steady-state solution

$$u(x) = \begin{pmatrix} 2U(1 - \frac{x_2^2}{L^2}) \\ 0 \\ 0 \end{pmatrix} \tag{26.7}$$

will be unstable. However, it is reasonable to expect that the averaging involved in the definitions of $\overline{u}$ and R_{ij} will restore the symmetries.

The average pressure $\overline{p}(x)$ is expected to depend on x_1, x_2, but not on x_3. (A pressure gradient in the x_1 direction is necessary to maintain the flow, and hence we expect the x_1- dependence.) The independence on x_3 is expected due to the invariance under translations in the x_3 direction and the fact that the mean flow is perpendicular to the x_3 direction.

Note that the mean flow $\overline{u}$ given by (26.3) satisfies $\overline{u}\nabla\overline{u} = 0$. Denoting by $'$ the derivative $\frac{\partial}{\partial x_2}$ and taking into account (26.6), the averaged equations (25.24) reduce to

$$\begin{aligned} -\nu\overline{u}_1'' + \overline{p}_{,x_1} + R_{12}' &= 0\,, \\ \overline{p}' + R_{22}' &= 0\,. \end{aligned} \tag{26.8}$$

From the second equation of (26.8) we see that

$$\overline{p}(x_1, x_2) = \pi(x_1) - R_{22}(x_2)\,. \tag{26.9}$$

Substituting this into the first equation and using the second equation again, we see easily that $\pi(x_1) = -Ax_1 + \text{const.}$ for some $A \in \mathbf{R}$ which we expect to be positive. Hence

$$\overline{p}(x_1, x_2) = -Ax_1 - R_{22}(x_2) + \text{const.} \tag{26.10}$$

We see that R_{22} lowers the pressure in the middle of the channel, similarly to what we saw in (25.30) or (25.31) in the context of the turbulent jet. We can now substitute $\overline{p}_{x_1} = A$ into the first equation and integrate once to obtain (after using $\overline{u}_1'(0) = R_{12}(0) = 0$ which follows from (26.6)),

$$R_{12} = Ax_2 + \nu\overline{u}_1'\,. \tag{26.11}$$

this expression will of course vanish if $u(x,t)$ is given by (26.7) and the flow is laminar. For turbulent flows the profile will of $\overline{u}_1(x_2)$ be much "flatter" away from the boundaries, changing only slowly as we move from the center of the channel to the boundary, with a quite sharp drop to zero once we get close to the boundary. This gives a good idea about what $R_{12}(x_2)$ is. However, we do not have enough equations to determine all the functions involved. If we know neither R_{12} nor $\overline{u}_1$, equation (26.11) does not say much. The closure problem appears again, and we do not know how to overcome it using only the "first principles", which in this case are expressed by the Navier-Stokes equation. There are various phenomenological theories and closure models which enable one to calculate $\overline{u}_1$ and lead to quite reasonable agreement of the calculated

profile $\overline{u}_1$ with measurements. However, even in this simple situation there is not a uniform opinion among experts about which models should be used.[140]

26.2 Energy dissipation

In lecture 12 we studied the energy conservation and the local flux of energy for Euler's equation. It is quite easy to adapt the formulae to the case of the Navier-Stokes equation. For viscous fluids we typically have loss of energy due to the internal friction and we expect that this will appear through the viscous stress tensor and the deformation tensor (see lecture 21) in the equation for the energy flux. In what follows we assume for simplicity that the fluid is incompressible and its density ρ is constant.
We write the Navier-Stokes equation as

$$\rho u_{it} + \rho u_j u_{i,j} + p_{,i} - \sigma_{ij,j} = 0\,, \qquad u_{i,i} = 0\,. \tag{26.12}$$

Multiplying the first equation by u_i and using the second equation we obtain

$$(\rho\frac{|u|^2}{2})_t + [u_j(\rho\frac{|u|^2}{2} + p) - \sigma_{ij}u_i]_{,j} = -\sigma_{ij}u_{i,j} = -\sigma_{ij}e_{ij} = -2\mu e_{ij}e_{ij}\,. \tag{26.13}$$

In comparison with equation (12.2) for Euler (with $f = 0$), we have the additional term $-\sigma_{ij}u_i$ which describes the work done by the viscous forces and the term $-2\mu\, e_{ij}e_{ij}$ which describes the local rate of energy loss due to the viscosity. The rate of energy loss on a domain $\mathcal{O}$ at a given time is

$$\int_{\mathcal{O}} 2\mu e_{ij}e_{ij}\, dx \tag{26.14}$$

and therefore we can call $2\mu e_{ij}e_{ij}$ the local rate of energy dissipation per unit volume. The physical dimension of this quantity is

$$[2\mu e_{ij}e_{ij}] = \frac{M}{LT^3} = \frac{\text{mass}}{(\text{length})\ (\text{time})^3}\,. \tag{26.15}$$

Often it is more convenient to work with the quantity

$$2\nu e_{ij}e_{ij} = 2\frac{\mu}{\rho}e_{ij}e_{ij} \tag{26.16}$$

which is the local rate of energy dissipation per unit mass. Its physical dimension is

$$[2\nu e_{ij}e_{ij}] = \frac{L^2}{T^3} = \frac{(\text{length})^2}{(\text{time})^3}\,. \tag{26.17}$$

[140] See for example the papers
Barenblatt G.I., Chorin A.J., Hald O.H., Prostokishin V.M., Structure of the zero-pressure-gradient turbulent boundary layer, Proc. Natl. Acad. Sci. USA 29:7817-19, 1997.
Alexander J. Smits, Beverley J. McKeon, and Ivan Marusic, HighReynolds Number Wall Turbulence, Annual Review of Fluid Mechanics, 2011, 43:353-75.

For a turbulent flow $u(x,t)$ we introduce a quantity

$$\epsilon = \overline{2\nu e_{ij} e_{ij}}\,. \tag{26.18}$$

This is the local average rate of dissipation per unit mass in the fluid, and often it is called just *dissipation*. In general we expect ϵ to depend on x but not on t (which is obvious if we use the averaging (25.15), assuming as always that the limit exists). We may write $\epsilon = \epsilon(x)$ if we wish to emphasize the dependence. For example, in the turbulent channel flow considered in the previous section we expect $\epsilon = \epsilon(x_2)$, with the profile nearly constant until we are close to the boundary, where it is less clear what to expect heuristically, as the changes in the flow near the boundary can be expected to be quite dramatic. (The region one has to worry about is only a thin boundary layer near the channel walls.) Note that unlike the Reynolds stresses R_{ij}, the dissipation ϵ typically does not vanish at the boundary. In the first approximation it is not unreasonable to assume that ϵ is not far from a constant up to the boundary.[141] (Note that for the laminar flow (26.7) ϵ is exactly constant for simple reasons. The reasons that it should also be (approximately) constant for turbulent flows are not so simple.) The situation with the pipe flow is similar: for large Reynolds number we expect ϵ to depend only on the distance from the pipe center, and, in fact, we expect the profile to be not far from a constant. Again, the situation can be somewhat more complicated near the pipe wall.

We should emphasize that the relative simplicity of the behavior of ϵ is expected only due to the averaging procedure in its definition. The non-averaged rate of energy dissipation $2\nu e_{ij} e_{ij}$ can be expected to oscillate wildly in both space and time once the flow is turbulent.

We now come to a key point in the phenomenological approach to turbulent flows. Namely, it is observed that

(P) *in many situations, once the Reynolds number is sufficiently high, the dissipation ϵ is quite independent of viscosity.*

This is by no means obvious and it might look even suspicious at first. For example, let us consider a pipe flow as in lecture 22. Assume the mean speed of the fluid is U, the radius of the pipe is R, and these quantities are fixed while we take smaller and smaller ν, so that the Reynolds number $\mathrm{Re} = \frac{UR}{\nu}$ becomes very large. The above principle (P) says that as we decrease the viscosity, the dissipation ϵ will not (significantly) change, once sufficiently large Reynolds numbers are reached. The idea is that the decrease of ν in the expression (26.18) will be compensated by the increase of $e_{ij}e_{ij}$. The smaller viscosity will result in steeper gradients ∇u and a larger magnitude of the deformation tensor e_{ij}, in such a way that the decrease of ν and the increase of $e_{ij}e_{ij}$ will cancel each other. It is not hard to understand that some effects in this direction should take

[141] These issues have of course been studied in detail experimentally and by numerical simulations. See for example the book of S. Pope "Turbulent Flows", Chapter 7.1

place, but the quantitative observation that the decrease of ν will be more or less exactly compensated by the increase in $e_{ij}e_{ij}$ so that ϵ will remain practically unchanged is harder to explain. We will accept it as an experimental fact. Of course, once we accept that ϵ is independent of ν, it is not hard to explain that the pressure drop P' in formula (22.15) should be quite independent of the Reynolds number, as the dissipation and the pressure drop are related to each other.

The above principle comes with some caveats (some of which we will discuss), but in many cases it works well and enables one to make useful predictions in problems which seem to be untractable by any other means.

27

11/14/2011

27.1 Kolmogorov-Richardson energy cascade

The velocity field in a turbulent flow is complex and one can see various structures at different scales. An example is provided by our everyday experience with meteorological flows. These flows have some two-dimensional features (because the relevant layer of the atmosphere is very thin in comparison with Earth's radius) and there are also some characteristic effects due to the Earth's rotation, but they still illustrate reasonably well some of the phenomena we wish to discuss. A large storm can have dimension of hundreds of kilometers and the motion of air associated with it can have a characteristic and easily distinguishable large-scale structure. It produces many local storms, and each of those can be seen as its own event, with clearly distinguished structures. Each local storm produces many wind gusts, and each of those has again its own characteristic structure and scale. This hierarchy continues up to a quite small scale, of orders of millimeters, where one could still distinguish individual vortex filaments produced by the flow, for example.[142] In a turbulent flow we can similarly discern various structures, parcels of fluid of various sizes which move for a while in a coordinated fashion, before morphing to other structures. It is customary to call these structures "eddies" or "whirls". Each eddy can be thought of as having a certain size and characteristic speed of the flow associated with it. It is not easy to formalize this notion mathematically. Usually the Fourier transform provides a good way to do it. The easiest situation is when we consider vector fields on a three dimensional torus of dimension $2\pi L > 0$

$$\mathbf{T}^3_L = \mathbf{R}^3/(2\pi L\mathbf{Z}^3), \tag{27.1}$$

which is the same as considering vector fields which are $2\pi L$ periodic in the direction of each coordinate axis. We can express each vector field $u(x)$ in $\mathbf{T}^3_L$ as

$$u(x) = \sum_{k\in \frac{\mathbf{Z}^3}{L}} \hat{u}(k)e^{ikx}\,, \tag{27.2}$$

and we can identify eddies of size l with the Fourier modes with wave numbers $k = (k_1, k_2, k_3)$ with $|k| \sim \frac{1}{l}$. We note that the physical dimension of the wave number k is

$$[k] = \frac{1}{\text{length}} = \frac{1}{L}\ .\ ^{143} \tag{27.3}$$

The normalization in (27.2) is chosen so that the energy per unit mass of the fluid at frequency k is $|\hat{u}(k)|^2$. Note that this is different from a normalization

[142] Of course, the meteorological flows include also various thermodynamical effects, such as evaporation, condensation, changes of temperatures which make them even more complicated. We will not discuss those effects here.

[143] We use L both as a specific length (in (27.1)) and a general symbol of length (in (27.2)). This should not lead to a conclusion as the context will always by clear.

which would be natural to use if we wished to pass to Fourier transform by taking $L \to \infty$. In that case one should write $u(x) = \frac{1}{(2\pi L)^3} \cdot \sum_{k \in \frac{\mathbf{Z}^3}{L}} \hat{u}(k) e^{ikx}$. In our normalization (27.2) the Fourier coordinate $\hat{u}(k)$ has the same physical dimension as $u(x)$, whereas its dimension would be $\frac{L^4}{T}$ if we used Fourier transformation.

The identification of the eddies with the Fourier components means that the quantities (such as the velocity or the vorticity) in the large eddies vary slowly. (Abrupt changes in $u(x)$ result in slowly decaying Fourier coefficients $\hat{u}(k)$.)

For vector fields $u(x)$ in a domain $\Omega \subset \mathbf{R}^3$ these notions can be defined by taking the Fourier transform of a suitable localization of $u(x)$, such as $\varphi(x)u(x)$, where φ is a suitable smooth cut-off function. We will not go into precise definitions here,[144] as the whole analysis will stay at a heuristic level, and therefore it is enough to use only a heuristic concept of the "wave number" k when talking about the fields in a domain (such as a pipe or a channel). We just keep in mind that the notion of "eddies of size l" in a vector field $u(x)$ in some subdomain $\mathcal{O}$ is roughly the same as the notion of the Fourier components of $\varphi(x)u(x)$ at wave numbers[145] $1/l$, where φ is a cut-off function adapted to $\mathcal{O}$.

For concreteness, let us consider a flow in a pipe oriented about the axis x_1. The flow is maintained by a pressure gradient in the x_1 direction, similar to (26.10). We can think of the pressure gradient as a constant force $\begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix}$. This is the force which supplies the energy which is dissipated by the flow. If we start action by this force on a fluid at rest, one can calculate the time development of the flow quite explicitly in terms of the heat equation. The solution will be of the form

$$u(x,t) = \begin{pmatrix} u_1(r,t) \\ 0 \\ 0 \end{pmatrix}, \qquad r = \sqrt{x_2^2 + x_3^2}\,, \tag{27.4}$$

with $u_1(r,t)$ approaching the Poiseulle's flow (22.9) as $t \to \infty$. (Note that the non-linear term vanishes on this solution.) The solution is unique, and therefore under ideal conditions no turbulence and the accompanying complicated eddies and oscillations in the flow will develop. However, there will always be some departure from the perfect symmetries, either in the force, or in the initial condition, or in the shape of the pipe, and the simple solution (27.4) will develop instabilities. The way this happens in a pipe can be quite complicated, and not quite clarified even today.[146] However, we can say that the instabilities will first

[144] This is a good exercise in Fourier transformation methods. In particular, one should establish that in the range of the frequencies we will be interested in, our notions do not depend on the details of the cut-off function φ assuming the function is chosen from an appropriate class.

[145] Often also called frequencies, or – more precisely – spatial frequencies

[146] See for example

T. Mullin, "Experimental Studies of Transition to Turbulence in a Pipe", Annu. Rev. Fluid Mech. 2011, 43:1-24 .

show as certain "eddies" in the flow, with the energy supplied to these eddies from the main flow. (If the loss of stability happens at a relatively low Reynolds number, of the order 10^3 or so, one will often observe only some sections of the pipe filled with regions of turbulence, forming "turbulent plugs". These turbulent regions need more energy to be sustained, and they can slow down the flow.) Eventually we have more and more eddies of various sizes appearing, and in the end some kind of equilibrium is reached in which the energy supplied by the force acting at low wave numbers and "macroscopic scale" is transmitted by the fluid into the small scales/high wave numbers. This creates high gradients, and the dissipation $2\nu e_{ij}e_{ij}$ dissipates energy at the level of these small scale eddies. In flows with boundaries, such as the pipe flows we discuss here, one has to distinguish between the flow close to the boundary, in the so-called boundary layer, and the "bulk flow" far away from the boundaries. The above description applies to the bulk flow, the boundary layer needs a separate analysis which we will not go into at this time.

We imagine that the energy mostly moves from the large eddies to the smaller eddies, but there can also be a non-zero transfer in the other direction. In terms of the Fourier picture one can see from the formula

$$\cos ax \, \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x] \tag{27.5}$$

that the quadratic interaction can change frequencies in both directions.

We imagine that there is some kind of a "statistical equilibrium" which the system reaches in which the energy supplied by the force into the large scales moves (in a non-local fashion) through the eddies of different sizes in a complicated way, with the net flux of energy being from the large scale towards the small scales. Note that this equilibrium is of a different nature than equilibria discussed, say, in the kinetic theory of gases,[147] in that the system is dissipative and requires a supply of energy.

Mathematically it is easier to replace the pipe flow by so-called Kolmogorov flows. These are flows on the torus $\mathbf{T}^3_L$ obtained from solving

$$u_t + u\nabla u + \frac{\nabla p}{\rho} - \nu\Delta u = f(x), \qquad \operatorname{div} u = 0, \qquad \int_{\mathbf{T}^3_L} u\,dx = 0\,.\ ^{148} \tag{27.6}$$

where

$$f(x) = \beta \begin{pmatrix} \cos k_2 x_2 \\ 0 \\ 0 \end{pmatrix} \tag{27.7}$$

for some fixed low k_2, with $\beta > 0$ being a parameter. There is a trivial steady-

[147] We will discuss these topics in more detail later.

[148] This last condition is useful to suppress the trivial non-uniqueness of the solutions caused by the fact that there is no boundary and if we change a solution by a constant, we still get a solution.

state solution

$$u(x) = \frac{\beta}{\nu k_2^2} \begin{pmatrix} \cos k_2 x_2 \\ 0 \\ 0 \end{pmatrix} . \tag{27.8}$$

This solution becomes unstable for sufficiently large β and the whole scenario described above again comes into play.[149] The advantage of this set up is that the definition of the eddies in terms of the Fourier coefficients $\hat{u}(k)$ in (27.2) is straightforward and there are no complications coming from the boundary effects.

One of the basic assumptions in the theory of turbulent flows is that for high Reynolds numbers the flow exhibit a certain universality. For example, if we watch a small area of the turbulent flow in a pipe and away from the boundary and subtract the mean velocity of the flow, what we see should not be that much different from watching, say, a small area of the Kolmogorov flow above (once it becomes turbulent), or say, a small area in a turbulent jet (once we subtract the mean velocity). In each case the macroscopic picture should be characterized only by a few quantities, such as the Reynolds number Re, energy dissipation ϵ, and the viscosity ν. The motion will of course be very complicated, but it will be "complicated in the same way" in all cases, once the few macroscopic parameters are the same.

In addition, the role of the viscosity ν is only in establishing a cut-off in the energy cascade due to the dissipation of the smallest eddies. The idea is that we are interested mostly in the large eddies, we do not really want to follow all the details of what all the small eddies do. The small eddies are important only to the degree that their behavior influences the large eddies. And – it is further assumed – for the behavior of the large eddies of size, say, l_0, it is not important whether the cut-off for the smallest eddies is at length $\frac{l_0}{10^4}$ or $\frac{l_0}{10^5}$ or even $\frac{l_0}{10^6}$. There might be a difference for the large eddies between the cut-off at $\frac{l_0}{10}$ and $\frac{l_0}{100}$, but not between $\frac{l_0}{10^5}$ and $\frac{l_0}{10^6}$. The large eddies will feel no difference between the cutoffs at such large Reynolds numbers, according the these ideas. This is the reason why the scale models we discussed in lecture 22 work better then one might naively expect.

In practice these ideas seem to be often confirmed, but one has to be quite careful and apply them correctly. Sometimes a small-scale phenomenon in a small area of the flow can significantly change the whole flow, as is the case with the phenomenon of the *drag crisis.*[150] There is a vague and somewhat

[149] This statement would not be true in dimension $n = 2$, it is important that we allow 3d perturbations. See the paper "An example of absence of turbulence for any Reynolds number" by C. Marchioro, Comm. Math. Phys. 105 (1986), 99-106.

[150] This is the effect that for the flows around bluff bodies (such as the sphere) the drag coefficient c in (21.9) suddenly drops by a significant factor (e. g. from .5 to .1 for a smooth sphere) at Reynolds numbers of order 10^5, to the degree that the force F itself *drops* if we increase the velocity. The effect was discovered by A. G. Eiffel (the architect of the Eiffel tower) in 1912, and explained in 1914 by L. Prandtl. The explanation is based in the changes in the flow in the very thin area of the boundary layer.

superficial similarity of such unexpected effects with phase transitions: one can have reasonable ideas about how molecular structure of, say, water affects its behavior in a qualitative way, but it is is not easy to calculate from the first principles when exactly will water start freezing and how brittle will ice be. In a similar vein, one can have reasonable qualitative ideas about what is going on in turbulent flows, but it is hard to predict from the Navier-Stokes equation when the drag crisis happens and how much it will reduce the drag.

27.2 The Kolmogorov length and the Kolmogorov-Obukhov law.

In the above picture the following questions are natural:

1. What is the size of the smallest eddies? (= the cut-off length)
2. What is the energy distribution of energy between the eddies of various sizes?

The answer to these questions is suggested by dimensional analysis, as first noticed by A.N. Kolmogorov and A. M. Obukhov in 1941.[151]
Let λ be the size of the smallest eddies, or the cut-off length. (Alternatively, $1/\lambda$ is the magnitude of the highest wave numbers which are needed to approximate u well by a Fourier series.) By the above considerations, λ should depend only on the dissipation ϵ and the kinematics viscosity ν, and possibly the density ρ. The physical dimensions of these quantities are

$$\begin{array}{lll} \lambda & ..... & L\,, \\ \epsilon & ..... & \frac{L^2}{T^3}\,, \\ \nu & ..... & \frac{L^2}{T}\,, \\ \rho & ..... & \frac{M}{L^3}\,. \end{array} \tag{27.9}$$

By the dimensional analysis as in lecture 1 it is easy to see that the only possible expression for λ is

$$\lambda = c\,\epsilon^{-\frac{1}{4}}\,\nu^{\frac{3}{4}}\,. \tag{27.10}$$

In the case of the pipe flow we know from our considerations above (see e. g. Principle P in the last lecture) that ϵ should depend only on U (the mean velocity) and R (pipe radius), and not an ν. By dimensional analysis we have

$$\epsilon = c\,\frac{U^3}{R}\,, \tag{27.11}$$

with possibly different value of c than in (27.10). In general, the value of c can change from line to line in what follows. In terms of the Reynolds number

[151] Kolmogorov, A.N., Local structure of turbulence in an incompressible fluid at very high Reynolds number, Dokl. Acad. Nauk SSSR, 30, No. 4, 299–303, 1941.
Obukhov, A.M., Spectral energy distribution in a turbulent flow, Dokl. Acad. Nauk. SSSR, 32, No. 1, 22–24, 1941.

$\mathrm{Re} = \frac{UR}{\nu}$ we can write

$$\lambda = c\frac{R}{\mathrm{Re}^{\frac{3}{4}}}\,. \tag{27.12}$$

In general, if we have some more complicated geometry but have some characteristic speed U [152]and characteristic length L, which can be used to define the Reynolds number

$$\mathrm{Re} = \frac{UL}{\nu}\,, \tag{27.13}$$

the above analysis can still be applied and we conclude that the cut-off length should be given by

$$\lambda = c\,\frac{L}{\mathrm{Re}^{\frac{3}{4}}}\,. \tag{27.14}$$

The length λ is called the *Kolmogorov length*. It is believed to give a reasonable estimate of the size of the smallest eddies, which is also the smallest scale which should be resolved in a numerical simulation. Therefore in dimension three the number N of grid points needed for simulating a flow should depend on the Reynolds number Re as $N \sim \mathrm{Re}^{\frac{9}{4}}$. In the physicist's terminology, the number of the degrees of freedom of a turbulent flow grows approximately as $\mathrm{Re}^{\frac{9}{4}}$ with the Reynolds number. We should emphasize that the conclusion is based on many assumptions and in practice the spacing of the grid may need to be even smaller if we wish to solve the equations precisely. The Kolmogorov distance λ represents the smallest scales under the assumptions that the intensity of the energy dissipation is, on average, uniformly spread throughout the space. In practice this may not be so and the significant dissipation may be concentrated in regions of relatively small volume, which might lead to smaller scales than anticipated by (27.14). Therefore the formula (27.14) should be considered with some caution.

Let us now turn to the question of how much energy is in the eddies of size l. It is more convenient to use the wave number magnitude $\kappa = 1/l$ rather than l. In terms of the Fourier representation (27.2) we have

$$\kappa = |k|\,. \tag{27.15}$$

Let $E(\kappa)$ be a function describing the average kinetic energy per unit mass in the fluid at wave numbers of magnitude κ, so that

$$\int_{\kappa_1}^{\kappa_2} E(\kappa)\,d\kappa \tag{27.16}$$

represents the average kinetic energy per unit mass in eddies with wave numbers with magnitude between κ_1 and κ_2. In terms of the Fourier representation (27.2)

[152]Strictly speaking, we really have to consider the size of the range of values of the velocity field, we should rule out the trivial situation where both the fluid and the boundaries move at a constant speed U, in which case we do not expect any instability or turbulence, of course.

we have

$$\int_{\kappa_1}^{\kappa_2} E(\kappa)d\kappa \sim \overline{\sum_{\kappa_1 \le |k| \le \kappa_2} |\hat{u}(k,t)|^2}\,, \tag{27.17}$$

where the "overbar" again means that we take an average, as in (25.16) or (26.18), and we denote explicitly the dependence of u on t.

Let $\kappa_K = \lambda^{-1}$ be the Kolmogorov cut-off frequency. By the discussion above we expect that for frequencies above κ_K the density $E(\kappa)$ is small, and – on average – the part of u coming from these high frequencies is negligible.[153] Beginning with some frequency κ_1 such that $\frac{1}{\kappa_1}$ is quite smaller than the size of the large eddies, the distribution of energy in the eddies should exhibit universal features. The interval of frequencies (κ_1, κ_K) is called *the inertial range*. The energy density $E(\kappa)$ in the inertial range should be independent of ν. It should depend only on ϵ and, of course, κ. Assuming this, it is easy to obtain the formula for $E(\kappa)$ from the dimensional analysis. The dimensions of the various quantities are

$$\begin{aligned} \kappa \quad &..... \quad \tfrac{1}{L}, \\ \epsilon \quad &..... \quad \tfrac{L^2}{T^3}\,, \\ E(\kappa)(d\kappa) \quad &..... \quad \tfrac{L^2}{T^2}\,, \\ E(\kappa) \quad &..... \quad \tfrac{L^3}{T^2}\,. \end{aligned} \tag{27.18}$$

The only way $E(\kappa)$ can depend on ϵ and κ in a dimensionally consistent manner is easily seen to be

$$E(\kappa) = c\ \epsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}\,. \tag{27.19}$$

This is the Kolmogorov-Obukhov law. This law does seem to be supported by experimental data. Various confirmations of it, deviations from it, and its modifications have been extensively discussed in the literature since 1950s, with the conclusion that the formula (27.19) does seem to capture an important part of the truth.

[153] Such a picture is incompatible with the possible existence of singularities in the Navier-Stokes solutions, a scenario which has not been ruled out mathematically.

28

11/16/2011

28.1 Time-scales and frequencies in turbulent flows

When considering the oscillations of the velocity in time, we have to distinguish between the Eulerian description and the Lagrangian description. In the Eulerian description we are interested in the time scales and frequencies of the functions

$$t \to u(x,t) \tag{28.1}$$

for fixed points x (and quantities obtained then by averaging over x). In flows with a non-zero mean speed U (such as the pipe flow, for example), these time-frequencies can depend on U. There is a heuristic principle due to G. I. Taylor, known as *the frozen turbulence hypothesis*, which says that in many cases (including the pipe flow and the channel flow) one can obtain the time-frequencies in (28.1) by simply assuming that vector fields with the spatial scales characteristic for the flow will move simply by translation by the mean speed U. In other words, we take a vector field $v(x)$ with spatial characteristics of the flow (such as $v(x) = u(x,t_1)$ and consider the field

$$w(x,t) = v(x - Ut)\,. \tag{28.2}$$

The assumption behind this principle is that the typical velocities of the "small eddies" (of size l such that the wave number $1/l$ is in the inertial range) are smaller than U, and the smaller eddies mostly move with the mean flow, so that the higher frequencies in (28.1) should be similar to the higher frequencies in $t \to w(x,t)$. This assumption seems to be confirmed by experimental results in certain simple flows, including the pipe flows and the channel flows.

Once we accept the frozen turbulence hypothesis, we can translate the information about the spatial behavior of $u(x,t)$ into the information about the temporal behavior and vice versa.

For example, if the Kolmogorov length for the flow is λ, the smallest time-scale relevant for (28.1) is clearly

$$\tau_e = \frac{\lambda}{U}\,, \tag{28.3}$$

where we use the subindex e to indicate that we are dealing with the time-dependence in the eulerian setting, as in (28.1). Combining this with (27.14), we can write

$$\tau_e = c\,\frac{R}{U}\,\mathrm{Re}^{-\frac{3}{4}}\,. \tag{28.4}$$

If we calculate the solution numerically on a fixed grid, that the relevant dependence on t is exactly (28.1) and the time-step should in our scheme should be at most of the size τ_e.[154]

[154] Of course, various numerical analysis considerations might suggest to take an even smaller step, the estimate above of the limit on the time-step by τ_e is independent of the choice of the numerical method.

From the frozen turbulence hypothesis and the Kolmogorov-Obukhov law (27.19) one can also obtain how much energy there is on average in the various frequencies of the function $t \to u(x,t)$. We simply replace $u(x,t)$ by $w(x,t)$ in (28.2), assuming that $v(x)$ is some typical field satisfying the Kolmogorov-Obukhov law. These calculations can be done with various degrees of rigor. The simplest rough reasoning would be that if we take a function with a spatial wave number of size κ, such as $f(x) = \sin \kappa x$ and consider $f(x - Ut)$, then we see a temporal frequency

$$\omega = \kappa U \tag{28.5}$$

Replacing κ by ω in the considerations leading of to (27.19), we obtain that the average energy per unit mass (in a fixed domain) which is contained in the part of velocity field with the *eulerian*[155] temporal frequencies between ω_1 and ω_2 is $\int_{\omega_1}^{\omega_2} E(\omega)\, d\omega$, with

$$E(\omega) = c\,(U\epsilon)^{\frac{2}{3}} \omega^{-\frac{5}{3}}\,. \tag{28.6}$$

Using the above one-dimensional model to conclude that we can just replace κ by ω using the relation (28.5) is really a simplification of a more involved calculation one should do. (In the above reasoning we did not take into account that the spatial wave numbers are vectors, and most of them will not be parallel to U, which leads to a modification of (28.5) to

$$\omega = \kappa U \cos \alpha\,, \tag{28.7}$$

where α is the angle between the direction of the velocity U and the direction of the wave number k (with $\kappa = |k|$). However, the more precise calculation (which can be done by using the Fourier transformation, for example), leads to the same conclusion as the simplified formula (28.5).[156]

[155] We have in mind the frequencies in (28.1) when x is held fixed. This should be distinguished from the Lagrangian frequencies, when we watch the temporal oscillations of the fluid from a coordinate system moving with fluid particles.

[156] If we assume

$$v(x) = \sum_{k \in \frac{\mathbf{Z}^3}{2\pi L}} \hat{v}(k) e^{ikx} \tag{28.8}$$

and consider $w(x,t) = v(x_1 - Ut, x_2, x_3)$ we have

$$w(x,t) = \sum_{k \in \frac{\mathbf{Z}^3}{2\pi L}} \hat{v}(k) e^{i(k_1(x_1 - Ut) + k_2 x_2 + k_3 x_3)} = \sum_{\omega \in \frac{U}{2\pi L}\mathbf{Z}} a(x,\omega) e^{-i\omega t}\,, \tag{28.9}$$

where

$$a(x,\omega) = \sum_{k'} \hat{v}(k_1, k') e^{ikx}\,, \tag{28.10}$$

with the obvious notation. We are interested in the average

$$b(\omega) = \frac{1}{(2\pi L)^3} \int_{\mathbf{T}_L^3} |a(x,\omega)|^2\, dx\,, \tag{28.11}$$

which can be evaluated from (28.9), leading to

$$b(Uk_1) = \sum_{k' \in \frac{\mathbf{Z}^2}{(2\pi L)^2}} |\hat{v}(k_1, k')|^2\,. \tag{28.12}$$

If we take the quantities ϵ and ν which characterize the shortest spatial scale, the Kolmogorov length λ((27.10)), we can from ϵ and ν a unique quantity with the dimension of time

$$\tau \sim \epsilon^{-\frac{1}{2}} \nu^{\frac{1}{2}} , \tag{28.13}$$

and a unique quantity with the dimension of velocity

$$v \sim \frac{\lambda}{\tau} \sim \epsilon^{\frac{1}{4}} \nu^{\frac{1}{4}} . \tag{28.14}$$

The time τ is called the Kolmogorov time. The three quantities λ, τ and v define the *Kolmogorov scales* of the turbulent motion, and they should describe some important features of turbulent motions. In the previous lecture have already identified λ as the size of the smallest eddies. The velocity v can be reasonably identified as a typical velocity within the smallest eddies (as watched from a coordinate system moving with the eddy we focus on), and the quantity τ can be reasonably identified as the timescale of the fluid oscillation within the smallest eddies (as watched from a coordinate system moving with the eddy we focus on). The fluid particles move along complicated trajectories. There is some mean velocity, and large deviations from the straightforward motion due to the large eddies. If we look closer, we see smaller-scale deviation due to the smaller eddies, and so on, up to the smallest eddies. The smallest eddies impose the "final level of oscillations" on the trajectories. These are small oscillations with amplitude λ and frequency $\frac{1}{\tau}$. None of this can be proved rigorously, but it is still remarkable that we can at least start discussing any plausible quantitative predictions for such complicated phenomena. The Kolmogorov theory brings at least some order (somewhat tentative, admittedly) into what would otherwise look as an untractable chaos.

The Reynolds number Re_λ corresponding to v, λ and ν is

$$\mathrm{Re}_\lambda = \frac{v\lambda}{\nu} = 1 , \tag{28.15}$$

which goes well together with the idea that most of the action of the molecular viscosity is takes place at the level of the small eddies.

28.2 Turbulent flow modeling

The ideas behind the Kolmogorov-Richardson cascade can be used in justifying various approaches to turbulent flow modeling and, as we already mentioned in the last lecture, they can also be used to explain why scale models are more useful than what we might first guess based on comparing the Reynolds numbers, see the discussion in lecture 22. Here we will only briefly discuss a simple

Assuming that $\hat{v}(k)$ is approximately isotropic for higher wave numbers, we obtain (28.6) after a simple calculation.

example of modeling. Assume that we wish to calculate the drag force, as in lecture 21 (section 21.3). Let us change the Navier-Stokes equation by considering the viscosity ν to be $x-$dependent:

$$u_{it} + u_j u_{i,j} + \frac{p_{,i}}{\rho} - (2\nu(x)\, e_{ij}(u)\,)_{,j} = 0, \qquad \operatorname{div} u = 0\ . \tag{28.16}$$

For a given $\nu(x)$ we can in principle calculate the drag force F. (Note that this includes some time averaging, see (21.23), and we assume that the averages give some well-defined drag force F.) We can write

$$F = F(R, U, \rho, [\nu(x)])\,, \tag{28.17}$$

where the square bracket $[\nu(x)]$ indicates a functional dependence on the whole function $\nu(x)$. We know that for water we have $\nu \sim 10^{-6}$, which leads to large Reynolds numbers for everyday flows, and therefore a detailed numerical calculation of these flows which would reproduce the field $u(x,t)$ in full detail may be very challenging, and in many cases impractical. We can now make considerations along the following lines. Let us have a look at the turbulent region behind the sphere (the turbulent wake). Perhaps the details of the motion in this region are not so important for the drag force, and we can try to cut the Kolmogorov-Richardson cascade shorter, to save computing power. This can be done by increasing the viscosity. We increase $\nu(x)$ in a smooth way by a factor of, say, 10 in parts of the turbulent wake which are at some distance from the sphere. (We assume that the Reynolds number is already quite high.) The effect of this is that in the region where ν was increased the length of the energy cascade was cut. We hope to find that this dramatic change in $\nu(x)$ does not have much effect of the resulting force F, while we can now do the calculation with less computing power, since our grid in the region of higher viscosity does not have to be so fine. With some experimentation, we can build up some expertise for where we can afford to increase $\nu(x)$ significantly without changing F too much, and where cannot do that without getting the answer completely wrong. Eventually we can build up enough experience to be able to calculate some important quantities (such as the drag force, or lift force) with enough precision, even though we do not really solve the full Navier-Stokes equation, but some model equation with artificial viscosity. There are many potential pitfalls in this approach. For example, to get the right values of the pressure in important areas, we may have to model the effect that the turbulent part of the velocity (some of which we have cut) lowers the pressure, as we discussed in lecture 25. This will require to bring in some auxiliary quantities which model effects of the velocity field oscillations which cannot be captured on our grid. You can imagine that the whole process is quite experimental, proceeding by trial and error, until we can tune our model so that we get reasonable answers from it. Because of the obvious practical importance of such calculations, a lot of effort has gone into building various models, based not only on the idea of artificial viscosity and Reynolds stresses, but also on many other ingenious ideas. There is a large literature on this subject, and we could spend the whole

year discussing various methods. Interested readers may consult for example the book "Turbulent Flows" by S. Pope. Even with the large computers we have today, getting reliable predictions in situations where we do not have a lot of previous experience (such as new geometrical arrangements of the flow) is challenging.

29

11/18/2011

29.1 Turbulence in dimension $n = 2$, preliminary considerations

In our considerations so far of the oscillatory Navier-Stokes solutions the dimension n of the space did not seemingly play much role, and therefore one might perhaps at first expect that the theory might also work in dimension $n = 2$. This turns out not to be the case. The behavior of 2d Navier-Stokes solutions at high Reynolds numbers is quite different. In fact, some experts believe that the term "2d turbulence" should be avoided, because turbulence really exists only in 3d. Nevertheless, the term is widely used.

The reason why the theory we have discussed in the last few lectures does not work in dimension $n = 2$ is that the basic tenet of the theory, Principle (P) formulated in lecture 26 is expected to fail in 2d. The energy dissipation behaves differently. The reason for this can be explained best in the situation of the free Navier-Stokes equation (with no forces) on a torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ or $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ (no boundaries and finite volume). We consider the initial-value problem

$$\begin{aligned} u_t + u\nabla u + \tfrac{\nabla p}{\rho} - \nu\Delta u &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \qquad \text{in } \mathbf{T}^n \times [0,\infty), \tag{29.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \mathbf{T}^n. \tag{29.2}$$

The initial condition $u_0(x)$ is assumed to be smooth but "generic", which means that it is "sufficiently complicated". In particular, we assume that it does not have any special features which would bring into play some extra properties of certain classes of solutions, such as for example being independent of one variable. In what follows we will assume that the problem (29.1), (29.2) has a smooth solution for our initial condition $u_0(x)$. This is only known in dimension $n = 2$, in dimension $n = 3$ the problem is open. [157] Formula (26.13) gives

$$\int_{\mathbf{T}^n} \frac{1}{2}|u(x,t)|^2\,dx + \int_0^t \int_{\mathbf{T}^n} \nu|\nabla u(x,s)|^2\,dx\,ds = \int_{\mathbf{T}^n} \frac{1}{2}|u_0(x)|^2\,dx\,. \tag{29.3}$$

Let us denote

$$D(t_1,t_2,u_0,\nu) = \int_{t_1}^{t_2} \int_{\mathbf{T}^n} \nu|\nabla u(x,s)|^2\,dx\,ds \tag{29.4}$$

and let us consider the behavior of $D(0,T,u_0,\nu)$ for some very large but fixed T as $\nu \to 0$. What is known/expected about this behavior when $n = 3$?

[157] See e. g. http : //www.claymath.org/millennium

1. If the problem (29.1), (29.2) with $\nu = 0$ (Euler's equation) has a smooth solution on the time interval $[0, T]$, then $D(0, T, u_0, \nu) \to 0$ as $\nu \to 0$. This is a rigorous result.

2. However, even if the assumption in 1. is satisfied, the convergence of $D(0, T, u_0, \nu)$ to 0 is expected to be very, very slow, as the corresponding solution of the Euler equation will presumably produce structures with extremely small scales.

3. It is conceivable (but unknown) that the Euler equation (corresponding to $\nu = 0$) does not have a smooth solution, in which case one might even speculate that $D(0, T, u_0, \nu) \geq \varepsilon > 0$ as $\nu \to 0$. This is an open problem.

The above is of course related to Principle (P) from lecture 26, although we should note that the set up here is different in that we have no external forcing (neither by a volume force nor from the boundary).

In dimension $n = 2$ the situation is completely different. We first note that for any smooth div-free field $v(x)$ in $\mathbf{T}^n$ we have

$$\int_{\mathbf{T}^n} |\nabla v(x)|^2 \, dx = \int_{\mathbf{T}^n} |\operatorname{curl} v(x)|^2 \, dx \,, \tag{29.5}$$

as one can see from integration by parts. Hence, denoting $\omega = \operatorname{curl} u$ as usual, we have

$$D(t_1, t_2, u_0, \nu) = \int_{t_1}^{t_2} \int_{\mathbf{T}^n} \nu |\omega(x, t)|^2 \, dx \, dt \,. \tag{29.6}$$

When $n = 2$ the vorticity ω is a scalar and we have

$$\omega_t + u \nabla \omega - \nu \Delta \omega = 0 \,. \tag{29.7}$$

The solutions of this equation satisfied a maximum principle. Therefore, denoting $M = \max_x |\omega_0(x)|$, we see that $|\omega| \leq M$ in $\mathbf{T}^2 \times [0, \infty)$, which gives

$$D(0, T, u_0, \nu) \leq \nu M^2 T \operatorname{area}(\mathbf{T}^2) \,. \tag{29.8}$$

Hence we see that, in the case of no forces and no boundaries, the dissipation must go to zero at rate $O(\nu)$ as $\nu \to 0$. Although the situation in which we have considered Principle (P) are more complicated than the simple example above in that boundaries and/or volume forces are involved, the calculation (29.8) casts serious doubt on Principle (P) in dimension $n = 2$.

Experimentally it is non-trivial to realize 2d flows, but it is reasonable to assume that the dynamics of large-scale meteorological objects of dimensions, say, hundreds of kilometers should exhibit some 2d features due to the relative thinness of the relevant part of the atmosphere in comparison with such large dimensions. Of course, locally the motion is still genuinely 3d, but one can expect that some averaged large-scale properties should exhibit 2d effects. It is non-trivial to make such speculations mathematically precise, of course, and at this point we will not attempt to do so.

Another situation when 2d features are promoted is the motion of fast rotating fluids. This is related to Problem 2 in Homework Assignment 2 and can be made rigorous.[158]

29.2 2d incompressible Euler equations – long time behavior

One can obtain a good idea about some of the features of the 2d turbulence from looking at the case $\nu = 0$, corresponding to the Euler equation. We will consider the Euler equation in a 2d domain Ω. As usual, we will denote by ω the vorticity and by ψ the stream function, see lecture 14 for details.

We will consider the Euler equation in the vorticity form

$$\omega_t + u\nabla\omega = 0\,, \qquad \text{in } \Omega \times [t_1, t_2], \tag{29.9}$$

where u is obtained from ω as discussed in lecture 14. We can take $t_1 = -\infty$ and/or $t_2 = \infty$. If we specify an initial condition of the form $\omega(x,0) = \omega_0$, where $\omega_0 \in L^\infty(\Omega)$, it can be shown[159] that there is a unique solution of (29.9) in $\Omega \times (-\infty, \infty)$ with $\omega(x,0) = \omega_0$. Since we are dealing with non-smooth functions, the exact uniqueness class and the sense in which the equation is satisfied and the initial condition is attained need to be specified with some care, but at the moment we will bypass these technical issues, referring the interested reader to the book of A. Majda and A. Bertozzi mentioned above. We will work with the non-smooth solutions with bounded vorticity because they are natural objects when studying the long-time behavior of the solutions, as we will see.

The solutions of (29.9) with $\omega(x,0) = 0$ satisfy the estimate

$$|\omega(x,t)| \le ||\omega_0||_{L^\infty(\Omega)}\,, \tag{29.10}$$

as one can see at a formal level from (2.17).[160]

Let us consider a set $A \subset \Omega$ with $|A| = \frac{1}{2}|\Omega|$. [161] We let $B = \Omega \setminus A$ and consider

$$\omega_0 = \chi_A - \chi_B\,, \tag{29.11}$$

where χ_X denotes the characteristic function of the set X.

To discuss the long-time behavior of the solution $\omega(x,t)$ of (29.7) with the initial datum (29.11), we recall the notion of weak* convergence in L^∞.

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[158] See for example

Babin, A., Mahalov, A., Nicolaenko, B., 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000). Indiana Univ. Math. J. 50 (2001), Special Issue, 135.

[159] This is a result of V.Yudovich from 1960s. A relatively short proof can be found for example in the book of A. Majda and A. Bertozzi "Vorticity and Incompressble Flow".

[160] This formula gives a proof for smooth solutions. The general case requires some additional reasoning.

[161] We use the notation $|A|$ for the Lebesgue maesuire of A.

We say that a sequence $\omega_j \in L^\infty(\Omega)$ converges weakly* to $\omega \in L^\infty$ if

$$\int_\Omega \omega_j f \, dx \to \int_\Omega \omega f \, dx \,, \qquad f \in L^1(\Omega) \,. \tag{29.12}$$

For sequences ω_j with $|\omega_j| \le C$ this notion is equivalent with a number of other notions of weak convergence, such as weak convergence in L^2 (in which case we take $f \in L^2$ in the above definition), weak convergence in distributions (when we take f smooth and compactly supported), etc. We will use the notation $\omega_j \rightharpoonup \omega$ to indicate any of these equivalent notions of weak convergence. We emphasize that the equivalence requires the assumption $|\omega_j| \le C$ (almost everywhere), where C can depend on the sequence. We also recall that the closed balls $\{\omega \in L^\infty \,,\ |\omega| \le C\}$ are weakly* compact, and that the weak* topology restricted to these balls is metrizable. [162]

Let us now return to the question of the long-term behavior of $\omega(x,t)$ with the initial condition (29.11). We can imagine that at the initial time we color the fluid particles in the set A red and the particles in the set B blue. As time progresses, the fluid particles are mixed by the flow, and naively we might think that, in most cases, it might be reasonable to conjecture the following:

$$\omega(t) \rightharpoonup 0 \,, \qquad t \to \infty \,, \tag{29.13}$$

or at least

$$\omega(t_j) \rightharpoonup 0 \,, \qquad \text{for some sequence } t_j \to \infty. \tag{29.14}$$

This conjecture can also be illustrated well in the Fourier space, where it also might look plausible at first. Let us consider the equation in the torus $\mathbf{T}^2$ and write $\Omega = \mathbf{T}^2$ for a moment. We have

$$\omega(x,t) = \sum_{k \in \mathbf{Z}^2} \hat\omega(k,t) e^{2\pi k x} \tag{29.15}$$

and the weak convergence (29.13) is equivalent to

$$\lim_{t\to\infty} \hat\omega(k,t) \to 0 \,, \qquad k \in \mathbf{Z}^2 \,. \tag{29.16}$$

This would mean that in the Fourier space the solution $\hat\omega(k,t)$ would move to increasingly higher frequencies as time progresses. This is also suggested in 3d by the Richardson-Kolmogorov cascade, and it might be a plausible conjecture for typical solutions of the 3d Euler equation (assuming the solutions exist). It is not known whether or not it is true (in dimension $n = 3$).

However, in dimension $n = 2$ the the behavior (29.13), (29.14), or (29.16) cannot occur, based on considerations of the energy. Recall that the kinetic energy is given by

$$E(\omega) = \int_\Omega \frac{1}{2} \rho |u|^2 \, dx = \int_\Omega -\frac{1}{2} \rho \psi \omega \, dx \,, \tag{29.17}$$

[162] Recall that the whole space $L^\infty(\Omega)$ with the weak* topology is not metrizable.

where u or ω are expressed through ω.[163] For example, on the torus $\mathbf{T}^2$ we have $\Delta\psi = \omega$, which determines ψ up to a constant, and hence $u = \nabla^\perp\psi$ is determined by ω uniquely.

Lemma 1
The energy $E = E(\omega)$ is weakly continuous on bounded sets of L^∞. In other words $|\omega_j| \le C$, $\omega_j \rightharpoonup \omega$ implies $E(\omega_j) \to E(\omega)$.*

Proof
Let us consider the corresponding stream functions ψ_j, ψ defined by $\Delta\psi_j = \omega_j$, $\Delta\psi = \omega$, together with appropriate boundary conditions. By elliptic regularity, the functions ψ_j, ψ have two derivatives in $L^2(\Omega)$ (assuming the boundary of Ω is smooth). Hence $\psi_j \to \psi$ uniformly by the Sobolev Imbedding Theorem (recall that we are in dimension $n = 2$) and $E(\omega_j) = \int_\Omega -\psi_j\omega_j\,dx \to \int_\Omega -\psi\omega\,dx = E(\omega)$ as $j \to \infty$. (There are many other ways to do the proof. Also, in the case $\Omega = \mathbf{T}^2$ one can do a simple proof "by hand" directly in the Fourier space.)

Lemma 1 obviously rules out the behavior (29.13), (29.14), or (29.16) above. The energy conservation provides an obstruction to a "complete mixing". We can try correct the conjecture as follows: perhaps typical solutions of the Euler equation (29.9) mix the vorticity to the maximal degree which is still consistent with the energy conservation? In the Fourier picture, this would correspond to "as much vorticity $\hat\omega(k)$ escaping to $k \sim \infty$ as possibly can be consistent with $E(\omega) = E(\omega_0)$. This suggests that $\overline{\omega}$ could perhaps be obtained from the principle

$$I(\omega) = \int_\Omega \frac{1}{2}\omega^2\,dx \to \min.\,, \qquad E(\omega) = E_0 = E(\omega_0)\,. \tag{29.18}$$

Note that both I and E are quadratic forms in ω, and one can obtain (at least formally) the equation characterizing the result of the minimization (29.18) by the usual Lagrange multiplier method: minimizing $I(\omega) - \lambda E(\omega)$ This leads to an equation

$$I'(\omega) - \lambda E'(\omega) = 0\,, \tag{29.19}$$

which is the same as

$$\omega + \tilde\lambda\psi = 0, \qquad E(\omega) = E_0\,, \tag{29.20}$$

where $\tilde\lambda = \lambda\rho$. We know that $\omega = \Delta\psi$, so that the minimizer $\overline{\omega}$ should be an eigenfunction of the Laplacian (with suitable boundary conditions). Moreover, the minimization in (29.18) dictates that $\tilde\lambda$ be the lowest possible nontrivial eigenvalue. In the case of $\Omega = \mathbf{T}^2$ all this can be again easily seen directly in the Fourier picture.

[163] We keep the constant density ρ in the formula so that we get the correct physical dimensions, although its specific value plays no role in our considerations here.

The above consideration suggest that as $t \to \infty$, *the solutions $\omega(t)$ should weakly* approach the finite-dimensional space generated by the eigenfunctions associated with the lowest non-trivial eigenvalue of the laplacian (with suitable boundary conditions).*

As we shall see this conjecture still needs further corrections, but in some situations (such as the initial data ω_0 with sufficiently small energy) it gives a prediction for the long-times behavior of $\omega(t)$ which is looks to some degree plausible. Moreover, one does see this behavior of the solutions numerically if instead of pure Euler equation (29.9) one adds a small viscosity $\nu > 0$ to the equation and looks at the solutions of

$$\omega_t + u\nabla\omega - \nu\Delta\omega = 0\,. \tag{29.21}$$

for times t of order $\frac{\sim c}{\nu}$, where c is a constant with the physical dimension of area. Therefore the above considerations, although non-rigorous, do seem to capture at least a part of the truth. However, it is still an over-simplification, as we shall see.

The integral $I(\omega)$ is called the *enstrophy*. The above considerations leading the principle of minimizing the enstrophy at a fixed energy lead to the following tentative conclusions: *in dimension $n = 2$ energy tends to be dissipated much more weakly than in dimension $n = 3$. Moreover, it tends to move to low spacial frequencies*. The flow of the energy to the low frequency modes is a particularly striking feature of the 2d flows, it is exactly the opposite of what is happening in 3d. It was first predicted by R. Kraichnan in 1967. At the level of the point-vortex model it was anticipated by L. Onsager in 1949. The term *inverse energy cascade* is often used in this context.

30

11/21/2011

30.1 The model equation $u_t + uu_x - \nu u_{xx} = 0$.

Let us consider the *viscous Burgers equation*

$$u_t + uu_x - \nu u_{xx} = 0 \tag{30.1}$$

in $\mathbf{R}^1 \times (t_1, t_2)$. As we discussed in lecture 2, the equation $u_t + uu_x = 0$ represents free particles. We can think of free particles which interact through viscosity, but a natural equation for such a system would not really be (30.1).[164] One can think of (30.1) simply as a model equation which can illustrate some types of behavior of PDE solutions. We allow $t_1 = -\infty$ and/or $t_2 = +\infty$. The equation can also be considered on $\mathbf{T}^1 \times (t_1, t_2)$, with similar results, but we will work in $\mathbf{R} \times (t_1, t_2)$, where one can write down more explicit solutions. The equation satisfies the obvious analogue of (29.3)

$$\int_{\mathbf{R}} \frac{1}{2} u(x, t_2)^2 \, dx + \int_{t_1}^{t_2} \nu |u_x|^2 \, dx \, dt = \int_{\mathbf{R}} \frac{1}{2} |u(x, t_1)|^2 \, dx \,, \tag{30.3}$$

and we can again define

$$D(t_1, t_2, u_0, \nu) = \int_{t_1}^{t_2} \nu |u_x|^2 \, dx \, dt \,, \tag{30.4}$$

where $u(x, 0) = u_0(x)$, and we assume $0 \leq t_1 \leq t_2$.

The equation has been studied in great detail in many papers, and in particular in E. Hopf's influential paper in the Communications on Pure and Applied Mathematics in 1950.[165] The reader interested in deeper study of the equation is refered to E. Hopf's paper, here we will only briefly illustrate that for the equation (30.1) scenario 3 from last lecture (section 29.1) is valid: the equation with $\nu = 0$ can develop singularities, and $D(0, T, u_0, \nu) \geq \varepsilon > 0$ as $\nu \to 0_+$ (under suitable assumptions). The reason for the development of the singularities when $\nu = 0$ was already discussed in lecture 2: if $u_0(x_1) > u_0(x_2)$ for some $x_1 < x_2$, we will have a situation that a faster particle is approaching a slower particle, and since there is no interaction and the particles move at constant speeds, sooner or later some two particles with different speeds will come into contact with each other, creating a discontinuity in u.

[164] The right equations describing such a system could be

$$\rho u_t + \rho u u_x = \sigma_x, \qquad \rho_t + (u\rho)_x = 0 \,, \tag{30.2}$$

where $\rho = \rho(x, t)$ is the density of the particles and σ is the viscous stress. We can assume for example that $\sigma = \sigma(u_x)$ or $\sigma = \sigma(u_x, \rho)$.

[165] Vol. 3, 1950, pp. 201–230

We note that if we start with a smooth and compactly supported u_0 at $t = 0$, then up to the time T^* of the first collision the solution of $u_t + uu_x = 0$ will be smooth and we will have $D(0, T, u_0, \nu) \to 0$ as $\nu \to 0_+$ for any $0 < T < T^*$.

If the inviscid equation (the case $\nu = 0$) develops a discontinuity at $x = x_1$ and time $t = t_1$ but is regular for $t < t_1$, the viscous equation will develop at times $t \in (t_1, t_1 + \delta)$ (for sufficiently small ν) a steep gradient of order $\frac{1}{\nu}$ in an interval of width of order ν, and hence the integral $\int_{t_1}^{t_1+\delta} \int_{\mathbf{R}} \nu |u_x|^2 \, dx \, dt$ will be at least of order $\sim b\delta/2$, where b is the jump of u across the discontinuity.

A remarkable feature of equation (30.1) is that it can be transformed to the linear heat equation

$$\varphi_t = \nu \varphi_{xx}. \tag{30.5}$$

This is achieved by the *Cole-Hopf transformation*

$$u = -2\nu \frac{\varphi_x}{\varphi}\,. \tag{30.6}$$

The Cole-Hopf transformation can be used to study the solutions of (30.1) in detail. Here we only mention the following properties of the solutions.

1. If $u(x, t)$ is a solution, so it $u(x - ct, t) + c$ for any $c \in \mathbf{R}$.

2. The equation has the same scaling symmetries and the Navier-Stokes equation, see (22.7), (22.8). For example, if $u(x, t)$ is a solution, so is $\lambda u(\lambda x, \lambda^2 t)$.

3. The function $u(x) = -\tanh \frac{x}{2\nu}$ is a solution. Together with 1. and 2. this can be used to construct traveling waves of the form $-\lambda \tanh\left(\frac{\lambda(x-ct)}{2\nu}\right) + c$.

4. More generally, if $A_k > 0, a_k, b_k \in \mathbf{R},\ k = 1, 2, \ldots m$, then the function

$$u(x, t) = \frac{\sum_{k=1}^m 2a_k A_k e^{\frac{-a_k(x-a_k t-b_k)}{\nu}}}{\sum_{k=1}^m A_k e^{\frac{-a_k(x-a_k t-b_k)}{\nu}}} \tag{30.7}$$

   is a solution of (30.1).

Points 1-3 above should give the reader a good idea as to what one can expect when a discontinuity (often called a shock) develops, and that $D(0, T, u_0, \nu)$ will not approach 0 with $\nu \to 0$ if T is larger than the lifespan of a smooth solution of the equation with $\nu = 0$. Taking $\nu \to 0_+$ in (30.7) gives and illustration of the behavior of the shocks and the dissipation of energy at the shocks.

The behavior of solutions of (30.1) is to some degree mimicked (with various complications) by compressible flows. The incompressible case is quite far away from (30.1).

31

11/23/2011

31.1 A simple example from Statistical Mechanics

In lecture 29 we tried to find the long-term behavior of the incompressible 2d Euler equation by minimizing the enstrophy $I = I(\omega)$ for a fixed energy $E(\omega) = E_0$. As we will see later, this procedure has still to be adjusted, but it already illustrates an important idea that we can try to bypass the often very difficult process of integrating the equations describing a physical system by using instead some simple phenomenological principle which might capture important features of the phenomena at hand. From a purely mathematical point of view this can only provide some more or less reasonable conjectures about the behavior of solutions, it is not a replacement for rigorously establishing that the solutions of our equations have the expected properties. However, from the point of view of physics such approach is often fruitful. In fact, as we have already mentioned before, in some cases it may turn out that the phenomenological principles on which our conjectures are based are more robust and perhaps even more fundamental than the equations themselves.[166]

Let us consider the following classical example from Statistical Mechanics. We consider a gas of total mass M in a smooth domain $\Omega \subset \mathbf{R}^3$ of a finite volume V. We assume the gas consists of a very large number N of very small particles. We can think of atoms, but our model is simpler, as we will assume that the particles are essentially point-particles, without any internal structure. Let the coordinates of the particles be $x^{(i)} \in \mathbf{R}^3, \quad i = 1, 2, \dots, N$. The velocities of the particles are $v^{(i)}, \quad i = 1, 2, \dots, N$. The mass of one particle is $m = \frac{M}{N}$. The total kinetic energy of the particles is

$$E = \sum_i \frac{1}{2} m \left| v^{(i)} \right|^2 . \tag{31.1}$$

We assume the particles interact with one another through collisions, and we will assume that the collisions are "elastic", in the sense that the kinetic energy is preserved during each collision. We also assume that the collisions of the particles with the boundary $\partial\Omega$ are elastic. We will adopt the notation

$$x_1 = x_1^{(1)},\ x_2 = x_2^{(1)}, x_3 = x_3^{(1)},\ x_4 = x_1^{(2)}, \dots,\ x_{3N} = x_3^{(N)} \tag{31.2}$$

[166] For example, the basic ideas about the equations of motion underwent a dramatic change after the discovery of the Quantum Mechanics, whereas the Statistical Mechanics, which was originally developed in the context of the Classical Mechanics, was in fact quite well prepared for the shift to Quantum Mechanics, as the main principles were not much dependent on the details of the equations of motion. In fact, one of the the important original impulses for the development of the Quantum Mechanics came from the problem of the black body radiation, where the (phenomenological) principle of the equipartition of energy between the different degrees of freedom in a physical system lead to a contradiction with ideas of the Classical Physics.

and

$$v_1 = v_1^{(1)}, v_2 = v_2^{(1)}, v_3 = v_3^{(1)}, v_4 = v_1^{(2)}, \ldots, v_{3N} = v_3^{(N)} \,. \tag{31.3}$$

If we let the system evolve for some time, we expect that it will reach some state of a "statistical equilibrium" as we see in the real gasses. What will be the distributions of the particles in Ω and the distribution of their velocities? Instead of trying to solve the equations of motion, we can follow the founders of Statistical Mechanics and make the following plausible conjectures:

(i) *The particles will be uniformly distributed in* Ω. This means that in a macroscopic domain $\Omega_1 \subset \Omega$ we will find approximately $N\frac{|\Omega_1|}{|\Omega|}$ particles at any given time, where $|\cdot|$ denotes the volume (= the Lebesgue measure).
(ii) *The vector* $v_1, \ldots, v_{3N}$ *will be uniformly distributed over the* $3N-1$ *dimensional sphere given by*

$$(v_1^2 + v_2^2 + \cdots + v_{3N}^2) = \frac{2E}{m} = \frac{2E}{M}N \,. \tag{31.4}$$

This means that the probability that the vector $v_1, \ldots, v_{3N}$ will be in a subset $\mathcal{O}$ of the sphere is $\frac{|\mathcal{O}|}{|\text{sphere}|}$, where $|\cdot|$ denotes the natural $3N-1$ dimensional measure on the sphere. In fact, it can be reasonably assumed that the same will be true in any macroscopic domain $\Omega_1 \subset \Omega$. (In this case we replace $3N$ by $3N\frac{|\Omega_1|}{|\Omega|}$, and we refer only to the particles which are in Ω_1.)

We will see below that based on these assumptions one can quite easily calculate the distribution of velocities of the particles and obtain the so-called Maxwell distribution, first calculated by Maxwell in 1859.

There are some important assumptions underlying the above conjectures. Roughly speaking, the conjectures say that the macroscopic behavior of the system is (after some time) determined only by the quantities $\frac{2E}{M}$ and N, regardless of other details of the initial data. Our assumption that the collisions are elastic implies that, no matter what the details of the collisions are, the energy E is conserved. The quantities M, and N are obviously also conserved in our model. To conjecture (i) and (ii), we should be confident that we did not miss any other conserved quantity which might affect the behavior envisaged by (i) and (ii). An additional conserved quantity has the potential of invalidating the conjectures. For example, let us assume that the domain Ω is bounded, smooth, and invariant under rotations about the x_3-axis. Then the x_3 component of the total angular momentum of the system

$$I_3 = \sum_i m\,(x^{(i)} \times v^{(i)})_3 \tag{31.5}$$

will be conserved by the evolution, and if $I_3 \neq 0$ for the initial data, the conjectures (i) and (ii) above cannot be correct, since one can easily see that they

imply $I_3 \to 0$ in the long-time limit.[167] If I_3 is the only additional quantity which is conserved, it is possible to adjust the conjectures (i) and (ii) in a way which takes the conservation of I_3 into account. However, the necessary adjustment is less obvious than (i) and (ii), although it is still possible to to come up with the right guess "by hand", without using the machinery of Statistical Mechanics.[168] In this case the density of the particles will not be independent of x, it will depend on the distance from the axis of symmetry. The local distribution of velocities will also depend on x, so that we will observe some macroscopic rotation of the gas about the x_3 axis.

Even when no other conservation laws are present, additional objections to conjectures (i) and (ii) can come from the reversibility of the equations of motion and the Poincaré recurrence theorem. These important issues, which have been discussed since the birth of Statistical Mechanics, are related to the validity of the Second law of Thermodynamics. We will not discuss them at the moment.

One can take the point of view that the experimental verification of (i) and (ii) is at least as good as the experimental verification of the equations of motion and promote (i) and (ii) to postulates, without worrying about the details of the equations of motion. That is in some sense the philosophy of Statistical Mechanics, where the analogues of (i) and (ii) are worked out and postulated in much more general situations, and an effective computational machinery for such considerations is developed.

In what follows we will assume that we do not have any additional conserved quantities which would invalidate (i) and (ii). Assuming (i) and (ii), let us calculate the distributions of the velocities of the particles. More precisely, we consider the following question. Given $a < b \in \mathbf{R}$, what is the probability that the x_1-component of velocity of a given particle of the gas is between a and b?

We first introduce some notation and recall some classical formulae.

S^{n-1}	.....	$(n-1)$-dimensional sphere $\{x \in \mathbf{R}^n, \lvert x\rvert = 1\}$.
$\Gamma(s)$	.....	Euler's Gamma function, so that $\Gamma(n) = (n-1)!$.
$\lvert S^{n-1}\rvert = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$	.....	$(n-1)$-dimensional volume of S^{n-1}.
σ_{n-1}	.....	the $(n-1)$-dimensional measure on S^{n-1} normalized to total volume 1

(31.6)

We also recall the limit

$$\frac{\Gamma(\frac{n}{2})}{\sqrt{\frac{n}{2}}\,\Gamma(\frac{n-1}{2})} \to 1, \qquad n \to \infty\,, \tag{31.7}$$

[167] In the context of the real physics of the gas molecules, it is presumably impossible to have an Ω which would have the above symmetry at the scale of the atomic dimensions, which is why the conservation of I_3 does not seem to present a problem for conjectures (i) and (ii) in axi-symmetric containers Ω. The symmetry is lost at the atomic scale and the angular momentum I_3 of the gas can change in the collisions with the walls of the container, even though they appear rotationally invariant to our eyes.

[168] With the use of the machinery the calculation is standard.

and the formula

$$\sigma_{n-1}\left(\{x \in S^{n-1},\ \alpha < x_1 < \beta\}\right) = \frac{|S^{n-2}|}{|S^{n-1}|}\int_\alpha^\beta (1-x_1^2)^{\frac{n-3}{2}}\, dx_1\,, \tag{31.8}$$

where $-1 \le \alpha \le \beta \le 1$.

In the context of (31.4), let us set $\epsilon = \frac{E}{M}$. This is the kinetic energy per unit mass due to the motion of the particles of the gas. We also set

$$v_i = y_i\sqrt{2\epsilon N}\,, \tag{31.9}$$

so that (31.4) becomes

$$y_1^2 + y_2^2 + \cdots + y_{3N}^2 = 1\,. \tag{31.10}$$

The condition

$$a < v_1 < b \tag{31.11}$$

is then equivalent to

$$\frac{a}{\sqrt{2\epsilon N}} < y_1 < \frac{b}{\sqrt{2\epsilon N}}. \tag{31.12}$$

We need to calculate

$$I(a,b) = \sigma_{3N-1}\left(\{y \in S^{3N-1},\, \alpha_N < y_1 < \beta_N\}\right), \tag{31.13}$$

where

$$\alpha_N = \frac{a}{\sqrt{2\epsilon N}}, \qquad \beta_N = \frac{b}{\sqrt{2\epsilon N}}\,. \tag{31.14}$$

By (31.8) we have

$$I(a,b) = \frac{|S^{3N-2}|}{|S^{3N-1}|}\int_{\alpha_N}^{\beta_N} (1-y_1^2)^{\frac{3N-3}{2}}\, dy_1\,. \tag{31.15}$$

Using (31.9) and (31.6), we have

$$I(a,b) = \frac{\Gamma(\frac{3N}{2})}{\sqrt{2\pi\epsilon N}\ \Gamma(\frac{3N-1}{2})}\int_a^b (1-\frac{v_1^2}{2\epsilon N})^{\frac{3N-3}{2}}\, dv_1\,. \tag{31.16}$$

We note that for $N \to \infty$ we have

$$I(a,b) = \int_a^b \frac{1}{\sqrt{4\pi\epsilon'}} e^{-\frac{v_1^2}{4\epsilon'}}\, dv_1\,, \tag{31.17}$$

where $\epsilon' = \frac{\epsilon}{3}$. We conclude that in the limit $N \to \infty$ the probability density that a given particle will have the x_1-component velocity v_1 is

$$\frac{1}{\sqrt{4\pi\epsilon'}} e^{-\frac{v_1^2}{4\epsilon'}}\,, \tag{31.18}$$

which is a form of the Maxwell distribution, first calculated by Maxwell in 1859 (by a somewhat different method). Note that ϵ' is a macroscopic quantity, one third of the energy per unit mass of the gas.

Remarks

1. Conjectures (i) and (ii) are very natural in the context of our example, where the geometry is so simple that we have no trouble guessing the "most uniform" distribution of x and v with the given energy level. However, in more complicated situations it can be less obvious to predict what the "most uniform" distribution should be and one has to come up with more sophisticated principles. This is the subject of Statistical Mechanics. One approach, which we will discuss later, is based on the introduction of the notion of *entropy*, and the "most uniform" distribution will be the one which will maximize the entropy, subject to given macroscopic constraints (such as the total energy, total momentum, etc.). In the language of the Statistical Mechanics, our calculation above was an elementary *microcanonical emsemble* calculation.

2. One could be tempted to imagine the particles as very small classical hard balls of some finite small radius, with a certain continuous distribution of mass over the ball. If we adopt such a picture, we should take into account that in addition to the translational motions, the balls can also spin, and the kinetic energy of the spinning contributes to the total energy of the system. The collision of two balls is now more complicated and we have to involve the energy contained in the spinning in our consideration. The analogy of conjecture (ii) is now somewhat harder to guess if we do not invoke some general principles of Statistical Mechanics.

3. If we imagine our balls as small elastic bodies, governed by the usual equations for elastic material, the deformations of the balls will have infinitely many degrees of freedom and the considerations based on the uniform distribution of energy between the modes of motion of the system will lead to the transparently wrong conclusion that the energy will move to higher and higher modes of the elastic oscillations of the balls, bringing the translational velocity essentially to zero after some time. We see that the idea of gas molecules as classical small balls with some elasticity leads to a contradiction. All these issues are fully resolved only at the level of Quantum Mechanics.

32

11/28/2011

32.1 Additional conserved quantities for 2d Euler flows

Let us now return to the principle (29.18) of minimizing the enstrophy $I(\omega)$ for a fixed energy level, which we used as our first conjecture about the long-time behavior of the Euler solutions. Based on this principle we would predict that after a long time, the vorticity $\omega(t)$ will be weakly* close to an eigenfunction of the Laplace operator, and perhaps we will have $\omega(t_j) \rightharpoonup \overline{\omega}$, where $\overline{\omega}$ is a suitable multiple of one of the eigenfunctions of the laplacian (and a suitable sequence $t_j \to \infty$), with $E(\overline{\omega}) = E(\omega_0)$. However, the equation

$$\omega_t + u\nabla\omega = 0 \tag{32.1}$$

conserves also other quantities than the energy, and this can invalidate the above prediction. For example, we know that

$$||\omega(t)||_{L^\infty} \leq ||\omega_0||_{L^\infty}, \tag{32.2}$$

and hence the prediction would be wrong if $||\overline{\omega}||_{L^\infty} > ||\omega_0||_{L^\infty}$. The situation is somewhat similar to what we saw last time with the example from Statistical Mechanics: if our system has a conservation law which we do not take into account in our conjecture concerning the long-time behavior, the conjecture may quite likely be wrong in cases where the extra conservation law plays an important role. The obvious additional conserved quantities of (32.1) are

$$I_f(\omega) = \int_\Omega f(\omega(x,t))\,dx\,, \tag{32.3}$$

where $f\colon \mathbf{R} \to \mathbf{R}$ is any continuous function. However, there is an additional complication: the convergence suitable for studying the long-time behavior seems to be the weak* convergence (in the sense that $\omega(t_j) \rightharpoonup \overline{\omega}$ for some $t_j \to \infty$, for example) and one can easily see that the quantities I_f are not weakly* continuous unless the function f is affine. Therefore we do not expect $I_f(\overline{\omega}) = I_f(\omega_0)$ in general for the possible long-time weak* limits $\overline{\omega}$. On the other hand, if f is convex, we do have

$$I_f(\overline{\omega}) \leq \liminf_{j\to\infty} I_f(\omega_j), \qquad \omega_j \rightharpoonup \overline{\omega}\,, \tag{32.4}$$

which is a constraint of the same nature as (32.2). The quantities I_f therefore have to be taken into account in the sense of (32.4). One way to formalize this is the following. For $\omega_0 \in L^\infty(\Omega)$ we set[169]

$$\mathcal{O}_{\omega_0} = \text{weak}^* \text{ closure of } \left\{ \omega_0 \circ h\,, \quad \begin{array}{l} h\colon \Omega \to \Omega \text{ is a volume} \\ \text{preserving diffeomorphism} \end{array} \right\}. \tag{32.5}$$

[169]The definition has been considered in the paper by A. I. Shnirelman "Lattice theory and flows of ideal incompressible fluid," Russian J. Math. Phys. 1 (1993), no. 1, 105–114.

The above constraints on the possible long-time limits $\overline{\omega}$ of $\omega(t)$ (in the sense that $\omega(t_j) \rightharpoonup \overline{\omega}$ for some $t_j \to \infty$) coming from the conservation properties of (32.1) viewed as a transport equation for ω and the condition $\operatorname{div} u = 0$ can then be summarized as

$$\overline{\omega} \in \mathcal{O}_{\omega_0} \,. \tag{32.6}$$

The sets $\mathcal{O}_{\omega_0}$ can have a simple characterization. For example, when $\omega_0 = \chi_A - \chi_{\Omega\setminus A}$ with $|A| = \frac{1}{2}|\Omega|$, then

$$\mathcal{O}_{\omega_0} = \{\, \omega \,,\ \int_\Omega \omega = 0 \,,\ \ -1 \le \omega \le 1\} \,. \tag{32.7}$$

In general, the sets $\mathcal{O}_{\omega_0}$ are always convex. This may not be immediately transparent from the definition, but the proof is not hard. We will return to this issue next time.

33

11/30/2011

33.1 Orbits and their weak* closures

We recall that we are dealing with bounded smooth domain $\Omega \subset \mathbf{R}^2$. We can also allow Ω to be a torus such as $\Omega = \mathbf{R}^2/\mathbf{Z}^2$ or, more generally $\Omega = \mathbf{R}^2/\Lambda$, where Λ is a lattice or rank 2 in $\mathbf{R}^2$. Last time we defined for $\omega_0 \in L^\infty(\Omega)$ the set

$$\mathcal{O}_{\omega_0} = \text{weak}^* \text{ closure of } \left\{ \omega_0 \circ h\,, \quad \begin{array}{l} h\colon \Omega \to \Omega \text{ is a volume} \\ \text{preserving diffeomorphism} \end{array} \right\}, \quad (33.1)$$

which is the weak* closure of the orbit of ω_0 under the action of the volume preserving diffeomorphisms.[170] It might at first appear that these sets are somewhat unwieldy, but in fact they are quite simple. It turns out that they are always convex, with a relatively easy characterization. We will illustrate this on the example

$$\omega_0 = \chi_A - \chi_B\,, \qquad |A| = \frac{1}{2}|\Omega|,\ B = \Omega \setminus A. \quad (33.2)$$

Lemma

For ω_0 given by (33.2) we have

$$\mathcal{O}_{\omega_0} = \{\, \omega\ ,\ \int_\Omega \omega = 0\ ,\ \ -1 \le \omega \le 1\}\,. \quad (33.3)$$

Proof

We say that ω is a *special simple function* if

$$\omega = \sum_j b_j \chi_{E_j}\,, \quad (33.4)$$

with $|b_j| \le C$ and E_j are mutually disjoint rectangles. We have to have $\Omega = \cup_j E_j$ modulo a set of measure zero. In what follows we will often ignore sets of measure zero when they are irrelevant for our considerations. It is not hard to see that it is enough to prove the lemma for the case what ω_0 is a special simple function. For this case we will show that if ω is a special simple function with $\int_\Omega \omega = 0$ and $-1 \le \omega \le 1$, then $\omega \in \mathcal{O}_{\omega_0}$. Again, it is not hard to see that this implies the lemma.

The key point of the proof is that one can quite easily approximate the bijective measurable maps $h\colon \Omega \to \Omega$ which preserve the measure (in the sense that

[170] In general we consider the diffeomorphisms to be orientation preserving.

$|h(E)| = E$ and $|h^{-1}(E)| = |E|$ for any measurable set E) by the diffeomorphisms in the L^1 metric. For the proof we will not need this approximation result in its full generality, we will only need to approximate maps which, roughly speaking, permute rectangles.

Let us consider two disjoint rectangles $Q_1, Q_2 \subset \Omega$ of the same volume. Let $h_{12}\colon Q_1 \to Q_2$ be a smooth volume-reserving diffeomorphism and let $h_{21} = h_{12}^{-1}$. Let us define a map $h\colon \Omega \to \Omega$ (not necessarily continuous) as follows

$$h(x) = \begin{cases} h_{12}(x) & x \in Q_1\,, \\ h_{21}(x) & x \in Q_2\,, \\ x & \text{elsewhere.} \end{cases} \tag{33.5}$$

For concreteness let us assume that Q_1, Q_2 are open. Let γ be a smooth curve joining Q_1 and Q_2, and let U an open neighborhood of γ. We can think of U as a thin strip along γ joining Q_1 and Q_2. We claim that for each compact set $K_1 \subset Q_1$ with $K_2 = h_{12}(K_1)$ there exists a smooth diffeomorphism $\tilde{h}\colon \Omega \to \Omega$ such that $\tilde{h} = h$ on $K_1 \cup K_2 \cup (\Omega \setminus (Q_1 \cup Q_2 \cup U))$. The main idea is simple: we think of the region $V = Q_1 \cup Q_2 \cup U$ as a container with an incompressible fluid. We imagine we color K_1 red and K_2 blue, and we have to move the fluid in the region V in a smooth fashion so that we switch the red and blue regions according to h_{12}, and the particles very close to ∂V do not move. The blue and red regions have to be passed through the thin connection U in an amoeba-like manner, exchanging their positions. At this point we will slightly cheat and omit the details of the proof of this step. (It does require some effort to work out a formal proof, and one way or another one has to use some non-trivial tools.)

From the above it is clear that h given by (33.5) can be approximated by smooth volume-preserving diffeomorphisms h_j with $\int_\Omega |f(h(x)) - f(h_j(x))|\,dx \to 0$ as $j \to \infty$ for each $f \in L^\infty(\Omega)$.

Once we can "switch" two rectangles in the above manner, it is clear that we can permute any finite number of mutually disjoint rectangles, as long as they have the same volume.

Let us now consider a particular special simple function $\omega \in \mathcal{O}_{\omega_0}$,

$$\omega = \sum b_j \chi_{E_j}\,, \qquad \beta_j = |E_j|\,. \tag{33.6}$$

The assumption $\omega \in \mathcal{O}_{\omega_0}$ is equivalent to

$$-1 \le b_j \le 1, \qquad \sum_j b_j \beta_j = 0\,. \tag{33.7}$$

We wish to construct an approximation of ω by $\omega_0 \circ h$ for a suitable volume volume preserving map h. Which functions η can be approximated by $\omega_0 \circ h$ (for some h)? By using the above construction with permuting the rectangles,

it is not hard to see that it is enough to approximate the function ω by simple special functions of the form

$$\eta = \sum_j (\chi_{F_j} - \chi_{G_j}) \,, \tag{33.8}$$

where F_j, G_j are mutually disjoint rectangles covering Ω (modulo a set of measure zero). In the first approximation, let us try to choose F_j, G_j so that $E_j = F_j \cup G_j$ and $|F_j| = t_j |E_j| = t_j \beta_j$, with

$$t_j = \frac{1 + b_j}{2} \,. \tag{33.9}$$

Note that $0 \leq t_j \leq 1$ by our assumptions, and that

$$\int_{E_j} (\chi_{F_j} - \chi_{G_j}) = \beta_j b_j \,. \tag{33.10}$$

The sets F_j have to be covered by images of mutually disjoint rectangles contained in the set $\{\omega_0 = 1\}$. Therefore the following compatibility condition must be satisfied:

$$\sum_j |F_j| = \frac{1}{2} |\Omega| \,. \tag{33.11}$$

This is easily seen to be a necessary and sufficient condition for our construction to be possible. We need to check that it is satisfied. We have

$$\sum_j |F_j| = \sum_j t_j \beta_j = \sum_j \frac{1}{2} \beta_j + \sum_j \frac{1}{2} b_j \beta_j \,. \tag{33.12}$$

Now $\sum_j \beta_j = |\Omega|$ as E_j cover Ω and $\sum_j b_j \beta_j = 0$ by (33.7), and therefore by (33.12) the compatibility condition (33.11) is satisfied. With the choice of F_j and G_j as above we approximated ω by a special simple function $\eta \in \mathcal{O}_{\omega_0}$ which in each E_j takes on the values 1 and -1 so that the average of η over E_j has the value b_j. To get approximations which weakly* converge to ω, we can cover each E_j by a large number of small rectangles $E_{jk}, \ldots$ and repeat the construction with the covering E_j replaced by E_{jk}. It is easy to see that in this way we can get a sequence $\eta_l \in \mathcal{O}_{\omega_0}$ converging weakly* to ω. This finishes the proof of the lemma.

Remark

For a general $\omega_0 \in L^\infty(\Omega)$ the set $\mathcal{O}_{\omega_0}$ can be characterized as

$$\mathcal{O}_{\omega_0} = \{\omega \in L^\infty(\Omega) \,,\ \int_\Omega \omega \, dx = \int_\Omega \omega_0 \, dx \ ,\ \int_\Omega (\omega - c)_+ \, dx \leq \int_\Omega (\omega_0 - c)_+ \, dx, \quad c \in \mathbf{R}\}, \tag{33.13}$$

where s_+ denotes the positive part of s. The main idea of the proof is similar to the proof of the lemma above, except that the "accounting" is somewhat more complicated.

33.2 Long-time behavior for 2d Euler - another attempt

In lecture 29 we explored the possibility of minimizing the enstrophy $I(\omega) = \int_\Omega \frac{1}{2}\omega^2\,dx$ at a given energy level $E(\omega) = E_0$ to predict the long time behavior of Euler solutions, and we saw in lecture 32 that this may lead in some cases to a transparently incorrect result, due to the constraints stemming from $\omega(t) \in \mathcal{O}_{\omega_0}$. We can now try to adjust this procedure by the following rule:

$$\textit{Minimize } I(\omega) \textit{ subject to the constraints } E(\omega) = E(\omega_0) \textit{ and } \omega \in \mathcal{O}_{\omega_0}\,. \tag{33.14}$$

In a domain without symmetries, this procedure takes into account all the conserved quantities we are aware of. In a domain which is rotationally invariant, the procedure misses the conservation of the moment of rotation of the fluid[171] and still needs to be adjusted (by simply adding the additional conserved quantity to our constraints).

Let us look again at the example (33.2) in the context of (33.14). In this case the constraint $\omega \in \mathcal{O}_{\omega_0}$ in (33.14) can be incorporated as follows. Let us define

$$g(\omega) = \begin{cases} \frac{1}{2}\omega^2, & |\omega| \le 1\,, \\ +\infty\,, & |\omega| > 1\,. \end{cases} \tag{33.15}$$

Then for our special case (33.2) the minimization (33.14) becomes

$$J(\omega) = \int_\Omega g(\omega) \to \min, \quad \text{subject to } E(\omega) = E(\omega_0), \quad \textstyle\int_\Omega \omega = 0\,. \tag{33.16}$$

The function $g(\omega)$ can be approximated by smooth finite functions approaching ∞ for $|\omega| > 1$. For a finite smooth g we can write the equation for the minimizers in (33.16) by the usual procedure of Lagrange multipliers. We consider recall that $E(\omega) = \int_\Omega -\frac{1}{2}\psi\omega$, where ψ is the stream function, and consider

$$L(\omega) = \int_\Omega \left(g(\omega) - \lambda(-\frac{1}{2}\psi\omega) - \mu\omega \right) dx. \tag{33.17}$$

The condition $L'(\omega) = 0$ gives

$$g'(\omega) + \lambda\psi - \mu = 0\,. \tag{33.18}$$

Keeping in mind that g is now a smooth finite approximation of the function (33.15), let us assume that g' is strictly increasing and set $F = g^{-1}$ (inverse function). Then (33.18) gives

$$\Delta\psi = \omega = F(-\lambda\psi + \mu). \tag{33.19}$$

[171] Given by $\int |x|^2\omega\,dx$ if the center of the rotational symmetries is at the origin.

As the functions g approach (33.15), the corresponding functions F approach the following function

$$F(y) = \begin{cases} y & |y| \leq 1 \\ 1 & y > 1 \\ -1 & y < -1\,. \end{cases} \tag{33.20}$$

Therefore the procedure (33.14) predicts that the Euler solutions will weakly* approach the set of solution of the equation (33.19) where F is given by (33.20), and the multipliers λ, μ are determined from the constraints $E(\omega) = E(\omega_0)$ and $\int_\Omega \omega = 0$. (We note that the solutions of (33.20) are steady-state solutions of the Euler's equation, see (14.31).)In comparison with our first guess (29.20), the equation is now non-linear, and the problem of finding the relevant solutions is non-trivial.

The prediction of the long-time behavior of the 2d Euler solutions via the solutions of (33.19) is certainly better than the prediction based only on the enstrophy we discussed in lecture 29, but it is still not optimal in that we do not really have any deeper justification for the finite part of the function g. In fact, and we can replace the finite part of g by any uniformly convex function and still get a similar prediction with a different function F in (33.20). Therefore we have many predictions. Which one is the most reasonable? We will discuss this issue soon. For now we note that we have to view all our prediction in this direction with a grain of salt. The mixing which is necessary to obtain the characterization of $\mathcal{O}_{\omega_0}$ in the lemma above is quite significant, as we can see from its proof, and we can have some doubt if the evolution by Euler's equation can really achieve that level of mixing. In reality this may not be the case, and therefore the principle (33.14), and also its still more sophisticated variants which we will discuss later, cannot be viewed with the same level of confidence as, say, the conjectures (i) and (ii) from lecture 31.

34

12/2/2011

34.1 2d Incompressible Euler as a dynamical system in a compact metric space

As we have seen in previous lectures, the space $L^\infty(\Omega)$ with the weak* topology seems to be a natural space for the vorticity ω when we consider the long-time evolution of 2d incompressible Euler solution. We have seen that the Euler solutions $\omega(t)$ satisfy $||\omega(t)||_{L^\infty} \leq ||\omega_0||_{L^\infty}$. Let $c = ||\omega_0||_{L^\infty}$ and let

$$X = X_c = \{\omega \in L^\infty(\Omega),\ ||\omega||_{L^\infty} \leq c\} \tag{34.1}$$

The set X equipped with the weak* topology is a compact metric space.[172] In what follows we will always consider X with the weak* topology, unless we explicitly state otherwise.

It is natural to ask if the evolution by Euler's equation defines a good dynamical system on X. This is a non-trivial issue. Note that so far we have always assumed that our initial condition ω_0 is "sufficiently regular" and the existence results we discussed in lecture 13 were also formulated in the context of smooth solutions. On the other hand, to be able to consider the Euler equation as a dynamical system in X, one should have the following:

1. (Existence and uniqueness) For each $\omega_0 \in X$ we have a unique solution of the 2d incompressible Euler equation (14.22) $t \to \omega(t)$ in X with $\omega(0) = \omega_0$.

2. (Continuity) The map $(t, \omega_0) \to \omega(t)$ from $R \times X \to X$ is continuous.[173]

These properties can be indeed established in our situation. Property 1 was established in the 1960s by V. Yudovich[174] and property 2 follows quite easily from Yudovich's results.

We note some technical points which come up in this context. First, the Euler equation

$$\omega_t + u\nabla\omega = 0 \tag{34.2}$$

should be reformulated so that it can be well-defined for vorticities $\omega(x,t)$ which are bounded but do not have any smoothness in x. (We assume that u is determined by ω as discussed in lecture 14.) Second, one must define in which

[172] A standard way to introduce a metric in X which gives the weak* topology on X is the following: consider a countable dense set f_j of functions in the unit ball of $L^1(\Omega)$, let $p_j(\omega) = |\int_\Omega f_j \omega\, dx|$, and set $\text{dist}(\omega', \omega'') = \sum_j \frac{2^{-j} p_j(\omega' - \omega'')}{1 + p_j(\omega' - \omega'')}$.

[173] The topology on $R \times X$ is taken to be the natural product topology: $(t_j, \omega_j) \to (t, \omega)$ if $t_j \to t$ and $\omega_j \to \omega$ in X.

[174] V. I. Yudovich, Non-stationary flow of an incompressible fluid, Zh. Vychisl. Mat. Mat. Fiz. **3**, 1032–1066, 1963.

sense the initial condition is attained. Both of these points are addressed by considering the *weak solutions* of (34.2). The key point in the definition of the weak solutions is that $\operatorname{div} u = 0$, and therefore (34.2) can be written as

$$\omega_t + \operatorname{div}(\omega u) = 0\,. \tag{34.3}$$

When ω and u are bounded measurable, the expression on the left-hand side of (34.3) is well-defined as a distribution, and that is the key to the definition of the weak solution. We will consider (34.3) with the natural boundary condition $u\,n = 0$ at the boundary $\partial\Omega$, where n is the unit normal to $\partial\Omega$ as usual. This condition on u is enforced via the boundary condition for the stream function when solving $\Delta\psi = \omega$. (One assumes that ψ is locally constant on the boudary, with a given constant for each component.)

If $\omega \in L^\infty(\Omega \times (t_1, t_2)$, the function $x \to \omega(x,t)$ is defined as an element of $L^\infty(\Omega)$ for a. e. $t \in (t_1, t_2)$ and for such t we can define the stream function $\psi(x,t)$ by solving $\Delta_x\psi(x,t) = \omega(x,t)$ in Ω, with the boundary condition as above. This determined $u(x,t) = \nabla^\perp\psi(x,t)$ a. e. in $\Omega \times (t_1, t_2)$. In what follows we will always consider u as determined by ω in this way.

We say that $\omega \in L^\infty(\Omega \times (t_1, t_2)$ is a weak solution of (34.3) in $\Omega \times (t_1, t_2)$ if

$$\int_{t_1}^{t_2} \int_\Omega -\omega\varphi_t - \omega u \nabla\varphi \;\, dx\, dt = 0 \tag{34.4}$$

for each smooth $\varphi\colon \overline{\Omega} \times (t_1, t_2)$ which is supported in $\overline{\Omega} \times [t_1 + \tau, t_2 - \tau]$ for some $\tau > 0$. (We note that we can also demand that φ be compactly supported in $\Omega \times (t_1, t_2)$, which leads to the same notion of solution, due to the fact that the condition $u\,n = 0$ is already enforced.)

If $f \in L^\infty(\Omega \times (t_1, t_2))$ satisfies for some $a = (a_1, a_2) \in L^\infty(\Omega \times (t_1, t_2))$

$$\int_{t_1}^{t_2} \int_\Omega -f\varphi_t - a\nabla\varphi\, dx\, dt = 0 \tag{34.5}$$

for each smooth φ compactly supported in $\Omega \times (t_1, t_2)$, the equation imposes some extra regularity on f, so that the function $x \to f(x,t)$ is well-defined as an element of $L^\infty(\Omega)$ for each $t \in [t_1, t_2]$. This can be seen from the fact that for a smooth $\varphi = \varphi(x)$ compactly supported in Ω the function

$$t \to \int_\Omega f(x,t)\varphi(x)\, dx \tag{34.6}$$

will satisfy

$$\int_{t_1}^{t_2} -\int_\Omega f(x,t)\varphi(x)\, dx\; \theta'(t)\, dt = \int_{t_1}^{t_2} \int_\Omega a(x,t)\nabla\varphi(x)\theta(t)\, dt \tag{34.7}$$

where $\theta(t)$ is smooth, compactly supported in (t_1, t_2). This shows that the function (34.6) has a bounded distributional derivative and therefore is Lispchitz.[175]

[175]Recall that a Lipschitz function $g\colon (t_1, t_2) \to \mathbf{R}$ is a function with $|g(t') - g(t'')| \leq C|t' - t''|, \quad t', t, \in (t_1, t_2)$ for some $C \geq 0$.

In particular, it is uniformly continuous in (t_1, t_2), and hence well-defined for each $t \in [t_1, t_2]$. As $\varphi = \varphi(x)$ was an arbitrary smooth compactly supported function, we see that $x \in f(x,t)$ is well-defined as an element of $L^\infty(\Omega)$ for each $t \in [t_1, t_2]$.

The above considerations show that *if $\omega \in L^\infty(\Omega \times (t_1, t_2))$ is a weak solution of (34.3), then $x \to \omega(x,t)$ is well-defined as an element of $L^\infty(\Omega \times (t_1, t_2))$ for each $t \in [t_1, t_2]$. Moreover, the function $t \to \omega(\cdot, t)$ is continuous as a function from (t_1, t_2) into $L^\infty(\Omega \times (t_1, t_2))$ with the weak* topology (and can be continuously extended to $[t_1, t_2]$. In other words, taking $c = ||\omega||_{L^\infty}$, the function $t \to \omega(\cdot, t)$ is a (uniformly) continuous function from (t_1, t_2) to the metric space X above.*

Assume now that $t_1 \leq 0 \leq t_2$. Taking into account the above remarks, it is clear that weak solutions provide a good framework to talk about the initial value problem

$$\omega_t + u\nabla\omega = 0, \qquad \omega(x,0) = \omega_0 \in L^\infty(\Omega). \tag{34.8}$$

The main result proved in the above quoted 1963 paper of Yudovic is the following:

Theorem 1

The initial value problem (34.8) has a unique weak solution in $L^\infty(\Omega \times (t_1, t_2))$ for any $t_1 \leq 0 \leq t_2$, including $t_1 = -\infty$ and $t_2 = \infty$.

We will not go into the proof of this theorem. We refer the reader to the book "Vorticity and Incompressible Flow" by A. Majda and A. Bertozzi, Chapter 8, or to the original paper of Yudovich. The main difficulty in the proof is the uniqueness part. The proof of the theorem in fact also gives the following result:

Theorem 2

The solution $\omega(x,t)$ in Theorem 1 depends continuously on the initial data in the following sense: if ω_0^j converges weakly* to ω_0 and ω^j is the solution corresponding to ω_0^j, then for each $t \in (t_1, t_2)$ the solutions $\omega^j(\cdot, t)$ converge weakly* to $\omega(\cdot, t)$.

Theorems 1 and 2 show that the Euler equation provides a good dynamical system in the metric space X, and we can use all the notions used in the study of abstract dynamical systems. For example, for each ω_0 we can define the $\omega-$limit set of the trajectory $t \to \omega(t)$ passing through ω_0 as

$$\Omega_+(\omega_0) = \cap_{t>0} \text{ weak}^* \text{ closure of } \{\omega(s),\, s \geq t\}\ , \tag{34.9}$$

where we use the notation $\omega(s) = \omega(\cdot, s)$. Next time we will look at the sets $\Omega_+(\omega_0)$ in more detail.

35

12/5/2011

35.1 Solutions with vorticity trajectories pre-compact in L^2

Having the possibility of considering the 2d Euler equation as a dynamical system on a compact metric has the advantage that we can apply general concepts of the theory of the dynamical systems, such as the ω-limit sets introduced last time, see (34.9). Going further in applying general conclusions which can be made for dynamical systems on compact metric spaces, we could for example construct measures on the $\omega-$limit sets which are invariant under the flow and study their ergodic properties. However, the information about the Euler solutions which one can get from the general principles concerning dynamical systems on compact metric spaces does not seem to be very deep, unless one uses some specific features of our situation. It is useful that we can define the sets $\Omega_+(\omega_0)$ (see (34.9)), but what can we say about these sets? Clearly $\Omega_+(\omega_0) \subset \mathcal{O}_{\omega_0} \cap \{E(\omega) = E(\omega_0)\}$. It is not clear how often it happens that $\Omega_+(\omega_0) = \mathcal{O}_{\omega_0} \cap \{E(\omega) = E(\omega_0)\}$. In the absence of some obvious additional conserved quantities[176], can we "typically" expect $\Omega_+(\omega_0) = \mathcal{O}_{\omega_0} \cap \{E(\omega) = E(\omega_0)\}$, or would such situation be exceptional/impossible? Questions of this type seem to be open.

Today we consider one simple result which is still based on fairly general arguments, but does seem to be of some interest in the context od 2d Euler solutions we have been studying.

Theorem

For any $\omega_0 \in L^\infty(\Omega)$ there exists $\overline{\omega}_0 \in \Omega_+(\omega_0)$ such that the trajectory $\overline{\omega}(t)$ passing through $\overline{\omega}_0$ is pre-compact in $L^2(\Omega)$. In particular, the $\omega-$limit set $\Omega_+(\overline{\omega}_0)$ is compact in L^2.

Proof

Consider the enstrophy $I(\omega) = \int_\Omega \frac{1}{2}|\omega|^2\,dx$. The functional I is sequentially weakly* lower semi-continuous on $L^\infty(\Omega)$, and hence weakly* lower semi-continuous on the metric space X (see (34.1)). The set $\Omega_+(\omega_0)$ is compact in X, and therefore I attains its minimum on it. Let $\overline{\omega}_0 \in \Omega_+(\omega_0)$ be such that $m = I(\overline{\omega}_0) \leq I(\omega)$ for each $\omega \in \Omega_+(\omega_0)$ and let $\overline{\omega}(t)$ be the trajectory with $\overline{\omega}(0) = \overline{\omega}_0$. Assume $\overline{\omega}(t_j)$ converge weakly* to ω_1. We have $\omega_1 \in \Omega_+(\omega_0)$ (relying on Theorem 2 from lecture 34) and therefore $I(\omega_1) \geq m$. On the other hand $I(\omega_1) \leq \liminf_{j\to\infty} I(\overline{\omega}(t_j)) = m$. Therefore $I(\overline{\omega}(t_j)) \to I(\omega_1)$, and together with the weak* convergence of $\overline{\omega}(t_j)$ to ω_1, this implies the strong convergence of $\overline{\omega}(t_j)$ to ω_1 in $L^2(\Omega)$. (It is not hard to see that the proof also works with I replaced by $I_f(\omega) = \int_\Omega f(\omega(x))\,dx$ as long as f is strictly convex.)

[176] such as $\int_\Omega |x|^2\omega\,dx$ when Ω is a disc centered at the origin

Remarks

1. In general, if we minimize I_f over $\mathcal{O}_{\omega_0} \cap \{E(\omega) = E(\omega_0)\}$ rather than $\Omega_+(\omega_0)$, we get a steady state solutions (which can depend on f). This follows from results in the paper of A. Shnirelman quoted in lecture 32 and can be also proved directly, by generalizing the procedure leading to (33.19) in lecture 33.

2. The solution $\overline{\omega}(t)$ from Theorem 1 seems to be relevant in the context of solutions we observe in long-time numerical integration of the 2d Euler equation. For concreteness, let us assume that the domain Ω is the two-dimensional torus $\mathbf{R}^2/2\pi\mathbf{Z}^2$ and let $\hat{\omega}(k,t)$ be the Fourier coefficients of ω (so that $\omega(x,t) = \frac{1}{(2\pi)^2}\sum_k \hat{\omega}(k,t)e^{ikx}$). A numerical simulation has a limit to its resolution, which can be represented by a cut-off in the frequency space, in the sense that we only consider frequencies $k \in \mathbf{Z}^2$ with $|k| \leq \kappa$, where κ is some large number.[177] We can imagine a cartoon picture in which the actual (non-truncated) solution is consisting of two parts. One which lives in "finite frequencies" (ideally $\leq \kappa$), and one which gradually drifts to the infinite frequencies. Both parts can have non-negligible L^2-norm. However, energy has to be conserved, and therefore the "finite frequency part" has to move somewhat towards the origin, to compensate for the loss of energy caused by the high frequency part moving to still higher frequencies. In our first attempt on the prediction of the long-time behavior in lecture 29 we assumed that the "finite frequency part" will move all the way down to the lowest possible frequencies. However, we can also imagine that this may not be the case, and on its way towards to low frequencies the "finite frequency part" can get stuck in some time-dependent regime, which will not descent all the way to the lowest modes. After a long time the high frequency part will be practically invisible, residing only in very high frequencies, and what we see will be only the "finite frequency part". There will be no further "leaking to infinity" from this part, and we can identify it with the solution $\overline{\omega}(t)$ above. The would be the simplest possible scenario. The reality is presumably more complicated, but the Theorem above is a (weak) statement in this direction.

3. If the scenario 2 above is correct, then we would expect that $\Omega_+(\omega_0)$ is in fact compact in L^2 for a "typical" ω_0. Whether or not this is true is unclear.

[177] On your laptop you can take easily $\kappa = 10^2$, on a larger machine one can take $\kappa \sim 10^4$, and $\kappa \sim 10^5$ is still realistic on today's big computers.

Homework Assignment 3
due December 21

Due one or more of the following five problems:

Problem 1

Verify that the Cole-Hopf transformation (30.6) takes the positive solutions of the heat equation (30.5) into the solutions of the viscous Burgers equation (30.1).

Problem 2

In the context of (33.13), prove that

$$\mathcal{O}_{\omega_0} \subset \{\omega \in L^\infty(\Omega)\,,\ \int_\Omega \omega\,dx = \int_\Omega \omega_0\ dx\ ,\ \int_\Omega (\omega - c)_+\,dx \le \int_\Omega (\omega_0 - c)_+\,dx, \quad c \in \mathbf{R}\}. \tag{35.1}$$

Problem 3

Calculate the fundamental solution of the linear steady Stokes system (25.11) in the cartesian coordinates.

Problem 4

Explain why blowing into a flute can generate sound.

Problem 5

Consider a flow of water in a 3/4 inch garden hose. Assume the stream of water leaving the hose can rise 20 feet high.
a) Estimate the Reynolds number of the flow inside the hose.
b) Estimate the size of the smallest eddies inside the hose.
c) Estimate the highest frequency with which the fluid particles oscillate around their mean trajectories.
d) Estimate how much power is needed to sustain the flow (not counting the power needed for accelerating the fluid from the state of rest) if the length of the hose is 100 feet.

36

11/7/2011

36.1 Maximizing the entropy

The idea behind the various conjectures for the long-time behavior of the 2d incompressible Euler solutions is mostly that, roughly speaking, the solution should mix the vorticity in the maximal possible way which is still consistent with the conservation of the conserved quantities.[178] One way to measure mixing is to look at values of integrals $I_f(\omega) = \int_\Omega f(\omega)\, dx$ for convex functions f. The smaller the value of $I_f(\omega)$ on $\omega \in \mathcal{O}_{\omega_0}$, the more mixing has to take place to produce ω from $\omega_0 \circ h$ (where h is a volume-preserving diffeomorphism) by weak* convergence. We have seen in the proof of the theorem in the last lecture that this idea works at some level, but there are clearly an ad hoc components in that approach. Today we look at the issue of how to measure the level of mixing in a more sophisticated way. We will again consider only the simple example[179]

$$\omega_0 = a\chi_A - a\chi_B, \qquad B = \Omega \setminus A, \qquad |A| = \frac{1}{2}|\Omega|\,, \tag{36.1}$$

where $a \geq 0$. [180] We will consider a simple approach which is sometimes used in introductory Statistical Mechanics.[181] Our goal is to illustrate the main ideas. Let us first consider the following discrete problem. We cover Ω (modulo a set of measure zero) by r mutually disjoint "boxes" $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ of the same measure $|\Omega|/r$. We divide each of these boxes into n mutually disjoint smaller boxes of measure $|\mathcal{B}_i|/n = |\Omega|/rn$. Consider now the vorticity functions ω which are constant constant on each box $\mathcal{B}_k$, denoting the value on $\mathcal{B}_k$ by ω_k. We assume that the value ω_k in $\mathcal{B}_k$ is a result of mixing values $+1$ and -1 in the small boxes contained in $\mathcal{B}_k$. In each of the small boxes the value of the function which we use for mixing is either $+1$ or -1. If in n_k^+ small boxes of $\mathcal{B}_k$ the value is $+1$ and in n_k^- of the small boxes of $\mathcal{B}_k$ the value is -1, then we have

$$n_k^+ + n_k^- = n, \qquad \omega_k = a \cdot \frac{n_k^+}{n} - a\frac{n_k^-}{n}\,. \tag{36.2}$$

Letting

$$\rho_k^\pm = \frac{n_k^\pm}{n}\,, \tag{36.3}$$

[178] As we have already mentioned in the last lecture, this conjecture may be too optimistic in that it over-estimates the amount of mixing the equation is able to provide. Nevertheless, it is interesting to work out the consequences of the "maxima possible mixing" hypothesis.

[179] It is not hard to generalize the approach we will consider to the general case.

[180] We could set $a = 1$ without loss of generality, but from the point of view of dimensional analysis it is better to use (36.1), where a is thought of having the same physical dimension as ω.

[181] See e. g. the text "Concepts of Modern Physics" by A. Beiser.

we can write

$$\rho_k^{\pm} = \frac{a \pm \omega_k}{2a} \,. \tag{36.4}$$

Imposing the constraint

$$\int_\Omega \omega = 0 \tag{36.5}$$

is clearly the same as demanding that the total number of the small boxes where we use $+1$ is the same as the total number of the small boxes where we use -1. We will only consider the configurations in which this constraint is satisfied. With all the boxes $\mathcal{B}_1, \ldots, \mathcal{B}_r$ and all the small boxed inside them fixed, how many configurations of $+1$ and -1 which "produce $\omega_1, \omega_2, \ldots \omega_r$ does there exist?[182] Recalling elementary combinatorics, we write the number of configurations as

$$K = K(\omega) = \Pi_{j=1}^{j=r} \, \frac{n!}{n_j^+! \; n_j^-!} \,. \tag{36.6}$$

We have

$$\log K = \sum_j \left[\log n! - \log n_j^+! - \log n_j^-! \right] \,. \tag{36.7}$$

Using the Stirling formula

$$\log m! = (m - \frac{1}{2}) \log m - m + \frac{1}{2} \log 2\pi + O(\frac{1}{m}), \qquad m \to \infty \tag{36.8}$$

and setting $n_j^{\pm} = \rho_j^{\pm} n$, we can write, assuming all $n_j^{\pm}$ are large enough,

$$\log K = \sum_j \left[n(-\rho_j^+ \log \rho_j^+ + \rho_j^- \log \rho_j^-) + \frac{1}{2} \log n + \frac{1}{2} \log \rho_j^+ + \frac{1}{2} \log \rho_j^- - \frac{1}{2} \log 2\pi \right] + R \,, \tag{36.9}$$

where R is an error term. Therefore

$$\frac{\log K}{nr} = \frac{1}{|\Omega|} \int_\Omega (-\rho^+ \log \rho^+ - \rho^- \log \rho^-) \, dx + \tilde{R} \tag{36.10}$$

where $\rho^{\pm} = \rho_j^{\pm}$ in $\mathcal{B}_j$ and $\tilde{R}$ is an error term. The integral on the right can be called *entropy*. This term is used in many situations in various ways. In the context of Statistical Physics it is often proportional to the logarithm of the number of ways in which some event can occur.[183] In our case it can be thought of as the logarithm of the number of ways in which the function ω can be mixed from $+1$ and -1, normalized per the small box. Note that

$$\rho^{\pm} = \frac{a \pm \omega}{2a} \tag{36.11}$$

[182] For a given ω the formulae may not give an integer $n_k^{\pm}$, but we will ignore this issue, as we are interested in the continuum limit anyway.

[183] For a rigorous treatment of probabilistic notions related to entropy we refer the reader to the book "Large deviation techniques and applications" by A. Dembo and O. Zeitouni. For an introduction to Information Theory, where the notion of entropy is also important, and the terms Shannon entropy or "measure of information" are also used, we refer the reader to the book "Information Theory" by R. B. Ash.

as in (36.4). Omitting a more detailed analysis of the error term[184] and passing formally to the continuum limit, we obtain that $\frac{\log K}{nk}$ approaches

$$\mathcal{S}(\omega) = \frac{1}{|\Omega|} \int_\Omega \frac{a+\omega}{2a} \log\left(\frac{a+\omega}{2a}\right) + \frac{a-\omega}{2a} \log\left(\frac{a-\omega}{2a}\right) \, dx \,. \tag{36.12}$$

We can call $\mathcal{S}(\omega)$ the entropy of the function ω (with respect to ω_0 given by (36.1), so we might also write $\mathcal{S}(\omega, \omega_0)$). The value $\mathcal{S}(\omega)$ should in some way quantify the among of mixing which is necessary to produce ω from ω_0.

We let

$$s(\omega) = \frac{a+\omega}{2a} \log\left(\frac{a+\omega}{2a}\right) + \frac{a-\omega}{2a} \log\left(\frac{a-\omega}{2a}\right) \,, \tag{36.13}$$

so that

$$\mathcal{S}(\omega) = \frac{1}{|\Omega|} \int_\Omega s(\omega) \, dx \,. \tag{36.14}$$

The function $s(\omega)$ is defined so far only for $\omega \in [-a, a]$ and it is natural to set $s(\omega) = -\infty$ when $|\omega| > a$.

Let us now consider the problem

$$\text{Maximize } \mathcal{S}(\omega) \text{ subject to } E(\omega) = E(\omega_0) \text{ and } \textstyle\int_\Omega \omega \, dx = 0. \tag{36.15}$$

Assuming that Ω is a torus or a simply connected domain,[185] we can write the equation for ω coming from (36.15) by using Lagrangian multipliers as usual. We maximize

$$\mathcal{S}(\omega) - \beta E(\omega) - \mu I(\omega) \,, \qquad E(\omega) = E(\omega_0) = E_0, \quad I(\omega) = \frac{1}{|\Omega|} \int_\Omega \frac{\omega}{a} \, dx = 0 \,, \tag{36.16}$$

where the physical dimension of β is 1/energy and μ is dimensionless. We have $E(\omega) = \frac{1}{2}\rho \int_\Omega -\psi\omega \, dx$, where ρ is the (constant) density of the fluid. Let $M = \rho|\Omega|$ be the total mass of the fluid. From the maximization of in (36.16) we obtain

$$s'(\omega) + \beta M \psi - \frac{\mu}{a} = 0 \,. \tag{36.17}$$

We have

$$s'(\omega) = \frac{1}{2a} \log\left(\frac{a+\omega}{a-\omega}\right) . \tag{36.18}$$

Inverting this expression, we obtain from (36.17)

$$\Delta\psi = \omega = a \tanh(\beta a M \psi + \mu) \,. \tag{36.19}$$

[184] In particular, in the derivation we should assume that the densities $\rho_j^\pm$ are not too close to zero, so that the Stirling approximation is still valid and the error term $\tilde{R}$ is small.

[185] The only issue in multiple connected domains is in recovering ψ from ω. One of course solves $\Delta\psi = \omega$, but the correct boundary condition on ψ needs some discussion. We will get to this issue later.

If the conjecture of "maximal possible mixing compatible with the given constraints" is correct, we can expect the solutions of Euler equation to approach (weakly*) the solutions of (36.19) for large times. The multipliers β and μ above are determined from $E(\omega) = E_0$ and $I(\omega) = 0$. The study of the relevant solutions of (36.19) (subject to $E(\omega) = E_0$ and $I(\omega) = 0$) is a nontrivial topic in its own right, and we will discuss some aspects of it soon.
The maximum value of $\mathcal{S}(\omega)$ among all ω subject to the constraints $\omega \in \mathcal{O}_{\omega_0}$ and $E(\omega) = E_0$ can be denoted by $S(E_0)$. This function $S = S(E)$ is the analogy of the entropy function used in Statistical Mechanics and Thermodynamics.

Historical comments

The Statistical Mechanics approach to 2d Euler equations has a long history, starting with a well-known 1949 paper by L. Onsager.[186] In 1970s a model based on the point-vortices approximation equations of the form $\Delta\psi = f(\beta\psi+\mu)$ with $f \sim \sinh$ was derived by G. Joyce and D. D. Montgomery.[187] Equation (36.19) (with a different normalization of the parameters) was first derived around 1990 (in a slightly different way) by J. Miller[188] and R. Robert[189], as a special case of a more general theory which instead of the special initial data (36.1) considers general ω_0. A further important contribution to the theory (which we will discuss next time) is due to B. Turkington.[190] The concept of mixing which we use was introduced in the 1993 paper by A. Shnirelman[191] quoted in lecture 32. The reader can also consult the book "Non-linear Statistical theories for Basic Geophysical Flows" by A. Majda and X. Wang", the book "Vorticity and Turbulence" by A. Chorin, or the book "Topological Hydrodynamics" by V. Arnold and B. Khesin.

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37

12/9/2011

37.1 Ideal gas revisited

In lecture 31 we calculated the velocity distribution of particles in an ideal gas (under some assumptions). That calculation was based on a conjecture that after letting the system evolve for some time, the probability of finding the vector of all the particle velocity coordinates to assume the value $(v_1, v_2, \dots v_{3N})$ is uniformly distributed over the given energy surface. Today we calculate the velocity distribution in a different way, using the notion of entropy. This will hopefully give some illustration of the notion of entropy in a situation which is simpler than the 2d Euler, and where one can actually explicitly calculate the analogue of the function $S = S(E)$ mentioned at the end of the last lecture.

The notion of entropy can be introduced in many ways. The way we chose here represents one of the more elementary approaches and our choice has been motivated by a desire to have a definition which closely follows our considerations for 2d Euler solutions in the previous lecture.

We again consider N particles of mass m in a domain $\Omega \subset \mathbf{R}^3$. The total mass is $M = Nm$. We will describe the state of the system in a way which is different from our description in lecture 31. Instead of using probability density in the "big space" $(x_1, \dots, x_{3N}, v_1, \dots, v_{3N})$ we will use the particle density in the 6−dimensional space $(x_1, x_2, x_3), (v_1, v_2, v_3)$. The density of the particles will be denoted by $\rho(x, v)$. For any box $\mathcal{B} \subset \mathbf{R}^3 \times \mathbf{R}^3$ the number of particles with coordinates $x = (x_1, x_2, x_3)$ and velocities $v = (v_1, v_2, v_3)$ which belong to $\mathcal{B}$ is

$$N \int_{\mathcal{B}} \rho(x, y) \frac{dx\, dv}{C}\,, \tag{37.1}$$

where C is a constant of dimension [length]^3 [velocity]^3 , which can be thought of as the volume of a reference box in $\mathbf{R}^3 \times \mathbf{R}^3$. The reason we introduce this factor is that we wish to keep the density ρ dimensionless. Our definition clearly implies the normalization

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3} \rho(x, v) \frac{dx\, dv}{C} = 1\,. \tag{37.2}$$

The energy E of the system of particles described by the density ρ will be assumed to be

$$E = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \rho \frac{1}{2} M |v|^2 \frac{dx\, dv}{C}\,. \tag{37.3}$$

This means that there is no potential or interaction energy. The particles are supposed to interact so that they can reach the "equilibrium state" which we are going to calculate based on some conjectures, so there is an idealization here: we assume that somehow the equilibrium state will be reached even when the energy is given by (37.3). We can imagine for example that the particles

are very small and they only interact through elastic collisions, and the total volume occupied by the particles is negligible in comparison with the volume they occupy. That is why we talk about "ideal gas". In a real gas there will be an additional term in (37.3) (not to speak about quantum effects and a number of other phenomena, which we also neglect). Nevertheless, in many situation the ideal gas provides a very good approximation of the real situation.

Assume now we take some large finite box in $\mathbf{R}^3 \times \mathbf{R}^3$ centered at $(0,0)$ and divide in into a large number r of small boxes $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r$ of the same volume b. We assume that the coordinates of all N particles are in the large box. Let n_j be the number of particles in $\mathcal{B}_j$. We let

$$p_j = \frac{n_j}{N}, \qquad p_j = \rho_j \frac{b}{C}. \tag{37.4}$$

Let K be the number of ways in which we can distribute the N particles between $\mathcal{B}_1, \ldots, \mathcal{B}_r$ so that the number of particles in $\mathcal{B}_j$ is n_j. We have[192]

$$K = \frac{N!}{n_1! n_2! \ldots n_r!} \tag{37.5}$$

Using Stirling's formula (36.8) together with $n_j = p_j N$, we obtain, similarly to the calculation (36.9) in the last lecture,

$$\log K = N(\sum_j -p_j \log p_j) + R\,, \tag{37.6}$$

where R is a remainder term which we will not study in detail. Hence

$$\frac{\log K}{N} \;\sim\; \int_{\mathbf{R}^3\times\mathbf{R}^3} -\rho \log(\rho \frac{b}{C}) \frac{dx\,dv}{C} \;=\; \int_{\mathbf{R}^3\times\mathbf{R}^3} -\rho \log \rho \, \frac{dx\,dv}{C} \;+\; \log(\frac{C}{b})\,. \tag{37.7}$$

The term $\log \frac{C}{b}$ approaches ∞ as $b \to 0_+$. However, we note that the term is independent of ρ, and in fact represents (modulo a finite term) the number $\frac{\log K}{N}$ for the special case $n_1 = n_2 = \cdots = n_r = \frac{N}{r}$. So we can remove this term with the understanding that our formula will represent not $\frac{\log K}{N}$ but rather the difference (up to some constant) between this quantity and the same quantity for the uniform distribution. Following the traditional notation, we will denote

$$H = H(\rho) = \int_{\mathbf{R}^3\times\mathbf{R}^3} -\rho \log \rho \, \frac{dx\,dv}{C}\;. \tag{37.8}$$

This integral, introduced by L. Boltzmann is sometimes called the entropy of the density ρ. It plays the same role as the function s in (36.12). The

[192]Here we assume that in principle it is possible to have some information which can distinguish the particles from one another. This is always possible in the classical picture of the world, where information is not "physical" and does not interact with our system. We know that in reality this is not the case, but for our purposes here it is still OK to assume that in principle we can tell one particle from another, e. g. by knowing their histories. However, we should keep in mind that at the level of the Quantum Mechanics this is impossible.

"most likely" density ρ corresponding to a given energy level E can now be predicted (following L. Boltzmann) to be given by maximizing $H(\rho)$ subject to the constraint (37.3) that the system has energy E and also the condition $\int_{\mathbf{R}^3\times\mathbf{R}^3} \rho \frac{dx\,dv}{C} = 1$. Using Lagrange multiplyers as in the last lecture, we see that we should maximize the functional

$$\int_{\mathbf{R}^3\times\mathbf{R}^3} \left(-\rho \log \rho - \beta\rho \frac{1}{2} M|v|^2 - \mu\rho \right) \frac{dx\,dv}{C} \,, \tag{37.9}$$

where β and μ are eventually determined by the constraints. Setting the variation of (37.9) equal to zero, one obtains

$$-\log\rho - \beta\frac{1}{2}M|v|^2 - \mu - 1 = 0 \tag{37.10}$$

Further calculation[193] leads to

$$\beta = \frac{3}{2E}\,, \qquad \rho(x,v) = \frac{C}{V(4\pi\epsilon')^{\frac{3}{2}}} e^{-\frac{|v|^2}{4\epsilon'}}\,, \tag{37.13}$$

where $V = |\Omega|$ and $\epsilon' = \frac{E}{3M}$, as in (31.18). This is again the Maxwell distribution of the velocities, this time expressed in a somewhat different way. We can now define the entropy $S = S(E,V)$ of the gas as the maximal value of $\int_{\mathbf{R}^3\times\mathbf{R}^3} -\rho\log\rho\,\frac{dx\,dv}{C}$ for the given constraints. Using (37.13) and calculating the integral, one obtains

$$S = S(E,V) = \log\left[\frac{V}{C} \left(\frac{E}{M} \right)^{\frac{3}{2}} \right] + \text{const.} \tag{37.14}$$

It is important to point out that this formula is not quite correct from the point of view of physics in that it does not give the correct dependence on M. (The dependence on V and E is correct, modulo the choice of normalization.) This is due to the fact that one should really do the calculation in the phase space (x,p), where $p = mv$. There are several reasons why the phase space (x,p) is the right object here, rather then the space (x,v), which we will discuss at some point. However, for our purposes of pointing out the analogy of our 2D Euler calculation with classical Statistical Mechanics the above calculation in (x,v) is adequate.

The above principle of entropy maximization can be used also in the cases when there are additional conserved quantities. We just maximize the entropy subject to the constraint that conserved quantities have certain given values.

[193] For the calculation use the identities

$$\int_{\mathbf{R}^3} e^{-\alpha\frac{1}{2}|x|^2}\,dx = \frac{1}{(2\pi\alpha)^{\frac{3}{2}}} \tag{37.11}$$

and

$$\int_{\mathbf{R}^3} \frac{1}{2}|x|^2 e^{-\alpha\frac{1}{2}|x|^2}\,dx = -\frac{d}{d\alpha}\ \frac{1}{(2\pi\alpha)^{\frac{3}{2}}}\,. \tag{37.12}$$

For example, in the hypothetical case when the vessel containing the gas would be axi-symmetric (even from the point of view of the atoms) so that the quantity I_3 given by (31.5) is conserved, we can calculate the statistical equilibrium distribution $\rho(x, v)$ by simply maximizing $H(\rho)$ subject to the constraints of fixed energy E and the x_3- component of the momentum $I_3 = I_3(\rho)$. The reader is encouraged to calculate the resulting distribution as an exercise.

Historical notes

The notion of entropy was first introduced by by R. Clausius in 1865. The original definition was phenomenological:

$$dS = \frac{dQ}{T}\,, \tag{37.15}$$

where dS is the (infinitesimal) change of entropy, dQ is the heat delivered to the medium and T is the temperature. The connection between the entropy and (the logarithm of) the number of states of the system was discovered by L. Boltzmann in 1870s, who pioneered the type of calculations discussed above and introduced a fundamental equation[194] for the evolution of the density $\rho(x, v)$ (which we did not discuss yet).

[194]The Boltzmann equation, in today's terminology. In our notation it reads $\rho_t + v\nabla_x\rho + F\nabla_v\rho = Q(\rho, \rho)$, although it is usually written in the phase-space variables (x, p).

38

12/12/2011

38.1 Turkington's entropy function for 2d Euler

We return to the 2d incompressible Euler with the initial condition

$$\omega_0 = a\chi_A + b\chi_B, \qquad B = \Omega \setminus A\,. \tag{38.1}$$

We assume $b \le a$. (We remark that all the principles we are discussing work also in the general case $\omega_0 \in L^\infty(\Omega)$ with not-so-hard adjustments. We work with (38.1) to focus on the main ideas in the simplest non-trivial situation.) Let

$$m = \frac{1}{|\Omega|}\int_\Omega \omega_0\,dx = a\frac{|A|}{|\Omega|} + b\frac{|B|}{|\Omega|}\,. \tag{38.2}$$

From our considerations in lecture 33 we see that

$$\mathcal{O}_{\omega_0} = \{\omega \in L^\infty(\Omega)\,,\, b \le \omega \le a\,,\, \frac{1}{|\Omega|}\int_\Omega \omega\,dx = m\}\,. \tag{38.3}$$

If Ω is a torus $\mathbf{R}^2/\Lambda$, we must take $m = 0$. For $\Omega \subset \mathbf{R}^2$ we will assume that Ω is simply connected for simplicity. In this case the stream function is determined by ω from

$$\Delta\psi = \omega, \qquad \psi|_{\partial\Omega} = 0\,. \tag{38.4}$$

The modification of the entropy function $s(\omega)$ in (36.13) to this situation is easily seen to be

$$s(\omega) = -\rho_a \log\rho_a - \rho_b\log\rho_b, \qquad \omega = a\rho_a + b\rho_b, \qquad 0 \le \rho_a, \rho_b \le 1\,, \tag{38.5}$$

and the "entropy of the function ω" is

$$\mathcal{S}(\omega) = \frac{1}{|\Omega|}\int_\Omega s(\omega)\,dx\,. \tag{38.6}$$

While the function $s(\omega)$ looks quite natural, it is not the only feasible candidate. Note that $s(\omega)$ is simply the "information entropy" $-p_a\log p_a - p_b\log p_b$ of the probability distribution (p_a, p_b) of the process where we choose a value of vorticity to be either a (with probability p_a) or b (with probability p_b), such that the mean value of the chosen vorticity is ω. This corresponds to "mixing the value ω" using only a, b. In the paper quoted at the end of lecture 36, B. Turkington argues that one should really think of ω as being "mixed" from all values between b and a, since – roughly speaking – in the process of mixing we can use some of the values which already have been mixed before. This corresponds to the idea that there can be many different scales on which the values b, a are

mixed, and not just the scale of the "small boxes" we used in lecture 36. One way to capture this idea mathematically is to replace the function $s(\omega)$ above by the following function

$$\begin{aligned}\tilde{s}(\omega) \quad = \quad & \sup\{\, \tfrac{1}{a-b}\int_b^a -\rho(y)\log\rho(y)\,dy\,, \\ & 0\le\rho\le 1,\, \tfrac{1}{a-b}\int_b^a \rho(y)\,dy = 1,\ \tfrac{1}{a-b}\int_b^a y\rho(y)\,dy = \omega\ \}.\end{aligned} \tag{38.7}$$

We are now allowing all "probability densities" $\rho(y)\frac{dy}{a-b}$ in (b,a) which give ω as their mean value and among those we choose the one with the maximal entropy. The value of the entropy at $\omega\in(b,a)$ is $\tilde{s}(\omega)$.

Note that $\tilde{s}(\omega)\to-\infty$ as $\omega\to a_-$ or $\omega\to b_+$, in contrast with the function $s(\omega)$ defined by (36.13). (We considered $s(\omega)$ only for $b=-a$, but the generalization to any $b<a$ is straightforward.) The states ω maximizing the entropy function

$$\tilde{\mathcal{S}}(\omega) = \frac{1}{|\Omega|}\int_\Omega \tilde{s}(\omega)\,dx \tag{38.8}$$

for the constraints of a given energy E and $\omega\in\mathcal{O}_{\omega_0}$ will satisfy the analogue of (36.17)

$$\tilde{s}'(\omega) + \beta M\psi - \mu = 0\,, \tag{38.9}$$

where this time we took μ to have the same dimension as ω. To express ω from (38.9) so that we get an equation of the form $\omega = \Delta\psi = F(\beta M\psi-\mu)$, we need to calculate the inverse function to the function $\omega\to\tilde{s}'(\omega)$. This can be done as follows. Using lagrange multipliers, one can see that the density $\rho(y)$ in (38.7) which achieves the maximal value of $\int_b^a -\rho\log\rho$ for the given constraints is

$$\rho(y) = e^{\alpha y+\gamma} \tag{38.10}$$

for some $\alpha,\gamma\in\mathbf{R}$. The values of α and γ are determined by the constraints. The calculation can be simplified by the following standard trick used in Statistical Mechanics. Let us set

$$Z(\alpha) = \frac{1}{a-b}\int_b^a e^{\alpha y}\,dy \;=\; \frac{e^{\alpha a}-e^{\alpha b}}{\alpha(a-b)} \tag{38.11}$$

and

$$f(\alpha) = \log Z(\alpha)\,. \tag{38.12}$$

We note that the function $f(\alpha)$ is convex[195] in $\mathbf{R}$, with

$$f'(\alpha)\to a,\quad \alpha\to\infty, \qquad \text{and } f'(\alpha)\to b,\quad \alpha\to-\infty. \tag{38.13}$$

The constraints can be expressed as

$$f'(\alpha) = \omega, \qquad f(\alpha) = -\gamma\,, \tag{38.14}$$

[195]This is a general property of the functions $\phi(\xi)$ which can be written as $\phi(\xi) = \log\int_X e^{\xi x}\,d\nu(x)$ for some probability measure ν.

and we have

$$\tilde{s}(\omega) = f(\alpha) - \alpha f'(\alpha), \qquad f'(\alpha) = \omega\,. \tag{38.15}$$

Alternatively, we have

$$\tilde{s}(\omega) = \inf_{\alpha} f(\alpha) - \omega\alpha. \tag{38.16}$$

This is an example of a *Legendre transformation*[196] An important fact concerning the Legendre transformation is the the relation

$$\omega = f'(\alpha) \tag{38.17}$$

is inverted as

$$\alpha = -\tilde{s}'(\omega)\,. \tag{38.18}$$

Vice versa, the relation (38.18) is inverted by (38.17). Recalling (38.9), we see that to write the equation for the stream function in the form $\Delta\psi = F(\beta M\psi - \mu)$, we do not have to calculate $\tilde{s}(\omega)$, it is enough to have $f(\alpha)$. In terms of $f(\alpha)$, the equation is

$$\omega = \Delta\psi = f'(\beta M\psi - \mu)\,. \tag{38.19}$$

This equation describes the conjectured "end-states" in the model of B. Turkington (in the special case (38.1)).

To make a comparison with (36.19), let us consider the case $b = -a$. Then

$$f(\alpha) = \log \frac{\sinh \alpha a}{\alpha a} \tag{38.20}$$

and

$$f'(\alpha) = aL(\alpha a)\,, \tag{38.21}$$

where $L(u) = \coth u - \frac{1}{u}$ is the so-called *Langevin function.* We note that $L(-u) = -L(u), L(0) = 0, L'(0) = 1, -1 < L(u) < 1,\ L'(u) > 0,\ L(u) \to \pm 1$ as $u \to \pm\infty$. Hence the function is somewhat similar to $\tanh u$ or the function F in (33.20), the difference being a slower approach to the limiting values when $u \to \pm\infty$.

To compare the three models given by $F, \tanh, L$ we note that each approaches the limiting values at $\pm\infty$ more slowly than the previous one, and hence predicts more "mixing" than the previous one. (As above, F is defined by (33.20).)

The conjecture that all solutions will approach the "end-states" predicted by any of these models is probably too optimistic. Many solutions will probably "get stuck" in regimes discussed in lecture 35.

[196]The usual definition is as follows. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a strictly convex C^1 function. Then the map $x \to f'(x)$ (which can be thought of as a map from $\mathbf{R}^n$ to its dual space, also identified with $\mathbf{R}^n$) is injective and maps $\mathbf{R}^n$ onto set $\mathcal{O} \subset \mathbf{R}^n$. (If f has uniform super-linear growth, then $\mathcal{O} = \mathbf{R}^n$.) Let $g(y) = \sup_x(yx - f(x))$. Then the map $x = g'(y)$ inverts the map $y = f'(x)$. The function g is called the Legendre transform of the function f. It comes up in many situations.

38.2 Open problems related to entropy maximization

Equations (38.19) and (36.19) may have many solutions and not all of them correspond to ω which maximizes the entropy $\tilde{S}(\omega)$ (resp. $\mathcal{S}(\omega)$) for the given constraints. In fact, a complete characterization of the maximizers seems to be an open problem. In the case $m = 0$ and for small values of energy all the maximizers should be close to the space of eigenfunctions corresponding to the first eigenvalue of the laplacian, as can be seen from the linearization of the equation. If $\Lambda \subset \mathbf{R}^2$ is the lattice generated by $(l, 0), (0, 1)$ with $l > 1$, $\Omega = \mathbf{R}^2/\Lambda$, and $m = 0$ (see (38.3)), it is quite conceivable that every $\omega \in \mathcal{O}_{\omega_0}$ which maximizes the entropy depends only on x_1, which would mean that the "maximal entropy" theories predict that in the limit the flow will always approach shear flows in the x_2 parallel to e_2. This is obviously not the case for shear flows parallel to e_1 (which are steady state solutions), but one might say that these would be exceptional cases. But what happens if we take a small perturbation of a shear flow parallel to e_1? Will the flow eventually end up being weakly* close to a shear flow parallel to e_2? That seems to be quite unlikely, although it has not been ruled our rigorously, as far as I know.[197] We can also study the functions

$$S = S(E, m) = \sup\{\mathcal{S}(\omega),\, \omega \in \mathcal{O}_{\omega_0},\, E(\omega) = E\}\,, \tag{38.22}$$

where $\mathcal{S}(\omega)$ is defined in terms of $s(\omega)$ in (38.5), (38.6), or the analogous functions

$$\tilde{S} = \tilde{S}(E, m) = \sup\{\tilde{\mathcal{S}}(\omega),\, \omega \in \mathcal{O}_{\omega_0},\, E(\omega) = E\}\,, \tag{38.23}$$

with $\tilde{S}$ defined by (38.8). The entropy functions arising through the canonical formalism of Statistical Mechanics (which we have not used or defined yet) are concave in E. Are the functions (38.22) or (38.23) concave in E? It seems this questions is open.

Let us consider how the function $E \to S(E) = S(E, 0)$ should look for a torus Ω (no boundary). First of all, it is not difficult to see that the energy function $\omega \to E(\omega)$ is bounded on $\mathcal{O}_{\omega_0}$. Let us denote the upper bound by $E_{\max}$. Since $\mathcal{O}_{\omega_0}$ is weakly* compact and $E(\omega)$ is weakly* continuous in ω, the maximum $E_{\max}$ is attained on some $\omega_1 \in \mathcal{O}_{\omega_0}$. One can show that ω_1 is a steady-state solution of the Euler's equation.[198] The states which maximize the energy on $\mathcal{O}_{\omega_0}$ are quite special and when we produce them, we expect we cannot really do any "mixing", as this would be wasting possibilities for increasing the energy. Therefore we expect that the entropy at $E_{\max}$ will either vanish (for the function $S(E, 0)$) or be $-\infty$ (for the function $\tilde{S}(E, 0)$. On the other hand, for $E = 0$ we can take $\omega = 0$ in (38.22) or (38.23), which has the maximal entropy of any

[197] Recall that our convention concerning flows on tori is that the velocity field is given by a stream function, $u = \nabla^\perp \psi$, and hence the mean velocity of the flow is always vanishes.

[198] In fact, ω_1 has some relatively strong stability properties, as the evolution preserves both $\mathcal{O}_{\omega_0}$ and the energy E. The is the main idea behind the so-called Arnold stability criterion, which we will discuss later.

function on $\mathcal{O}_{\omega_0}$, due to Jensen's inequality. Hence we expect that $E \to S(E,0)$ and $E \to \tilde{S}(E,0)$ will be decreasing functions of E, defined on the finite interval $[0, E_{\max}]$. This is in contrast with the entropy function $S(E,V)$ of the ideal gas we calculated in lecture 37 (see (37.14)) which is increasing in E (and concave in E, as expected). The thermodynamical formula (37.15) motivates the definition of temperature T in Statistical Mechanics as[199]

$$\frac{1}{T} = \frac{\partial S}{\partial E}\,. \tag{38.24}$$

We see that in the case of the torus $\Omega = \mathbf{R}^2/\Lambda$ and ω_0 given by (38.1), the temperature will be negative (assuming our conjectures above are correct). The existence of state with negative temperature in statistical mechanics on 2d Euler's equation was predicted already in 1949 by L. Onsager in the paper quoted at the end of lecture 36.

Let us now look at a simply connected case $\Omega =$ disc in $\mathbf{R}^2$ and $b = 0, a = 1$ in (38.1), with $b < m < a$ in (38.2). In this case it is not hard to see that the energy on $\mathcal{O}_{\omega_0}$ cannot reach zero, but it attains a minimal value $E_{\min} > 0$ on $\mathcal{O}_{\omega_0}$. It also attains a finite maximal value $E_{\max}$ on $\mathcal{O}_{\omega_0}$, similarly as in the case of the torus discussed above. The energy minimizing configuration should be achieved when all the vorticity $+1$ is concentrated in a strip near the boundary of the disc Ω, whereas the energy maximizing configuration should be attained when all the vorticity $+1$ is in a disc centered at the center of the disc Ω. These configurations should in fact be unique (for the disc), and hence we expect $S(E_{\min}) = S(E_{\max}) = 0$ and $\tilde{S}(E_{\min}) = \tilde{S}(E_{\max}) = -\infty$ in this case. The functions $S, \tilde{S}$ will then be increasing up to $\overline{E} = E(\omega_m)$, where $\omega_m \equiv m$, before becoming decreasing. Therefore the temperature defined by (38.24) should be positive in $(E_{\min}, \overline{E})$ and negative in $(\overline{E}, E_{\max})$. The maximal entropy achieved at $\overline{E}$ will be $S(\overline{E}) = s(m)$ and $\tilde{S}(\overline{E}) = \tilde{s}(m)$ respectively.

It would be interesting to prove the above picture rigorously. As far as I know, it has not been done.

[199] Of course, this temperature has nothing to do with the usual temperature of the fluid.

39

12/14/2011

39.1 Kraichnan's 2d turbulence theory

We have seen in lecture 29 that the dissipation of energy characteristic for 3d turbulence theory (lecture 27) must have quite different features in dimension $n = 2$, and our study of the long-time behavior of the solution of the 2d incompressible Euler equation in the last few lectures confirmes this expectation. The solutions of the 2d equations become "chaotic" in a quite different sense than the 3d solutions. We expect to see some local "disorder" and randomness at the level of the vorticity field ω, but at the same time the integrals I_f in (32.3) are conserved by the equation, and the energy of the solutions survives taking weak* limits. As a consequence we expect the appearance of the large-scale structures, a purely 2d phenomena which has no analogy in 3d turbulence.

Let us now consider the 2d Navier-Stokes equation with a forcing term, written in terms of the vorticity $\omega(x,t)$.

$$\omega_t + u\nabla\omega - \nu\Delta\omega = g(x,t) \tag{39.1}$$

in a 2d domain Ω. For simplicity we will consider the case $\Omega = \mathbf{R}^2/L\mathbf{Z}^2$, a "periodic box" of size L. We can write

$$\omega(x,t) = \sum_{k\in 2\pi\mathbf{Z}^2/L} \hat{\omega}(k,t)e^{ikx}, \tag{39.2}$$

so that we have

$$\frac{1}{|\Omega|}\int_\Omega |\omega(x)|^2\,dx = \sum_k |\hat{\omega}(k)|^2\,. \tag{39.3}$$

We will always assume $\hat{\omega}(k)|_{k=0} = 0$. The corresponding velocity fields $u(x)$ will also be assumed assumed to satisfy $\hat{u}(k)|_{k=0} = 0$, corresponding to the condition $\int_\Omega u(x)\,dx = 0$ (which is preserved in the time evolution by Euler's equations). The energy per unit volume of the velocity field in "eddies" of the sizes $l \in (1/\kappa_1, 1/\kappa_2)$ is given by

$$\sum_{\kappa_1\le|k|<\kappa_2} |\hat{u}(k)|^2 \tag{39.4}$$

and we will write it as

$$\int_{\kappa_1}^{\kappa_2} E(\kappa)\,d\kappa\,. \tag{39.5}$$

In this setting the measure $E(\kappa)\,d\kappa$ is of course discrete, but as is usually done (and as we did in lecture 27) we will deal with $E(\kappa)$ as if it were a continuous function (which is the case in the limit $L\to\infty$).[200]

[200] You can think of replacing $E(\kappa)$ by $\int E(\kappa-\kappa')\phi(\kappa')d\kappa'$ where ϕ is a mollifier with support spread about a suitable distance.

The Fourier components of the function $x \to g(x,t)$ are denoted by $\hat{g}(k,t)$ (or just $\hat{g}(k)$ if the dependence on t is not emphasized). We assume that $\hat{g}(k)$ are non-zero only for frequencies of size close to some fixed size κ_{in}. Assume that ν is very small. As is the 3d case, we can ask the following natural questions:

1. What is the smallest length scale (or the highest spatial frequencies) observed in $\omega(x,t)$?

2. How does the function $E(\kappa)$ look like?

The basic assumption in the 3d case which enabled us to make reasonable conjectures concerning these questions was that the rate of energy dissipation in the fluid per unit mass ϵ (see (26.18)) was independent of ν. See lecture 26. In view of our considerations in lecture 29, this is no longer a good assumption in dimension $n = 2$. Following R. Kraichnan[201], we will replace this assumption by an assumption concerning the (mean) rate of dissipation of the enstrophy $\int_\Omega \frac{1}{2}\omega^2\, dx$. From (39.1) we have

$$\frac{\partial}{\partial t}\frac{1}{2}\omega^2 + \operatorname{div}[u\frac{1}{2}\omega^2] - \nu\Delta\frac{1}{2}\omega^2 = -\nu|\nabla\omega|^2 + g\omega \ . \tag{39.6}$$

Let

$$\eta = \overline{\nu|\nabla\omega|^2}\,, \tag{39.7}$$

where the notation $\overline{f}$ means a suitable average, as in lecture 26. Kraichnan suggested to replace the (3d) principle P in lecture 26 by the following assumption:

(P_2) In many situations, once the Reynolds number is high, the quantity η is quite independent of ν.

We will see that one has to take this principle with many caveats, and even more caution is necessary than in the case of the 3d principle (P) is lecture 26. Nevertheless, if applied correctly, the principle seems to contain some part of the truth.

Let us now combine dimensional analysis and P_2 to get feasible conjectures for answers to questions 1 and 2 above. Let λ denote the minimal length scale from question 1 (the 2d Kolmogorov length). The physical dimensions of the relevant quantities are

$$\begin{array}{rll} \lambda & \ldots\ldots\ldots & L \\ \omega & \ldots\ldots\ldots & \frac{1}{T} \\ \nu & \ldots\ldots\ldots & \frac{L^2}{T} \\ \nabla\omega & \ldots\ldots\ldots & \frac{1}{LT} \\ \eta & \ldots\ldots\ldots & \frac{1}{T^2} \\ \epsilon & \ldots\ldots\ldots & \frac{L^2}{T^3} \\ \kappa & \ldots\ldots\ldots & \frac{1}{L} \\ E(\kappa) & \ldots\ldots\ldots & \frac{L^3}{T^2}\,. \end{array} \tag{39.8}$$

[201] The Physics of Fluids, Vol. 10., No. 7, July 1967

Here ϵ has the same meaning as in lecture 27: the average rate of energy dissipation per unit mass of fluid.

If we assume that λ and $E(\kappa)$ depend only on η and $E(\kappa)$ depends only on η and κ, the dimensional analysis shows

$$\lambda = c\eta^{-\frac{1}{6}}\nu^{\frac{1}{2}} \tag{39.9}$$

and

$$E(\kappa) = c\eta^{\frac{2}{3}}\kappa^{-3}\ , \tag{39.10}$$

where the value of the dimensionless constant c can change from line to line as usual. Equation (39.10) is assumed to be true in the "inertial range" $\kappa \in (\kappa_{\rm in}, \kappa_{\rm max})$, with $\kappa_{\rm max} \sim 1/\lambda \sim c\eta^{\frac{1}{6}}\nu^{-\frac{1}{2}}$. As a consistency check, we can calculate the rate of enstrophy dissipation over the inertial range from (39.9) and (39.10). We have

$$\int_{\kappa_{\rm in}}^{\kappa_{\rm max}} \nu E(\kappa)\kappa^4\, d\kappa \sim c\eta^{\frac{2}{3}}\eta^{\frac{2}{6}} \sim c\eta\,, \tag{39.11}$$

as should be the case, where we have used $\frac{1}{|\Omega|}\int_\Omega |\nabla\omega|^2\, dx \sim \int_0^\infty E(\kappa)\kappa^4\, d\kappa$.

Let us now look at the rate of energy dissipation per unit mass due to the dissipation in the inertial range:

$$\epsilon_{\rm inertial} = \int_{\kappa_{\rm in}}^{\kappa_{\rm max}} \nu E(\kappa)\kappa^2\, d\kappa \sim c\nu\eta^{\frac{2}{3}}\log(\frac{\eta^{\frac{1}{6}}}{\kappa_{\rm in}\nu^{\frac{1}{2}}})\ . \tag{39.12}$$

If η stays fixed, this quantity clearly approaches zero as $\nu \to 0_+$. This is consistent with our considerations is lecture 29. We see that the picture above cannot be complete, as the inertial range does not account for the energy dissipation which is necessary to have a steady influx of energy at frequencies $\sim \kappa_{\rm in}$. What happens with the excess energy? If our hypotheses above are correct, the excess energy cannot go to high frequencies (where it would be dissipated as in 3d) the only way it can go is to the low frequencies. For this to be possible, we must have enough "space" at the low frequencies for the energy to go to. If we asume that the dimension L of our periodic box is very large, with $\kappa_0 = \frac{1}{L} << \kappa_{\rm in}$, one can imagine, following Kraichnan, the situation where the inertial range regime is established, and after that the energy gradually cascades down to the low frequencies, in a striking reversal of the 3d situation. During the time before this cascade eventually reaches the lowest frequecies, a "quasi-steady" state is established: there is a lowest "active" frequency depending on time, $\kappa(t) > \kappa_0$, such that the frequencies below it contain only a small amount of energy. The energy in the range $(\kappa_1(t), \kappa_{\rm in})$ where $\kappa_1(t)$ is slightly larger than $\kappa(t)$ is distributed according to the Kolmogorov law:

$$E(\kappa) \sim \epsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}, \qquad \kappa \in (\kappa_1(t), \kappa_{\rm in}) \tag{39.13}$$

an in this interval of frequencies the energy moves downward. For frequencies $\kappa > \kappa_{\text{in}}$ we have the steady inertial range characterized by (39.10). The rate at which $\kappa(t)$ decreases can be estimated from energy conservation:

$$\int_{\kappa(t)}^{\kappa_{\text{in}}} E(\kappa)\, d\kappa \sim t\varepsilon\,, \tag{39.14}$$

where we have ignored the difference between $\kappa(t)$ and $\kappa_1(t)$. This gives

$$\kappa(t) \sim \epsilon^{-\frac{1}{2}} t^{-\frac{3}{2}}\,. \tag{39.15}$$

If $L = \infty$ so that $\kappa_0 = 0$, this process can continue indefinitely. If L is finite, then $\kappa_0 > 0$ and we can consider the following scenarios:

1. Once $\kappa(t)$ reaches κ_0, the energy starts accumulating in the low modes $\kappa \sim \kappa_0$ and the energy in these modes will become very large. This will of course have a ripple effect on the higher modes: they will also gradually receive more energy than in the quasi-steady state above, until the total dissipation reaches a level which is enough to dissipate the energy being delivered to the system. The arguments above do not predict what the final (and presumably steady) distribution of energy $E(\kappa)$ will be. There is a result from the theory of attractors (which we have not discussed yet), which seems to be relevant for the considerations concerning this final stage. If λ is the minimal length scale in the flow, we expect the flow will have $\sim (\frac{L}{\lambda})^2$ degrees of freedom. That means that in a computer simulation we would need roughly $\sim (\frac{L}{\lambda})^2$ modes to capture the flow. If (39.9) was correct also for the ultimate equilibrium state, it would mean that the number of degrees of freedom of the system is $\sim \frac{\eta^{\frac{1}{3}} L^2}{\nu}$. However, the theory of attractors suggest[202] that it is in fact higher, of order $\sim c\nu^{-\frac{4}{3}}$ which presumably can be explained by the changes in the asymptotics after the downward cascade reaches the lowest frequency κ_0.

2. If there is a mechanism for removing energy in the low modes with frequencies $\kappa \sim \kappa_0$, the above "quasi-steady" situation can be stable: the energy will cascade from κ_{in} down to $\sim \kappa_0$ where is will be dissipated, and (39.13) together with (39.10) (for $\kappa > \kappa_0$) will describe the energy distribution.

The scenarios described above are not really universally accepted, and there are not many rigorous results concerning these topics. Opinions of experts concerning the validity of these conjectures are not uniform. In any case, it seems to be clear that the phenomenological formulae (1.1) and (1.4) will not be valid in dimension $n = 2$. Based on the above it is reasonable to conjecture that for the flows in dimensions $n = 2$ the constants c in those formulae will approach 0 as the Reynolds number increases to ∞. The reasoning behind formulae (1.1) and (1.4) did not explicitly use that we were in dimension $n = 3$, and therefore the approximate correctness of the formulae for $n = 3$ was in part a lucky coincidence.

[202] P. Constantin, C. Foias, R. Temam, On the Dimension of Attractors in Two-Dimensional Turbulence, Physica D 30, 1988, 284–296.

40

1/18/2012

40.1 Stability

We have used quite a few times the phrase "the flow becomes unstable", without really defining this terminolgy precisely in mathematical terms. Let us discuss this issue in some more detail, still somewhat loosely. In the next lecture we will discuss a model equation by E. Hopf and for this model we will be able to do explicit calculations. For the Navier-Stokes equation the corresponding calculations more involved, and without using numerical simulation on computers they can be done only in a few simple cases and not without considerable effort. Our approach will be to describe first only qualitatively what one observes for the "real flows", so that we can appreciate the model of Hopf, where some of these effects can be seen in a situation where calculations are not difficult.

Example 1: The Taylor-Couette flow[203]

We have briefly discussed this flow in lecture 22. Let $0 < R_1 < R_2$ and consider the domain

$$\Omega = \{x \in \mathbf{R}^3,\, R_1 < \sqrt{x_1^2 + x_2^2} < R_2\,,\, 0 < x_3 < L\}. \tag{40.1}$$

Let (r, θ, z) be the cylindrical coordinates[204], and let $e_\theta = e_\theta(x)$ be the vector with cartesian coordinates $(-\frac{x_1}{r}, \frac{x_2}{r}, 0)$, often denoted by $\frac{\partial}{r\partial\theta}$. We consider the Navier-Stokes equation[205]

$$u_t + u\nabla u + \frac{\nabla p}{\rho} - \nu\Delta u = 0, \qquad \operatorname{div} u = 0 \tag{40.2}$$

in Ω with the boundary conditions

$$u(x) = \begin{cases} U e_\theta & \text{when } x \in \partial\Omega \text{ and } r = \sqrt{x_1^2 + x_2^2} = R_1, \\ 0 & \text{when } x \in \partial\Omega \text{ and } r > R_1. \end{cases} \tag{40.3}$$

In other words, we assume that the inner cylinder rotates with angular velocity ω given by $U = \omega R_1$, while the rest of the boundary is stationary. We assume that $L >> R_2$, so that the cylinder is long.[206]

[203] See for example the book: P. Chossat and G. Ioos, The Couette-Taylor Problem, Springer, Berlin, (1994).

[204] given as usual by $x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z$

[205] As usual, we assume that the density ρ is constant, and with a suitable choice of units we can assume $\rho = 1$.

[206] Note that boundary condition (40.3) is not continuous at the intersection of the "lids" and the inner cylinder. This probably does not cause any serious problems, although I am not sure whether the question was rigorously investigated.

The boundary conditions (40.3) are imposed in experiments, but for mathematical analysis one imposes slightly different conditions. Namely, one assumes that u is defined in the domain

$$\tilde{\Omega} = \{x \in \mathbf{R}^3,\ R_1 < r < R_2\}, \qquad r = \sqrt{x_1^2 + x_2^2} \tag{40.4}$$

with

$$u(x) = \begin{cases} Ue_\theta & \text{for } r = R_1 \\ 0 & \text{for } r = R_2 \end{cases} \tag{40.5}$$

and the additional condition

$$u(x_1, x_2, x_3 + L) = u(x_1, x_2, x_3)\,, \qquad x \in \tilde{\Omega}\,. \tag{40.6}$$

It turns out that for $L >> R_2$ the conditions (40.5), (40.6) are, for many purposes, a good approximation of (40.3), up to possible disturbances near the ends $x_3 = 0,\ x_3 = L$. [207]

The equation (40.2) in $\tilde{\Omega}$ with the boundary conditions (40.5),(40.6) has a simple steady state solution

$$\overline{u}(x) = \left(\frac{a}{r} + br\right) e_\theta \tag{40.7}$$

where $a, b \in \mathbf{R}$ are easily calculated from the boundary conditions.

In experiments this flow is only observed for low velocities $U = \omega R_1$, where the notion of "low" depends also on R_1, R_2. The right way to formulate the stability condition for $\overline{u}$ is in terms of the so-called *Taylor number*[208]

$$\mathrm{Ta} = \frac{\omega^2 R_1 (R_2 - R_1)^3}{\nu^2}\,, \tag{40.8}$$

which is similar to (but not the same as) the square of the Reynolds number. The definition takes into account the role of the centrifugal force, a factor which makes the situation different from, say, the simple shear flow $\overline{u}(x) = (U\frac{x_3}{L}, 0, 0)$ in the (infinite) channel $\{x = (x_1, x_2, x_3);\ 0 < x_3 < L\}$. The simple flow (40.7) is observed up to Taylor numbers of about 1700. For higher Taylor number the trivial solution is still present, but is unstable, and experimentally one observes different solutions.

The loss of stability is analyzed mathematically by searching the Navier-Stokes solutions in the form

$$u(x,t) = \overline{u}(x) + v(x,t)\,. \tag{40.9}$$

[207] This is confirmed experimentally. I am not sure if a rigorous mathematical proof of this statement is available.

[208] It is interesting to compare the definition with the Reynolds number. Since we have two lengths R_1, R_2, in addition to U or ω, there is not a unique way to define the Reynolds number. For example, when R_1 is large in relative to $R_2 - R_1$, it is reasonable to choose the Reynolds number as $\mathrm{Re} = \frac{U(R_2-R_1)}{\nu} = \frac{\omega R_1 (R_2-R_1)}{\nu}$.

The perturbation v vanishes at the boundary and, in the case when condition (40.6) is considered, also satisfies this condition. The equations for $v(x,t)$ are

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + v\nabla v + \frac{\nabla q}{\rho} - \nu\Delta v = 0, \qquad \operatorname{div} v = 0\,. \tag{40.10}$$

Mathematically, one can define the notion of stability in several ways. One notion of the stability of $\overline{u}$ stability of $\overline{u}$ is that any solution of v (with any sufficiently regular initial condition $v(x,0) = v_0(x)$) converges to 0 as $t \to \infty$.

Usually one first studies the so-called *linearized stability*, which amounts to studying the linear part of equation (40.10),

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + \frac{\nabla q}{\rho} - \nu\Delta v = 0, \qquad \operatorname{div} v = 0\,. \tag{40.11}$$

The idea is that for small perturbation the term $v\nabla v$ can be neglected. We can write this linear equation as

$$v_t = Lv\,, \qquad \operatorname{div} v = 0\,. \tag{40.12}$$

where $L = L(\overline{u})$ a linear operator.

We can compare this equation with a (finite dimensional) ODE for $x(t) = (x_1(t), \dots, x_m(t))$

$$\dot{x} = Ax\,, \tag{40.13}$$

where A is an m by m matrix. The general solution of (40.13) is

$$x(t) = e^{tA}\overline{x}, \qquad x(0) = \overline{x}. \tag{40.14}$$

Let $\sigma(A) \subset \mathbf{C}$ be the spectrum of the matrix A. For a finite matrix this is of course the same as the set of all the eigenvalues of A. We know that if

$$\sigma(A) \subset \{z \in \mathbf{C},\, \operatorname{Re} z < 0\} \tag{40.15}$$

then all solutions of (40.13) converge exponentially to zero. If we have $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda > 0$, we have an exponentially growing solution, and for $\lambda \in \sigma(A)$ with $\operatorname{Re}\lambda = 0$ we have a solution which does not converge to zero.

Let us consider a non-linear ODE

$$\dot{x} = f(x)\,, \qquad x = (x_1, \dots, x_m), \quad f\colon \mathbf{R}^m \to \mathbf{R}^m, \tag{40.16}$$

where f is assumed to be smooth. Let $\overline{x}$ be an equilibrium point, i. e.

$$f(\overline{x}) = 0\,. \tag{40.17}$$

The linearized equation at $\overline{x}$ is

$$\dot{y} = Df(\overline{x})y\,, \tag{40.18}$$

where $Df(\overline{x})$ is the jacobi matrix $\frac{\partial f_i}{\partial x_j}(\overline{x})$. In the theory of ODE it is proved that if

$$\sigma(Df(\overline{x})) \subset \{z \in \mathbf{C}\,,\ \mathrm{Re}\, z < 0\} \tag{40.19}$$

then the solutions $x(t)$ of (40.16) with $x(0)$ close to $\overline{x}$ converge exponentially to $\overline{x}$ as $\to -\infty$. Hence the linearized (exponential) stability at a rest point $\overline{x}$ implies the local non-linear (exponential) stability of $\overline{x}$.

A similar result is true also in the context of the Navier-Stokes equations, although one has to be more careful with the formulation, as the notion of "close" can have many meanings in function spaces. At this stage we will not go into technical details, and simply state that the local (exponential) stability of a general steady solution $\overline{u}$ is decided by the spectrum of the linearized operator L, similarly to the finite dimensional case. If

$$\sigma(L) \subset \{z \in \mathbf{C}\,,\ \mathrm{Re}\, z < 0\} \tag{40.20}$$

then the steady solution $\overline{u}$ is stable. The equation determining the spectrum is

$$-\nu\Delta v + \overline{u}\nabla v + v\nabla\overline{u} + \frac{\nabla q}{\rho} = \lambda v, \qquad \mathrm{div}\, v = 0, \qquad \text{in } \Omega, \tag{40.21}$$

with the corresponding boundary conditions at $\partial\Omega$. In the case with $\tilde{\Omega}$ considered above the boundary conditions are $v = 0$ at $\tilde{\Omega}$ and $v(x_1, x_2, x_3 + L) = v(x_1, x_2, x_3)$. The analysis of the spectrum in the context of (40.21) is quite non-trivial even in simple geometries, such as in the case of the solution (40.7). Nevertheless, for (40.7) the solution can be reduced to an ODE by the method of the separation of the variables (since the base solution $\overline{u}$ is independent of θ and z in the cylindrical coordinates). The calculations can be found for example in a famous paper by G. I. Taylor.[209] The calculations show that there is a critical value $\mathrm{Ta}_{\mathrm{crit}}$ of the Taylor number such that for $\mathrm{Ta} < \mathrm{Ta}_{\mathrm{crit}}$ we have

$$\sigma(L) \subset \{z \in \mathbf{C}\,,\ \mathrm{Re}\, z < 0\} \tag{40.22}$$

and at $\mathrm{Ta} = \mathrm{Ta}_{\mathrm{crit}}$ a real eigenvalue is crossing the imaginary axis, and becomes positive for $\mathrm{Ta} > \mathrm{Ta}_{\mathrm{crit}}$. The simple solution $\overline{u}$ becomes unstable, and other stable steady solutions solutions appear. The new stable solution, say, $\tilde{u}$ is more complicated than $\overline{u}$. It is still axi-symmetric, but it is not independent of z. If you do an online image seach for "Taylor vortices", you will get some good pictures of the solution.

In this special case the loss of stability is via a bifurcation to a more complicated steady solution. The original solution is still present, but it looses stability. A cartoon picture of this can be given even in a 1d model. Consider the ODE for a scalar function $x = x(t)$ given by

$$\dot{x} = f(x, \lambda) = -x(x^2 - \lambda)\,. \tag{40.23}$$

[209] "Stability of a Viscous Liquid contained between Two Rotating Cylinders", Philosophical Transactions of the Royal Society of London, 1923.

For any $\lambda \in \mathbf{R}$ we have the trivial solution $\overline{x} = 0$. The linearized "operator" at $\overline{x}$ is

$$Ly = f_x(\overline{x}, \lambda)y = \lambda y \tag{40.24}$$

and $\sigma(L) = \{\lambda\}$. We see that $\overline{x}$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. (The case $\lambda = 0$ is left to the reader as an exercise.) For $\lambda > 0$ two new stable solutions appear

$$x = \pm\sqrt{\lambda}\,, \tag{40.25}$$

and the solutions of (40.23) starting at any $x \neq 0$ will converge to one of those steady solutions.

The situation with the Navier-Stokes solution $\overline{u}$ above is more complicated than this cartoon picture, as in the case when the explicit calculations can be done – the boundary conditions (40.5), (40.6) – the new stable equilibria form a 1d manifold (obtained by shifting one of the solutions along the z-axis), but for the moment we will ignore these more subtle points.

The loss of stability due to a bifurcation to a more complicated steady solution is only one of the several ways in which stability can be lost. Another important way of the stability loss is due to appearance of a periodic solution via the so-called Hopf bifurcation. In this case the steady solution starts oscillating. This will be illustrated on a model equation due to Hopf in the next lecture. The Hopf bifurcation corresponds to situations when the spectrum of the linearized operator crosses the imaginary axis with a pair of non-zero complex-conjugate eigenvalues. The Hopf bifurcation appears to describe for example the loss of stability of a steady flow around the cylinder.[210] In this case the parameters are U (the velocity as $x \to \infty$), R (the radius of the ball) and ν (the kinematic viscosity of the fluid). The behavior of the solution is characterized by the Reynolds number $\mathrm{Re} = \frac{UR}{\nu}$. The Hopf bifurcation of the steady to the so-called von Karman vortex street appears to occur for $\mathrm{Re} \sim 40$, when one can start observing some oscillation of the wake behind the cylinder.[211]

We should emphasize that the general theory of the Hopf bifurcation is somewhat more complicated then what is indicated in the brief comments above. In particular, one has to distinguish the so-called super-critical and sub-critical cases. We will discuss the details later. The above comments pertain mostly to the super-critical case.[212]

[210] In this one has to calculate the corresponding solution and the spectrum of the linearized operator numerically on a computer. The solution appears to be too complicated for a "paper and pencil" calculation.

[211] See for example some of the photographs in the book "An album of Fluid Motion" by M. Van Dyke.

[212] For more details the reader can consult the original 1942 papers of E. Hopf (*Ber. Math.-Phys. Kl. Säch Acad. Wiss. Leipzig, 94, 1-22* and *Ber. Verh. Acad. Wiss. Leipzig Math.-Nat. Kl., 95 (1), 3-22* or various book. The book "Nonlinear Oscillations, Dynamical systems, and Bifurcations of Vector Fields", by J. Guckenheimer and P. Holmes contains an excellent exposition of the topics.

As we increase the Reynolds number (or the Taylor number), the solutions bifurcating from a simple solution can undergo further bifurcations: for example, the "Taylor vortices solution" can bifurcate to an even more complicated steady solution, which in turn can later bifurcate to a periodic solution. Periodic solution can lose stability via a secondary oscillation at a (usually faster) frequency, so that it becomes "doubly periodic", similarly to the function $A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}$. With further increase of the Reynolds number (or the Taylor number) new and new frequencies can appear, so that the solution could resemble a function of the form

$$\sum_{k=1}^{m} A_k e^{i\omega_k t} \tag{40.26}$$

with m frequencies $(\omega_1, \omega_2, \ldots, \omega_m)$.

When the frequencies $(\omega_1, \ldots, \omega_m)$ are linearly independent over the rational numbers and the amplitudes $A_1, \ldots, A_m$ do not vanish, the closure of the trajectory (40.26) will be an $m-$dimensional torus. One can imagine a (deformed) version of such a torus imbedded in the space of the div-free vector fields as an $m-$ dimensional manifold X_m and the solution $u(x,t)$ "winding" around X_m. We can also imagine that the manifold X_m will attract all solutions $u(x,t)$, starting from any (sufficiently regular) initial data. This is essentially the picture proposed by L. Landau[213] in 1944 and also by E. Hopf[214] in 1948, and is often referred to as the Landau-Hopf model of turbulence. One can envisage that for large Reynolds numbers the dimension m will become very large, and such a scenario does have the potential to explain many observations concerning turbulent flows.[215] When computers enabled researchers to perform numerical simulations of dynamical systems, it was discovered that the loss of stability can happen also in other ways (which were quite unexpected at the time), and the so-called *strange attractors* were discovered.[216] It is natural to assume that the Landau-Hopf scenario has to be generalized to incorporate these additional ways in which stability can be lost. This is discussed in the well-known 1971 papers by D. Ruelle and F. Takens,[217] where a rigorous mathematical treatment of some (finite-dimensional) examples of bifurcations to strange attractors is given. A good discussion of the topics discussed in this lecture can also be found in the newer editions of the Landau-Lifschitz's "Fluid Mechanics".

[213] Doklady Akademii Nauk SSSR 44: 339-342.

[214] Communications on Pure and Applied Mathematics 1: 303-322.

[215] It is worth noting that at some point it was not generally believed that the turbulent flow is described by the Navier-Stokes equations, and Landau's insight that turbulence might be simply a dynamical effect of the equations of motions was not universally accepted.

[216] The best know "strange attractor" is probably the *Lorenz attractor*, see Edward N. Lorenz, "Deterministic Nonperiodic Flow", Journal of the Atmospheric Sciences 20 (2): 130141, (1963).

[217] "On the nature of turbulence." Commun. Math. Phys. 20, 167-192(1971) and 23, 343-344(1971).

41

1/20/2012

41.1 E. Hopf's model equation

Today we will explain the main features of the model from the 1948 paper of E. Hopf mentioned in the last lecture. We will closely follow E. Hopf's paper.

We consider 2π-periodic functions and $\mathbf{R}$, which is the same as considering the function on the unit circle S^1. Following Hopf's paper, we will use the following conventions.

For a sufficiently regular $f\colon S^1 \to \mathbf{C}$ we will write its Fourier series as

$$f(x) = \sum_k f_k e^{ikx}\,, \tag{41.1}$$

with

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{ikx}\,dx\,, \qquad k \in \mathbf{Z}\,. \tag{41.2}$$

We define

$$f \circ g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+y)g(y)\,dy\,, \tag{41.3}$$

which is a variant of the usual convolution $f * g(x) = \int f(x-y)g(y)\,dy$.
We denote by $\overline{f}$ the function $\overline{f}(x) = \overline{f(x)}$.
We note that

$$(f \circ \overline{g})_k = f_k \overline{g}_k\,. \tag{41.4}$$

Our equations will be for two (sufficiently regular) function $u, z\colon S^1 \times [0,\infty) \to \mathbf{C}$. We will write $u = u(x,t), z = z(x,t)$. The equations will contain a given function $F\colon S^1 \to \mathbf{C}$, with $F = F(x)$. The specific form of the equations is as follows:

$$\begin{aligned} u_t &= -z \circ \overline{z} - u \circ 1 + \mu u_{xx}\ , \\ z_t &= z \circ \overline{u} + z \circ \overline{F} + \mu z_{xx}\ , \end{aligned} \tag{41.5}$$

where $\mu > 0$ is a parameter analogous to fluid viscosity.

We will make the following additional assumptions

1. The functions u, z are even in x, e. i. $u(-x) = u(x),\ z(-x) = z(x)$. In terms of the Fourier coefficients this means that $u_k = u_{-k},\ z_k = z_{-k}$.

2. The function u is real, i. e. $\overline{u} = u$. In terms of the Fourier coefficients this means $\overline{u}_k = u_{-k}$.

With these assumptions is is easy to verify that the quadratic terms in the equation satisfy the identity

$$\int_0^{2\pi} -(z \circ \overline{z})u + (z \circ u)\overline{z}\; dx = 0\,, \tag{41.6}$$

similar to the identity

$$\int_\Omega (u\nabla u + \nabla p)u\, dx = 0, \qquad (\text{assuming } \operatorname{div} u = 0) \tag{41.7}$$

which is important in connection with the energy estimate for the Euler or the Navier-Stokes equations, see lectures 11 and 12.

In the Fourier variables z_k, u_k the system (41.5), under our additional symmetry assumptions 1. and 2. above, can be written as

$$\begin{array}{rcl} \dot{u}_0 &=& -z_0\overline{z}_0 - u_0\ , \\ \dot{z}_0 &=& z_0 u + \overline{F}_0 z_0\ , \end{array} \tag{41.8}$$

and

$$\begin{array}{rcll} \dot{u}_k &=& -z_k\overline{z}_k - k^2\mu u_k, & \\ \dot{z}_k &=& z_k\overline{u}_k + \overline{F}_k z_k - k^2\mu z_k\,, & k \neq 0\,. \end{array} \tag{41.9}$$

The key point is that in the Fourier variables the system (41.5) decomposes into a family of mutually non-interacting 3 by 3 ode systems of a similar nature, and – as we will see – the general solutions of each of the system can be studied in detail. Therefore the system (41.5) is in some sense too simple. Nevertheless, the behavior of its solution is still instructive.

Let us look more closely at (41.9) for some fixed $k \neq 0$. We set

$$u_k = u, \quad z_k = z = x + iy, \quad F_k = F = a + ib, \quad k^2\mu = \nu\,. \tag{41.10}$$

We obtain

$$\begin{array}{rcl} \dot{u} &=& -z\overline{z} - \nu u, \\ \dot{z} &=& z\overline{u} + \overline{F}z - \nu z\,. \end{array} \tag{41.11}$$

The system (41.11) has an obvious equilibrium point $(u, z) = (0, 0)$. Its linearization at this equilibrium is

$$\begin{array}{rcl} \dot{u} &=& -\nu u, \\ \dot{z} &=& \overline{F}z - \nu z\,, \end{array} \tag{41.12}$$

or

$$\frac{d}{dt}\begin{pmatrix} u \\ x \\ y \end{pmatrix} = \begin{pmatrix} -\nu & 0 & 0 \\ 0 & a-\nu & b \\ 0 & -b & a-\nu \end{pmatrix}\begin{pmatrix} u \\ x \\ y \end{pmatrix} = A\begin{pmatrix} u \\ x \\ y \end{pmatrix}\,. \tag{41.13}$$

The eigenvalues of the matrix A can be easily seen to be

$$-\nu, \quad -\nu + a + ib, \quad -\nu + a - ib\,. \tag{41.14}$$

The trivial solution $u = 0, z = 0$ will be linearly stable if and only if $-\nu + a < 0$. (Recall that we always assume $\nu > 0$.) If $a > 0$, then as we change ν from large to small, for $\nu = a$ the eigenvalues of the matrix A will cross the imaginary axis at $\pm ib$ and the trivial solution becomes unstable.

It is in fact possible to analyze the global behavior of the general solution of (41.11) in all detail. Following E. Hopf, we use the cylindrical coordinates r, θ, u, writing

$$z = re^{i\theta}\,. \tag{41.15}$$

The system (41.11) then becomes

$$\begin{aligned} \dot{u} &= -r^2 - \nu u\,, \\ \dot{r} &= r(u + a - \nu)\,, \\ \dot{\theta} &= -b\,. \end{aligned} \tag{41.16}$$

The general solution of the third equation is

$$\theta(t) = \theta_0 - bt\,, \tag{41.17}$$

and the first two equations do not contain θ. Therefore it is enough to study the system

$$\begin{aligned} \dot{u} &= -r^2 - \nu u\,, \\ \dot{r} &= r(u + a - \nu)\,, \end{aligned} \tag{41.18}$$

which should be considered for $u \in \mathbf{R}, r \geq 0$. The system has a trivial equilibrium $u = 0, r = 0$ which is stable linearly stable for $a - \nu < 0$ and linearly unstable for $a - \nu > 0$. For $a - \nu > 0$ the system has a second equilibrium, given by

$$u = -a + \nu\,, \qquad r = \sqrt{-\nu u}\,. \tag{41.19}$$

This equilibrium is linearly stable, as one can easily check. To understand the global behavior of a general solution, we eliminate u from (41.18) by expressing it in terms of r and $\dot{r}$ from the second equation, and substituting the expression into the first equation. We obtain

$$\frac{d}{dt}\left(\frac{\dot{r}}{r}\right) = -r^2 - \nu\left(\frac{\dot{r}}{r} - a + \nu\right)\,. \tag{41.20}$$

Letting $r = e^q$, we obtain

$$\ddot{q} + \nu\dot{q} = -e^{2q} + \nu(a - \nu) = -\frac{\partial V(q)}{\partial q}\,, \tag{41.21}$$

where

$$V(q) = \frac{1}{2}e^{2q} - \nu(a - \nu)q\,. \tag{41.22}$$

Equation (41.21) represents a motion of a particle (with coordinate q) in the potential $V(q)$ and friction $\nu\dot{q}$. We distinguish tow cases:

Case 1: $a - \nu \leq 0$.
In this case the potential V is strictly increasing and the particle will always approach $q \to -\infty$ as $t \to \infty$. This shows that in the case $a-\nu < 0$ the solutions of the original system will always approach the trivial equilibrium $u = 0, z = 0$, while "orbiting" around the $u-$ axis at a constant angular velocity $-b$.

Case 2: $a - \nu > 0$.
In this case the potential V is convex, decreasing for large negative q and increasing for large positive q, and attains its minimum at $q = q_0 = \frac{1}{2}\log(\nu(a - \nu))$. Hence the general solution $q(t)$ will be given by exponentially decaying oscillations around the equilibrium q_0, with $q(t) \to q_0$ as $t \to \infty$.

Going back to the original system (41.11), we see from the above that for $a-\nu \leq 0$ and $b \neq 0$ any solution will approach the trivial equilibrium $u = 0, z = 0$. For $a - \nu > 0$ any solution starting away from the line $z = 0$ will approach (possibly with some oscillations) the unique non-trivial periodic orbit of the system, characterized in the cylindrical variables r, θ, u by equations (41.19) and the equation $\dot{\theta} = -b$. The solutions starting on the line $z = 0$ will approach the trivial equilibrium $u = 0, z = 0$.

We have thus obtained a very good picture of the behavior of the solutions of (41.11).

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1/23/2011

E. Hopf's model equation (continued)

The function F in (41.5) can be thought of as a parameter (somewhat similar to a "right-hand side" of an equation, although it is really a coefficient of a linear term). Recall that we write its Fourier coefficients as $F_k = a_k + ib_k$, with $a_k, b_k \in \mathbf{R}$. Last time we analyzed the system (41.9). We note that the analysis of the equations (41.8) for the $k = 0$ modes is similar. In fact, if we take $\nu = 1$ and $F = F_0 + 1$ in (41.11), we get (41.8). In particular, the trivial solution $u_0 = 0, z_0 = 0$ of (41.8) will be globally stable for $a_0 < 0$. We will assume this for simplicity, so that in the rest of our discussion we do not have to pay attention to the modes with $k = 0$. (The modifications one has to make in the general case are self-evident.)

According to our calculations last time, the trivial equilibrium $u_k = 0, z_k = 0$ of the $k-$th mode equations (41.9) will be globally stable if and only if

$$a_k - k^2\mu \leq 0\,. \tag{42.1}$$

We can choose a smooth function F with $a_k > 0$ and $b_k \neq 0$ for all $k \neq 0$. (Note that the smoothness of F implies that a_k, b_k approach quickly zero as $|k| \to \infty$.) Let us set

$$\mu_k = \frac{a_k}{k^2} \quad \text{and } S = \{\mu_k,\ k \in \mathbf{Z},\ k \neq 0\}\ . \tag{42.2}$$

Clearly $\mu_k \to 0$ as $|k| \to \infty$.

If we now consider the solutions of (41.5) for general smooth initial data and vary $\mu > 0$, we will see the following behavior. (Recall that we consider only the solutions which are even in x and for which $u(x,t)$ is a real-valued function.)

1. For large μ all solutions $u(x,t), z(x,t)$ will converge to the trivial solution $u = 0, z = 0$ as $t \to \infty$.

2. As we decrease $\mu > 0$ to smaller values (still positive), we can cross points of S. The modes k with $\mu_k > \mu$ will become unstable, and – except for the very special case when $z_k|_{t=0} = 0$ – will approach a non-trivial periodic solution. The amplitude of the oscillation of the $k-$th mode with $\mu_k > \mu$ depends on μ as $k^2\sqrt{\mu(\mu_k - \mu)}$, while their frequency is proportional to b_k. Therefore the "end-state" of a generic solution for a given $\mu > 0$ is that it is "winding up" around a torus in the phase space, the dimension of which coincides with the number of points in the set $S \cap (\mu, \infty)$ (counted with their multiplicities). Other non-trivial but unstable solutions can be obtained by choosing some of the amplitudes of the modes with $\mu_k > \mu$ to vanish, while keeping the other non-trivial.

We see that the solutions do exhibit some features which we expect in turbulent solutions. Some features of the model which do not correspond to what we expect for the Navier-Stokes solutions are for example the following:

1. The Fourier modes u_k, z_k and u_l, z_l do not interact for $k \neq l$ (unless $|k| = |l|$). This means that the model cannot have any Richardson-Kolmogorov cascade as discussed in lecture 27. The energy does not move from low to high frequencies in contrast to what is envisaged for the 3d Navier-Stokes equations.

2. When we decrease the viscosity in the Navier-Stokes equations and new and new frequencies appear we expect that these new frequencies will most of the time be higher than the "old" frequencies. This is clearly not the case in the Hopf model.

Nevertheless in spite of these (and some additional) drawbacks the model does provide some insight in what we might expect for the Navier-Stokes solutions.

42.1 Steady Solutions

So far we have discussed essentially two ways in which the stability of a solution can be lost when a flow parameter, such as viscosity, or the magnitude of force is changed. One was the loss of stability of the elementary solution (40.7) of the Taylor-Couette flow, and one was Hopf's example, where we have observed the Hopf bifurcation. We mentioned that the numerical evidence suggests that the loss of stability of the steady flow around the cylinder to a periodic solution (approximated by the so-called Karmán vortex street) is due to a Hopf bifurcation, similar to what we have seen in Hopf's example.[218] Both of these examples share the feature that the "old solution" which loses stability is still present even after the stability loss, we just do not typically see it because of the instability. Equation (40.23) provides one of the simplest examples of this situation. There are other ways in which stability can be lost. As a toy example, the reader can consider the 1d equation

$$\dot{x} = f(x, \lambda) = -\frac{\partial W(x, \lambda)}{\partial x} \,, \tag{42.3}$$

with

$$W(x, \lambda) = \frac{x^4}{4} - \frac{a^2 x^2}{2} - \lambda x \,, \tag{42.4}$$

where $\lambda \in \mathbf{R}$ is the "bifurcation parameter" and $a > 0$ is considered as fixed. (One can also consider a as a bifurcation parameter, of course.)

To get some idea of what one might expect of the actual solutions of the Navier-Stokes equations, it is useful to have a closer look at the set of the steady state

[218] Of course, the coordinates in which the similarity will be apparent have to be suitably chosen.

solutions.[219] We will consider the following situation. We consider a smooth bounded domain $\Omega \subset \mathbf{R}^3$ and a smooth function $f\colon \Omega \to \mathbf{R}^3$. We consider the problem of finding a smooth $u\colon \Omega \to \mathbf{R}^3$ with

$$\begin{aligned} -\nu\Delta u + u\nabla u + \nabla p &= f(x)\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{42.5}$$

These are of course the steady states of the Navier-Stokes equation with the right-hand side f (and the Dirichlet boundary condition $u|_{\partial\Omega} = 0$). For most real situations such steady states are observed when ν is sufficiently large, but are not observed for small ν. Mathematically, the existence of such steady states for small ν was not clear until the 1930s, when J. Leray proved the existence of the steady states for any $\nu > 0$.[220]

Instead of considering a bounded domain $\Omega \subset \mathbf{R}^3$, we could also consider the case $\Omega = \mathbf{R}^3/\mathbf{Z}^3$, when Ω can be thought of as a 3d torus, and one imposes additional conditions $\int_\Omega f\,dx = 0$ and $\int_\Omega u\,dx = 0$. This case is in fact simple in some details.

All proofs of the existence of the steady states are based one way or another on the identity

$$\int_\Omega (u\nabla u + \nabla p)u\,dx = 0\,. \tag{42.6}$$

which – as we have already seen in the last semester (see, for example, lecture 12) – is true for any smooth div-free field $u\colon \Omega \to \mathbf{R}^3$ with $u|_{\partial\Omega} = 0$.

We will recall the "weak Poincare inequality": for each smooth $u\colon \Omega \to R^3$ with $u|_{\partial\Omega} = 0$ we have

$$\int_\Omega |u|^2\,dx \le C\int_\Omega |\nabla u|^2\,, \tag{42.7}$$

where $C = C(\Omega)$. The inequality also holds for $\Omega = \mathbf{R}^3/\mathbf{Z}^3$ when $\int_\Omega u\,dx = 0$, which immediately follows from the Fourier series representation of u. (The condition $\operatorname{div} u = 0$ is not necessary for (42.7), although in some cases it may be used to lower the constant C. In our case the size of C is not important.)

Let us denote

$$||\nabla u||^2 = \int_\Omega |\nabla u|^2\,dx, \qquad ||f||^2 = \int_\Omega |f|^2\,dx. \tag{42.8}$$

We note that (42.6) and (42.7) imply that

$$\begin{aligned} \textstyle\int_\Omega(\nu\Delta u - u\nabla u - \nabla p + f)u\,dx &= \textstyle\int_\Omega(-\nu|\nabla u|^2 + fu)\,dx \\ &\le -\nu||\nabla u||^2 + C||f||\,||\nabla u|| \\ &< 0 \qquad \text{when } ||\nabla u|| > \nu^{-1}C||f||\,. \end{aligned} \tag{42.9}$$

[219] In general, when investigating the solutions of $\dot{x} = f(x,\lambda)$, it is always a good idea to check the steady states, given by $f(x,\lambda) = 0$. In the context of PDEs the characterization of the steady states might be a non-trivial task.

[220] J. Leray, Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. Pures Appl. 12, 182, (1933).

A finite-dimensional analogy of the situation is the following. Assume we have a smooth vector field $b = b(x)$ in $\mathbf{R}^m$ (where m can be large). Assume that

$$b(x)x = \sum_{j=1}^{m} b_j(x)x_j < 0 \qquad \text{on the sphere } \{|x| = r\}. \tag{42.10}$$

Then the field b has to vanish inside the ball $\{|x| < r\}$. In other words, we assume (in (42.10)) that the trajectories given by the equation

$$\dot{x} = b(x) \tag{42.11}$$

intersect the boundary $\{|x| = r\}$ of the ball $B_r = \{|x| < r\}$ transversally, with all the boundary points entering the inside of the ball as time increases. In particular, the ball in invariant under the flow. Under these conditions one can see from Browder's fixed point theorem that the field b has to vanish inside the ball B_r . [221] Inequality (42.9) is analogous to (42.10), so if we could pretend that we are in a finite dimension, we would conclude that $-\nu\Delta u + u\nabla u + \nabla p$ has to vanish for some u, p with $||\nabla u|| < \nu^{-1}C||f||$. The problem is to make sure that in the case we are considering this argument works in the infinite dimensional setting of the vector fields u in Ω. (In general, Browder's fixed point theorem can fail in infinite dimensions.) This line of reasoning can indeed be made into a rigorous proof, but we will use another argument, which perhaps gives more insight into the situation.

Let us first consider the linear equation for $u\colon \Omega \to \mathbf{R}^3$ given by

$$\begin{aligned} -\nu\Delta u + \nabla p &= f\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{42.12}$$

This is the steady Stokes problem. This equation is relatively well understood.[222] The problem can be formulated as a minimization problem: among all div-free fields with $u|_{\partial\Omega} = 0$ (and sufficient regularity) find the one which minimizes the integral

$$\int_\Omega (\frac{1}{2}\nu|\nabla u|^2 - fu)\; dx\,. \tag{42.13}$$

It can be shown that the problem behaves in many respects similarly to the Laplace equation, if we disregard the scalar features of the Laplace equation (such as the maximum principle and the Harnack inequality).[223]

[221] Consider the map $x \to \epsilon b(x)$ and show that for a sufficiently small $\epsilon > 0$ it maps $\overline{B}_r$ into itself. By Browder's theorem this means that the map has a fixed point, i. e. $x + \epsilon b(x) = x$, which is the same as $b(x) = 0$., i. e. the flow has to have a rest point inside $\overline{B}_r$. Since b does not vanish at the boundary of B_r, the fixed point must by in B_r.

[222] There are still quite a few open problems about it, such as optimal regularity of solutions in Lipschitz domains.

[223] This is usually a safe heuristics for the steady problem (42.12), but it can be dangerous to use a similar comparison between the corresponding time-dependent problem $u_t - \nu\Delta u + \nabla p = f$, $\operatorname{div} u = 0$ and the heat equation $u_t - \nu\Delta u = f$. Although many similarities exist also for the time-dependent problems, one has to be cautious.

Let G_ν be the solution operator for (42.12). This means that for a given f we denote the unique solution u of (42.12) by $G_\nu f$. Note that $G_\nu f = \frac{1}{\nu} G f$, where $G = G_1$.

We can rewrite (42.5) as

$$u + G_\nu(u\nabla u) = G_\nu f\,, \tag{42.14}$$

or

$$u + \frac{1}{\nu} G(u\nabla u) = \frac{1}{\nu} G f\,. \tag{42.15}$$

The natural scalar product in the context of this equation is

$$((u, v)) = \int_\Omega \nabla u \nabla v \, dx\,. \tag{42.16}$$

Identity (42.6) implies

$$((Gu, u)) = 0 \qquad \text{when } \nabla u \in L^2(\Omega) \text{ and } u|_{\partial\Omega} = 0. \tag{42.17}$$

This will play an important role in solving (42.15).

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Steady state solutions (continued)

Using the notation from the last lecture, let us set

$$G(u\nabla u) = N(u), \quad Gf = F, \quad \frac{1}{\nu} = \lambda\,. \tag{43.1}$$

Equation (42.15) can then be written as

$$u + \lambda N(u) = \lambda F\,. \tag{43.2}$$

To solve this equation for a specific $\lambda = \lambda_0$, we will find solutions for all $\lambda \in [0, \lambda_0]$ by continuation from $\lambda = 0$. In the case $\lambda = 0$ the equation is trivial and we have clearly have a unique solution $u = 0$.

We will denote by H the space of all div-free vector fields in Ω with $\nabla u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$. Recall that for $u, v \in H$ we defined the scalar product

$$((u, v)) = \int_\Omega \nabla u \nabla v\, dx\,. \tag{43.3}$$

The space H equipped with this scalar product is a Hilbert space. We will use the notation

$$||u||_H = \sqrt{((u, u))}. \tag{43.4}$$

Our first step is the following:

Lemma 1

Equation (43.3) is uniquely solvable in H for sufficiently small λ.

Remark: Our goal is of course to extend the existence part of the lemma to any $\lambda \in [0, \lambda_0]$. (We will see that it is in fact true for any $\lambda \in \mathbf{R}$.) For the uniqueness part of the lemma the smallness assumption on λ is essential. In general, uniqueness can fail when λ is not sufficiently small.

We postpone the proof of the lemma and state the next main point of our method, and a-priori estimate for the solutions of (43.3). (The estimate is in fact a version of (42.9).)

Lemma 2

For any solution $u \in H$ of (43.2) with $\lambda \geq 0$ we have

$$||u||_H \leq \lambda ||F||_H\,. \tag{43.5}$$

Proof

Taking the scalar product of both sides with u and recalling (42.17), we have

$$||u||_H^2 = \lambda((F,u)) \leq \lambda||F||_H||u||_H\,, \tag{43.6}$$

and (43.5) follows.

Summarizing the situation, we see that

1. Our equation has a unique solution $u = u(\lambda)$ for sufficiently small λ. (In fact, u depends smoothly on λ for these small λ.)
2. When $\lambda \in [0, \lambda_0]$, our equation has no solutions u with $||u||_H > \lambda_0||F||_H$.

Let us investigate what to expect in this situation if H were finite-dimensional. Let us consider a smooth map

$$\phi\colon \mathbf{R}^m \times [0,\lambda_1] \to \mathbf{R}^m\,. \tag{43.7}$$

We will write x for elements of $\mathbf{R}^m$, so that we have $\phi = \phi(x,\lambda)$. Assume that $\phi(x,0) = x$ and that the equation $\phi(x,\lambda) = 0$ has no solutions for when $|x| \geq r$, $\lambda \in [0,\lambda_1]$ for some $r > 0$. By the Implicit Function Theorem we know that for sufficiently small λ the equation $\phi(x,\lambda) = 0$ has a unique solution $x = x(\lambda)$ depending smoothly on λ (when λ stays small).

Assume ϕ satisfies an additional "non-degeneracy assumption"

$$\operatorname{rank} D_{x,\lambda}\,\phi(x,\lambda) = m \text{ for each } |x| \leq r, \lambda \in [0,\lambda_1]. \tag{43.8}$$

Here we denote by $D_{x,\lambda}\phi$ the $(m+1)\times m$ Jacobi matrix of all partial derivatives of ϕ, with respect to both x and λ.

The Implicit Function Theorem implies that if (43.8) is satisfied and $(\overline{x},\overline{\lambda})$ is a solution of $\phi(x,\lambda) = 0$ with $|\overline{x}| \leq r$ and $\overline{\lambda} \in (0,\lambda_1)$, then in a sufficiently small ball $B \subset \mathbf{R}^m \times \mathbf{R}$ centered at $(\overline{x},\overline{\lambda})$, the solutions $(x,\lambda) \in B$ of $\phi(x,\lambda) = 0$ form a smooth curve (not necessarily parametrized by λ). In other words, the set $X = \{(x,\lambda), \phi(x,\lambda) = 0, |x| \leq r, \lambda \in (0,\lambda_1)\}$ is a smooth one-dimensional submanifold of $\{|x| \leq r, \lambda \in (0,\lambda_1)\}$. Note that by our assumptions the set X stays away from $\{(x,\lambda), |x| \geq r, \lambda \in [0,\lambda_1]\}$.

For the general ϕ satisfying our assumptions, the manifold X can be complicated. It can have many connected components, some of them being closed curves, some of them being curves starting at (x',λ_1) for some $x' \in \mathbf{R}^m$ with $|x'| < r$ and tracing a segment in $\{(x,\lambda)\,,\, |x| < r, \lambda \in (0,\lambda_1)\}$ before returning to some point (x'',λ_1), with $|x''| < r$. We emphasize again that under our assumptions X cannot have any branch points.

We will now focus on the connected component of X which contains the curve of solutions $x(\lambda)$ near zero (defined for small λ), with $x(0) = 0$. Let Y be the connected component of X containing the curve $x(\lambda)$ for small λ. Let $\pi\colon X \to [0,\lambda_1]$ be defined by

$$\pi(x,\lambda) = \lambda\,. \tag{43.9}$$

Lemma 3
We have $\pi(Y) = (0, \lambda_1)$.

Proof:

Assume $\pi(Y)$ does not contain $(0, \lambda_1)$. In that case the set $Y \cup \{(0, 0\}$ is compact (since X is a is locally a submanifold), and it is in fact a curve emanating from $(0, 0)$ which can be parametrized by length. Note that the curve cannot return to the area of small λ, due to the uniqueness of the solutions for those λ. Let $\gamma(s) = (x(s), \lambda(s)), \quad s \in [0, L)$ be the parametrization by length (with $|\gamma'(s)| = 1$). Clearly L has to be finite (otherwise X would not be a submanifold). Due to the condition $|\gamma'(s)| = 1$ the limit $\gamma(L) = \lim_{s \to L} \gamma(s)$ exists, and at $\gamma(L)$ we get a contradiction with the fact that X is a submanifold, as the curve "ends" at this point.

The proof shows that for any $\lambda_0 < \lambda_1$ the curve starting of solutions starting at $(0, 0)$ will eventually reach some point of the form (x, λ_0) with $|x| < r$. We emphasize that the curve may not be globally a graph of a function $\lambda \to x(\lambda)$. It starts as such a function, but it may later "turn back". A simple example of this situations is presented by our earlier example (42.3). At the "turning points" the tangent line to the manifold $\phi(x, \lambda) = 0$ is perpendicular to the line $x = 0$, and cannot be parametrized by λ. The matrix $D_x\phi(x, \lambda)$ is singular at such points (while the the full matrix $D_{x,\lambda}\phi(x, \lambda)$ has rank m). The curve Y can be calculated numerically by continuation, but near the turning points we cannot use λ to parametrize it. We can use for example length as an alternative parameter.

Assume we are in the situation described above and $(\overline{x}, \overline{\lambda})$ is a turning point where the curve "turns back" (so that in a parametrization by length $s \to (x(s), \lambda(s))$ we have $\overline{\lambda} = \lambda(\overline{s})$, with the function $\lambda(s)$ having a local maximum at $\overline{s}$). Assume moreover that the solutions $(x(s), \lambda(s))$ for $s \in (\overline{s} - \delta, \overline{s})$ are stable, in the sense that the eigenvalues of $D_x\phi(x, \lambda)$ are in $\{z \in \mathbf{C}\,,\ \mathrm{Re}\, z < 0\}$. Then we can locally write $x = x(\lambda)$ for the curve. What happens to the solutions $x(\lambda)$ if we increase λ just above $\overline{\lambda}$ and solve the equation $\dot{x} = \phi(x, \lambda)$ with $x(0) = \overline{x}$, say? There is not much we can say in general, since the trajectory $x(t)$ may get far away from $\overline{x}$ local features of $\phi(x, \lambda)$ near $(\overline{x}, \overline{\lambda})$ are insufficient to determine what happens. As an exercise the reader can analyze the toy model (42.3) in this context.

The above analysis used in a crucial way the assumption that rank $D_{x,\lambda}\phi(x, \lambda) = m$ when $\phi(x, \lambda) = 0$. What happens when this conditions is not satisfied? The following theorem shows that an arbitrary small perturbation of the equation can fix a possible failure of the condition rank $D_{x,\lambda}\phi(x, \lambda) = m$.

Sard's Theorem[224]

[224] A. Sard, The measure of the critical values of differentiable maps, Bull. Am. Math. Soc. vol. 48 (1942) pp. 883-890. For a comprehensive treatment see for example H. Federer's book "Geometric measure Theory", Chapter 3.4.

Let $\mathcal{O} \subset \mathbf{R}^m$ be an open set and let $f\colon \mathcal{O} \to \mathbf{R}^l$ be a smooth map, where $l \leq m$. Let for almost every $y \in \phi(\mathcal{O})$ (with respect to the standard $l-$dimensional measure $\mathbf{R}^l$) the condition $\operatorname{rank} Df(x) = l$ *is satisfied everywhere on $f^{-1}(y)$.*

Returning to the situation above with the map $\phi(x, \lambda)$, let us now consider the general situation when condition (43.8) is not necessarily satisfied. Let us again set $X = \{(x, \lambda), |x| < r, \lambda \in [0, \lambda_1), \phi(x, \lambda) = 0\}$ and let Y again be the connected component of X containing the solutions $x(\lambda)$ for small λ. The difference this time is that X, Y may not be submanifolds. All we can say is that these sets are compact in $\{(x, \lambda), |x| < r, \lambda \in [0, \lambda_2]$ for each $\lambda_2 < \lambda_1$, and that Y is connected. We claim that Lemma 3 is still valid:

Lemma 3'

In the situation above we still have $\pi(Y) = (0, \lambda_1)$.

Proof

Let $y_j \in \mathbf{R}^m$, $|y_j| < r$, $y_j \to 0$ be such that $\operatorname{rank} D_{x,\lambda}\phi(x, \lambda) = m$ when $\phi(x, \lambda) = y_j$. Such y_j exist in view of Sard's theorem and the assumption $\phi(x, 0) = x$. Let X_j and Y_j be defined by replacing 0 by y_j in the definitions of X and Y above. The same reasoning as in the proof of Lemma 3 shows $\pi(Y_j) = (0, \lambda_1)$ for each j. Let $\tilde{Y}$ a limit point of the sequence of sets Y_j in the so-called Hausdorff metric.[225] Clearly $\pi(\tilde{Y}) = (0, \lambda_1)$ and $\tilde{Y} \subset Y$. Therefore $\pi(Y) = (0, \lambda_1)$ and the proof is finished.[226]

Remarks:

1. We recommend that the reader goes through this proof with $\phi(x, \lambda)$ replaced by $f(x, \lambda)$ from example (40.23), where the sets X_j, Y_j and X, Y can be easily visualized.

2. Replacing the set X which may possibly have branching points by regular manifolds X_j may sometimes be a good strategy in numerical continuation of solutions, as we can avoid difficulties related to passing through branch points of X.

3. Even in the situation when X is a smooth manifold and away from the turning points, the steady solution can still lose stability to a time-periodic solution, or a more complicated time-dependent behavior. For example, in the case of a flow around a cylinder, it appears numerically that the curve of solutions obtained

[225] The Hausdorff metric on the set of compact subsets of a compact metric space has several equivalent definitions. We recall for example the following one. Let M be a compact metric space. For any compact set $K \subset M$ we define $f\colon M \to R$ by $f_K(x) = \operatorname{dist}(x, K)$. We can define a distance function d on the set $\mathcal{X}$ of all compact subsets of M by $d(K_1, K_2) = \sup_x |f_{K_1}(x) - f_{K_2}(x)|$. This is one of the several possible definitions of the so-called *Hausdorff distance* between two compact sets. It is not hard to show that $\mathcal{X}$ equipped with this distance function is again a compact metric space.

[226] An interesting question is the following: does there exist a "regular curve" $Z \subset Y$ with $\pi(Z) = (0, \lambda_1)$? This should be a decidable question, and some variant of it may have been studies in the literature, although I do not know a reference.

by the continuation from the small Reynolds numbers is a regular curve without branching or turning points. Its stability is lost to a periodic solution (via a Hopf bifurcation), when a pair of eigenvalues of (an analogue of) $D_x\phi(x,\lambda)$ crosses the imaginary axis away from the origin.

Next we need to establish how to make these finite-dimensional results relevant for the infinite-dimensional setting of the Navier-Stokes equations.

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Steady solutions (continued)

Our considerations above were in finite dimensions. The steady Navier-Stokes equation in H is of course an infinite-dimensional problem, and therefore the finite-dimensional considerations cannot be directly applied to equation (43.2). There are several ways to overcome this difficulty. We will present the so-called *Galerkin method*, which is based on finite-dimensional approximations. The method is also suitable for calculating the solutions numerically on a computer.

The Galerking method appears most naturally in the context of variational problems. Let us consider for example the steady linear Stokes system

$$\begin{aligned} -\nu\Delta u + \nabla p &= f\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{44.1}$$

We have mentioned in lecture 42 that this problem is equivalent to minimizing the functional

$$J(u) = \int_\Omega (\frac{\nu}{2}|\nabla u|^2 - fu)\,dx \tag{44.2}$$

over the space H of the div-free vector fields in Ω with $\int_\Omega |\nabla u|^2\,dx = 0$ with $u|_{\partial\Omega} = 0$, see (43.2), (43.3).

Let V be a finite-dimensional subspace of H. Instead of minimizing J over H, we can minimize it over V. Finding a minimum of J over V is a task which is well-suited for computer simulations.[227] The condition that u be a critical point of J on V is

$$J'(u)v = \int_\Omega (\nu\nabla u\nabla v - fv)\,dx = 0, \qquad v \in V\,. \tag{44.3}$$

This is often called the variational formulation of (44.1). For any V as above it obviously has a unique solution $u_V \in V$. This notion immediately generalizes to the steady Navier-Stokes by simply replacing f with $f - u\nabla u$. Hence the steady-Navier-Stokes equation "truncated to V" is the system

$$\int_\Omega (\nu\nabla u\nabla v + u\nabla uv - fv)\;dx = 0\,, \qquad v \in V\,, \tag{44.4}$$

[227] One should emphasize that effective construction of good finite-dimensional subspaces V for general domains Ω is non-trivial, especially when we wish that the condition $\operatorname{div} u = 0$ be exactly satisfied. It is of course easy to write-down many div-free vector fields (such as $u = \operatorname{curl} A$), but for a numerical simulation we wish that our space V has some additional properties which may not be easy to accommodate together with the condition $\operatorname{div} u = 0$. In the case when $\Omega = \mathbf{R}^3/\mathbf{Z}^3$ (and also for more general tori), the most natural finite-dimensional approximations are by truncated Fourier series, which are of course easy to generate.

for which we seek a solution $u_V \in V$. Note that (44.4) can be written again in the form of (43.2), by simply inverting the linear part of the equation. Let $P_V \colon H \to V$ be the orthogonal projection (where we use the scalar product (43.3) in H). The system (44.4) is nothing but

$$u + \lambda P_V N(u) = \lambda P_V F \,. \tag{44.5}$$

Note that for each $u \in V$ we have

$$((P_V N(u), u)) = ((N(u)u, P_V u)) = ((N(u), u)) = 0 \,. \tag{44.6}$$

Hence the same reasoning as in Lemma 2 in the previous lecture gives

$$||u_V||_H \leq \lambda ||P_V\, F||_H \leq \lambda ||F||_H \tag{44.7}$$

for any solution u_V of (44.5). An equivalent estimate

$$\nu ||\nabla u_V||_H \leq C||f||_{L^2(\Omega)} \tag{44.8}$$

can be obtained by setting $v = u_V$ in (44.4), similarly to (42.9). Assume that $v_1, \ldots, v_m$ is a basis of V. Writing $u = \xi_1 v_1 + \cdots + \xi_m v_m$, we obtain and equation for $\xi \in R^m$ of the form

$$\phi(\xi, \lambda) = 0 \,. \tag{44.9}$$

with the same properties as the map $\phi(x, \lambda)$ we studied in the previous lecture. The scalar product of $\xi = (\xi_1, \ldots, \xi_m)$ and $\eta = (\eta_1, \ldots, \eta_m)$ is now of course given by

$$(\xi, \eta) = \sum_{i,j=1}^{m} \xi_i \eta_j \int_\Omega \nabla v_i \nabla v_j \, dx \,. \tag{44.10}$$

By Lemma 3' in the previous lecture we have

Proposition

For each finite-dimensional subspace $V \subset H$ the problem (44.6) has at least one solution u_V. Moreover, any such solution u_V satisfies (44.7) or (equivalently) (44.8).

Let us now recall basic facts concerning weak convergence in Hilbert spaces. We recall that a sequence $u_j \in H$ converges weakly to $u \in H$ if $((u_j, v)) \to ((u, v))$ for each $v \in H$.[228] The weak convergence will be denoted by $\rightharpoonup$. The same definition works in any Hilbert space. We recall that in any Hilbert space X we have the following:

$$x_j \rightharpoonup x, \quad y_j \to y \quad \Rightarrow \quad (x_j, y_j) \to (x, y) \,. \tag{44.11}$$

[228] In Functional Analysis it is proved that this implies that the sequence u_j is in fact bounded in H. If we already know that u_j is a bounded sequence, then in the definition of the weak convergence it is enough to require $((u_j, v)) \to ((u, v))$ for any v from some dense subspace of H.

Recall also that every bounded sequence in a separable[229] Hilbert space contains a weakly converging subsequence.

We will also need the following case of the Sobolev inequality inequality. If $\Omega \subset \mathbf{R}^3$ is a domain with a smooth boundary, then for each $u \in H$ we have

$$||u||_{L^6(\Omega)} = \left(\int_\Omega |u(x)|^6 \, dx \right)^{\frac{1}{6}} \leq C \left(\int_\Omega |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} = C||\nabla u||_{L^2(\Omega)} = C||u||_H \,, \tag{44.12}$$

where C is some constant depending on Ω. In addition, we have the *Rellich's Theorem*:

$$u_j \rightharpoonup u \quad \text{in } H \quad \Rightarrow \quad u_j \to u \quad \text{in } L^p(\Omega) \text{ for each } p < 6 \text{ (strong convergence)} \,. \tag{44.13}$$

With these preliminaries, it is now easy to pass to the limit in our construction of the solutions in H. Let us consider a sequence

$$V_1 \subset V_2 \subset \cdots \subset H \tag{44.14}$$

of finite dimensional subspaces of H with the following property:

$$v \in H \quad \Rightarrow \quad \text{there exists } v_j \in V_j \text{ with } ||v - v_j||_H \to 0 \,. \tag{44.15}$$

Let us write for simplicity u_j for u_{V_j}, the solutions of (44.5) with $V = V_j$. Due to (44.7) we know that the sequence is bounded in H and hence we can choose a subsequence weakly converging to a $u \in H$ with $||u||_H \leq \lambda ||F||_H \leq \frac{C}{\nu}||f||_{L^2}$. Skipping some members of the sequence V_j if necessary, we can assume that $u_j \rightharpoonup u$. Let $v \in H$ and let $v_j \in V_j$ with $v_j \to v$ in H. We have

$$\int_\Omega \nabla u_j \nabla v_j + u_j \nabla u_j v_j - f v_j = 0 \tag{44.16}$$

for each j. We note that by Rellich's Theorem (42.13) the sequence u_j converges strongly to u in the space $L^4(\Omega)$. For the same reason (or by Sobolev's Inequality (44.12)) the sequence v_j converges to v in $L^4(\Omega)$. From this and from (44.13) it is now easy to see that we can pass to the limit in (44.16), obtaining

$$\int_\Omega \nabla u \nabla v + u \nabla u v - f v = 0 \,. \tag{44.17}$$

Since $v \in H$ is arbitrary, we see that u is a variational solution of the steady state problem (42.5). This type of solution is also sometimes called *weak solution*. Our reasoning above establishes the following result

[229] Recall that a Hilbert space is separable if it contains a countable dense set.

Theorem (Existence of steady state solutions)

For each $f \in L^2(\Omega)$, the steady Navier-Stokes equation (42.5) has at least one weak solution $u \in H$. Moreover, any such solution satisfies $||u||_H \leq \frac{C}{\nu}||f||_{L^2(\Omega)}$.

Note that at this point we have not established that u is smooth when f is smooth. We will discuss this point next time. In fact, it is natural to ask if with a suitable choice of the subspaces V_j the sequence u_j will approach u smoothly (assuming $u_j \rightharpoonup u$ and f being smooth). This should be the case, although I do not know a reference for such a result.

45

1/30/2012

Steady solutions (continued)

We will first establish the uniqueness of the steady state solutions for small Reynolds numbers.

We will use the following special case of the so-called Sobolev Inequality. As above, $\Omega \subset \mathbf{R}^3$ is a smooth bounded domain. Let u be a vector field in Ω with $\nabla u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$. Then we have

$$\left(\int_\Omega |u|^6 \, dx\right)^{\frac{1}{6}} \leq C \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}} \tag{45.1}$$

for a suitable constant C. (The constant C is in fact independent of Ω in this case, and its best possible value can be determined quite explicitly, if we do not impose the div-free constraint on u.)

Remark: If you see this type of inequality for the first time, a good way to remember the exponents (and to have a good rule of thumb of what they are in other cases) is to use dimensional analysis. If we think of u as a velocity field the physical dimension of which is $\frac{L}{T}$, we check easily the following dimensions

$$\begin{array}{rcl}
\int_\Omega \, dx & ..... & L^3 \\
u & ..... & LT^{-1} \\
\nabla & ..... & L^{-1} \\
\nabla u & ..... & T^{-1} \\
\int_\Omega |\nabla u|^2 \, dx & ..... & L^3T^{-2} \\
(\int_\Omega |\nabla u|^2 \, dx)^{\frac{1}{2}} & ..... & L^{\frac{3}{2}}T^{-1} \\
\int_\Omega |u|^6 \, dx & ..... & L^9T^{-6} \\
\left(\int_\Omega |u|^6 \, dx\right)^{\frac{1}{6}} & ..... & L^{\frac{3}{2}}T^{-1} \\
\int_\Omega |u|^p \, dx & ..... & L^{p+3}T^{-p} \\
\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} & ..... & L^{1+\frac{3}{p}}T^{-1}
\end{array} \tag{45.2}$$

Here we used the dimension LT^{-1} for u only because we have in mind the Navier-Stokes equation. If the viscosity in the Navier-Stokes equation is fixed, we can use units in which $\nu = 1$. Since the dimension of ν is L^2T^{-1}, this means that we can count T as L^2. In this case the dimensions in the table will be just

powers of L:

$$\begin{array}{rcl} \int_\Omega \, dx & ..... & L^3 \\ u & ..... & L^{-1} \\ \nabla & ..... & L^{-1} \\ \nabla u & ..... & L^{-2} \\ \int_\Omega |\nabla u|^2 \, dx & ..... & L^{-1} \\ (\int_\Omega |\nabla u|^2 \, dx)^{\frac{1}{2}} & ..... & L^{-\frac{1}{2}} \\ \int_\Omega |u|^6 \, dx & ..... & L^{-3} \\ \left(\int_\Omega |u|^6 \, dx\right)^{\frac{1}{6}} & ..... & L^{-\frac{3}{2}} \\ \int_\Omega |u|^p \, dx & ..... & L^{-p+3} \\ \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} & ..... & L^{-1+\frac{3}{p}} \end{array} \tag{45.3}$$

Since in this case the dimension is determined only by a single exponent, we often identify the dimension with the exponent. For example, we can say that the dimension of u is -1, the dimension of ∇u is -2, etc.

When dealing with inequalities of the form (45.1), with the same degree of homogeneity in u on both sides, it does not really matter what the dimension of u is, and we can also consider it dimensionless. In this case the only terms contributing to the dimension will be dx (which has dimension 3) and ∇ (which has dimension -1).

No matter which of the above ways we used to count the dimension, the basic rule of thumb is that the dimension of both sides of the inequality should be the same. This determins the exponents in (45.1) and other Sobolev inequalities.

If we are interested in an inequality of the for (45.1) with an exponent $1 \leq p \leq 6$, we can write

$$\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}} \leq \tilde{C} L^{\frac{1}{2}-\frac{3}{p}} \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}} , \tag{45.4}$$

where L is some natural characteristic of Ω having the dimension of length, such as its diameter. If we write the inequality in this form, then $\tilde{C}$ will depend on the shape of (When L is the diameter of Ω, the inequality will be true with a $\tilde{C}$ independent of Ω, although $\tilde{C}$ may not be chosen so that it is optimal for all the domains when $p < 6$.)

For general treatment of Sobolev inequalities the reader can consult for example the book "Sobolev Spaces" by V. G. Mazja (Springer-Verlag, New York, 1985), where a geometric approach is adopted. For a Harmonic Analysis approach, the reader can consult for example the book "Singular Integrals and Differentiability properties of Functions" by E. Stein (Princeton University Press, 1970).

With (45.4) one can easily prove uniqueness of the steady state solutions of (42.5) for large viscosity. Assume

$$\begin{array}{rcl} -\nu\Delta u + u\nabla u + \nabla p & = & f(x) \, , \\ \operatorname{div} u & = & 0 \, , \\ u|_{\partial\Omega} & = & 0 \, . \end{array} \tag{45.5}$$

and

$$\begin{aligned} -\nu\Delta v + v\nabla v + \nabla q &= f(x)\,, \\ \operatorname{div} v &= 0\,, \\ v|_{\partial\Omega} &= 0\,. \end{aligned} \tag{45.6}$$

Letting $w = u - v$ and subtracting the two equations, we obtain

$$\begin{aligned} -\nu\Delta w + w\nabla u + v\nabla w + \nabla(p-q) &= 0\,, \\ \operatorname{div} w &= 0\,, \\ w|_{\partial\Omega} &= 0\,. \end{aligned} \tag{45.7}$$

Multiplying the first equation of (45.7) by w and integrating by parts, we obtain

$$\int_\Omega \left(\nu|\nabla w|^2 + (w\nabla u)w\right)\, dx = 0\,. \tag{45.8}$$

Writing $L = \operatorname{diam}\Omega$, we have

$$\int_\Omega |(w\nabla u)w|\, dx \le \left(\int_\Omega |w|^4\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}} \tag{45.9}$$

and hence from (45.4)

$$\int_\Omega |(w\nabla u)w|\, dx \le \tilde{C}^2 L^{-\frac{1}{2}} \left(\int_\Omega |\nabla w|^2\, dx\right) \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}\,. \tag{45.10}$$

By (43.5) we have

$$||\nabla u||_{L^2} \le C\nu^{-1}||f||L\,, \tag{45.11}$$

where we have explicitly restored the "dimensional balance" into the weak Poincare inequality (42.7) by writing it in the form (45.4) with $p = 2$. It is now easy to see from (45.8) that $w = 0$ when

$$\nu > \tilde{C}^2 L^{-\frac{1}{2}} C\nu^{-1} L||f||\,, \tag{45.12}$$

or

$$C_1 L^{\frac{1}{2}}\nu^{-2}||f|| < 1\,, \tag{45.13}$$

where $C_1 = \tilde{C}^2 C$. Hence we have shown

Theorem (Uniqueness of the steady solutions for small Reynolds number)
Condition (45.13) guarantees that the steady Navier-Stokes problem (42.5) has a unique solution in H.

We now return to the form (43.1) of (42.5), and establish properties of N from which the uniqueness result as well as the existence result for small Reynolds number will also be transparent.
We recall that $N(u) = G(u\nabla u)$. We let

$$B(u,v) = G(u\nabla v) + G(v\nabla u) = DN(u)v\,, \tag{45.14}$$

where DN denotes the derivative (in the space H).

Lemma 1
The bilinear form B is continuous on H. In other words, we have

$$||B(u,v)|| \le C||u||_H||v||_H\,, \qquad u,v \in H\,. \tag{45.15}$$

We start by the following natural energy estimate for the steady Stokes problem. Assume

$$\begin{aligned} -\Delta u + \nabla p &= \operatorname{div} f\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,, \end{aligned} \tag{45.16}$$

where $f = \{f_{ij}\}$ and $\operatorname{div} f$ is the vector $f_{ij,j}$. Then

$$||\nabla u|| \le ||f||\,, \tag{45.17}$$

where, as usual, $||\cdot||$ means the L^2-norm. This is the most natural inequality for the system (**??**). For the proof we just multiply the first equation of (45.16) by u and integrate by parts. We obtain

$$||\nabla u||^2 = \int_\Omega |\nabla u|^2\,dx = \int_\Omega -f\nabla u \le ||f||\,||\nabla u||\,, \tag{45.18}$$

which implies (45.17).
The function $w = B(u,v)$ is obtained as a solution of

$$\begin{aligned} -\Delta w_i + \partial_i p &= \partial_j(u_i v_j + u_j v_i)\,, \\ \operatorname{div} w &= 0\,, \\ w|_{\partial\Omega} &= 0\,, \end{aligned} \tag{45.19}$$

and hence by (45.17) we have

$$||\nabla w|| \le ||u\otimes v + v\otimes u|| \le 4||u||_{L^4}||v||_{L^4}\,, \tag{45.20}$$

where we use the usual notation $u\otimes v$ for the matrix $u_i v_j$ and

$$||f||_{L^p} = \left(\int_\Omega |f|^p\,dx\right)^{\frac{1}{p}}\,. \tag{45.21}$$

Using (45.4), we conclude

$$||B(u,v)||_H \le C^2 L^{-\frac{1}{2}}||u||_H||v||_H\,. \tag{45.22}$$

We will treat (43.2) as an abstract equation, writing it in the form

$$u + \frac{\lambda}{2}B(u,u) = \lambda F\,. \tag{45.23}$$

The existence and uniqueness of the solutions for small λ can now be derived from the following simple abstract result.

Lemma 2

Let B be a continuous bilinear form on a Banach space X satisfying $||B(x,y)|| \leq c||x||\,||y||$. Let $z \in X$ such that $||z|| < \frac{1}{4c}$ and let $0 < \xi_1 < \xi_2$ be the roots of the equation $\xi = ||z|| + c\xi^2$. Then the equation $x + B(x,x) = z$ has a solutions $\overline{x}$ satisfying $||\overline{x}|| \leq \xi_1$. Moreover, the solutions is unique in the open ball $\{x \in X,\, ||x|| < \xi_2\}$.

Proof

Consider the map $f(x) = z - B(x,x)$. It is not hard to see that for each $x \in X$ with $\xi_1 < ||x|| < \xi_2$ we have $||f(x)|| < ||x||$ and that the iterates $f(x), f^2(x) = f(f(x)), \ldots, f^k(x), \ldots$ approach the ball $\{x, ||x|| \leq \xi_1\}$. Moreover, we have

$$||f(x) - f(y)|| \leq c||x-y||\,(||x|| + ||y||)\,, \tag{45.24}$$

which shows that f is a contraction on any ball of radius $\xi \in [\xi_1, \frac{\xi_1+\xi_2}{2})$. Hence for any $x \in \{x, ||x|| < \xi_2$ the iterates $f^k(x)$ must approach a fixed point. the fixed point has to be unique in any ball on which f is a contraction, and hence, in particular, in $\{x, ||x|| < \xi\}$ whenever $\xi \in [\xi_1, \frac{\xi_1+\xi_2}{2}\})$.

Note that the proof is practically the same for $X = \mathbf{R}$ and the case when X is a general Banach space.

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2/1/2012

46.1 Regularity

The solutions u of the steady Navier-Stokes problem (42.5) we have obtained in the previous lectures satisfy the equation belong to the space H of div-free vector fields in Ω with $\nabla u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$, and satisfy the equation in the sense

$$\int_\Omega (\nu \nabla u \nabla v + u \nabla u\, v - f v)\, dx = 0, \qquad v \in H\,. \tag{46.1}$$

Are the solutions smooth? Obviously, the solutions can be only as smooth as f allows. For example, when f is discontinuous, and we have $-\nu\Delta u + u\nabla u + \nabla p = f$, then some of the terms on the left-hand side must be discontinuous. The best we can expect is that the second derivatives of u are "as good as f". This of course should be made more precise, and the precise formulation requires some definitions of functions spaces which "measure" the regularity of functions.

Details of the regularity theory for the problem above are somewhat technical, but the main ideas are not hard to understand. We will illustrate them on the following model problem. Consider vector valued functions $u\colon (a,b) \to \mathbf{R}^m$. We will write $u = u(x)$, with $x \in (a,b)$ and $u(x) = (u_1(x), \dots, u_m(x))$. Let us consider a quadratic form $B(u,u) = \sum b_{ij} u_i u_j$ on $\mathbf{R}^m$ and the equation

$$u' + B(u,u) = f\,, \tag{46.2}$$

where $f\colon (a,b) \to \mathbf{R}^m$ is a given integrable function. An analogy of the definition (46.1) in the context (46.2) is the following: we say that u is a weak solution of (46.2) in (a,b) if u is a locally integrable function such that $B(u,u)$ is locally integrable and for each smooth compactly function $\varphi\colon (a,b) \to \mathbf{R}^m$ we have

$$\int_a^b (-u\varphi' + B(u,u)\varphi - f\varphi)\, dx = 0\,. \tag{46.3}$$

We would like to show that any weak solution (46.2) are "as regular as f allows". This can be done as follows:

1. For any locally integrable $g\colon (a,b) \to \mathbf{R}^m$ show that any weak solution[230] of

$$v' = g \tag{46.4}$$

is actually the usual solution: namely, v is absolutely continuous and $v' = g$ almost everywhere in (a,b). In particular, when g is continuous, v is continuously differentiable and satisfies $v' = g$ everywhere in (a,b).

2. If g is differentiable, we can take the derivative of (46.4) and repeat the argument. Hence if g' is locally integrable, v' is absolutely continuous and

[230] The definition is as expected from the above: v is locally integrable and $\int_a^b -u\varphi' - g\varphi = 0$ for any smooth, compactly supported φ.

$v'' = g'$. This can be iterated as many times as allowed by g. In particular, if g is k times continuously differentiable, then v is $k+1$ times continuously differentiable.[231]

3. We can now apply the above conclusions for the linear equation $v' = g$ to the non-linear problem (42.6). First, since f and $B(u,u)$ are locally integrable, we see that u is absolutely continuous and $u' + B(u,u) = f$. This means that the term $B(u,u)$ is in fact absolutely continuous, with $[B(u,u)]' = B(u',u) + B(u,u')$, which is a locally integrable function (as u' is locally integrable and u is continuous). This means that if we can differentiate f, we can differentiate the whole equation and the derivative u' is a weak solution of

$$u'' = f' - [B(u,u)]' = f' - B(u',u) - B(u,u') \tag{46.5}$$

and hence u' is absolutely continuous. It is clear that we can continue this procedure and take as many derivatives of the equation as allowed by the regularity of f.

The above procedure by which we establish regularity by iterating a linear regularity argument, using at each step the information we have from previous steps, is usually called *bootstrapping*.

One can use essentially the same procedure to prove that the solutions of (46.1) are as smooth as f allows, if we measure regularity in the right way. The role of the linear equation $v' = g$ will now be played by the linear Stokes problem (44.1), and the role of locally integrable functions will be played by various function spaces[232] , which are designed so that the solutions of the linear equation "gain regularity", similarly to what we have seen in the simple example above.

At the level of PDEs there is one very important phenomenon our model ODE problem (46.2) fails to reproduce (at least in the formulation above). It is the notion of *criticality*, which expresses the fact that we need a certain level of regularity to start the bootstrapping procedure. Let us illustrate this notion in a simple PDE setting.

We consider the PDE

$$-\Delta u = |u|^{2\sigma} u \tag{46.6}$$

for a scalar function u in a domain $\Omega \subset \mathbf{R}^n$, where $\sigma > 0$. For our purposes here we can take $\Omega = \{x, |x| < 1\}$. By analogy with (46.3), a weak solution of (46.5) in Ω is a locally integrable function u in Ω such that $|u|^{2\sigma}u$ is also locally integrable in Ω and

$$\int_\Omega \left(-u\Delta\varphi - |u|^{2\sigma} u\,\varphi\right)\, dx = 0 \tag{46.7}$$

[231] This may seem to be obvious tautology, but note that we define the derivatives via the weak formulation: $\int(-v\varphi' - g\varphi)\,dx = 0$ for each smooth compactly supported φ.

[232] such as the Sobolev spaces $W^{k,p}$ or the Hölder spaces $C^{k,\alpha}$

for each smooth compactly supported function $\varphi\colon \Omega \to \mathbf{R}$.
For simplicity we will consider only positive solutions, so that the non-smoothness of the function $u \to |u|^{2\sigma}u$ at $u = 0$ can be ignored and we can write the equation simply as

$$-\Delta u = u^{2\sigma+1}\,. \tag{46.8}$$

When are the (positive) solutions of this equation smooth? Unlike in the case (46.2), we can now write down solutions which are not smooth. Namely, it is easy to check that for $\sigma > \frac{1}{n-2}$ and $A_\sigma = [\frac{1}{\sigma}\sigma(n-2-\frac{1}{\sigma})]^{\frac{1}{2\sigma}}$, the function

$$w = \frac{A_\sigma}{r^{\frac{1}{\sigma}}} \tag{46.9}$$

is a weak solution of (46.8). For this function we see explicitly that the non-linear operation $w \to w^{2\sigma+1}$ results in a loss of regularity, which the inverting of the laplacian exactly compensates for, but does not improve.[233]
Based on this, one can conjecture that the function w represents a "bordeline for regularity": if a (positive) solution u of (46.8) is "slighly more regular regular" than w given by (46.9), it has to be smooth. There are various ways of making the notion of "sligtly more regular" precise. For example, in many cases the condition[234] $u \in L^p$ for a suitable p works well. If the function w above is indeed the borderline for regularity in terms of the L^p spaces, then we can expect the following result

Assume $\sigma > \frac{1}{n-2}$. Let u be a weak (positive) solution of (46.8). If $u \in L^p(\Omega)$ for some $p \geq \sigma n$, then u is smooth.

This can indeed be proved. It is quite easy for $p > \sigma n$, when a direct bootstrapping argument along the lines of the ODE example above works (with the necessary adjustments taking into account that instead of the simple operator $\frac{d}{dx}$ we invert $-\Delta$).[235] The case $p = \sigma n$ is more subtle, and one has to supplement the simple bootstrapping by an additional idea.

[233] It is instructive to look at the (nonlinear) operation $T : v \to (-\Delta)^{-1}v^{2\sigma+1}$ for functions of the form $ar^{-\alpha}$. Assuming that $2 < \alpha(2\sigma+1) < n$, we have $T(ar^{-\alpha}) = cr^{(-2\sigma\alpha-\alpha+2)}$, which can be interpreted as a gain in regularity when $-\alpha < -2\sigma\alpha - \alpha + 2$, or, equivalently, $\alpha < \frac{1}{\sigma}$. The case $\alpha = \frac{1}{\sigma}$ is the borderline for a regularity gain by T.

[234] Recall that we use $L^p(\Omega)$ to denote the space of all measurable functions f in Ω such that $\int_\Omega |f|^p\,dx$ is finite. For such functions we define $||f||_{L^p} = (\int_\Omega |f|^p\,dx)^{\frac{1}{p}}$

[235] We should remark that although this process is by now standard in the theory of PDE, it took some time to find the right technical set up. This was achieved in the first half of the 20th century (for the elliptic equations we consider here). It turned our that the spaces of continuous functions, or functions with continuous derivatives (usually called C^k today) are not suitable for most PDE arguments, and neither are the spaces of integrable functions or spaces with integrable derivatives (usually called $W^{k,1}$ today). The continuity has to be replaced by Hölder continuity (the spaces $C^{k,\alpha}$, and the integrability has to be replaced by the "L^p integrability" (the spaces $W^{k,p}$) for $1 < p < \infty$. The reason why the spaces C^k are not suitable is the following: if $\Delta v = f$ and f is (locally) continuous, then the second derivatives $\nabla^2 u$ may not be continuous. Similarly, if $f \in L^1$ (locally), then $\nabla^2 u$ may not be locally in L^1.

We see that there is a certain threshold regularity effect for the solutions of (46.8). In general one can have irregular (weak) solutions, but once the solution has a certain critical level of regularity, it has to be smooth. On the scale of the L^p spaces, the space $L^{\sigma n}$ plays a distinguished role for the weak solutions of (46.9), and is often called the *critical space*.

Equation (46.8) is invariant under the scaling symmetry $u(x) \to u_\lambda(x) = \lambda^{\frac{1}{\sigma}} u(\lambda x), \quad \lambda > 0$ Note that $w_\lambda = w$ for each $\lambda > 0$. We say that the solution is scale invariant and its singularity at $x = 0$ is self-similar. Note also the the L^p- norm for the critical $p - \sigma n$ is scale invariant: when $p = \sigma n$ and $v \in L^p(\mathbf{R}^n)$, then $||v_\lambda||_{L^p} = ||v||_{L^p}$ for $\lambda > 0$.

The regularity question can also come up in a slightly different context. Suppose we know from some other argument that the solutions we are interested in satisfy, say, $\nabla u \in L^2(\Omega)$. When will such solutions be regular? Clearly, if the solution w above satisfies $\nabla w \in L^2$, then the condition ∇u is insufficient for regularity. The condition $\nabla w \in L^2$ is equivalent to $\sigma > \frac{2}{n-2}$, which is the same as $2\sigma + 1 > \frac{n+2}{n-2}$. In this case it can be proved that the exponent $2\sigma + 1 = \frac{n+2}{n-2}$ is critical for regularity (with the assumption $\nabla u \in L^2$) and with this exponents the solutions are still regular. The critical exponent can also be from the scaling invariance: the right value of σ is the one for which the controlled quantity, in this case $\int |\nabla v|^2 \, dx$ is invariant under the scaling $v \to v_\lambda$.

For the steady Navier-Stokes equation one can adopt a similar approach. It can be used to show relatively easily (once the linear theory for the linear problem is worked out) that the solutions of (46.1) are smooth, the condition $\nabla u \in L^2$ turns out to be quite better than "critical" in dimension $n = 3$ in this case. (It turns out to be exactly critical in dimension $n = 4$.)

However, unlike in the simple example (46.8), for the Navier-Stokes problem we do not have simple explicit solutions which would be plausible candidates for determining the borderline cases. The notion of "critical" in this case is more based on the method of proof, and the critical spaces are the borderline spaces for the bootstrapping argument. For solutions in the spaces with less regularity than the critical spaces the bootstrapping argument does not work, but we do not have any examples which would show that the solutions can actually be singular. Let us illustrate this by mentioning the following problem. Consider the solution of $-\Delta u + u\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0$ in a domain $\Omega \subset \mathbf{R}^n$. It is easy to see that for a regular solution u one has the following: if $\varphi \colon \Omega \to \mathbf{R}^n$ is a compactly supported smooth vector field with $\operatorname{div} \varphi = 0$, then[236]

$$\int_\Omega (-u_i \Delta \varphi_i - u_i u_j \varphi_{i,j}) \, dx = 0 \,. \tag{46.10}$$

The integral (46.10) is well-defined whenever $u \in L^2(\Omega)$. Therefore one can consider the div-free vector fields $u \in L^2(\Omega)$ satisfying (46.10) for each smooth

[236] We sum over the repeated indices and use the notation $\varphi_{i,j} = \partial_j \varphi_i$.

div-free vector field φ compactly supported in Ω. Such solutions are sometimes call *very weak solutions*, as the term "weak solution" is usually reserved for solutions $u \in H$ defined by (46.1). What is the borderline for regularity for the very weak solutions? The bootstrapping argument, in its more sophisticated version suitable for the critical cases, gives the following:

Assume $n \geq 3$. Then any very weak solution of (46.10) with $u \in L^n(\Omega)$ is smooth in Ω.[237]

It is not known if this result is optimal.[238] We have no example of a very weak solution which would not be smooth. It also appears to be open if the result remains true for $n = 2$. (In that case we know it is true if we replace L^n by $L^{n+\delta}$ for some $\delta > 0$.)

Note that the norm in the space L^n is invariant under the scaling $u \to u_\lambda(x) = \lambda u(\lambda x)$: if $u \in L^n(\mathbf{R}^n)$, then $||u_\lambda||_{L^n} = ||u||_{L^n}$.

In dimension $n = 3$ the condition $u \in H$ comfortably implies that $u \in L^3(\Omega)$ (for bounded domains). In dimension $n = 4$ the condition $\nabla u \in L^2$ still implies $u \in L^4$, although it is now a borderline case. In dimension $n = 5$ the condition $\nabla u \in L^2$ no longer implies $u \in L^5$, and it is an open question whether all very weak solutions with $\nabla u \in L^2$ are regular.

We will now show a heuristic argument which illustrates the notion of criticality from a slightly different angle. The argument can be made rigorous, but we will ignore the technicalities at this stage. Assume that we solve the problem depending on a parameter $\kappa > 0$

$$\begin{aligned} -\Delta u + u\nabla u + \nabla p &= \kappa f \\ \operatorname{div} u &= 0 \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{46.11}$$

in a bounded smooth domain $\Omega \subset \mathbf{R}^n$. The function f is assumed to be smooth. Assume we have a smooth solutions for all $\kappa \in [0, \overline{\kappa})$. Moreover, assume that the solutions satisfy a uniform bound

$$\int_\Omega |\nabla u|^2 \, dx \leq C \tag{46.12}$$

for all $\kappa \in [0, \overline{\kappa})$. Can these solutions become unbounded as $\kappa \to \overline{\kappa}$?

For a given solution let $M = \max_x |u(x)|$. We can imagine that the modulus $|u|$ of the velocity field u has a peak of hight M at some point $\overline{x} \in \Omega$. We will

[237] This is an interior regularity result, we say nothing about the regularity at the boundary. In fact, since we make no assumptions on u at the boundary, the solution may not be smooth up to the boundary.

[238] It is clear, though, that our current methods cannot prove a better result (at least at the scale of the L^p spaces).

say that the peak has width $r > 0$ if $|u| > M/2$ in the ball $B_{\overline{x},r} \cap \Omega$, where $B_{\overline{x},r} = \{x, |x - \overline{x}| < r\}$.

Claim

If M is sufficiently large, the width of the peak is at least of order $1/M$.

In other words, there exists $\varepsilon > 0$ such that whenever M is sufficiently large, then the width of the peak is $\geq \frac{\varepsilon}{M}$.
A rigorous proof requires some work, but we can at least outline the main idea. Define $v(y) = \frac{1}{M} u(\overline{x} + \frac{y}{M})$. Note that v is defined on a large domain $\tilde{\Omega} = M(\Omega - \overline{x})$, and $|v| \leq 1$ in $\tilde{\Omega}$. The field v satisfies the eqution (46.11) with Ω replaced by $\tilde{\Omega}$ and f replaced by $\tilde{f}(y) = \frac{\kappa}{M^3} f(\overline{x} + \frac{y}{M})$. Note that $\tilde{f}$ is small (point-wise), together with is derivatives up to a given order, when M is sufficiently large. In this situation the linear theory[239] gives a bound on the modulus of continuity of v,[240] which means that $|v| > \frac{1}{2}$ is some fixed neighborhood of 0 which is independent of M. Going back from v to u we get the result.

Note that the above claim is exactly what one would expect from the dimensional analysis. Since the viscosity is now fixed to $\nu = 1$, we can use (45.3) to determine the dimension of various quantities. The velocity u has dimension -1, the peak width has dimension 1. The only dimensionally consistent way to estimate the width r through the maximum of $|u|$ (denoted by M) is that r has to at least of order $\frac{1}{M}$.

The above claim quantitatively captures the heuristics that the peak cannot be too thin, due to the effect of the viscosity. The viscosity (which we normalized to $\nu = 1$ in our setting here) is trying to keep the velocities of the nearby fluid particles the same, so any very thin peak will be washed out by it. High peaks can only appear if they are sufficiently massive.

In dimension $n = 3$ we have the Sobolev inequality (45.1), and in view of (46.13) we see that

$$\int_\Omega |u|^6 \, dx \leq \tilde{C} \tag{46.13}$$

for some $\tilde{C}$ independent of M. Using the estimate of the width of our peak, we obtain

$$\int_\Omega |u|^6 \, dx \geq \epsilon_1 M^3 \tag{46.14}$$

for some $\epsilon_1 > 0$ which is independent of M. This gives a contradiction with (46.13) once M is sufficiently large. Hence we must conclude that the functions u stay uniformly bounded as for $\kappa \in [0, \overline{\kappa})$. In this way we see that as we increase κ to some given value $\overline{\kappa}$ (with a bound depending on $\overline{\kappa}$, of course).

[239] We write the equation as $-\Delta v + \nabla q = \tilde{f} - \operatorname{div}(v \otimes v), \quad \operatorname{div} v = 0$.

[240] The rule of thumb is that if v is of order 1, then the linear theory implies that ∇v will also be of order 1. We omit technical details, which do require some work.

It is interesting to check what happens with this argument in dimensions $n = 4$ and $n = 5$. It turns out that dimension $n = 4$ is critical for this argument to work: it still works, but without any margin to spare. In dimension $n = 5$ the argument no longer works. The existence of the high peaks is compatible with both the linear theory and the energy estimates.[241] It may be incompatible with some other estimate which we have not yet discovered [242], but based only on the energy estimate and the linear theory we cannot tell.

Similar considerations apply also to the time-dependent problem. At the moment we will not go into the details, but we just mention the following. In dimension $n = 3$ (now for $u = u(x, t)$) the existence of arbitrary large "peaks" is compatible we both the linear theory and the energy estimate, so we cannot rule out singular behavior based on those two ingredients alone. There might be some additional estimates, hitherto undiscovered, which might rule out the singular behavior. Alternatively, the singular behavior can perhaps occur. We do not know which scenario is correct. This is the famous Navier-Stokes regularity problem, for the solution of which the Clay Mathematical Institute is offering $1M, see
`http://www.claymath.org/millennium/Navier-Stokes_Equations/`

[241] This can again be seen also from behavior of the controlled quantity $\int |\nabla u|^2 \, dx$ under the Navier-Stokes scaling $u(x) \to \lambda u(\lambda x)$. For a function u on $\mathbf{R}^n$ with $\nabla u \in L^2$ we have $\int_{\mathbf{R}^n} |\nabla u_\lambda(x)|^2 \, dx = \lambda^{4-n} \int_{\mathbf{R}^n} |\nabla u|^2 \, dx$. In the critical case the controlled quantity is invariant under the natural scaling of the equation.

[242] In fact, if we restrict our attention to the areas away from the boundary, such estimates have been discovered by J. Frehse and M. Ruzicka in the 1990s, see for example their paper "Existence of Regular Solutions to the Stationary Navier-Stokes Equations", Math. Ann. 302 (1995), pp. 699-717.

47

2/3/2012

47.1 A classical example of a stability calculation

We will discuss a classical example of a linearized stability calculation, due to Rayleigh[243]. There many other classical calculations.[244] We chose Rayleigh's example due to its role in the discovery of the *Lorenz attractor* which is one of the important early examples of chaotic solutions of dynamical systems. We will discuss this connection later.

We consider fluid in a tank heated from below. We consider the usual quantities describing the fluid: the density ρ, the pressure p and the velocity field u. In addition to these quantities we will also consider the temperature of the fluid, which we will denote by Θ. We wish to capture the effect that hot fluid rises up (as can be seen when watching a fire, for example). We introduce the coefficient of the thermal expansion α of the fluid defined by

$$\frac{\delta\rho}{\rho} = -\alpha\ \delta\Theta\,, \tag{47.1}$$

where $\delta\rho$ and $\delta\Theta$ are small changes of ρ and Θ respectively. In general α is not constant, it depends on ρ, θ, and the pressure p. In the regimes we will consider here we will assume α to be constant. It would seem that with (47.1) we have no choice but to assume that the fluid is compressible. However, there is an approximation of the precise equations, due to Boussinesq, which captures the main phenomena while still keeping the flow incompressible and the density constant. We will only consider the two-dimensional problem, for simplicity. Assume the fluid occupies a domain Ω. Let $e_2 = (0,1)$. The gravitational force per unit mass of the fluid is assumed to be given by $-ge_2$, where g is the acceleration due to gravity. We will assume that the flow is incompressible with $\rho = \text{const.}$ (so that, in contrast to (47.1), the temperature does not change the density). Instead, we will assume that the temperature of a fluid particle directly affects the gravitational force on it so that the resulting equations (due to Boussinesq)[245] are

$$\begin{aligned} u_t + u\nabla u + \tfrac{\nabla p}{\rho} - \nu\Delta u &= -ge_2 + g\alpha(\Theta - \Theta_0)e_2\,, \\ \operatorname{div} u &= 0\,, \\ \Theta_t + u\nabla\Theta - \kappa\Delta\Theta &= 0\,. \end{aligned} \tag{47.2}$$

In this system the density ρ is assumed to be constant, and all the other coefficients g, α, ν, κ are also assumed to be constant. The coefficient κ is the heat

[243] "On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side", Philosophical Magazine, Vol. XXXII. pp. 529–546, 1916. Available online, `http://www.archive.org/details/scientificpapers06rayliala`, p. 432.

[244] such as Kelvin's "Vibrations of a columnar vortex", Phil. Mag., x. 1880, pp. 155–168.

[245] Théorie de l'écoulement tourbillonnant et tumultueux des liquides dans les lits rectilignes a grande section (1897), available online
`http://www.archive.org/details/thbeoriedelbeco01bousrich`

conduction coefficient in the fluid and ν is the kinematic viscosity of the fluid. While (47.2) is only an approximation of the exact equations which would involve a variable ρ, it does capture the main features of the phenomena we are interested in. (It turns out the equations also have some resemblance with the axi-symmetric Euler/Navier-Stokes equations away from the axis of rotation, with the temperature Θ playing the role of the component u^θ of the velocity, see lecture 17.)

We will consider (47.2) in a 2d domain Ω with coordinates $x = (x_1, x_2)$. We will assume that the x_1 direction is "periodic", i. e. all quantities we consider are periodic in x_1 with a period L. In the direction x_2 the domain is characterized by $0 < x_2 < H$. We can also think of Ω as $\Omega = S^1_{\frac{L}{2\pi}} \times [0, H]$, where S^1_r denotes a circle of radius r. The boundary $\partial\Omega$ consists of two components, $\Gamma_0 = \{x_2 = 0\}$, and $\Gamma_1 = \{x_2 = L\}$. The boundary conditions are as follows

$\Theta = \Theta_0$ at Γ_0 ,
$\Theta = \Theta_1 = \text{const.}$ at Γ_1 ,
$u_2 = 0$ at $\partial\Omega$ when $\nu = 0$ (Euler's equations) .

In the case $\nu > 0$ (Navier-Stokes equations) we would ideally like to impose the condition $u = 0$ at $\partial\Omega$. With this condition some of the explicit calculations become unfeasible, and therefore we will replace it (following Rayleigh) by a somewhat less physical condition that $u_2 = 0$ at $\partial\Omega$ and the tangential component of the viscous force along $\partial\Omega$ vanishes (the so called Navier boundary condition), which amounts to the conditions $u_2 = 0$ and $\omega = \operatorname{curl} u = 0$ at $\partial\Omega$.

We make the following additional simplification. We will seek Θ in the form

$$\Theta = \Theta_0 + (\Theta_1 - \Theta_0)\frac{x_2}{H} + \theta \tag{47.3}$$

We let

$$\beta = \frac{\Theta_1 - \Theta_0}{H} \,. \tag{47.4}$$

We can also write

$$p = -g(x_2 - H) + \frac{\beta}{2}(x_2 - H)^2 + q. \tag{47.5}$$

With these substitutions, equations (47.2) become

$$\begin{aligned} u_t + u\nabla u + \tfrac{\nabla q}{\rho} - \nu\Delta u &= g\alpha\,\theta\, e_2 \,, \\ \operatorname{div} u &= 0 \,, \\ \theta_t + u\nabla\theta + \beta u_2 - \kappa\Delta\theta &= 0 \,, \end{aligned} \tag{47.6}$$

with the boundary conditions

$$\begin{aligned} \theta &= 0 \,, \\ u_2 &= 0 \,. \end{aligned} \tag{47.7}$$

at $\partial\Omega$ for $\nu = 0$, and the boundary conditions

$$\begin{aligned} \theta &= 0 \,, \\ u_2 &= 0 \,, \\ \operatorname{curl} u &= 0 \,. \end{aligned} \tag{47.8}$$

at $\partial\Omega$ when $\nu > 0$. The system (47.6) with the boundary conditions (47.7) or (48.8) has a trivial solution $u = 0, \theta = 0$ and our goal is to investigate the linearized stability of this trivial solution. The linearized equations at $u = 0, \theta = 0$ are

$$\begin{aligned} u_t + \tfrac{\nabla q}{\rho} - \nu\Delta u &= g\alpha\,\theta\, e_2\,, \\ \operatorname{div} u &= 0\,, \\ \theta_t + \beta u_2 - \kappa\Delta\theta &= 0\,. \end{aligned} \tag{47.9}$$

We can now decompose all the fields in the Fourier series in the x_1 variable:

$$\begin{pmatrix} u_1(x_1,x_2,t) \\ u_2(x_1,x_2,t) \\ q(x_1,x_2,t) \\ \theta(x_1,x_2,t) \end{pmatrix} = \sum_{k\in\frac{2\pi\mathbf{Z}}{L}} \begin{pmatrix} \hat{u}_1(k,x_2,t) \\ \hat{u}_2(k,x_2,t) \\ \hat{q}(k,x_2,t) \\ \hat{\theta}(k,x_2,t) \end{pmatrix} e^{ikx_1} \tag{47.10}$$

Since the system (48.9) is linear and invariant under the translation along x_1, it has to be satisfied separately by each summand of the Fourier series. We will fix $k \in \frac{2\pi}{L}$ and write $u_1(x_2,t)$ for $\hat{u}_1(k,x_2,t)$, and similarly for u_2, q, θ. At a fixed k we get the following system of equations

$$\begin{aligned} u_{1t} + ik\tfrac{q}{\rho} - \nu u_1'' + \nu k^2 u_1 &= 0\,, \\ u_{2t} + \tfrac{q'}{\rho} - \nu u_2'' + \nu k^2 u_2 &= g\alpha\,\theta\,, \\ iku_1 + u_2' &= 0\,, \\ \theta_t + \beta u_2 - \kappa\theta'' + \kappa k^2\theta &= 0\,, \end{aligned} \tag{47.11}$$

where we denote by f' the derivative $\partial_{x_2} f$. The reader can check that for $k = 0$ all solutions of this system satisfying our boundary conditions approach 0 for $t \to \infty$, (when $\kappa > 0$ and $\nu > 0$) or stay bounded (when $\nu = 0$ or $\kappa = 0$) and hence the mode $k = 0$ is stable. (Note that in the case $\kappa = 0$ and $\nu = 0$ there is no energy dissipation in the system, and therefore the solutions of (48.9) cannot approach 0 when the initial condition does not vanish.) Therefore in what follows we will assume $k \neq 0$ without loss of generality.

One way of approaching the stability problem is to view system (48.11) as a linear system $z_t = Lz$ for an unknown (vector-valued) function z and to calculate the eigenvalues of the operator L. This amounts to searching the solutions in the form $z(x_2)e^{\lambda t}$. In general, this is a sound procedure for finite-dimensional systems or for parabolic systems (corresponding to the case $\nu > 0, \kappa > 0$ in the example above). For linearized problems coming from the theory of Euler's equation this method has to be considered with some care, as it may miss effects associated with the possible continuous spectrum of L, if the continuous spectrum is present. In the case at hand this does not happen, but we will still try avoid making the Ansatz $z(x_2)e^{\lambda t}$ and see how far we can get without it.

Our boundary conditions still are (47.7) or (48.8) above, depending on whether ν vanishes. Note that the 4th equation of (48.11) enforces the condition $\theta'' = 0$ for $x_2 = 0$ and $x_2 = H$. This is the usual compatibility issue arising in the context of the heat equation. If the condition is not satisfied for the initial

condition $\theta(x,0)$ at time $t=0$, the equation will still enforce it for $t>0$, at the cost of discontinuity of the derivatives $\theta_t, \theta_{x_2x_2}$ at time $t=0$.

We can proceed with the calculation of the solutions of (48.11) as follows

Step 1. Eliminate u_2 by using the 4th equation:

$$u_2 = \frac{1}{\beta}(-\theta_t + \kappa\theta'' - \kappa k^2\theta)\,. \tag{47.12}$$

Step 2. Eliminate q using the first equation:

$$q = \frac{i\rho}{k}(u_{1t} - \nu u_1'' + \nu k^2 u_1)\,. \tag{47.13}$$

Step 3. Eliminate u_1 using the third equation:

$$u_1 = \frac{i}{k}u_2'\,. \tag{47.14}$$

This leaves us with a single equation for the function θ:

$$\theta_{tt} - \frac{1}{k^2}\theta_{tt}'' + 2\nu k^2\theta_t - (\kappa+3\nu)\theta_t'' + \frac{\kappa+\nu}{k^2}\theta_t'''' + \\ + (\beta g\alpha + \nu\kappa k^4)\theta - 3\nu\kappa k^2\theta'' + 3\nu\kappa\theta'''' - \frac{\nu\kappa}{k^2}\theta^{(6)} = 0\,. \tag{47.15}$$

This equation can luckily be again solved by writing θ as a Fourier series. When

$$\theta(x_2,t) = \sum_{l\in\frac{\pi\mathbf{N}}{H}} \theta_l(t)\sin lx_2\,, \tag{47.16}$$

each summand has to satisfy the equation. A key point is that the representation (47.16) (together with (47.14) and (48.12)) is consistent with the boundary conditions (48.8), as the reader can easily check. (This would not be so if we imposed the Dirichlet boundary conditions $u=0$ on $\partial\Omega$.) Fixing l, dropping the index l in $\theta_l(t)$, we get an ODE of the form

$$A\ddot\theta + B\dot\theta + C\theta = 0\,, \tag{47.17}$$

where

$$\begin{aligned} A &= 1+\tfrac{l^2}{k^2}\,,\\ B &= (\kappa+3\nu)l^2 + (\kappa+\nu)\tfrac{l^4}{k^2}\,,\\ C &= \beta g\alpha + \nu\kappa k^4(1+\tfrac{l^2}{k^2})^3\,. \end{aligned} \tag{47.18}$$

Clearly $A>0$. The coefficient B always satisfies $B\geq 0$, with $B>0$ when $\nu>0$ or $\kappa>0$. We see that the stability of the solutions is decided by the coefficient C. If $C<0$ then the trivial solution $\theta=0$ is unstable, if $C>0$ the trivial solution is stable. When $C=0$ and $B>0$ the solution may not approach 0, but it stays bounded. For $C=B=0$ the trivial solution is unstable.

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2/6/2012

A classical example of a stability calculation (continued)

Let us first return to equation (47.17), and consider the special case $\nu = 0, \kappa = 0$. The equation then simplifies to

$$\theta_{tt} - \frac{1}{k^2}\theta''_{tt} + \beta g\alpha\, \theta = 0\,. \tag{48.1}$$

Let us denote by $G_k f$ the solution of the problem

$$L_k v = v - \frac{1}{k^2}v'' = f, \qquad v(0) = v(H) = 0\,. \tag{48.2}$$

In other words, G_k is the Green's function of the operator L_k with the (homogeneous) Dirichlet boundary conditions. We can write (48.1) as

$$\theta_{tt} + \beta g\alpha G_k\theta = 0. \tag{48.3}$$

This should be compared with the classical wave equation

$$\vartheta_{tt} + \varkappa(-\vartheta'') = 0 \qquad \vartheta(0) = \vartheta(H) = 0. \tag{48.4}$$

Both G_k and $\vartheta \to -\vartheta''$ (with the Dirichlet boundary conditions) are positive definite operators. We see that for $\beta > 0$ we can think of (48.3) as an equation for waves with unusual dispersion. For $\beta < 0$ the typical solutions of (48.1) exhibit exponential growth in t, but the usual initial-value-problem is still well-posed. (By contrast, for $\varkappa < 0$ the initial-value problem for (48.4) is ill-posed.)

Let us now consider the stability condition $C > 0$ for (47.17) in more detail. The trivial solution $(u, \theta) = (0, 0)$ of (48.9) will be stable if $C = C(k, l) > 0$ for all $k \in \frac{2\pi\mathbf{Z}}{L}$, $l \in \frac{\pi\mathbf{N}}{H}$. Let us determine when this is the case.

Since $C = C(k, l)$ is increasing in l, it is enough to consider the case $l = \bar{l} = \frac{\pi}{H}$. For a given k this is clearly the most unstable mode. Let us set

$$a = \frac{\bar{l}}{k} = \frac{L}{2H}\frac{1}{m}\,, \tag{48.5}$$

where $m \in Z$. Since C depends only on k^2, we can consider $m \in \mathbf{N}$ without loss of generality. The condition $C(k, \bar{l}) > 0$ can be written as

$$-\frac{\beta g\alpha H^4}{\nu\kappa} < \pi^4 a^{-4}(1 + a^2)^3\,. \tag{48.6}$$

The expression on the left is called the *Rayleigh number*, and denoted by R,

$$R = -\frac{\beta g\alpha H^4}{\nu\kappa} = \frac{(\Theta_0 - \Theta_1)g\alpha H^3}{\nu\kappa}\,, \tag{48.7}$$

where Θ_0 is the temperature at $x_2 = 0$ and Θ_1 is the temperature at $x_2 = H$. The reader can check that R is dimensionless, independent of the choice of units of length, time, and temperature. The stability condition $C(k,l) > 0$ for each k, l can now be written as

$$R < \pi^4 \min_{a \in \{\frac{L}{2Hm},\, m \in \mathbf{N}\}} a^{-4}(1+a^2)^3 \,. \tag{48.8}$$

Clearly a sufficient condition for stability will be

$$R < \pi^4 \min_{a>0} a^{-4}(1+a^2)^3 \,. \tag{48.9}$$

The reader can check that

$$\min_{a>0} a^{-4}(1+a^2)^3 = a^{-4}(1+a^2)^3|_{a^2=2} = \frac{27}{4} \,. \tag{48.10}$$

Hence we arrive at the *Rayleigh stability criterion*

$$R < \frac{27\pi^4}{4} \,. \tag{48.11}$$

We note that the minimum in (48.8) is well-approximated by the minimum in (48.9) when the length of the fluid layer L is large in comparison with its height H. From this we can see that the criterion is sharp in the limit $L \to \infty$, and still quite close to optimal for when L is large in comparison with H.

From the above calculations we also see that the most unstable modes will be of the form

$$\cos\left(\frac{\pi(x_1 - \overline{x}_1)}{aH}\right) \sin\left(\frac{\pi x_2}{H}\right) \,, \tag{48.12}$$

where a is the minimizer in (48.9) (well-approximated by $\sqrt{2}$ for L/H large) and $\overline{x}_1$ is arbitrary. The corresponding flow pattern can be seen from (48.12) and (47.14) and can be easily visualized by calculating the stream function of the flow. The flow pattern will show characteristic "convective cells", quite similar to the Taylor vortices (also called Taylor cells) observed in the Taylor-Couette experiment. When the geometry is not really periodic in x_1 but the width of the layer is large in comparison with H, we will still see convective cells similar to (47.13) away from the vertical boundaries, and $\overline{x}_1$ will be fixed by the position of the vertical boundaries. A precise investigation of the influence of the boundaries and the type of boundary conditions which are chosen would be non-trivial.

49

2/8/2012

49.1 Galerkin approximation and the Lorenz system

Computer simulations aiming to investigate what happens with the solutions of the *non-linear* Boussinesq system (47.6) after the trivial solution $u = 0, \theta = 0$ loses stability lead in the early 1960s to the discovery of the *Lorenz attractor*, one of the canonical examples of the surprising chaotic behavior of solutions of quite simple dynamical systems.[246]

Before explaining the connection, we first discuss the Galerkin approximation of evolution PDEs. We have already used this type of approximation in the context of steady solutions of the Navier-Stokes equation in lecture 44. Let now discuss the approximation in the context of the time-dependent Navier-Stokes equations (in a smooth domain $\Omega \subset \mathbf{R}^3$, or a torus $\Omega = \mathbf{R}^3/\frac{2\pi\mathbf{Z}^3}{L}$ for some $L > 0$.)

We wish to find a finite-dimensional approximation of the time-dependent Navier-Stokes in Ω

$$\begin{aligned} u_t + u\nabla u + \tfrac{\nabla p}{\rho} - \nu\Delta u &= f(x,t)\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,, \qquad (\text{if } \partial\Omega \neq \emptyset)\,. \end{aligned} \tag{49.1}$$

Let us chose a finite-dimensional space $V \subset H$, where H is again the set of all div-free vector fields with $u|_{\partial\Omega} = 0$ and finite $\int_\Omega |\nabla u|^2$. The procedure is similar to (44.4): we simply consider the term $u_t + u\nabla u$ as a part of f in the linear problem (44.1), with the understanding that we solve the problem for each time. In other words, we are seeking a function $u\colon [0,T) \to V$ such that

$$\int_\Omega (u_t v + \nu\nabla u\nabla v + u\nabla u\, v - fv)\, dx = 0, \quad v \in V, \quad t \in [0,T). \tag{49.2}$$

If $v^{(1)}, \dots, v^{(m)}$ is a basis of V, we can write

$$u(x,t) = \xi_1(t)v^{(1)}(x) + \cdots + \xi_m(t)v^{(m)}(x) \tag{49.3}$$

and (49.2) can then be expressed in the form

$$\dot\xi_i = -\nu a_{ij}\xi_j + b_{ijk}\xi_j\xi_k + c_i\,, \tag{49.4}$$

where the summation convention is understood, the matrix a_{ij} is positive definite[247] and $b_{ijk}\xi_i\xi_j\xi_k = 0$ for any $\xi \in \mathbf{R}^m$. Multiplying (49.4) by ξ_i, we obtain

[246] Edward N. Lorenz, "Deterministic Non-periodic Flow", Journal of the Atmospheric Sciences 20 (2): 130141, (1963).
Barry Saltzman, "Finite amplitude free convection as an initial value problem - I", Journal of the Atmospheric Sciences 19: 329–341, (1962).

[247] When Ω is a torus the matrix can have a non-trivial one-dimensional kernel, but this does not cause problems and for the moment we ignore it.

(at least in the case when Ω is a bounded domain with boundary)

$$\frac{d}{dt}|\xi|^2 \le -2\gamma|\xi|^2 + |c||\xi| \le -\gamma|\xi|^2 + \frac{1}{4\gamma}|c|^2 \tag{49.5}$$

for some $\gamma > 0$, which shows that when the function $t \to |c(t)|^2$ is integrable on $[0,T)$, the solution of (49.4) is well-defined on $[0,T)$ for any initial condition $\overline{\xi} \in \mathbf{R}^m$. The same calculation can be made at the level of (49.2). Since (49.2) is satisfied for each t for each $v \in V$, we can replace v with $u(x,t)$. Assuming we can use (42.7), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u(x,t)|^2\,dx + \nu\int_\Omega |\nabla u(x,t)|^2\,dx &= \int_\Omega f(x,t)u(x,t)\,dx \\ &\le \left(\int_\Omega |f(x,t)|^2\,dx\right)^{\frac{1}{2}}\left(C\int_\Omega |\nabla u(x,t)|^2\,dx\right)^{\frac{1}{2}} \\ &\le \frac{\nu}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + \frac{C}{2\nu}\int_\Omega |f(x,t)|^2\,dx\,. \end{aligned}\tag{49.6}$$

(This calculation has to be adjusted when Ω is a torus $\int f(x,t)\,dx \neq 0$, but we will ignore this detail at this point.)

If the initial condition for (49.1) is $u_0(x)$, the natural condition for (49.2) or (49.4) is given by

$$\int_\Omega u(x,0)v(x)\,dx = \int_\Omega u_0(x)v(x)\,dx\,. \tag{49.7}$$

In other words, $u(x,0) = u_V(x,0)$ is the L^2 projection of u_0 in V.

What happens if, for a given u_0 and f we take a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset H$ with the property (44.15), and consider the sequence of the solution $u_j = u_{V_j}$? Do the Galerkin solutions u_j converge? This is an important and difficult open problem closely related to the regularity problem. Partial answers are provided by the theory of weak solutions, due to Leray and Hopf, which we will discuss at some point. For our purposes at the moment the information we have obtained above about the Galerkin approximations is enough.

We now turn our attention to the Boussinesq system (47.6). We can choose a suitable finite-dimensional space V of div-free fields for the velocity field u and a suitable space W for the temperature, and the definition of the Galerkin approximation is similar to (49.2). However, if we wish to impose the Navier boundary condition $u_2 = 0, \operatorname{curl} u = 0$ at $\partial\Omega$ we have to make some adjustments. Let us first consider the linear problem (44.1) with the Navier boundary condition. The variational formulation is: minimize

$$\tilde{J}(u) = \int_\Omega (\nu e_{ij}(u)e_{ij}(u) - fu)\,dx \tag{49.8}$$

over the space $\tilde{H}$ of div-free vector fields u in Ω with $u_2|_{\partial\Omega} = 0$ finite $\int_\Omega |\nabla u|^2\, dx$, where e_{ij} is the deformation tensor defined in lecture 21

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})\,. \tag{49.9}$$

This leads to the weak formulation: find $u \in \tilde{H}$ such that

$$\int_\Omega (2\nu e_{ij}(u)e_{ij}(v) - fv)\; dx = 0, \quad v \in \tilde{H}\,. \tag{49.10}$$

Let H_0 be the space of all functions θ with $\int_\Omega |\nabla\theta|^2\, dx$ finite and $\theta|_{\partial\Omega} = 0$.

The variational formulation of the Boussinesq system (47.6) then is

$$\begin{array}{rcll} \int_\Omega (u_t v + u\nabla u v + 2\nu e(u)_{ij}e(v)_{ij} - g\alpha\theta v_2\, dx & = & 0\,, & v \in \tilde{H}\,, \\ \int_\Omega (\theta_t\vartheta + u\nabla\theta\vartheta + \kappa\nabla\theta\nabla\vartheta + \beta u_2\vartheta & = & 0\,, & \vartheta \in H_0. \end{array} \tag{49.11}$$

For a Galerkin approximation we choose a finite-dimensional subspaces $V \subset \tilde{H}$ and $W \subset H_0$, and seek

$$u\colon [0,T) \to V, \quad \theta\colon [0,T) \to W \tag{49.12}$$

such that

$$\begin{array}{rcll} \int_\Omega (u_t v + u\nabla u v + 2\nu e(u)_{ij}e(v)_{ij} - g\alpha\theta v_2\, dx & = & 0\,, & v \in V\,, \\ \int_\Omega (\theta_t\vartheta + u\nabla\theta\vartheta + \kappa\nabla\theta\nabla\vartheta + \beta u_2\vartheta & = & 0\,, & \vartheta \in W. \end{array} \tag{49.13}$$

For finite-dimensional V, W this will be a system of $ODEs$. The initial condition for the ODE can be given by prescribing $u(x,0)$ and $\theta(x,0)$.

The system (47.2) has some natural estimates: for example, by the maximum principle for the heat equation (with drift), the solution θ will be bounded (if it is bounded initially) and from this we can get a bout for u, similarly as in (49.6). We will not discuss this in detail for now, but instead we focus our attention on a particular finite-dimensional approximation, following E. Lorenz.

Let us now consider wave numbers $k \in \frac{2\pi\mathbf{Z}}{L}$ and $\frac{l\in\pi\mathbf{N}}{H}$. We take the space V to be one-dimensional, generated by most unstable mode of the linearized system (48.9):

$$V = \mathbf{R}(\frac{l}{k}\cos kx_1 \cos lx_2, \sin kx_1 \sin lx_2) = \mathbf{R}\overline{u}\,. \tag{49.14}$$

The space W will be chosen two-dimensional. First, we include in it the temperature field θ associated (up to a multiplicative factor) with the mode $\overline{u}$ above in the context of the linearized analysis:

$$\overline{\theta}_1 = \sin kx_1 \sin lx_2\,. \tag{49.15}$$

Second, we add the field

$$\overline{\theta}_2 = \sin 2lx_2\,. \tag{49.16}$$

The space W will be given as

$$W = \mathbf{R}\overline{\theta}_1 + \mathbf{R}\overline{\theta}_2 \,. \tag{49.17}$$

We search the solution (u, θ) as

$$u = A(t)\overline{u}(x), \qquad \theta = B(t)\overline{\theta}_1(x) + C(t)\overline{\theta}_2(x) \,. \tag{49.18}$$

The Galerkin approximation will now be an ODE for the functions $A(t), B(t), C(t)$. After some calculation, we get

$$\begin{array}{rcl} \dot{A} &=& -\nu(k^2+l^2)A + g\alpha\frac{k^2}{k^2+l^2}B \,, \\ \dot{B} &=& -\beta A - \kappa(k^2+l^2)B + lAC \,, \\ \dot{C} &=& -4\kappa l^2 C - \frac{l}{2}AB \,. \end{array} \tag{49.19}$$

Note that the linear part of the system splits into a 2×2 system for A, B and a separate equation for C, reflecting the fact that at the linear level the different Fourier modes do not interact. The stability condition for the trivial solution of the linearized system for (A, B) is exactly Rayleigh condition

$$\nu\kappa k^4(1 + \frac{l^2}{k^2})^3 + \beta g\alpha > 0 \,, \tag{49.20}$$

as it should be.

By a change of variables of the form

$$A(t) = ax(\gamma t), \quad B(t) = by(\gamma t), \quad C(t) = cz(\gamma t) \,, \tag{49.21}$$

where a, b, c, γ are suitable parameters, we can transform system (49.19) into a "canonical form". We use the traditional notation (σ, ρ, β) for the parameters in the canonical form, with the understanding that the β does not have the same meaning as in (49.19) or (47.6). The canonical form is[248]

$$\begin{array}{rcl} \dot{x} &=& -\sigma x + \sigma y \,, \\ \dot{y} &=& \rho x - y - xz \,, \\ \dot{z} &=& -\beta z + xy \,. \end{array} \tag{49.22}$$

In this system some of the features of the original Boussinesq system (47.6) are still visible:

- the terms $-\sigma x, -y$ and $-\beta z$ play the role of $-\nu\Delta u$ and $-\kappa\Delta\theta$.
- the term σy plays the role of the buoyancy $g\alpha\theta e_2$
- the term ρx plays the role of the "background convective term" $-\beta u_2$. (Recall that the β in (47.6) has a different meaning β in (49.22).) In particular a situation when the fluid layer is heated from below corresponds to $\rho > 0$, with larger ρ corresponding to higher temperature at the bottom.
- the nonlinear terms $-xz$ and zy represent the convective term $u\nabla\theta$. The convective term $u\nabla u$ does not appear in the Galerkin truncation we use here. It produces terms which are orthogonal to the 1d space of velocities we are using.

[248] See the original paper of E. Lorenz quoted above.

50

2/9/2012

50.1 The Lorenz system

We will have a closer look at the Lorenz system

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y\,, \\ \dot{y} &= \rho x - y - xz\,, \\ \dot{z} &= -\beta z + xy\,. \end{aligned} \tag{50.1}$$

The role of the various terms was explained last time. We assume that the parameters σ, ρ, β are all positive and satisfy some further constraints which will be specified during our calculations. The "classical values" of the parameters for which Lorenz observed to now well-known chaotic behavior of the solutions are

$$\sigma = 10, \qquad \beta = \frac{8}{3}, \qquad \rho = 28\,. \tag{50.2}$$

All trajectories are attracted to some bounded region

We note that the change of variables

$$(x, y, z) \to (x, y, z + z_0) \tag{50.3}$$

changes the system to

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y\,, \\ \dot{y} &= (\rho + z_0)x - y - xz\,, \\ \dot{z} &= -\beta z + xy\,. \end{aligned} \tag{50.4}$$

Taking $z_0 = -\rho - \sigma$, the system will be of the form

$$\dot{\xi}_i = -a_{ij}\xi_j + a'_{ij}\xi_j + b_{ijk}\xi_j\xi_k + c_i \tag{50.5}$$

with a_{ij} positive definite, a'_{ij} anti-symmetric, and $b_{ijk}\xi_i\xi_j\xi_k = 0$. One can now use (49.5) to see that all trajectories of (50.5) are attracted to some fixed ball. Going back to the original variables (x, y, z) we see that all trajectories are attracted to a ball centered at $(0, 0, z_0)$. (Note that this change of variables is somwhat similar to the change of variables $\theta \to \Theta$ for the original Boussinesq system. It is easier to see the energy estimate for (47.2) than for (47.6).)

The trivial equilibrium and its stability

The point $0 = (0, 0, 0)$ is trivially an equilibrium. The linearization at this equilibrium is given by the matrix

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix} \tag{50.6}$$

Under our assumption $\sigma, \beta, \rho > 0$, the eigenvalues of this matrix are in the half-plane $\operatorname{Re} z < 0$ if and only if

$$\rho < 1\,. \tag{50.7}$$

This corresponds to the Rayleigh stability criterion (48.6). It is easy to see that for $\rho \leq 1$ the trivial equilibrium is the only equilibrium of the system. One can also show that in this case the dynamics is very simple: all trajectories are attracted to the trivial equilibrium.

Non-trivial equilibria and their stability

For $\rho > 1$ the system has (under our assumptions) exactly three equilibria. The trivial one, and

$$q = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1), \qquad q' = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1)\,. \tag{50.8}$$

The linearization of the system at q, q' is given respectively by the matrices

$$L = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -a \\ a & a & -\beta \end{pmatrix}\,, \qquad L' = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & a \\ -a & -a & -\beta \end{pmatrix}\,, \qquad a = \sqrt{\beta(\rho-1)}\,. \tag{50.9}$$

We have

$$\det(\lambda I - L) = \det(\lambda - L') = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\,, \tag{50.10}$$

with

$$a_1 = (\sigma+\beta+1) > 0\,, \qquad a_2 = (\rho+\sigma)\beta > 0\,, \qquad a_3 = 2\sigma\beta(\rho-1) > 0\,. \tag{50.11}$$

It is easy to check that for ρ just above 1, the polynomial (50.10) has three strictly negative roots. This means that for ρ just above 1, the equilibria q, q' are linearly stable.

The equilibria q, q' correspond to the classical picture of "convective cells" in the fluid (similar to Taylor cells in the Taylor-Couette flow) , with the velocity field given respectively by $\varepsilon\overline{u}$ or $-\varepsilon\overline{u}$ for $\varepsilon \sim \sqrt{\rho-1}$.

What happens to the roots of (50.10) as we increase ρ? Can they cross from $\operatorname{Re} z < 0$ into $\operatorname{Re} z > 0$? Clearly the polynomial (50.10) always has at least one strictly negative root, and $\lambda = 0$ is not a root for $\rho > 1$. Therefore the only way the roots can cross the imaginary axis when $\rho > 1$ is that a pair of two distinct complex conjugate roots $\lambda, \overline{\lambda}$ crosses the imaginary axis away from zero. If this happens, we have $\lambda = i\tau, \overline{\lambda} = -i\tau$ for some $\tau > 0$ and

$$\tau^2 = a_2, \qquad \tau^2 = \frac{a_3}{a_1}, \tag{50.12}$$

and hence $a_3 = a_1 a_2$. When $\sigma - \beta - 1 > 0$, which we will assume, this amounts to

$$\rho = \rho_{\text{crit}} = \frac{\sigma(\sigma+\beta+3)}{(\sigma-\beta-1)}\,. \tag{50.13}$$

For this value of ρ we will have pair of complex conjugate roots $\lambda, \overline{\lambda}$ on the imaginary axis. To calculate what happens with them as we change ρ, we can calculate

$$\dot{\lambda} = \frac{d\lambda}{d\rho} \tag{50.14}$$

by taking a derivative of

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0\,. \tag{50.15}$$

For the range of parameters we will be interested in we obtain

$$\operatorname{Re}\dot{\lambda} > 0\,, \tag{50.16}$$

and we see that at ρ given by (50.13) the root cross from $\operatorname{Re} z < 0$ into $\operatorname{Re} z > 0$ with $\operatorname{Re}\dot{\lambda} > 0$ (for the range of parameters we are investigating).

Hence at ρ_{crit} given by (50.13) we expect a Hopf bifurcation.

One can also calculate the value $\rho_{\text{r}} \in (1, \rho_{\text{crit}})$ of ρ for which the roots of (50.10) cease to be real. With complex eigenvalues the equilibria q, q' will be approached via damped oscillations around them. To calculate ρ_{r}, one can use the condition that the cubic discriminant

$$a_1^2 - 4a_2^2 - 4a_1^3a_3 - 27a_3^2 + 18a_1a_2a_3 \tag{50.17}$$

vanishes, which gives a quadratic equation for ρ.

In the regime $\rho \in (1, \rho_{\text{crit}})$ the global dynamics is still relatively simple. "Most trajectories" approach either q or q'. There is a 2d stable manifold of the trivial equilibrium, which separates the two basins of attraction. However, we may also have unstable periodic orbits, which of course would not be attracted to any of the equilibria. Hence the statement that the three equilibria attract all the trajectories, which can be found in some texts, may be inaccurate. (A standard numerical simulation would typically not detect unstable periodic orbits, at least when we do not expect them.)

The Hopf bifurcation at $\rho = \rho_{\text{crit}}$

From the examples of fluid flows we have discussed previously, it would not be unreasonable to expect a loss of stability of q, q' to a periodic orbit close to them, similarly to what happens in Hopf's model discussed in lectures 41, 42. This may be the case for some values of σ and β, but it does not happen for many other values of those parameters, including the classical values $\sigma = 10, \beta = \frac{8}{3}$. For those values the Hopf bifurcation at ρ_{crit} is subcritical: an unstable periodic orbit exists for ρ just below ρ_{c}, and as ρ approaches ρ_{crit}, the orbit shrinks to 0 (at the rate $\sim \sqrt{\rho_{\text{crit}} - \rho}$). We will discuss this type of Hopf bifurcation somewhat more in the next lecture. For now we just accept the conclusion that there is no stable periodic orbits near q, q' for ρ just above ρ_{crit}.

The behavior of solutions for ρ just above ρ_{crit}, first observed by E. Lorenz in early 1960s, is now one of the classical classical examples of "Chaos". Typical

solutions may first be somewhat attracted to q or q' along their stable manifolds, but eventually they are repelled along the unstable manifolds, gradually swirling away from the corresponding equilibrium (due to the complex eigenvalues). Typical trajectories visit neighborhoods of both q and q' in a seemingly random fashion. If you type "Lorenz attractor" into an image search engine, you will get many pictures of the trajectories.

51

2/15/2012

51.1 Hopf bifurcations

In general, the theory of the Hopf bifurcation is concerned with the following situation: we have a system

$$\dot{x} = f(x, \varepsilon) \tag{51.1}$$

where $x \in \mathbf{R}^n$ and f is a (sufficiently regular) vector field depending (sufficiently smoothly) on a parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, with

$$f(0, \varepsilon) = 0. \tag{51.2}$$

For simplicity we will not discuss the exact regularity requirements, we can simply assume that f is smooth in (x, ε). One can also consider the situation when $\mathbf{R}^n$ is replaced by an infinite-dimensional space, but we will not consider this generalization here. The linearization of the system (51.1) at the trivial equilibrium $x = 0$ is

$$\dot{x} = L(\varepsilon)x\,, \tag{51.3}$$

where $L(\varepsilon)$ is the matrix $D_x f(x, \varepsilon)|_{x=0}$. The matrix $L(\varepsilon)$ is obviously real, and hence its non-real eigenvalues come in pairs $\lambda, \overline{\lambda}$. We will assume that the spectrum of $L(0)$ contains exactly one such pair $\lambda_0, \overline{\lambda}_0$ on the imaginary axis, and that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ these eigenvalues can be "continued in ε", so that the matrix $L(\varepsilon)$ has a pair of eigenvalues $\lambda(\varepsilon), \overline{\lambda}(\varepsilon)$ depending smoothly on ε, which coincide with $\lambda_0, \overline{\lambda}_0$ at $\varepsilon = 0$. The other eigenvalues of $L(\varepsilon)$ are assumed to stay away from the imaginary axis $\mathbf{R}\sqrt{-1}$ for $\varepsilon \in (\varepsilon_0, \varepsilon_0)$. Moreover, we assume that

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\ \mathrm{Re}\,\lambda(\varepsilon) > 0\,, \tag{51.4}$$

so that the curve $(\varepsilon, \lambda(\varepsilon))$ in $(-\varepsilon_0, \varepsilon_0) \times \mathbf{C}$ is smooth near $\lambda(0)$ and crosses the imaginary axis transversally. Since our statements will be local, we can in fact assume that

$$\frac{d}{d\varepsilon}\,\mathrm{Re}\,\lambda(\varepsilon) \geq \delta > 0\,, \qquad \varepsilon \in (-\varepsilon_0, \varepsilon_0)\,. \tag{51.5}$$

Example 1

Let $\lambda = \lambda(\varepsilon)$ be as above and consider the dynamical system in $\mathbf{R}^2 \sim \mathbf{C}$ given by

$$\dot{z} = (\lambda + c|z|^2)z\,, \tag{51.6}$$

with

$$c = a + bi,\ a, b \in \mathbf{R}, \qquad \lambda(0) = i\omega,\ \omega > 0\,. \tag{51.7}$$

This is a special case of the situation considered above, with $n = 2$. Writing $z = re^{i\theta}$, we obtain from (51.6)

$$\begin{aligned} \dot{r} &= (\operatorname{Re}\lambda + ar^2)r\,, \\ \dot{\theta} &= \operatorname{Im}\lambda + br^2\,. \end{aligned} \tag{51.8}$$

The second equation describes the rate of rotation of the trajectory about the origin and is easy to understand. The first equation, which is independent of θ, is crucial. By re-parametrizing ε if necessary, we can write without loss of generality

$$\operatorname{Re}\lambda = \varepsilon, \qquad \varepsilon \in (-\varepsilon_0, \varepsilon_0) \tag{51.9}$$

and hence

$$\dot{r} = (\varepsilon + ar^2)r\,. \tag{51.10}$$

This equation has to be considered in the region $r \geq 0$. It is natural to distinguish three cases

Case 1: $a < 0$ *super-critical Hopf bifurcation*

In this case the equation (51.10) has a unique equilibrium $r = 0$ for $\varepsilon < 0$, and (still for $\varepsilon < 0$), the equilibrium is stable. For $\varepsilon > 0$ we have two equilibria: $r = 0$, which is unstable, and $r = \sqrt{-\frac{\varepsilon}{a}}$, which is stable. In terms of the original system (51.6): for $\varepsilon < 0$ all trajectories spiral towards the origin. For $\varepsilon > 0$ the origin becomes unstable and the trajectories starting close to it will spiral away from and will approach the stable periodic orbit circling along the circle $r = \sqrt{-\frac{\varepsilon}{a}}$. The trajectories starting with large r will also approach this orbit. In this scenario we say that the trivial equilibrium $z = 0$, which is stable for $\varepsilon < 0$, loses its stability to a periodic orbit for $\varepsilon > 0$, with the amplitude of the oscillations of order $\sqrt{\varepsilon}$.

Case 2: $a > 0$ *sub-critical Hopf bifurcation*

In this case the trivial equilibrium $z = 0$ is again stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$. However, for $\varepsilon > 0$ we have no other equilibrium and all trajectories starting away from the unstable equilibrium will spiral away to ∞. For $\varepsilon < 0$ the equation (51.10) has an unstable equilibrium at $r = r_1 = \sqrt{-\frac{\varepsilon}{a}}$, which represents an unstable periodic orbit. Trajectories starting at $r < r_1$ are attracted by the trivial equilibrium, while the trajectories starting at $r > r_1$ spiral away to ∞.

Case 3: $a = 0$

This is a degenerate case when $\dot{r} = \varepsilon r$. It is not really representative of general systems (51.1). A more interesting equation would be

$$\dot{z} = (\lambda + c_1|z|^2 + c_2|z|^4)z, \qquad c_j = a_j + b_j i\,. \tag{51.11}$$

with $a_1 = 0, a_2 \neq 0$, which the reader can analyze as an exercise.

It can be shown that Example 1 is in some sense representative of the general situation concerning (51.1) described above, with the understanding that the degenerate case corresponding to $a = 0$ can be more complicated.

More precisely, one can show the following:[249]

For (51.1), in the situation described above, one can find in the space (x, ε) a smooth three dimensional submanifold which is invariant under the flow[250] *and system of coordinates (locally, near (0,0)) preserving the planes $\varepsilon =$ const. such that the flow on the submanifold is given by (51.6) up to the terms of order 4.*

In the non-degenerate cases $a > 0$ or $a < 0$ this information is sufficient to determine the flow: up to small deformations, it will be the same as in Example 1. When $a = 1$ one needs to consider higher-order terms, such as in (51.11) (and there are many more possibilities).

The proof of theorem is based on two steps: 1. construction of the invariant manifold and 2. reduction of the resulting 2d equation to the *normal form*.[251]

For the Lorenz system (50.1) with the classical values of the parameters $\sigma = 10, \beta = \frac{8}{3}$, it turns out that the Hopf bifurcation at $\rho = \rho_{\text{crit}}$ discussed in the last lecture is sub-critical. This is not an easy calculation, see the book of Marsden and McCracken quoted above. (For some values of σ, β to bifurcation can be super-critical, see also the book of Marsden and McCracken.) This leads us to two non-trivial conclusions: 1. For ρ close to ρ_{crit} and $\rho < \rho_{\text{crit}}$ the system has non-trivial periodic solutions, and hence not all solutions are attracted to one of the three equilibria (even though two of them are stable).
2. When ρ crosses ρ_{crit}, the solutions may have no stable periodic orbit to bifurcate to (and we already know that there is no stable equilibrium in this regime). This opens a possibility of some interesting behavior of solutions.[252]

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[249] See, for example, Marsden, J. E., McCracken, M., The Hopf Bifurcation and Its Applications, Springer, 1976. This book also contains a translation of Hopf's original 1942 paper.

[250] $\dot{x} = f(x, \varepsilon), \dot{\varepsilon} = 0$

[251] We recommend the book "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields" by J. Guckenheimer and P. Holmes, Springer 1983 (second printing).
The idea of a normal form of an equation $\dot{x} = Lx + g(x)$ where L is a matrix and $g(x)$ is of order at least 2 goes back to Poicaré. One can ask if the equation can be reduced to $y = Ly$ (up to terms of a given order) and try to achieve the reduction by a change of variables $x = h(y)$. In the y variable the equation is $\dot{y} = [Dh(y)]^{-1}(Lh(y) + g(h(y))$ and one can try to get the right-hand side as close as possible to the form Ly by a suitable choice of h. We can try $h_i(y) = y_i + a_{ij}y_j + b_{ijk}y_jy_k + \dots$ and calculate a_i, b_{ij}, etc. Whether or not this is possible depends on the spectral properties of L. If, at a given order, a complete reduction is not possible, one can still try to eliminate as many terms as possible. See the above book for more details.

[252] We should mention that the nature of the Hopf bifurcation at $\rho = \rho_{\text{crit}}$ was probably first calculated analytically only in the book of Marsden and McCracken quoted above, while the interesting behavior for $\rho > \rho_{\text{crit}}$ was discovered numerically by E. Lorenz in 1963. Since then a number of possible types bifurcations and routes to chaos have been investigated. The reader can for consult the paper by Ruelle and Takens quoted in lecture 40, for example.

52

2/15/2012

52.1 Simple models of "Chaos"

The chaotic behavior of the solutions of the Lorenz system mentioned briefly at the end of lecture 50 can be somewhat understood in terms of relatively simple 1d models which we will now discuss.[253] In what follows we have in mind the Lorenz system with the classical values of the parameters $\sigma = 10, \beta = \frac{8}{3}, \rho = 28$. In numerical simulation it is observed that most trajectories approach an object Σ reminiscent of a branched surface. (It is not really a branched surface in the usual sense, though. The object is more complicated and has fractal dimension just above 2. Nevertheless, we will use the term "surface" in what follows.) We can imagine that the action of the flow near that surface typically is as follows:[254]

1. The flow pushes trajectories towards to surface.
2. Once close to the surface, the trajectories are repelled from one another in the direction tangential to the surface.

Let us cut the "surface" Σ by a plane Π transversal to it, and let us consider a "return map" associated with our plane: we take a point $x \in \Pi$ which is close to $\Pi \cap \Sigma$ and move along a trajectory starting at x until we return to Π. The return point is denoted by $f(x)$. We emphasize that we do not intend to define these objects rigorously, there are quite a few "loose ends" in our description at this point.[255] Our intention is only to give a relatively rough idea where the "chaos" is coming from.

Let us now take in Π a very thin strip S containing $\Sigma \cap \Pi$. (We can think of it as a quite thin rectangle of length of order 1, perhaps deformed.) The position of a point in this strip can be quite precisely (although not completely precisely) described by one coordinate (along the length of the strip), let us call it ξ. Assume $\xi \in [a, b] \subset \mathbf{R}$. Let us take a point x with coordinate ξ and consider $f(x)$. Assume that $f(x)$ is again in our strip S and its (approximate) 1d coordinate is η. We will write $\eta = \varphi(\xi)$. (The map φ is only defined up to a small error, but this is sufficient for our purposes.)

[253] There are much more sophisticated and precise models, which we will not discuss, but we can refer the reader to the book of J. Guckenheimer and P. Holmes mentioned in the last lecture and to the work of W. Tucker in which it was proved (by a computer-assisted proof) that the geometric model proposed by J. Guckenheimer does capture the dynamics. See for example `http://www2.math.uu.se/~warwick/main/papers/comptes.pdf`

[254] Here we do not define the notion of "typically" precisely. What we have in mind is that the net result of the flow, when followed along a trajectory over a certain part of its journey, can be modeled by these effects. To make these notions more precise, one should calculate the eigenvalues and eigenvectors of the linearization of the system along typical trajectories.

[255] For example: what exactly is Σ, how do we choose Π, how do we know that a trajectory starting at x will return to Π, ect?

The main point is that the function $\xi \to \varphi(\xi)$ may not be monotone. This can be explained as follows: the trajectories starting in S are pushed away from each other in the direction tangential to S, and hence the original strip can stretch under the evolution. In addition to being stretched, the strip can also be deformed by the flow in other ways, and when the trajectories come back to Π, it will not only have stretched, but it may also have "folded"[256] so that $f(S)$ can be thought of as the original S first stretched and then folded inside (or close to) the original S. We can think of a one dimensional version of the dough preparation for a croissant: we roll the dough flat, then we fold it, roll it out flat again , fold again, and keep repeating the process. The dough has some thickness and is three dimensional, but we can introduce an approximate 2d coordinate $\tilde{\xi}$ and a $2d$ "fold and roll map" $\tilde{\xi} \to \tilde{\varphi}(\tilde{\xi})$, which will not be injective, although, strictly speaking, the real 3d map which describes the process precisely is injective. However, to be able to invert the precise map, we would have to know a position of a particle in the thin direction very precisely.

To get an idea about the long-time dynamics of the trajectories, it is therefore sufficient to investigate the map $\xi \to \varphi(\xi)$. It may be hard to calculate φ from the ODE with some precision without numerical simulation (and, in fact, at this point φ is not even precisely defined), but one can get a reasonably good qualitative idea about what is going on by looking at some specific models.

We see that one can hope for the following picture:

- The original system is a flow in $\mathbf{R}^3$. The "flow map" $x \to \phi^t(x)$ is of course invertible (although not necessarily volume preserving).
- The "return map" f which maps some domain $S \subset \Pi \sim \mathbf{R}^2$ into itself is still injective, but it now represents a discrete dynamical system. While flows defined by $\dot{x} = g(x)$ cannot exhibit chaotic behavior in dimension 2, discrete dynamics defined by (injective) maps $\mathbf{R}^2 \to \mathbf{R}^2$ can.
- The "approximate map" $\varphi\colon [a,b] \to [a,b]$ is 1d, but not injective. While (smooth) injective maps $[a,b] \to [a,b]$ cannot exhibit chaotic behavior,[257] smooth non-injective maps can.

We can summarize this with the following table:

	lowest dimension allowing "chaos"
smooth ODE	$n = 3$
smooth discrete injective dynamics	$n = 2$
smooth discrete non-injective dynamics	$n = 1$

Let us look at the dynamics of 1d mappings. We will change our notation and set $I = [0,1]$, write x for a typical point of I, and consider continuous functions $f\colon I \to I$. We start with the following simple example

[256] because the stretched object is assumed to be pushed to $\Sigma \cap \Pi$ as is approaches Π

[257] The reader can check this as an exercise.

Example 1

$$f(x) = \begin{cases} 2x\,, & 0 \le x \le \frac{1}{2}\,, \\ 2(1-x)\,, & \frac{1}{2} \le x \le 1\,. \end{cases} \tag{52.1}$$

This is perhaps the simplest possible example of "stretching and folding" by a continuous map. Let us denote by $\tilde{f}$ the 1-periodic extension of f to $\mathbf{R}$. It is not hard to see that

$$f^k(x) = \overbrace{f(f...(f(x)))}^{k \text{ times}} = \tilde{f}(2^{k-1}x)\,. \tag{52.2}$$

We see that the position of x after k iteration of f (given by $f^k(x)$) is independent of the first $k-1$ terms of the dyadic expansion $\varepsilon_1\varepsilon_2\varepsilon_3\ldots$ of x. Therefore to predict $f^k(x)$, we have to know x roughly to $k-1$ dyadic places. If we can know x only up to some error δ, we can only predict $f^k(x)$ when

$$2^{k-1}\delta << 1\,. \tag{52.3}$$

This is where the "chaos" and "unpredictability" of $f^k(x)$ for large k comes from.

Example 2 (Logistic map)

$$f(x) = f(x,a) = ax(1-x)\,, \tag{52.4}$$

where $a \in [0,4]$ is a given parameter. The map was introduced in 1845 by Verhulst to model the population of some biological species. Its dynamics was studied in depth in well-known papers by M. Feigenbaum in the 1970s,[258] in which some unexpected and deep universal features of the dynamics of 1d maps were discovered. Here we will only sketch some of the main properties of the map depending on the parameter a. The example is more subtle than Example 1, as the map does not always stretch. (Note that a smooth non-monotone cannot always stretch.)

For $a \le 1$ the sequence $f^k(x)$ approaches 0 for any $x \in I$. For $a > 0$ the map f has a non-trivial fixed point at

$$\overline{x} = \frac{a-1}{a}\,. \tag{52.5}$$

The fixed point will be (locally) stable (for the iteration $f(x), f^2(x), \ldots$) if

$$|f'(\overline{x})| < 1\,, \tag{52.6}$$

which amounts to

$$a < 3\,. \tag{52.7}$$

[258] Journal of Statistical Physics, Vol. 19, No. 1 (1978) and Vol 21, No. 6, 1979

It can be shown that for $a < 3$ the iterations $f^k(x)$ will converge to $\overline{x}$ for any $x \in (0,1)$. Let us set $a_1 = 3$.

For a just above 3 we will observe that $f^k(x)$ typically converges to an orbit of period 2. Such orbits correspond to stable fixed points of the map f^2. These can be calculated, and one can also calculate the condition for their stability. One finds that the 2-periodic orbit is stable up to $a = a_2 = 1 + \sqrt{6}$. At this value of a it loses stability and a stable $4-$periodic orbit appears. The stability of the $4-$periodic orbit persists up to a_3, when it loses stability to a $8-$periodic orbit etc. The effect when the stability of an orbit is lost to an orbit with twice the period is usually called *period doubling* (both for discrete and continuous dynamical systems). The sequence $a_1 < a_2 < a_3, \dots$ can be continued indefinitely, and for $a \in (a_n, a_{n+1})$ the map has a stable 2^n-periodic orbit, which attracts typical iteration sequence $f^k(x)$. There are the unstable fixed point and the unstable 2^k-periodic orbit for $k = 1, \dots, n-1$.

It turns out that

$$a_k \to \overline{a} \sim 3.5700\,. \tag{52.8}$$

The situation for $a > \overline{a}$ is complicated: one can see "chaos" for most (but not all) values of a. If the reader types "logistic map" into an image search engine, she will get many nice pictures of the famous bifurcation diagram capturing the period doubling process and the chaos setting in for most $a > \overline{a}$.

In summary, the logistic map exhibits a surprising level of complexity when viewed from the point of view of dynamical systems.

We should mention the special case $a = 4$. In this case the dynamics $x, f(x), f^2(x), \dots$ is essentially the same as in Example 1, as observed by von Neumann. Let us write F for the map (52.1). We introduce a change of variables

$$x = \sin^2 \frac{\pi}{2}\,\theta, \qquad \theta \in [0,1]\,. \tag{52.9}$$

Then

$$4x(1-x) = 4\sin^2 \frac{\pi}{2}\,\theta\ \cos^2 \frac{\pi}{2}\,\theta = \sin^2 \frac{\pi}{2}\,2\theta = \sin^2 \frac{\pi}{2} F(\theta)\,. \tag{52.10}$$

This shows that for $a = 4$ the logistic map is, modulo a change of variables, the same as Example 1.

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2/17/2012

53.1 Bifurcations and their classification

So far our approach to the loss of stability of the steady solutions of the Navier-Stokes equation, and other systems (including the finite-dimensional ODE systems) was the study of specific examples. One can adapt a more general approach, and try to find in a systematic way various scenarios in which stability can be lost. For concreteness, let us consider a finite-dimensional ODE system

$$\dot{x} = f(x, \varepsilon)\,, \qquad x = (x_1, \dots, x_n)\,, \qquad \varepsilon \in (-\varepsilon_0, \varepsilon_0)\,, \tag{53.1}$$

where f is assumed to be smooth. We assume that we have a curve of equilibria $\overline{x}(\varepsilon)$ defined for $\varepsilon \le 0$, i. e. $f(\overline{x}(\varepsilon), \varepsilon) = 0,\ \varepsilon \le 0$. Let

$$L = L_\varepsilon = D_x f(\overline{x}(\varepsilon), \varepsilon)\,. \tag{53.2}$$

We have a good idea about the behavior of the solutions of the linearized equation

$$\dot{y} = Ly\,. \tag{53.3}$$

It is given by the spectral decomposition of L. In the case when the spectrum does not intersect the imaginary axis, we can decompose $\mathbf{R}^n$ into the spaces of stable and unstable directions respectively, with the space Y_{s} of the stable directions generated by the (generalized) eigenspaces of eigenvalues λ with $\operatorname{Re}\lambda < 0$, and the space Y_{u} of the unstable directions generated by the (generalized) eigespaces of eigenvalues λ with $\operatorname{Re}\lambda > 0$. If we only slightly perturb such a matrix L, the spaces Y_{s} and Y_{u} can get perturbed only slightly, and from the point of view of stability of the solutions, the perturbed situation will be similar to the unperturbed situation.

Equilibria $\overline{x}$ for which the spectrum of L lies away from the imaginary axis are called *hyperbolic equlilibria*. They are in some sense the simplest equilibria. [259]

As we change ε, the most significant changes in the behavior of the solutions of (53.1) (such as loss of stability) near an equilibrium $\overline{x}(\varepsilon)$ will occur when one (or more) of the eigenvalues of L_ε will reach the imaginary axis (and possibly crosses it, although we should also think about the scenario where the curve $\overline{x}(\varepsilon)$ "turns back" at $\varepsilon = 0$, and there may be no equilibria for $\varepsilon > 0$). In general, when this happens, the changes in the dynamics can be complicated. It is reasonable to start the study under suitable "non-degeneracy assumptions" (in a similar way as it is reasonable to start the study of spectral properties of matrices with the case when all eigenvalues are different, and worry about what

[259] For example, one has the *Hartman-Grobman Theorem* stating that locally near a hyperbolic equilibrium $\overline{x}$, the dynamics of (53.1) is topologically conjugate to the linearized dynamics.

happens with multiple eigenvalues only after we understand what happens with simple eigenvalues.) Sometimes we also use the term "transversality conditions".

One can consider the following assumptions

A1 As $\varepsilon \to 0$, only one eigenvalue $\lambda = \lambda(\varepsilon)$ will approach the imaginary axis, with all the other eigenvalues staying away from it. (In particular, this means that the eigenvalue is real, as the complex eigenvalues come in conjugate pairs $\lambda, \overline{\lambda}$.) Moreover, we assume the eigenvalue is simple.[260]

A2 The Hopf bifurcation scenario described in lecture 51.

Scenario A2 was discussed in some detail in lecture 51, and therefore we focus on scenario A1. Note that the in the "generic case" the spectrum should reach/cross the imaginary axis either by scenario A1 or scenario A2. (The most common way when the assumption of "genericity" is violated in a natural setting is when the situation at hand has some symmetries which lead to a nontrivial multiplicity of eigenvalues. These situations are more difficult than the "generic case", and will not be discussed here.)

The bifurcations due to scenario A1 are usually called *co-dimension one bifurcations*. An important point of the theory is that the study of co-dimension one bifurcations can be largely reduced to the study of the case when the space $\mathbf{R}^n$ is one-dimensional, $n = 1$. At a heuristic level, the reason is as follows: under assumption A1 the most important change of the dynamics when the eigenvalue passes through the imaginary axis involves the direction of the corresponding eigenvectors. In this direction we go from the attraction towards the equilibrium to the repulsion from the equilibrium and this is clearly a complete change of behavior. By comparison, the changes affecting the other directions should be less significant. Without going to details, we just state that this heuristics can be made rigorous. This is one of the important applications of the so-called *center manifold theory*.[261]

The simplest co-dimension one bifurcations under assumption A1 in their $n = 1$ representation by differential equations are as follows

$$\dot{x} = \varepsilon + x^2 \quad \text{(saddle-node)} \tag{53.4}$$

$$\dot{x} = \varepsilon x - x^2 \quad \text{(transcritical)} \tag{53.5}$$

$$\dot{x} = \varepsilon x - x^3 \quad \text{(pitchfork)} \tag{53.6}$$

The saddle-node bifurcation can occur when a curve of solutions $x = x(\varepsilon)$ of $f(x, \varepsilon) = 0$ "turns back" at some point ε_1 (with $\varepsilon_1 = 0$ in (53.4)), and cannot be locally continued beyond ε_1. In some sense the this the least degenerate

[260] In the sense that the eigenspace $E(\lambda)$ is one-dimensional and we have the decomposition $\mathbf{R}^n = E(\lambda) \oplus Y(\lambda)$, where $Y(\lambda)$ is invariant under L.

[261] See for example the book of Guckenheimer and Holmes quoted in the previous lecture.

bifurcation, as the rank of the matrix $D_{x,\varepsilon}f$ is the maximal possible at all points of the curve.

The pitchfork bifurcation occurs for example when the trivial equilibrium of the Lorenz system loses its stability at $\rho = 1$, see lecture 50.

We have not really seen a transcritical bifurcation in our previous lectures.

Usually one includes also the Hopf bifurcations discussed in lecture 51 among the co-dimension one bifurcations.

We emphasize that the above list is not a complete list of possible co-dimension one bifurcations. If the first non-zero terms in the Taylor expansion of $f(x, \varepsilon)$ are of higher order than in these examples, one may get more complicated bifurcations (possibly with relaxed assumptions about the way in which the imaginary axis is crossed).

53.2 Periodic solutions and their stability

The stability of periodic solutions can be treated in terms of the so-called Poincaré return map. Let $\overline{x}(t, \varepsilon)$ be a periodic solution of

$$\dot{x} = f(x, \varepsilon)\,. \tag{53.7}$$

Assume the minimal period of the solution is $T > 0$. Let us take a plane Π transversal to the curve $\overline{x}(t)$ at, say, $\overline{x}(0)$. Let y be the coordinates in the plane, with the origin $y = 0$ corresponding to $\overline{x}(0)$. If we take $y \in \Pi$ close to $y = 0$ and take the trajectory of (53.7) starting at y, the trajectory will be close to $\overline{x}(t)$, and hence will intersect Π close to $\overline{x}(0)$ at a time T_1 close to the period T of $\overline{x}(t)$. Let us denote by $F(y) = F(y, \varepsilon)$ this intersections point. By definition, $F(y) = 0$. It is clear that F is well-defined in some neighborhood of the origin $y = 0$. Also, if f is smooth, the map F will be smooth (in a neighborhood of $y = 0$). One can now ask what happen to the periodic orbit when we change ε. Let

$$L = D_y F(0)\,. \tag{53.8}$$

We will also write $L = L(\varepsilon)$ if we wish to emphasize the dependence of L on ε.

If the spectrum of L is inside the unit disc $\{|\lambda| < 1\}$, the periodic orbit will be stable and trajectories of (53.7) starting close to it will be attracted to it. Also, if we slightly change ε, the orbit will deform slightly to a periodic orbit of the system with the new value of ε.

The stability of the orbit can be lost when the spectrum of L crosses the boundary of the unit circle.

Let us assume that at $\varepsilon = 0$ we have a stable periodic orbit, and we start changing ε. The simplest ways the stability of the orbit can be lost is the following:

1. A pair of complex-conjugate eigenvalues $\lambda, \overline{\lambda}$ reaches (or crosses) the circle $\{|\lambda| = 1\}$ away from 1 or -1, while the rest of the spectrum stays inside some smaller circle.

2. One real simple eigenvalue reaches or crosses the circle $\{|\lambda| = 1\}$ at -1, while the rest of the spectrum stays inside some smaller circle.

3. One real simple eigenvalue reaches or crosses the circle $\{|\lambda| = 1\}$ at 1, while the rest of the spectrum stays inside some smaller circle.

The first possibility is an analogue of the Hopf bifurcation. For example, when the stability of the orbit is lost to small secondary oscillations of the solution about the orbit, this corresponds to the super-critical Hopf bifurcation. As the matrix L does not become singular, the equation $F(x) = x$ will still be solvable when the stability is lost, and the original orbit will "survive" (typically with some deformation and a small period change), except that it will become unstable.

The second possibility is called "period doubling". Note that $Ly = -y$ means that $LLy = L^2y = y$, and one can expect a periodic solution with period close to $2T$. This new solution can at first be stable, but can later become unstable to another period doubling. The situation can be similar to the loss of stability of the periodic orbits in the logistic mapping discussed in lecture 52. However, other scenarios are possible. Similarly to the first possibility, the "old orbit" will survive (typically with some deformation and a small change in its period), but will become unstable.

Finally, the third possibility may occur for example in a situation when the periodic orbit will disappear under a small change of the parameter, somewhat similar to the loss equilibria when a curve of solutions "turns back".

Instead of using the linearization of the Poincare map F, one can also work of the linearized equation about the periodic solution. The linearized equation is of the form

$$\dot{\xi} = A(t)\xi\,, \tag{53.9}$$

where the matrix A is periodic with period T. One seeks solutions of (53.9) in the form $\xi(t) = \eta(t)e^{i\omega t}$, where $\eta(t)$ is periodic with period T. This is the topic of the *Floquet theory*,[262] which we however will not discuss at this point, in spite of its importance. Our goal in this lecture was only to show that there is a well-developed theory into which many of our examples and scenarios fit.

[262]Floquet, Gaston, "Sur les équations différentielles linéaires à coefficients périodiques", Annales de l'École Normale Supérieure 12: 47–88, (1883).

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2/20/2012

54.1 Limits of Galerkin approximations of the Navier-Stokes equations.

We return to the situation considered in lecture 49.
Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary or the torus $\mathbf{R}^3/2\pi\mathbf{Z}^3$ [263]
Let a div-free field u_0 in Ω be given. For simplicity we can think of u_0 as being smooth, div-free, and vanishing at $\partial\Omega$, but it is in fact sufficient to assume that $u_0 \in L^2$, div-free and the normal component $u_0 n$ vanishes at the boundary.[264]

For a fixed $T > 0$ we consider the problem of determining a vector field $u = u(x,t)$ in $\Omega \times [0,T]$ which solves (for a suitable p)

$$\begin{aligned} u_t + u\nabla u + \tfrac{\nabla p}{\rho} - \nu\Delta u &= f(x,t)\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,, \qquad (\text{if } \partial\Omega \neq \emptyset)\,, \\ u(x,0) &= u_0(x)\,, \qquad \text{in } \Omega. \end{aligned} \tag{54.1}$$

Ideally we would like $u(x,t)$ to be smooth. However, it is known that for trivial reasons the problem may not have a solution which is smooth at $\partial\Omega \times \{0\}$, similarly to what happens for the heat equation.[265] This is a minor issue which we will discuss later.

We do not know any obstacle which would prevent the existence of a solution which is smooth for $t > 0$, if $f(x,t)$ is smooth. At the same time, we do not know if such a smooth solution exists.

Let us chose a finite-dimensional space $V \subset H$, where H is again the set of all div-free vector fields with $u|_{\partial\Omega} = 0$ and finite $\int_\Omega |\nabla u|^2$. As in lecture 49, we can consider the following problem

$$\int_\Omega (u_t v + \nu\nabla u\nabla v + u\nabla u\, v - fv)\,dx = 0, \quad v \in V, \quad t \in [0,T), \tag{54.2}$$

with the initial condition

$$\int_\Omega u(x,0)v(x)\,dx = \int_\Omega u_0(x)v(x)\,dx \qquad v \in V\,. \tag{54.3}$$

The last condition means that $u(\cdot,0)$ is an L^2 orthogonal projection of u_0 onto V.

[263] More general tori $\mathbf{R}^3/\Lambda$ with rank 3 lattices Λ can also be considered.

[264] For $u_o \in L^2(\Omega)$ with $\operatorname{div} u_0 = 0$ the normal component $u_0 n$ is well-defined, as the reader can check from the formula $\int_{\partial\Omega} u_0 n\varphi\,dx = \int_\Omega u_0\nabla\varphi$.

[265] If $u_t = \Delta u$ in $\Omega\times[0,T]$ and u is smooth, then $\Delta u = 0$ at $\partial\Omega$, and this may not be satisfied at $t = 0$ for $u(x,t)|_{t=0} = u_0(x)$.

The energy inequality (49.6) implies

$$\int_\Omega \frac{1}{2}|u(x,t)|^2\,dx + \int_0^t \int_\Omega \frac{\nu}{2}|\nabla u(x,s)|^2\,dx\,dt \\ \leq \int_\Omega \frac{1}{2}|u_0(x)|^2\,dx + \frac{C}{2\nu}\int_0^t |f(x,s)|^2\,dx\,ds\,. \quad (54.4)$$

Strictly speaking, the energy inequality needs a minor adjustment when Ω is a torus and $\int_\Omega f(x,t)\,dx \neq 0$ or $\int_\Omega u_0 \neq 0$, which we leave to the reader as an exercise.

Let us take a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset H$ with the property (44.15), and consider the sequence of the solution $u^j = u^{V_j}$? Does this sequance converge? This is an important and difficult open problem closely related to the regularity problem. We aim to show that some subsequence of u^j converges weakly to some function u (in suitable spaces) which solves the equation in a weak sense.

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2/22/2012

Limits of Galerking approximations (continued)
Let us set

$$Q = \Omega \times (0,T), \qquad L^{\infty,2} = L^{\infty,2}(Q) = L^\infty(0,T;L^2(\Omega)). \tag{55.1}$$

Recall that

$$H = H(\Omega) = \{u\colon \Omega \to \mathbf{R}^3\,,\ \nabla u \in L^2(\Omega)\,,\ \operatorname{div} u = 0\,,\ u|_{\partial\Omega} = 0\}\,, \tag{55.2}$$

so that we can also write, in terms of the usual Sobolev space notation,

$$H = \{u \in W_0^{1,2}(\Omega, \mathbf{R}^3)\,,\ \operatorname{div} u = 0\}\,. \tag{55.3}$$

We also set

$$L^2_t H_x = L^2_t H_x(Q) = L^2(0,T;H(\Omega))\,. \tag{55.4}$$

The sequence of the approximate solutions u^j constructed at the end of the last lecture is uniformly bounded in both spaces $L^{\infty,2}(Q)$ and $L^2_t H_x$. Therefore we can choose a subsequence, still denoted by u^j, such that for some function $u \in L^{\infty,2} \cap L^2_t H_x$ the (sub)sequence u^j converges weakly* in $L^{\infty,2}$ and weakly in $L^2_t H_x$ to u. (It is easy to see that for sequences bounded in $L^{\infty,2} \cap L^2_t H_x$ the weak* convergence in $L^{\infty,2}$ is equivalent to the weak convergence in $L^2_t H_x$ and this in turn is equivalent to the convergence in distributions.)

We now aim to investigate the consequences of the equation (54.2) satisfied by the approximate solutions for the limit u. The situation is somewhat similar to what we have seen in lecture 44 in the context of steady solutions, except for a complication coming from the fact that we do not have a simple control of the time derivative u^j_t, at least not at the same level as for the spatial gradient ∇u^j.

Let us first slightly reformulate (54.2). We consider a test function $v\colon [0,T] \to V$ which is smooth in t. We will write $v = v(x,t)$. Integrating (54.2) over a time interval $(t_1,t_2) \subset [0,T]$, we obtain

$$\int_\Omega uv\,dx\,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega [-uv_t + u\nabla uv + \nu\nabla u\nabla v - fv]\,dx\,dt = 0\,. \tag{55.5}$$

Integration by parts gives

$$\int_\Omega uv\,dx\,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega [-uv_t - u_j u_i v_{i,j} + \nu\nabla u\nabla v - fv]\,dx\,dt = 0\,. \tag{55.6}$$

Let us set

$$I(t_1,t_2) = I(t_1,t_2;u,v) = \int_{t_1}^{t_2}\int_\Omega \left[-uv_t - u_j u_i v_{i,j} + \nu\nabla u\nabla v - fv\right]\,dx\,dt\,. \tag{55.7}$$

Our goal is to study the continuity properties of I with respect to t_1. We will see soon why this is important. For $0 \le t_1 \le t_2 \le T$ we clearly have

$$I(t_1,t_2) + I(t_2,t_3) = I(t_1,t_3)\,, \tag{55.8}$$

and hence it is enough to look at $I(t_1,t_2)$ for as $(t_2-t_1)\searrow 0$.
One can do the estimate for $I(t_1,t_2)$ with various degree of sophistication, depending on what we are willing to assume about v. For the purpose of establishing an existence theorem for weak solutions, one needs only quite elementary estimates. For example, if we assume that

$$|v_t| \le A\,, \qquad |\nabla v| \le B\,, \qquad |v| \le C\,, \tag{55.9}$$

we have the simple estimates

$$\int_{t_1}^{t_2}\int_\Omega |uv_t|\,dx\,dt \;\le\; A(t_2-t_1)|\Omega|^{\frac12} \sup_{t_1\le t\le t_2}\left(\int_\Omega |u(x,t)|^2\,dx\right)^{\frac12} \tag{55.10}$$

$$\int_{t_1}^{t_2}\int_\Omega |u_j u_i v_{i,j}|\,dx\,dt \;\le\; B(t_2-t_1)\sup_{t_1\le t\le t_2}\int_\Omega |u(x,t)|^2\,dx\,, \tag{55.11}$$

$$\int_{t_1}^{t_2}\int_\Omega \nabla u\nabla v\,dx\,dt \;\le\; B|\Omega|^{\frac12}(t_2-t_1)^{\frac12}\left(\int_{t_1}^{t_2}\int_\Omega |\nabla u|^2\,dx\,dt\right)^{\frac12} \tag{55.12}$$

$$\int_{t_1}^{t_2}\int_\Omega fv\,dx\,dt \;\le\; C|\Omega|^{\frac12}(t_2-t_1)^{\frac12}\left(\int_{t_1}^{t_2}\int_\Omega f^2\,dx\,dt\right)^{\frac12}. \tag{55.13}$$

This shows that with the bounds (54.4) and (55.9), the function $t_1 \to I(t_1,t_2)$ is uniformly continuous in t_1, independently of t_2.

Going back to (55.6) and replacing $v(x,t)$ by $v(x)\eta(t)$ for a $v \in V$ and a suitable smooth function $\eta(t)$, we see that the function

$$t \to \int_\Omega u(x,t)v(x)\,dx \tag{55.14}$$

is uniformly continuous in t.

We now explain at a heuristic level why this conclusion is important. Roughly speaking, the energy estimate (54.4) controls the size of u and the size of possible oscillations of u in the $x-$direction. It does not a-priori give any estimate on the possible oscillations of u in t. Even if our approximate solutions u^j were

bounded with bounded ∇u^j but were wildly oscillating in t, we would have a serious problem in passing to the limit in the equation. We can assume u^j converge weakly to u, but we need to pass to the limit in the non-linear expression $u^j_k u^j_l$ which appears in the equation in one form or another. It is not hard to see that oscillation in t can spoil the convergence $u^j_k u^j_l$ to $u_k u_l$.

The uniform continuity in t of the functions (55.14) puts a constraint on the oscillations in t which will trurn out to be sufficient to pass to the limit.

Note that the estimate for the uniform continuity was obtain from the equation, even though at first the expression for the derivative u_t obtained from the equation does not look very promising. The key point is that test function v in (55.14) can be taken relatively smooth, and the spatial derivatives of u appearing in $\int_\Omega u_t v$ after using the equation can be moved to v by integration by parts.

This is a quite general principle used in evolution PDEs. It was used by J. Leray in his classical 1934 paper as well as by E. Hopf in his well-known 1950 paper on Navier-Stokes. Later the idea was generalized to an abstract setting by J. L. Lions and T. Aubin, and its abstract version is known as Aubin-Lions Lemma.[266]

Assumptions (55.9) are too strong if one wishes to work with general finite-dimensional subspaces $V \subset H$. In that can one can still take for $v(x,t)$ a field $v(x)\eta(t)$, where $v \in V$ and η is a smooth function of t. One can replace estimate (55.11) by

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |u_j u_i v_{i,j}(x)\eta(t)|\,dx\,dt &\le ||\nabla v||_{L^2}||\eta||_{L^\infty}\int_{t_1}^{t_2}||u(t)||^2_{L^4}\\ &\le ||\nabla v||_{L^2}||\eta||_{L^\infty}\int_{t_1}^{t_2}||u(t)||^{\frac{3}{2}}_{L^6}||u(t)||^{\frac{1}{2}}_{L^2}\\ \le C||\nabla v||_{L^2}||\eta||_{L^\infty}\sup_t||u(t)||^{\frac{1}{2}}_{L^2}(t_2-t_1)^{\frac{1}{4}}&\left(\int_{t_1}^{t_2}||\nabla u(t)||^2_{L^2}\,dt\right)^{\frac{3}{4}}, \quad (55.15)\end{aligned}$$

where we have used (45.1) to estimate $||u(t)||_{L^6}$. The replacement for (55.10), (55.12), (55.13) is even easier and is left to the reader as an exercise.

[266] Let $X_1 \subset X_2 \subset X_3$ be Banach spaces such that the unit ball of X_1 is compact in X_2. If a set M of functions $f\colon [0,T] \to X_1$ is bounded in $L^p(0,T;X_1)$ and the set of their derivatives $\{\frac{df}{dt},\ f \in M\}$ is bounded in $L^1(0,T;X_3)$, then the set M is pre-compact in $L^p(0,T,X_2)$.
A typical application is with $X_1 = W^{1,2}, X_2 = L^2, X_2 = W^{-k,2}$ for some large k. The application to the Navier-Stokes equation is not immediate due to the non-local nature of the pressure term.

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Limits of Galerking approximations (continued)

We now use the information about the uniform continuity of the functions (55.14) to show that the weak limit u of the (sub)sequence $u^j = u^{V_j}$ considered in the beginning of lecture 55 gives a solution of the problem (54.1) in a suitable weak sense. In what follows we assume that the whole sequence u^j converges weakly to u. This is no loss of generality, as we can always replace our original sequence by a weakly converging subsequence.

Let us fix $j_0 \in \mathbf{N}$ and set $V = V_{j_0}$. Similarly to (55.5) and (55.6), for any (smooth) test function $v\colon [0,T] \to V$ and any $(t_1, t_2) \subset [0,T]$ we have for $j \geq j_0$

$$\int_\Omega u^j v \, dx \,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_\Omega \left[-u^j v_t + u^j \nabla u^j v + \nu \nabla u^j \nabla v - f v \right] dx \, dt = 0 \,. \quad (56.1)$$

and by integration by parts

$$\int_\Omega u^j v \, dx \,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_\Omega \left[-u^j v_t - u^j_k u^j_l v_{l,k} + \nu \nabla u^j \nabla v - f v \right] dx \, dt = 0 \,. \quad (56.2)$$

Roughly speaking, our goal is to show that as $j \to \infty$ the limit of all the terms in the last equation is obtained simply by replacing u^j by u.

While it is clear that

$$\int_{t_1}^{t_2} \int_\Omega -u^j v_t \, dx \, dt \to \int_{t_1}^{t_2} \int_\Omega -u v_t \, dx \, dt, \qquad j \to \infty \quad (56.3)$$

and

$$\int_{t_1}^{t_2} \int_\Omega \nu \nabla u^j \nabla v \, dx \, dt \to \int_{t_1}^{t_2} \int_\Omega \nu \nabla u \nabla v \, dx \, dt, \qquad j \to \infty \,, \quad (56.4)$$

the convergence of the other terms with u^j is not obvious.

Following the ideas of Leray and Hopf, we will first deal with $\int_\Omega u^j(x,t) v(x) \, dx$ and then use the result, together with additional considerations, to pass to the limit in the non-linear term $\int_{t_1}^{t_2} \int_\Omega u^j_k u^j_l v_{l,k} \, dx \, dt$.

Let us take $v = v(x) \in V$ and set

$$h^j(t) = \int_\Omega u^j(x,t) v(x) \, dx \,, \qquad h(t) = \int_\Omega u(x,t) v(x) \, dx \,. \quad (56.5)$$

Note that while the functions $h^j(t)$ are defined for each t, the function h is at the moment defined only for almost every t. Also note that the functions h^j, h are uniformly bounded as u^j are uniformly bounded in $L^\infty_t L^2_x$.

From the definition of the weak convergence we know that

$$\int_0^T h^j(t)\eta(t)\,dt \to \int_0^T h(t)\eta(t)\,dt, \qquad j\to\infty \tag{56.6}$$

for each smooth test function η. At the same time, in the last lecture we established that the functions h^j are equicontinuous (i.e. they have a common modulus of continuity). Together with their uniform boundedness this implies, by the Arzela-Ascoli Theorem, that the sequence h^j is pre-compact in the topology of uniform convergence. We also know that h^j converge to h weakly* in L^∞ (or weakly in L^2), by (56.6) and the uniform boundedness of h^j. We conclude that

$$\sup_{t\in[0,T]} |h^j(t)-h(t)| \to 0, \qquad j\to\infty\,. \tag{56.7}$$

This conclusion was reached for any fixed $v\in V_{j_0}$. Since j_0 can be chosen arbitrarily and we assume that $\cup_j V_j$ is dense in H, we see that for each t and each $v\in H$

$$\int_\Omega u^j(x,t)v(x)\,dx \to \int_\Omega u(x,t)v(x)\,dx, \qquad j\to\infty\,. \tag{56.8}$$

Together with the uniform boundedness of u^j in $L^\infty_t L^2_x \cap L^2_t H_x$ and the condition $\operatorname{div} u^j = 0$, this implies that in fact

$$\int_\Omega u^j(x,t)b(x)\,dx \to \int_\Omega u(x,t)b(x)\,dx, \qquad j\to\infty, \qquad b=(b_1,b_2,b_3)\in L^2(\Omega)\,. \tag{56.9}$$

(Sketch of proof: we can assume that b is smooth. Let $b = a+\nabla\pi$ be the Helmholtz decomposition of b, with $\operatorname{div} a = 0$ and $a\,n = 0$ at $\partial\Omega$, where n is the normal to $\partial\Omega$. We can replace b by a in (56.9), as $\operatorname{div} u^j = 0$, $\operatorname{div} u = 0$. We claim that H is L^2-dense in $L^2_{\rm div} = \{a\in L^2(\Omega),\ \operatorname{div} a = 0,\ a\,n = 0 \text{ at } \partial\Omega\}$. Otherwise there would be an $a\in L^2_{\rm div}(\Omega)$, $a\neq 0$, such that $\int_\Omega av\,dx = 0$ for each $v\in H$. This means that $a=\nabla\alpha$ for some function $\alpha\in W^{1,2}(\Omega)$ with $\frac{\partial\alpha}{\partial n}=0$ at $\partial\Omega$. Now $\operatorname{div} a = 0$ means that $\Delta\alpha = 0$ and we see that $\alpha = $ const. by the uniqueness for the Neumann problem. This means $a=0$, a contradiction,and the proof of (56.9) is finished.)

We see that the function $x\to u(x,t)$ is well-defined as an L^2 function for *each* $t\in[0,T]$, and, moreover

$$u^j(\cdot,t) \rightharpoonup u(\cdot,t) \quad \text{(weak } L^2 \text{ convergence) for } \textit{each } t\in[0,T]\,. \tag{56.10}$$

It also shows that

$$t\to u(\cdot,t) \quad \text{is continuous as a function from } [0,T]\to(L^2,\text{weak topology})\,. \tag{56.11}$$

All these statements reflect the additional information obtained from the fact that u^j not only satisfy the energy bounds, but they also satisfy an equation

which gives higher regularity in the t direction than the mere boundedness in the energy norm

Concerning the passage to the limit in (56.2) as $j \to \infty$, the above considerations take care of the term $\int_\Omega u^j v \, dx \,|_{t=t_1}^{t=t_2}$. To handle the nonlinear term $\int_{t_1}^{t_2} \int_\Omega u_k^j u_l^j v_{l,k} \, dx \, dt$, we will indicate two approaches. The first approach, due to E. Hopf relies on the following

Lemma (E. Hopf)
Let $Q = \Omega \times [0,T]$. Assume that a sequence of functions $w^j \colon Q \to \mathbf{R}^m$ is bounded in $L_t^\infty L_x^2(Q)$ and $L_t^2 H_x(Q)$ and converges weakly in $L_t^\infty L_x^2$ to a function w. In addition, assume that*

$$w^j(\cdot,t) \rightharpoonup w(\cdot,t) \text{ weakly in } L^2(\Omega) \text{ for each } t \in [0,T]\,. \tag{56.12}$$

Then

$$w^j \to w \text{ strongly in } L^2(Q). \tag{56.13}$$

Proof
The proof follows quite easily from the following inequality due to K. O. Friedrichs: for each $\varepsilon > 0$ there exist finitely many smooth, compactly supported functions $a_k \colon \Omega \to \mathbf{R}^m$, $k = 1, 2, \ldots, r$ such that for each $z \colon \Omega \to \mathbf{R}^m$, $z \in W^{1,2}(\Omega)$ we have

$$\int_\Omega |z|^2 \, dx \le \sum_{k=1}^r |\int_\Omega a_k z \, dx|^2 \, dx + \varepsilon \int_\Omega |\nabla z|^2 \, dx\,. \tag{56.14}$$

Using the inequality with $z(x) = w^j(x,t) - w(x,t)$, for each t and integrating over $t \in [0,T]$, we obtain

$$\int_0^T \int_\Omega |w^j - w|^2 \, dx dt \le \int_0^T \sum_{k=1}^r |\int_\Omega [a_k(w^j - w)] dx|^2 dt + \varepsilon \int_0^T \int_\Omega |\nabla(w^j - w)|^2 \, dx dt\,. \tag{56.15}$$

It is clear that our assumptions imply that the expression on the right-hand side can be made small for high j by choosing first a small ε, then a sufficiently high j.

The lemma shows that, under our assumptions, we can pass to the limit in the term

$$\int_{t_1}^{t_2} \int_\Omega -u_k^j u_l^j v_{l,k} \, dx \, dt\,. \tag{56.16}$$

of (56.2), which was the last term where the passage to the limit was not clear. We see that in the limit $j \to \infty$ we can simply replace u^j by u in (56.2).

Leray used another technique for proving the strong convergence of the approximate solutions u^j. (He worked with different approximations, but his method

can also be applied to the Galerking approximations.) We briefly explain the main idea of Leray's method, which does not rely on Hopf's lemma above.

Let us set

$$e^j(t) = \int_\Omega |u^j(x,t)|^2\,dx\,, \qquad e(t) = \int_\Omega |u(x,t)|^2\,dx\,. \tag{56.17}$$

An important observation from the energy inequality is the following:

$$\textit{The total variation } \int_0^T |\tfrac{de^j}{dt}(t)|\,dt \textit{ of the functions } e^j \textit{ is uniformly bounded.} \tag{56.18}$$

This is obvious in the case $f = 0$ as the functions $e^j(t)$ are monotone (energy is not increasing), and follows quite easily from the energy inequality (54.4) in the general case (under some natural assumptions on f which are satisfied in the situation we consider here).

We can therefore assume, after perhaps passing to another subsequence, if necessary, that the functions e^j converge point-wise to a function e^*:

$$e^j(t) \to e^*(t)\ , \quad j \to \infty, \quad \text{for each } t \in [0,T]. \tag{56.19}$$

Clearly

$$\int_0^T e^j\,dt \to \int_0^T e^*\,dt\,, \qquad j \to \infty\,. \tag{56.20}$$

If we can show that

$$e^*(t) = e(t) \text{ for almost every } t \in [0,T], \tag{56.21}$$

we will know that

$$\int_Q |u^j|^2\,dx\,dt \to \int_Q |u|^2\,dx\,dt\,, \qquad j \to \infty\,, \tag{56.22}$$

and the strong convergence $u^j \to u$ in $L^2(Q)$ will follow.

Let

$$D^j(t) = \int_\Omega |\nabla u^j(x,t)|^2\,dx\,, \qquad D^*(t) = \liminf_{j\to\infty} D^j(t)\,. \tag{56.23}$$

We know from the energy inequality that

$$\int_0^T D^j(t)\,dt \le C < \infty, \qquad j = 1,2,\dots \tag{56.24}$$

and hence

$$\int_0^T D^*(t)\,dt = \int_0^T \liminf_{j\to\infty} D^j(t)\,dt \le \liminf_{j\to\infty} \int_0^T D^j(t)\,dt \le C < \infty \tag{56.25}$$

by Fatou's lemma. In particular, the function $D^*(t)$ is finite for almost every $t \in [0,T]$. If $D^*(t) < \infty$, then since we know that $u^j(\cdot,t) \rightharpoonup u(\cdot,t)$ in L^2 (weak convergence), we can infer from Rellich's compactness theorem that $u^{j'}(\cdot,t) \to u(\cdot,t)$ in $L^2(\Omega)$ for some subsequence of $u^{j'}(\cdot,t)$ of the sequence $u^j(\cdot,t)$. We conclude that $e(t) = e^*(t)$ for each t for which $D^*(t) < \infty$, and this establishes (56.21) and hence also (56.22).

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Limits of Galerkin approximations (continued)
Last time we have shown that for each $v\colon [0,T] \to V_{j_0}$ (sufficiently regular in t) we can pass to the limit $j \to \infty$ in (56.2) and the limit function u will satisfy

$$\int_\Omega uv\,dx\,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega [-uv_t - u_k u_l v_{l,k} + \nu\nabla u\nabla v - fv]\,dx\,dt = 0\,, \quad (57.1)$$

where u is a weak limit of the Galerkin approximations u^j. Note that it is not clear to what degree u is uniquely determined by the procedure, as passing to suitable subsequences is involved. We know, however, that u is sufficiently regular for all the terms in (57.1) to be well-defined. In particular, $u(\cdot,t)$ is well-defined for each $t \in [0,T]$ as an element of $L^2(\Omega)$.

We would now like to remove the restriction $v(\cdot,t) \in V_{j_0}$ for each t and replace it by a more natural assumption. For example, it is natural to consider the class $\mathcal{C}^1_{\mathrm{div},0}$ of test functions $v(x,t)$ which are once continuously differentiable in $\overline{\Omega} \times [0,T]$, div-free, and vanish on $\partial\Omega \times [0,T]$. We wish to show that

Assume that $\cup_j V_j$ is dense in H. Then (57.1) is satisfied for each $v \in \mathcal{C}^1_{\mathrm{div},0}$.

Proof

It is not hard to see that it is enough to prove the statement for $v(x,t)$ of the form

$$v(x,t) = \sum_{k=1}^r \eta^k(t) w^k(x)\,, \quad (57.2)$$

where $w^1,\dots,w^r \in H$ and η^k are $\mathcal{C}^1$ functions $[0,T] \to \mathbf{R}$. We approximate each w^k in H by a sequence $w^{k,j} \in V_j$ and let

$$v^j(x,t) = \sum_{k=1}^r \eta^k(t) w^{k,j}(x)\,. \quad (57.3)$$

Clearly

$$v^j \to v \text{ in } L^\infty_t H_x \quad (57.4)$$

and

$$\int_\Omega uv^j\,dx\,|_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2}\int_\Omega \left[-uv^j_t - u_k u_l v^j_{l,k} + \nu\nabla u\nabla v^j - fv^j\right] dx\,dt = 0\,. \quad (57.5)$$

Passing to the limit $j \to \infty$ we obtain the desired result. The passage $j \to \infty$ is obvious in all terms except perhaps for the term $\int_{t_1}^{t_2}\int_\Omega -u_k u_l v^j_{l,k}\,dxdt$. To see that we can pass to the limit in this term it is enough to take into account that the function $u_k u_l$ is bounded in $L^1_t L^3_x$. (In fact, we have a stronger information:

the function is bounded in $L^\infty_t L^1_x \cap L^1_t L^3_x$, and hence also in $L^p_t L^q_x$ for $1 \le p \le \infty$, $2/p + 3/q = 3$.)

The function u we constructed above is usually called a *Leray-Hopf solution* of the problem (54.1).

It is not hard to see that

If a Leray-Hopf solution is a sufficiently regular function, then it is a classical solution of (54.1) for a suitable pressure function $p(x,t)$.

Sketch of proof

Reversing the integrations by parts involved in the contruction, we obtain that

$$\int_\Omega [u_t(x,t) + u(x,t)\nabla u(x,t) - \nu\Delta u(x,t)]v(x)\eta(t)\,dx\,dt = 0 \tag{57.6}$$

for each smooth, compactly supported, div-free vector field v and each smooth $\eta\colon [0,T] \to \mathbf{R}$. Hence, for each $t \in [0,T]$ and each v as above we have

$$\int_\Omega [u_t(x,t) + u(x,t)\nabla u(x,t) - \nu\Delta u(x,t)]v(x)\,dx = 0\,. \tag{57.7}$$

This shows that

$$u_t(x,t) + u(x,t)\nabla u(x,t) - \nu\Delta u(x,t) = -\nabla p(x,t) \tag{57.8}$$

for some function $p(x,t)$ (which is determined up to an arbitrary function of t).

The condition $u(x,0) = u_0(x)$ follows from the construction of u: for $t_1 = 0$ and $t_2 = T, v(x,T) = 0$ the identity (57.5) is satisfied with the first term replaced by $\int_\Omega u_0(x)v(x,0)\,dx$, and the integration by parts then shows that $u(x,0) = u_0(x)$.

We state without proof the following result which was proved due to contributions by Leray, Ladyzhenskaya, Prodi, Serrin, Caffarelli-Kohn-Nirenberg, and Seregin.

If a Leray-Hopf solution u belongs to $L^5_{t,x}(Q)$, then it is "as regular in $\overline{\Omega} \times [0,T]$ as f allows".[267] *In particular, if f is smooth in $\overline{\Omega} \times [0,T]$, then u is smooth in $\overline{\Omega} \times (0,T]$.*

In general it is not known if the Leray-Hopf solutions constructed for smooth data u_0, f will be smooth. Perhaps an even more serious drawback is that *it is not known if the Leray-Hopf solutions are unique, even when u_0 and f are*

[267] In the context of the usual parabolic equations this could be taken locally, at least at the level of C^∞ regularity (but not necessarily analytic regularity). The situation with the Navier-Stokes equations is more non-local, and one has to be somewhat careful concerning local regularity. However, if f is be everywhere smooth then u will also be everywhere smooth in $\overline{\Omega} \times (0,T]$, as long as the assumption $u \in L^5_{t,x}(Q)$ is satisfied.

smooth. Without uniqueness we cannot be sure how much predictive power the equations really have.

It is not known how to rule out some seemingly bizarre scenarios. For example, let us consider the following situation. Recall that in the last lecture we used the Galerkin approximations to define the functions $e(t)$ and $e^*(t)$, see (56.17) and (56.19). *Assume that $e^*(t)$ is a smooth decreasing function on $[0, T]$. Is it possible that $e(t) = e^*(t)$ at all points except some $t_1 \in (0, T)$ where we have $e(t_1) = 0$?*

It seems that the answer to this question is not known. It is known that such scenario cannot take place if $u \in L^4_{t,x}(Q)$, for example.

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Limits of Galerkin approximations (continued)

Let us in more detail at the energy inequality for the weak solution u. For the Galerkin approximations u^j we have, as can be seen for example from (56.1),

$$\int_\Omega \frac{1}{2}|u^j(x,t_1)|^2\,dx = \int_\Omega \frac{1}{2}|u^j(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega [\ \nu|\nabla u^j|^2 + f u^j\ ]\,dx\,dt\,. \quad (58.1)$$

Using the notation (56.17), it is clear that if

$$e(t_1) = e^*(t_1), \quad (58.2)$$

we obtain from (58.1) as $j \to \infty$

$$\int_\Omega \frac{1}{2}|u(x,t_1)|^2\,dx \geq \int_\Omega \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega [\ \nu|\nabla u|^2 + f u\ ]\,dx\,dt\,. \quad (58.3)$$

From this it is easily seen that under the assumption (58.2) we have

$$\limsup_{t\to {t_1}^+} e(t) \leq e(t_1)\,. \quad (58.4)$$

At the same time we have from the weak lower semi-continuity of the norm and the weak convergence $u(\cdot,t) \rightharpoonup u(\cdot,t_1)$ as $t \to t_1$

$$\liminf_{t\to t_1} e(t) \geq e(t_1)\,. \quad (58.5)$$

In other words, the function $t \to e(t)$ is always lower semi-continuous. We conclude that the function $e(t)$ will continuous from the right at any point t with $e(t) = e^*(t)$. Clearly this also means that the function

$$t \to u(\cdot,t) \quad (58.6)$$

will be continuous from the right as a function $[0,T] \to L^2(\Omega)$ (with the norm topology) at such points. In particular, these considerations imply that

$$u(\cdot,t) \to u_0, \qquad t \to 0^+ \qquad \text{(strong convergence in } L^2\text{)}. \quad (58.7)$$

Considerations concerning the energy inequality (58.3) and the energy identity obtained from it by replacing the inequality by identity can be linked to condition $u \in L^4_{t,x}(Q)$. In particular, it can be shown that

If $u \in L^4_{t,x}(Q)$, then (58.3) holds true with equality for each $0 \leq t_1 \leq t_2 \leq T$.

The proof of this statement can be derived from the linear theory of the Stokes system

$$\begin{aligned} u_t + \tfrac{\nabla p}{\rho} - \nu\Delta u &= \operatorname{div} f\,, \\ \operatorname{div} u &= 0\,, \\ u|_{\partial\Omega} &= 0\,, \qquad (\text{if } \partial\Omega \neq \emptyset)\,, \\ u(x,0) &= u_0(x)\,, \qquad \text{in } \Omega. \end{aligned} \tag{58.8}$$

Here we use the usual notation $f = \{f_{kl}\}_{k,l=1}^{3}$ and $(\operatorname{div} f)_k = f_{kl,l}$.
This form of the equation, with the right-hand side in the "divergence form", is quite natural in the context of the energy identity

$$\int_\Omega \frac{1}{2}|u(x,t_1)|^2\,dx = \int_\Omega \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega [\,\nu|\nabla u|^2 + f\nabla u\,]\,dx\,dt\,. \tag{58.9}$$

The identity suggests that $f \in L^2_{t,x}(Q)$ should a natural condition if we wish to estimate the solution u in $L^\infty_t L^2_x \cap L^2_t H_x$. This is easy for solutions which have enough regularity so that the usual derivation of (58.9) is rigorous. It can be also shown that (58.9) is still true for distributional solutions of (58.8) which belong to $L^\infty_t L^2_x \cap L^2_t H_x$ (and, in fact, to $L^2_{t,x} \cap L^2_t H_x$). In this case one can show that the function $t \to u(\cdot,t)$ is continuous as a function from $[0,T]$ to $L^2(\Omega)$, in the strong topology and one has (58.9). This is a non-trivial exercise in the linear theory which we will not pursue. The reader interested in these issues in the context of parabolic equations can consult the book of Ladyzhenskaya-Solonnikov-Uraltseva on parabolic equations. (We also discussed some of these topics in the PDE course last year, see the online course notes, lecture 60, p. 237.)

If we now have a weak solution of the Navier-Stokes equation with $u \in L^4_{t,x}(Q)$, we can think of it as a weak solution of (58.8) with $f_{kl} = -u_k u_l$ (at least when the right-had side f in the Navier-Stokes vanishes). The condition $u \in L^4_{t,x}(Q)$ ensures $f_{kl} \in L^2_{t,x}$ and the validity of (58.9). With the condition $u \in L^\infty_t L^2_x \cap L^2_t H_x \cap L^4_{t,x}$ it is not hard to see that

$$\int_{t_1}^{t_2}\int_\Omega -u_k u_l u_{k,l}\,dx = 0\,, \tag{58.10}$$

and hence (when the right-hand side in the Navier-Stokes vanishes)

$$\int_\Omega \frac{1}{2}|u(x,t_1)|^2\,dx = \int_\Omega \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega \nu|\nabla u|^2\,dx\,dt\,. \tag{58.11}$$

It is easy to incorporate the Navier-Stokes right-hand side and derive

$$\int_\Omega \frac{1}{2}|u(x,t_1)|^2\,dx = \int_\Omega \frac{1}{2}|u(x,t_2)|^2\,dx + \int_{t_1}^{t_2}\int_\Omega [\,\nu|\nabla u|^2 + fu]\,dx\,dt\,. \tag{58.12}$$

for (54.1) (when $u \in L^\infty_t L^2_x \cap L^2_t H_x \cap L^4_{t,x}$ and $f \in L^2_{t,x}$). In this case the energy identity holds exactly, even though the regularity of the solutions (and also their uniqueness) is unknown. With the current knowledge one can only obtain regularity/uniqueness with additional assumptions, such as $u \in L^5_{t,x}$.

58.1 Imbeddings of the energy space

It is useful to recall the imbeddings

$$L_t^\infty L_x^2 \cap L_t^2 H_x \hookrightarrow L_t^p L_x^q \,. \tag{58.13}$$

Let us first consider the case of dimension $n = 3$. We have for $2 \le q \le 6$

$$||v||_{L_x^q} \le ||v||_{L_x^2}^\alpha ||v||_{L_x^6}^{1-\alpha} \le C^{1-\alpha} ||v||_{L_x^2}^\alpha ||v||_{H_x}^{1-\alpha}\,, \quad \alpha \in [0,1], \quad \frac{1}{q} = \frac{\alpha}{2} + \frac{1-\alpha}{6}\,. \tag{58.14}$$

Hence

$$\int_0^T ||u(t)||_{L_x^q}^p \, dt \le C^{(1-\alpha)p} ||u||_{L_t^\infty L_x^2}^{\alpha p} \int_0^T ||u(t)||_{H_x}^{(1-\alpha)p} \, dt \,. \tag{58.15}$$

Taking

$$(1-\alpha)p = 2, \quad \frac{1}{q} = \frac{\alpha}{2} + \frac{1-\alpha}{6}\,, \tag{58.16}$$

we obtain that (58.13) is valid for

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \qquad 2 \le q \le 6\,. \tag{58.17}$$

For $p = q$ we obtain

$$p = q = \frac{10}{3}\,. \tag{58.18}$$

Hence

$$L_t^\infty L_x^2 \cap L_t^2 H_x \hookrightarrow L_{t,x}^{\frac{10}{3}}\,. \tag{58.19}$$

This is optimal at the level of the $L_{t,x}^p$ spaces. We see that the information from the energy space is insufficient to obtain $u \in L_{t,x}^4$ which – as we have seen above – would be sufficient for the validity of the exact energy identity.

Next time we will discuss dimension $n = 2$, where the situation is much more favorable.

59

3/7/2012

59.1 Weak solutions in dimension $n = 2$

We will use the notation

$$Q_t = \Omega \times (0, t) \tag{59.1}$$

and

$$[\![u]\!] = [\![u]\!]_{Q_t} = ||u||_{L_t^\infty L_x^2 \cap L_t^2 H_x} , \tag{59.2}$$

which is the same as

$$[\![u]\!]^2 = [\![u]\!]^2_{Q_t} = \operatorname*{ess\,sup}_{t' \in [0,t]} \int_\Omega |u(x,t)|^2 \, dx + \int_0^t \int_\Omega |\nabla u(x,t')|^2 \, dx \, dt' . \tag{59.3}$$

In dimension $n = 2$ we have an important inequality[268]

$$||u||_{L^4_{t,x}(Q_T)} \leq C [\![u]\!]_{Q_T} , \qquad u \in L_t^\infty L_x^2 \cap L_t^2 H_x \ . \tag{59.4}$$

Sketch of proof

By Gagliardo-Nirenberg inequality and Cauchy-Schwartz inequality we have (when $n = 2$) for each t:

$$\int_\Omega |u|^4 \, dx = ||uu||^2_{L^2_x} \leq c_2 ||\nabla(uu)||^2_{L^1_x} \leq 4c_2 ||u||^2_{L^2_x} ||\nabla u||^2_{L^2_x} . \tag{59.5}$$

Integrating over t we obtain

$$\int_0^T \int_\Omega |u|^4 \, dx \, dt \leq 4c_2 ||u||^2_{L_t^\infty L_x^2} ||\nabla u||^2_{L_t^2 L_x^2} . \tag{59.6}$$

and the result follows.

We come to the important conclusion that in dimension $n = 2$ the natural energy norm controls the $L^4_{t,x}$ norm. Therefore, by our considerations in the last lecture, in dimension $n = 2$ the weak solution satisfy the energy identity (58.12) exactly, for each $0 \leq t_1 \leq t_2 \leq T$. In fact, similar consideration can be used to obtain the following important result, essentially due to O. A. Ladyzhenskaya:

Theorem

In dimension $n = 2$ the Leray-Hopf weak solution u of (54.1) with the properties established above is unique. If $\cup_j V_j$ is dense in H, then the Galerkin approximations u^j converge to u in the energy norm, without the need to pass to a subsequence.

[268] due to O. A. Ladyzhenskaya, who used the term "multiplicative inequality"

Sketch of proof

Assume we have two solutions u, v and set $w = u - v$. Then $u, v, w \in L^\infty_t L^2_x \cap L^2_t H_x$, $w(x,0) = 0$ and

$$w_t - \Delta w + \nabla q = -\operatorname{div}(u \otimes w + w \otimes v + w \otimes w)\,. \tag{59.7}$$

The energy consideration concerning the linear problem (58.8) together with the fact that $u \otimes w, w \otimes v, w \otimes w$ belong to $L^4_{t,x}$ imply that

$$\int_\Omega \frac{1}{2}|w(x,t)|^2\,dx + \int_0^t \int_\Omega |\nabla w|^2\,dx\,dt' = \int_0^t \int_\Omega w_j v_i w_{i,j}\,dx\,dt'\,. \tag{59.8}$$

Note that all integrals in this identity are well-defined. From (59.8) we infer

$$[\![w]\!]^2_{Q_t} \le c||v||_{L^4_{t,x}(Q_t)}[\![w]\!]^2_{Q_t} \tag{59.9}$$

for some fixed constant $c > 0$. The function $\phi(t) = ||v||_{L^4_{t,x}(Q_t)}$ is uniformly continuous in t on $[0,T]$ and vanishes at $t = 0$. We see that that $w = 0$ in some interval $[0, t_1]$ where $t_1 \ge \tau > 0$, with τ depending only on c and the modulus of continuity of ϕ. The procedure can be repeated with $t = 0$ replaced by $t = t_1$ and we conclude that w vanishes on $[0, t_2]$ with $t_2 \ge 2\tau$. It is clear that after finitely many steps we obtain $w = 0$ in $[0, T]$

It turns out that in the case $n = 2$ which we are considering one can in fact show that the solution u is "as smooth as the data allow". In particular, if f is smooth in Q_T then u will be smooth in $\Omega \times (0, T]$. The precise proof of this result in domains with boundaries requires some work. We will discuss the reasons behind this result in the next lectures (although we will not go into all technical details).

Regardless of the regularity result, the uniqueness theorem shows that for $n = 2$ we have identified the right class of solutions. In general, if one can show for an evolution PDE both existence and uniqueness of solutions in a suitable class (and sufficiently general initial data), we know that the equation can be used to make predictions.

Regularity is also related to uniqueness due to the so-called weak-strong uniqueness theorems (the first version of which can already be found in Leray's 1934 paper). In dimensions $n \le 3$ one can show if the equation has a regular solution, then any Leray-Hopf weak solution has to coincide with the regular solution. (This means, among other things, that the Galerking approximation will converge to the solution in the energy norm and we do not have to pass to a subsequence, assuming that $\cup_j V_j$ is dense in H, of course).

59.2 Green's function for the Stokes system in $\mathbf{R}^n \times (0, \infty)$.

When discussing regularity, it is usually easier to work in the whole space $\mathbf{R}^n$ rather than in domains with boundaries, as boundary regularity brings additional difficulties. In addition to the advantage of not having to deal with the boundary, the case $\Omega = \mathbf{R}^n$ also has the advantage that the solution of the linear problem (58.8) (called the Cauchy problem when $\Omega = \mathbf{R}^n$) can be represented by quite explicit kernels.

Let us first consider the Cauchy problem in $\mathbf{R}^n \times (0, \infty)$

$$\begin{aligned} u_t + \nabla p - \Delta u &= f\,, \\ \operatorname{div} u &= 0\,, \\ u(x,0) &= u_0(x)\ . \end{aligned} \tag{59.10}$$

The system (59.10) has to be supplemented by additional conditions on u and/or p to ensure uniqueness. without additional conditions the system admits "parasitic solutions" of the form

$$u(x,t) = \nabla h(x,t), \quad \Delta_x h(x,t) = 0, \quad p(x,t) = -h_t(x,t)\,, \tag{59.11}$$

where the dependence of h on t can be arbitrary (and can be chosen so that $u(x,0) = 0$). If one works with functions f which have some decay as $x \to \infty$, one can impose the condition

$$u(x,t) \to 0, \qquad x \to \infty \tag{59.12}$$

which will obviously rule out the parasitic solutions (59.11). We will first derive the representation formula for the solutions of (59.10) under the assumption that f has some decay as $x \to \infty$ and (59.12).

One can consider three special cases of (59.10).

Case 1: $f = 0$.
In this case the solution can be obtained by setting $p = 0$ and setting u to be equal to the solution of the heat equation with the initial condition $u(x,0) = u_0(x)$.

$$u(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t) u_0(y)\, dy\,, \tag{59.13}$$

where Γ is the heat kernel

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\, e^{-\frac{|x|^2}{4t}}\ . \tag{59.14}$$

Case 2: $u_0 = 0\,, \quad f(x,t) = \nabla \phi_x(x,t)$.
In this case we have $u = 0$ and $p(x,t) = \phi(x,t) + c(t)\,.$

Case 3: $u_0 = 0$, div $f = 0$.
In this case the solution is again given by the heat kernel, via the Duhamel's principle applied to Case 1:

$$u(x,t) = \int_0^t \int_{\mathbf{R}^n} \Gamma(x-t, t-s) f(y,s)\, dy\, ds\,. \tag{59.15}$$

The general case can be considered as a superposition of these three cases. For a general f we write (for each t)

$$f = Pf + \nabla_x \phi\,, \tag{59.16}$$

where $f \to Pf$ is the Helmholtz projection on the div-free fields (see lecture 6). We can write

$$u(t) = \Gamma(t) * u_0 + \int_0^t \Gamma(t-s) * Pf(s)\, ds, \qquad \nabla p(t) = (f(t) - Pf(t))\ , \tag{59.17}$$

where we use $*$ to denote the spatial convolution:

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y) g(y)\, dy\,. \tag{59.18}$$

The kernel of the operator $f \to \int_0^t \Gamma(t-s) * Pf(s)\, ds$ can be written as follows. Let $G(x)$ be the fundamental solution of the Laplace operator and set

$$\Phi(x,t) = \int_{\mathbf{R}^n} G(y) \Gamma(x-y, t)\, dy\,. \tag{59.19}$$

Note that for $n \geq 3$ we have

$$\Phi(x,t) = \frac{1}{t^{\frac{n-2}{2}}} F\left(\frac{|x|}{\sqrt{t}}\right)\,, \tag{59.20}$$

where $F(r)$ is s smooth function with decay $\sim r^{-(n-2)}$ as $r \to \infty$.

The reader can check that (59.17) can be written as

$$u_i(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t) u_{0i}(y)\, dy + \int_0^t \int_{\mathbf{R}^n} k_{ij}(x-y, t-s) f_j(y,s)\, dy\, ds\ , \tag{59.21}$$

with

$$k_{ij}(x,t) = \left(-\delta_{ij}\Delta + \frac{\partial^2}{\partial x_i \partial x_j}\right) \Phi(x,t)\,. \tag{59.22}$$

It is easy to see that

$$|k_{ij}(x,t)| \leq \frac{C}{(|x|^2 + t)^{\frac{n}{2}}} \tag{59.23}$$

and

$$|\nabla_x k_{ij}(x,t)| \leq \frac{C}{(|x|^2 + t)^{\frac{(n+1)}{2}}}\,. \tag{59.24}$$

Homework Assignment 4

due March 29, 2012

Let $L > 0$ and let $\Omega = [0, L]$. Consider the equation

$$u_t = u_{xx} + au - bu^3$$

for $u = u(x, t)$ in $\Omega \times (0, \infty)$ with the boundary condition $u(0, t) = u(L, t) = 0,\ t \geq 0$. We assume $a, b > 0$.

a) Investigate the linearized stability of the trivial steady-state solution $u = 0$.

b)* (Optional) Describe all steady-state solutions and determine their stability. (Hint: write the equation for the steady states as $u'' = -\frac{\partial V}{\partial u}$, in which x can be thought of as time and u as a position of a particle of unit mass in potential field V. The solutions can be written in terms of elliptic functions, but you do not have to give these formulae. It is enough to describe the solutions qualitatively, e. g. by their oscillations.)

c)* (Optional) Show that the equation has no solutions which would be periodic in t other then the steady states. (Hint: note also that if we let $I(u) = \int_\Omega (\frac{1}{2}|u_x|^2 - \frac{a}{2}u^2 + \frac{b}{4}u^4)\, dx$, we can think of our equations as $\dot{u} = -\text{grad } I(u)$.)

d)** (Optional) Try to guess when the equation has bounded solutions defined in $\Omega \times (-\infty, \infty)$ which are not steady states. (Hint: think about the same question for an ODE of the form $\dot{x} = -\nabla f(x)$, where f is a function on $\mathbf{R}^n$ with $f(x) \to \infty$ when $x \to \infty$.)

60

3/9/2012

60.1 Estimates for solutions of the linear Stokes Systems

We have seen Navier-Stokes equation (with zero right-hand side) can be written as (59.10) with $f_i = -\partial_j(u_i u_j)$. Our goal is to obtain certain a-priori estimates from bounded solutions of the Navier-Stokes equations. Therefore it is natural to consider solutions of (59.10) when f is replaced by $\operatorname{div} F$, where F is a bounded function. We will write $F = \{F_{ij}\}$. For $f = \operatorname{div} F$ formula (59.21) can be written as

$$u_i(x,t) = \int_{\mathbf{R}^n} \Gamma(x-y,t)u_{0i}(y)\,dy + \int_0^t \int_{\mathbf{R}^n} K_{ijl}(x-y,t-s)F_{jl}(y,s)\,dy\,ds\ , \tag{60.1}$$

where

$$K = \{K_{ijl}\} = \{k_{ij,l}\}\,, \tag{60.2}$$

or, more symbolically,

$$K = \nabla k\,. \tag{60.3}$$

We will now obtain estimates for u from (60.1), assuming both that the initial condition u_0 and the function F are bounded, and that $t = T$ for some fixed $T > 0$ which will be assumed to be of order $T \sim 4$. The first term behaves represent the solution of the heat equation with the initial data u_0, and its derivatives are therefore easily seen to be bounded by

$$|\partial_t^l \nabla^k u(x,t)| \le \frac{C_{k,l}}{t^{\frac{k}{2}+l}} ||u_0||_{L^\infty}, \quad x \in \mathbf{R}^n,\ \ t > 0 \quad k,l = 0,1,2,\dots \tag{60.4}$$

We now wish to estimate contribution to

$$|u(x,t) - u(x',t')| \tag{60.5}$$

coming from the second term. We will assume $t > t'$ and estimate

$$|u(x,t) - u(x',t)|, \qquad |u(x',t) - u(x',t')|\,. \tag{60.6}$$

For the first expression, we can assume $x = 0$ and $x' = \alpha e$, where $|e| = 1$. Clearly it is enough to work under the assumption

$$|F| \le 1 \tag{60.7}$$

and estimate

$$I = \int_0^t \int_{\mathbf{R}^n} |K(-y,s) - K(\alpha e - y,s)|\,dy\,ds\,. \tag{60.8}$$

We use the scaling invariance of K

$$K(\lambda x, \lambda^2 t) = \lambda^{-n-1} K(x,t) \tag{60.9}$$

and set

$$y = \alpha z, \quad s = \alpha^2 \tau \tag{60.10}$$

to obtain

$$I = \alpha \int_0^{\frac{t}{\alpha^2}} \int_{\mathbf{R}^n} |K(-z,\tau) - K(e-z,\tau)|\, dz\, d\tau\,. \tag{60.11}$$

For $0 \le \tau \le 2$ we will write

$$|K(-z,\tau) - K(e-z,\tau) \le |K(-z,\tau)| + |K(e-z,\tau)| \tag{60.12}$$

while for $\tau > 2$ we can write

$$|K(-z,\tau) - K(e-z,\tau)| \le \frac{C}{(|z|^2+\tau)^{\frac{n+2}{2}}}\,. \tag{60.13}$$

Hence (setting $z = \sqrt{\tau}\, z'$ when integrating (60.13)) we see that

$$I \le \int_0^2 + \int_2^{\frac{t}{\alpha^2}} \cdots \le C_1\alpha + C_2\alpha \int_2^{\frac{t}{\alpha^2}} \frac{d\tau}{\tau}\,. \tag{60.14}$$

For $t = T \sim 4$ this means that

$$I \le C\alpha \left(1 + \log_+ \frac{1}{\alpha}\right)\,. \tag{60.15}$$

In the calculation of $|u(x',t) - u(x',t')|$ we can assume $x' = 0$. Letting $t' = t - \tau$, we obtain

$$|u(0,t) - u(0,t-\tau)| \le J = \int_0^t \int_{\mathbf{R}^n} |K(-y,t) - K(-y,t-\alpha^2)|\, dy\, dt\ , \tag{60.16}$$

where we set $K = 0$ for $t < 0$. We set $t = \alpha^2\tau$ and $y = \alpha z$. We obtain

$$J = \alpha \int_0^{\frac{t}{\alpha^2}} \int_{\mathbf{R}^n} |K(-z,\tau) - K(-z,\tau-1)|\, dz\, d\tau. \tag{60.17}$$

Writing again

$$J = \alpha \left[\int_0^2 \cdots + \int_2^{\frac{t}{\alpha^2}} \ldots\right] \le \alpha \left[\int_0^2 \cdots + \int_2^{\infty} \ldots\right] \tag{60.18}$$

and using

$$|K(-z,\tau) - K(-z,\tau-1)| \le \frac{C}{(|z|^2+\tau)^{\frac{n+3}{2}}}\ , \qquad \tau > 2 \tag{60.19}$$

(for a suitable $C > 0$), we obtain that

$$J \le \tilde{C}\alpha\,. \tag{60.20}$$

We see that the function $u(x,t)$ given by (60.1) and $t \sim 4$ may fail to be Lipschitz only by a logarithmic factor, and that for a given $R > 0$ it is $\alpha-$Hölder continuous for $t \in [2,4], x \in B_R$ for any exponent $\alpha \in (0,1)$.

61

3/19/2012

61.1 The local well-posedness of the Navier-Stokes equations in subcritical spaces

The representation formula (60.1) together with estimates (59.23) and (59.24) can be also used for an easy proof of existence and uniqueness of the so-called *mild solutions* of the Cauchy problem in $\mathbf{R}^n \times [0,T)$ for

$$\begin{aligned} u_t + u\nabla u + \nabla p - \Delta u &= f\,, \\ \operatorname{div} u &= 0\,, \\ u(x,0) &= u_0(x)\ . \end{aligned} \tag{61.1}$$

The mild solutions are as smooth as the data allow (they have the same smoothness as the solutions of the linear Stokes system with the same data) and are unique in some natural classes of solution. However, it is only known how to prove their local-in-time existence, the global existence remains open, in general. Mild solution differ by definition from the *strong solutions* [269], but in practice they are more or less the same. (This must of course be proved from the precise definitions.)

In this section we will assume $f = 0$ for simplicity. The reader can adjust the proofs to cover the case $f \neq 0$ as exercise. To motivate the definition, we will write (61.1) with $f = 0$ as

$$\begin{aligned} u_t + \nabla p - \Delta u &= -\operatorname{div}(u \otimes u)\,, \\ \operatorname{div} u &= 0\,, \\ u(x,0) &= u_0(x)\ . \end{aligned} \tag{61.2}$$

Let

$$U(t) = \Gamma(t) * u_0 \tag{61.3}$$

be the solution of the heat equation with the initial condition u_0. (Here we think of U and the heat kernel Γ as functions of t with values in a suitable space of functions of x. We will often use this notation in what follows.) Using the notation (60.2), we can re-write (61.2) as

$$u(t) = U(t) + \int_0^t K(t-s) * [-u(s) \otimes u(s)], ds\ , \tag{61.4}$$

where $*$ denotes the spatial convolution.[270] For functions u, v on $\mathbf{R}^n \times [0,T]$

[269] which might be defined as functions for which the derivatives entering the equations are continuous and the equation is satisfied point-wise, together with some requirements for behavior as $|x| \to \infty$

[270] $f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\,dy\,.$

(the exact class of the function will be specified later) we set

$$B(u,v)(t) = \int_0^t K(t-s) * [-u(s) \otimes v(s)]\, ds \tag{61.5}$$

and re-write (61.2) as

$$u = U + B(u,u)\,. \tag{61.6}$$

This will be considered as an abstract equation in a suitable function space of functions on the space-time $\mathbf{R}^n \times (0,T)$.

Our approach will be based on the following well-known abstract lemma.

Lemma

Let X be a Banach space and let $B\colon X \times X \to X$ be a continuous bilinear form on X with

$$||B(x,y)|| \le \gamma ||x||\, ||y||\,. \tag{61.7}$$

For $a \in X$ consider the equation

$$x = a + B(x,x)\,. \tag{61.8}$$

When

$$4\gamma\, ||a|| < 1\ , \tag{61.9}$$

the equation (61.8) has a unique solution $\overline{x}$ in the ball

$$\left\{ x \in X,\ ||x|| < \frac{1+\sqrt{1-4\gamma||a||}}{2\gamma} \right\}\,. \tag{61.10}$$

Moreover,

$$||\overline{x}|| \le \frac{1-\sqrt{1-4\gamma||a||}}{2\gamma}\,. \tag{61.11}$$

Remark

The less precise statement that under the condition (61.9) the equation (61.8) has a unique solution in the ball

$$\left\{ x,\ ||x|| \le \frac{1}{2\gamma} \right\} \tag{61.12}$$

is often sufficient.

Sketch of proof of the Lemma

Essentially the proof in the general case can be understood by analyzing the case $X = \mathbf{R}$ and $B(x,y) = \gamma xy$. In that case our equation is

$$x = a + \gamma x^2\,. \tag{61.13}$$

The necessary and sufficient condition for (61.13) to have a solution is of course

$$4\gamma a < 1 \tag{61.14}$$

in which case the roots are

$$\xi_{1,2} = \frac{1 \pm \sqrt{1 - 4\gamma a}}{2\gamma} \,. \tag{61.15}$$

For general Banach space X we first note that for each (small) $\delta > 0$ there exists $\varepsilon > 0$ such

$$||a + B(x,x)|| \leq ||a|| + \gamma ||x||^2 < ||x|| - \varepsilon \tag{61.16}$$

when

$$\xi_2 + \delta < ||x|| < \xi_1 - \delta \tag{61.17}$$

with

$$\xi_{1,2} = \frac{1 \pm \sqrt{1 - 4\gamma ||a||}}{2\gamma} \,. \tag{61.18}$$

Moreover, we have

$$||a+B(x,x)-a-B(y,y)|| = ||B(x-y,x)+B(y,x-y)|| \leq \gamma(||x||+||y||)\,||x-y|| \,. \tag{61.19}$$

This shows that the map

$$\phi \colon X \to X \tag{61.20}$$

defined by

$$\phi(x) = a + B(x,x) \tag{61.21}$$

is a contraction on the ball $\{x,\, ||x|| \leq \rho\}$ for any $\xi_2 \leq \rho < \frac{1}{2\gamma}$. We see that for any $x \in X$ with $||x|| < \xi_1$ the iteration

$$x_0 = x, \quad x_{k+1} = \phi(x_k) \tag{61.22}$$

will produce a sequence converging to a fixed point $\overline{x}$ with $||\overline{x}|| \leq \xi_2$. Due to (61.16) and (61.19) this fixed point is unique in $\{x,\, ||x|| < \xi_1\}$, and the lemma is proved.

To apply the lemma to (61.1), we have to specify the space X. There are many choices, with various levels of sophistication.[271] Here we will consider only the very simple case

$$X = X_T = L^\infty(\mathbf{R}^n \times (0,T)) \,. \tag{61.23}$$

In this case we can consider $u_0 \in L^\infty(\mathbf{R}^n)$ and by maximum principle for the heat equation we have

$$||U||_X \leq ||u_0||_{L^\infty} \,. \tag{61.24}$$

[271] For an elegant choice based on more advanced harmonic analysis estimates see for example the paper by H. Koch and D. Tataru, Adv. Math. 157 (2001), no. 1, 22–35.

For $u, v \in X$ we have

$$||B(u,v)||_{X_T} \leq ||u||_{X_T} ||v||_{X_T} \int_0^T \int_{\mathbf{R}^n} |K(x,t)|\, dx\, dt \tag{61.25}$$

Using (59.24), a calculation similar to those we did in the last lecture (but easier) shows that

$$\int_0^T \int_{\mathbf{R}^n} |K(x,t)|\, dx\, dt \leq C\sqrt{T}\,, \tag{61.26}$$

for a suitable $C > 0$, the exact value of which is not important to us at this point. Hence

$$||B(u,v)||_{X_T} \leq C\sqrt{T}||u||_{X_T} ||v||_{X_T}\,, \tag{61.27}$$

and we see that the lemma above can be applied when

$$4C\sqrt{T}||u_0||_{L^\infty} < 1\,. \tag{61.28}$$

The lemma can now be used to obtain a local-in-time existence (and uniqueness) of the solution of (61.6) in the space X_T. We note that in the case $X_T = L^\infty(\mathbf{R}^n \times [0,T])$ we are considering our local-in-time solution will in general not satisfy

$$||u(t) - u_0||_{L^\infty(\mathbf{R}^n)} \to 0, \qquad t \to 0_+\,. \tag{61.29}$$

This cannot be satisfied even for the solution of the linear heat equation $U(t)$ if the initial datum u_0 is not uniformly continuous. However, the possible failure of (61.19) is only due to the first (linear) term in the decomposition $u = U + B(u,u)$. The second term will approach 0 in $L^\infty(\mathbf{R}^n)$ as $\sim \sqrt{t}$ as $t \to 0_+$ by (61.25). Also, by calculations from the last lecture we know that $B(u,u)$ will be Hölder continuous.

The uniqueness statement in the lemma, together with suitable "localization in time" can be used to obtain the following statement:

For each $u_0 \in L^\infty(\mathbf{R}^n)$ and $U(t) = \Gamma(t) * u_0$ the equation $u = U + B(u,u)$ has at most one solution in X_T for any $T > 0$ (not necessarily small).

Sketch of proof

Let u, v be two different solutions in X_T. The uniqueness statement in the lemma implies that $u = v$ in $\mathbf{R}^n \times (0, t_1)$ for some $t_1 = t_1(||u||_{X_T}, ||v||_{X_T})$. The main point now is that this argument can be "continued in t". Heuristically this is clear: since the solutions coincide at $t = t_1$, we can take t_1 as a new initial time and the same argument should give that $u = v$ on $[t_1, 2t_1]$. We can now continue this argument until we reach T. In this argument we treat (61.6) essentially as an ODE in t. We have to show that this is indeed justified. This

amounts to showing the following statement: assume that $0 < t_1 < t_2$ and that a function $u \in L^\infty(\mathbf{R}^n \times (0, t_2)$ solves

$$u(t) = \Gamma(t) * u(0) + \int_0^t K(t-s) * F(s)\, ds, \quad t \in [0, t_1]\,, \tag{61.30}$$

$$u(t) = \Gamma(t-t_1) * u(t_1) + \int_{t_1}^t K(t-s) * F(s)\, ds\,, \quad t \in [t_1, t_2]\,, \tag{61.31}$$

where $F \in L^\infty(\mathbf{R}^n \times (0, t_2))$. Then

$$u(t) = \Gamma(t) * u_0 + \int_0^t K(t-s) * F(s)\, ds, \quad t \in [0, t_2]\,. \tag{61.32}$$

Roughly speaking, this says that solutions of (61.6) can be "glued together" on adjacent time intervals $[0, t_1], [t_1, t_2]$, assuming their value at $t = t_1$ is the same. We leave the proof of the statement to the interested reader as an exercise. (Note that by the remarks above concerning the regularity of the solutions, even when we originally assume only $u \in L^\infty(\mathbf{R}^n \times (0, t_1))$, a class for which $u(t_1)$ is not well-defined, the equation gives us extra regularity which is more than enough for $u(t_1)$ to be well-defined.

The above "gluing procedure" also implies that for each $u_0 \in L^\infty(\mathbf{R}^n)$ we can define the maximal interval of existence T^* of the solution u starting at u_0 in a way which is similar to the usual ODE definition. As an exercise you can show that

$$||u(t)||_{L^\infty} \geq \frac{\varepsilon_1}{\sqrt{T^* - t}}, \qquad t \to T^*_- \tag{61.33}$$

for some $\varepsilon_1 > 0$.

It can also be shown that any mild solution $u \in X_T$ is smooth in $\mathbf{R}^n \times (0, T]$, with $t^{\frac{k}{2}} \nabla^k u(t) \in X_T$.

Above we showed that (61.1) is locally-in-time well-posed for $u_0 \in L^\infty(\mathbf{R}^n)$. One can show the same statement for $u_0 \in L^p(\mathbf{R}^n)$ when $p \geq 3$. The case $p > n$ can be done in a way which is quite similar to the case $p = \infty$. The case $p = n$ is the "critical case" and a slightly different method has to be used.

Note that the above constructions are completely independent of the energy identity. In fact, a-priori it is not completely clear that when $u_0 \in L^2 \cap L^\infty$, then the (unique) mild solution u will satisfy the energy identity. As the reader might expect the identity will be satisfied, but it has to be proved.

62

3/21/2012

62.1 Some blow-up criteria

Let us consider a mild solution of the Cauchy problem (61.1) with $f = 0$ and $u_0 \in L^\infty_x \cap L^2_x$. Let $(0, T)$ be the maximal interval of existence of the mild solution, and assume that $T < 0$. Whether or not this can happen for $n = 3$ is a famous open question. We know by (61.33) that $||u(t)||_{L^\infty} \to \infty$ as $t \to T_-$. Let us consider a sequence $0 < M_1 < M_2 < \cdots \to \infty$, where M_1 is sufficiently large. For $j = 1, 2, \dots$ let us denote by t_j the first time the function $t \to ||u(t)||_{L^\infty}$ takes on the value M_j. Let $x_j \in \mathbf{R}^n$ be such that

$$|u(x_j, t_j)| = M_j \,. \tag{62.1}$$

By definition of t_j, we have

$$|u(x,t)| \le M_j, \qquad 0 \le t \le t_j \,, \qquad x \in \mathbf{R}^n \,. \tag{62.2}$$

Heuristically one expects that the sequence x_j should be bounded in $\mathbf{R}^n$. This can indeed be shown, but we will not go into the proof at this point.

Let us define the functions v_j by

$$v_j(y,s) = \frac{1}{M_j} u(x_j + \frac{y}{M_j}, t_j + \frac{s}{M_j^2}), \qquad y \in \mathbf{R}^n, \quad s \in [-M_j^2 t_j, M_j^2 (T - t_j)] \,. \tag{62.3}$$

By the definitions we have

$$|v_j(y,s)| \le 1, \qquad y \in \mathbf{R}^n, \quad s \in [-M_j^2 t_j, 0] \tag{62.4}$$

and

$$|v(0,0)| = 1 \,. \tag{62.5}$$

By the Hölder estimates in lecture 60 we can conclude that

$$|v_j| \ge \tfrac{1}{2} \text{ in } B_\rho \times [-\rho^2, 0] \tag{62.6}$$

for some fixed $\rho > 0$. Letting $z_j = (x_j, t_j)$ and

$$Q_j = Q_{z_j, \frac{\rho}{M_j}} = B_{x_j, \frac{\rho}{M_j}} \times (t_j - \frac{\rho^2}{M_j^2 t_j}, t_j) \,, \tag{62.7}$$

we see that

$$|u(x,t)| \ge \frac{M_j}{2}, \qquad (x,t) \in Q_j \,. \tag{62.8}$$

We can interpret the situation in the following way: in $\mathbf{R}^n \times (0, t_j)$ the function $|u(x,t)|$ has a "peak" at $z_j = (x_j, t_j)$. The "height" of the peak is M_j. Before

the value of $|u(z)|$ drops below $\frac{M_j}{2}$, the (space-time) point z has to be away from Q_j. We can say that a peak of height M has to be at least $\sim \frac{1}{M}$ "wide" and its "duration" is at least $\sim \frac{1}{M^2}$.

It is easy to see that

$$\int_{Q_j} |u|^{n+2}\, dx\, dt \geq \rho^{n+2}\,. \tag{62.9}$$

By passing to a subsequence, if necessary, we can assume that the parabolic balls Q_j are disjoint. It is then clear that for each $\tau > 0$ we have

$$\int_{T-\tau}^{T} \int_{\mathbf{R}^n} |u|^{n+2}\, dx\, dt = +\infty\,. \tag{62.10}$$

This is a special can of the so-called Ladyzhenskaya-Serrin-Prodi criterion. A slightly more general case is the following:

Assume that a mild solution u blows up at a finite time T. Let $q > n, p \geq 2$ be such that

$$\frac{2}{p} + \frac{n}{q} = 1\,. \tag{62.11}$$

Then for each $\tau > 0$ we have

$$\int_{T-\tau}^{T} \left(\int_{\mathbf{R}^n} |u(x,t)|^q\, dx \right)^{\frac{p}{q}} dt = +\infty\,. \tag{62.12}$$

The calculation is analogous to (62.9).

When $n = 2$ and u satisfied the energy identity, we know from lecture 59 (see (59.4)) that

$$\int_0^T \int_{\mathbf{R}^n} |u|^4\, dx\, dt \leq C < +\infty \tag{62.13}$$

and hence $T = \infty$. In other words, the mild solution is defined globally.[272]

[272]The regularity for $n = 2$ can also be shown by using the vorticity equation $\omega_t + u\nabla\omega - \Delta\omega = 0$.

63

3/23/2012

63.1 Ruling out self-similar singularities

Assume that a mild solution of the Cauchy problem starting from some nice initial datum u_0 develops develops a finite-time singularity and the maximal time of existence T is finite. It can be shown that when $u_0 \in L^\infty \cap L^2$ (for example), then a finite-time singularity can be the only reason for any local-in-time mild solution failing to be global in time. Roughly speaking, the behavior of the mild solutions does not exhibit any surprises near the spatial infinity as long as the initial datum u_0 has some reasonable decay as $x \to \infty$. (A rigorous proof of this result requires some work, and we will omit it at this stage.) We can change coordinates so that the first singularity occurs at the origin of the space-time. This shifted solution will of course start at some negative time $t_0 < 0$. Let us consider such a solution u. We assume that u is defined on $\mathbf{R}^3 \times [t_0, 0)$ and that there exists a sequence $(x_j, t_j) \to (0, 0),\ t_0 < t_1 < t_2 < \dots 0$ such that $|u(x_j, t_j)| \to \infty$.

We will now do a slightly different re-scaling of u than the one used in the last lecture. For $\lambda > 0$ we set

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)\,. \tag{63.1}$$

The field u_λ is defined on $\mathbf{R}^3 \times (\lambda^{-2} t_0, 0)$. When λ is small we can think of u_λ as a "magnification" of u: we watch the same solution, but we re-scaled the unit of length by a factor λ and the unit of time by a factor λ^2, so we can think of watching u in slow motion under a microscope. What happens as $\lambda \to 0_+$?

This process of "magnifying singularities" has been used in various situations. For example, if we study some manifold $\Sigma \subset \mathbf{R}^m$ which contains 0 and, moreover, 0 is its "singular point", it often turns out to be useful to consider the limit of $\Sigma_\lambda = \lambda^{-1}\Sigma$ (taking in a suitable sense) as $\lambda \to 0_+$. This again "magnifies" what is going on near the singularity. This has proved to be very useful in studying singularities of minimal surfaces. The procedure is in fact interesting already for algebraic surfaces.

Returning to the Navier-Stokes equation, the scaling (63.1) is of course dictated by our desire to preserve the equation: the fields u_λ and u both satisfy the same equation. Although the pressure does not enter the definition of the mild solutions, it is still interesting to check how it scales. The reader can verify that the right scaling for the pressure is

$$p \to p_\lambda, \qquad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)\,. \tag{63.2}$$

The corresponding scaling of the initial data is

$$u_0 \to u_{0\lambda}\,, \qquad u_{0\lambda}(x) = \lambda u_0(\lambda x)\,. \tag{63.3}$$

We note that in general dimension n we have

$$\int_{\mathbf{R}^n} |u_{0\lambda}|^2 \, dx = \lambda^{2-n} \int_{\mathbf{R}^n} |u_0|^2 \, dx \ . \tag{63.4}$$

We see that in dimensions $n \geq 3$ the energy of $u_{0\lambda}$ grows as λ^{2-n} as $\lambda \to 0_+$. That is another manifestation of the "super-criticality" of the Navier-Stokes equations in dimension $n \geq 3$.

What happens with u_λ as $\lambda \to 0+$? If $(0,0)$ were a regular point of the solution of u, then of course $u_\lambda \to 0$ on compact subsets of $\mathbf{R}^n \times (-\infty, 0)$ as $\lambda \to 0_+$. On the other hand, if $(0,0)$ is a singular point of the solution[273], many other scenarios could be possible. For example, with our current knowledge we cannot rule out the scenario that

$$|u_\lambda(x,t)| \to \infty \,, \quad \lambda \to 0_+ \,, \quad (x,t) \in \mathbf{R}^n \times (-\infty, 0) \,. \tag{63.5}$$

Let us *assume* that we have

$$u_\lambda \to \overline{u}, \qquad \lambda \to 0_+ \tag{63.6}$$

where $\overline{u}$ is some "nice" function on $\mathbf{R}^n \times (-\infty, 0)$. We have not specified the nature of convergence in (63.6). This is in fact a non-trivial point: our ability to prove certain results may depend on the exact nature of the convergence assume in (63.6). For example, heuristically one should have the following:

Under the assumption (63.6), when the convergence is sufficiently strong[274] *and* $\overline{u} = 0$*, then* $(0,0)$ *is not a singular point of* u*.*

Such statements can indeed be proved, but at the moment we will not go into the details. Instead, we will assume that the limit (63.6) exists (in a sufficiently strong sense) and is non-trivial. If this is the case, one sees that

$$(\overline{u})_\lambda = (\lim_{\lambda' \to 0_+} u_{\lambda'})_\lambda = \lim_{\lambda' \to 0_+} u_{\lambda\lambda'} = \overline{u} \,. \tag{63.7}$$

Note that in this identity the existence of the "full" limit (63.6) is crucial. It is not enough to take limits over subsequences (assuming they exist), for example.

Identity (63.7) says that the solution $\overline{u}$ is scale-invariant:

$$\lambda \overline{u}(\lambda x, \lambda^2 t) = \overline{u}(x,t) \,, \qquad \lambda > 0. \tag{63.8}$$

Let $\kappa > 0$. For a fixed $t < 0$ let $\lambda = \frac{1}{\sqrt{2\kappa(-t)}}$ and apply (63.8) to obtain

$$\overline{u}(x,t) = \frac{1}{\sqrt{2\kappa(-t)}} \ \overline{u} \left(\frac{x}{\sqrt{2\kappa(-t)}}, -\frac{1}{2\kappa} \right) \,. \tag{63.9}$$

[273] We emphasize that it is unknown if such points exist.

[274] For example, convergence in $L^3(B_\rho \times (-\rho^2, 0))$ for each $\rho > 0$.

We see that under the assumptions above, there should be a singular solution of the Navier-Stokes equations of the form

$$\overline{u}(x,t) = \frac{1}{\sqrt{2\kappa(T-t)}}\, U(\frac{x}{\sqrt{2\kappa(T-t)}})\,, \tag{63.10}$$

where T is the "blow-up time" (taken as $T = 0$ in (63.9)), and U is the "self-similar profile function" (taken as $\overline{u}(\,\cdot\,, -\frac{1}{2\kappa})$ in (63.9)). The parameter $\kappa > 0$ is introduced only for convenience and could be taken as $\kappa = 1$ or $\kappa = \frac{1}{2}$ without loss of generality.

The regularity we can assume about U depends on the nature of convergence in (63.6). It is not hard to check that the function U must satisfy

$$-\Delta U + \kappa x \nabla U + \kappa U + U\nabla U + \nabla P = 0, \qquad \operatorname{div} U = 0\ , \tag{63.11}$$

for a suitable function P. This is an elliptic equation which implies full regularity of U once some critical regularity condition is satisfied. For (63.11) it can be shown that *a (very) weak solution of (63.11) which is in $L^3_{\rm loc}(\mathbf{R}^3)$ is smooth.*

One can imagine situations when the condition $U \in L^3_{\rm loc}(\mathbf{R}^3)$ is not "automatically" satisfied. For example, if the convergence in (63.6) is only in $L^2_{\rm loc}(\mathbf{R}^3 \times (-\infty, 0))$, then $\overline{u}$ will be a (very) weak solution of the Navier-Stokes equation which is scale invariant (in the sense of (63.8)), and the function U will be well-defined as an element of $L^2_{\rm loc}(\mathbf{R}^3)$, and will satisfy (63.11) in the sense that

$$\int_{\mathbf{R}^3} [-U_i \Delta\varphi_i - \kappa U_i (x_j\varphi_i)_{,j} + \kappa U_i\varphi_i - U_i U_j \varphi_{i,j}]\, dx = 0\,, \tag{63.12}$$

for each smooth, compactly supported vector field φ with $\operatorname{div}\varphi = 0$. Under these condition it is not known whether U will be smooth.

The suggestion that solutions of the form (63.10) may be relevant was made already in 1934 by J. Leray in his well-known *Acta Math.* paper quoted earlier. The main result about such solutions is the following

Theorem[275]

Let $p \in [3, \infty]$. Assume that $U \in L^p(\mathbf{R}^3)$ is a very weak solution of (63.11), in the sense of (63.12). Then U is constant. In particular, when $p < \infty$, then $U \equiv 0$.

We will show the main idea of the proof of the theorem next time.

[275] For $p = 3$ see Necas, J.; Ruzicka, M.; Sverak, V., On Leray's self-similar solutions of the Navier-Stokes equations. Acta Math. 176 (1996), no. 2, 283–294.
For $p > 3$ see Tsai, Tai-Peng, On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates. Arch. Rational Mech. Anal. 143 (1998), no. 1, 29–51.

Remarks

1. The most natural requirement on the function U at $x \to \infty$ is that

$$U(x) = O(|x|^{-1}), \qquad x \to \infty\,. \tag{63.13}$$

This is because it can be shown that under natural assumptions on the solution (63.10) the limits

$$\lim_{t \to T_-} \overline{u}(x,t) \tag{63.14}$$

will exist for $x \neq 0$ and be finite. This is a non-trivial result which follows from the partial regularity theory.[276] Therefore the most interesting case of the above theorem is when $p > 3$.

2. The case $p < 3$ of the theorem is open, due to the possible lack of the local regularity of U.

3. The theorem can be considered as a Liouville-type theorem for (63.11). It is interesting to compare the result with the case $\kappa = 0$ (steady Navier-Stokes). In that case it is not known if a bounded solution has to be constant, or if a locally regular solution with $\nabla U \in L^2$ and $U \in L^6$ vanishes. It turns out that the terms with κ are quite helpful in proving the theorem, and in this sense the case $\kappa > 0$ is in fact easier than the case $\kappa = 0$, contrary to what one might initially expect.

The proof of the Theorem will be given in the next lecture.

[276] Caffarelli, L.; Kohn, R.; Nirenberg, L. Partial regularity of suitable weak solutions of the Navier-Stokes equations. Comm. Pure Appl. Math. 35 (1982), no. 6, 771–831.

64

3/26/2012

Ruling out self-similar singularities (continued)

We now sketch the proof of the main theorem of the last lecture. We will essentially follow the proof of T.P.Tsai quoted last time, which uses a maximum principle from the NRS paper. As a preparation, we recall the following classical fact. Let u be a solution of the steady Navier-Stokes equation

$$-\Delta u + u\nabla u + \nabla p = 0\,, \qquad \operatorname{div} u = 0\,. \tag{64.1}$$

Let

$$H = \frac{1}{2}|u|^2 + p\,. \tag{64.2}$$

Then

$$-\Delta H + u\nabla H = -|\omega|^2\,, \qquad \omega = \operatorname{curl} u\,. \tag{64.3}$$

The calculation is easy. Clearly

$$-\Delta H + u\nabla H = -u\Delta u - |\nabla u|^2 - \Delta p + u\nabla\frac{|u|^2}{2} + u\nabla p = -|\nabla u|^2 - \Delta p\,, \tag{64.4}$$

and now we use

$$-\Delta p = u_{i,j}u_{j,i}\,. \tag{64.5}$$

Writing

$$\nabla u = S + A\,, \tag{64.6}$$

where S is the symmetric part of ∇u and A is the anti-symmetric part, we note that

$$-|\nabla u|^2 - \Delta p = -|S|^2 - |A|^2 + |S|^2 - |A|^2 = -2|A|^2 = -|\omega|^2 \tag{64.7}$$

and (64.3) follows.

Let us now return to equation (63.11) and write

$$\tilde{U} = U + \kappa x, \qquad \tilde{P} = P - \frac{1}{2}\kappa^2|x|^2\,. \tag{64.8}$$

The vector field $\tilde{U}$ describes the classical particle trajectories in the self-similarity coordinates. The reader can easily check that (63.11) is equivalent to

$$-\Delta\tilde{U} + \tilde{U}\nabla\tilde{U} + \nabla\tilde{P} = 0, \qquad \operatorname{div}\tilde{U} = 3\kappa\,. \tag{64.9}$$

Let us set

$$\Pi = \frac{1}{2}|\tilde{U}|^2 + \tilde{P} = \frac{1}{2}|U|^2 + P + \kappa x U\,. \tag{64.10}$$

A calculation similar to that leading to (64.3) shows that (64.9) implies

$$-\Delta\Pi + (U + \kappa x)\nabla\Pi = -|\Omega|^2\,, \qquad \Omega = \operatorname{curl} U\,. \tag{64.11}$$

Let L be the linear operator given by

$$L = -\Delta + (U + \kappa x)\nabla \quad . \tag{64.12}$$

We claim that for small $\varepsilon > 0$ the function

$$f(x) = f_\varepsilon(x) = e^{\varepsilon \frac{|x|^2}{2}} \tag{64.13}$$

is a super-solution of L in $\{x,\ |x| > R\}$ when R is sufficiently large. To see this, we calculate

$$f_{,j} = \varepsilon x_j f, \qquad \Delta f = \varepsilon n f + \varepsilon^2 |x|^2 f\,, \tag{64.14}$$

and hence

$$Lf = (-\varepsilon n - \varepsilon^2|x|^2 + \varepsilon(Ux) + \kappa\varepsilon|x|^2)f\,. \tag{64.15}$$

We see that $Lf \geq 0$ when x is large, $\varepsilon < \kappa$ and

$$|U(x)| = o(|x|),\ |x| \to \infty\,. \tag{64.16}$$

The last condition is obviously satisfied when $U \in L^\infty$, and can be proved to be always satisfied under the assumptions of the theorem, we refer the reader to T. P. Tsai's paper for details.

Let

$$M(r) = \sup_{|x|\leq r} \Pi(x)\,, \tag{64.17}$$

let $r_0 > 0$ be sufficiently large so that $Lf \geq 0$ in $\mathcal{O}_{r_0} = \{x\,,\ |x| \geq r_0\}$ and let $M_0 = M(r_0)$. For $\delta > 0$ let us set

$$F = F_{\varepsilon,\delta,M_0} = M_0 + \delta f_\varepsilon\,, \tag{64.18}$$

where f_ε is the function above (see (64.13)). We have

$$LF > 0 \text{ in } \mathcal{O}_{r_0}, \quad F \geq \Pi \text{ at } \partial\mathcal{O}_{r_0}, \quad F \geq \Pi \text{ near } \infty. \tag{64.19}$$

The last condition follows from (64.16) and the fact that P can have at most polynomial growth at ∞, which one can prove by using elliptic estimates, see the paper of T. P. Tsai.

Since we have $L\Pi \leq 0$, we see that (64.19) imples

$$\Pi(x) \leq F(x), \qquad x \in \mathcal{O}_{r_0}\,. \tag{64.20}$$

Taking $\delta \to 0$, we see that

$$\Pi(x) \leq M_0, \qquad x \in \mathbf{R}^3\,. \tag{64.21}$$

The strong maximum principle for the operator L now implies that

$$\Pi(x) = M_0, \qquad x \in \mathbf{R}^3\,. \tag{64.22}$$

We see that $L\Pi = 0$ in $\mathbf{R}^3$ and hence and (64.11) gives $\operatorname{curl} U = 0$ in $\mathbf{R}^3$. Since also $\operatorname{div} U = 0$ by our assumptions, we see from the Liouville theorem that U is constant.

65

3/28/2012

65.1 Some model equations

Assume that a mild solution u of the Navier-Stokes equation discussed in lecture 61 "blows up" at time $t = T$. In other words, $(0, T)$ is the maximal interval of existence of the solution. We say that the blow-up is of *Type I*, if for some $\tau > 0$

$$\sup_{t\in(T-\tau,T)} \sqrt{T-t}\ ||u(t)||_{L^\infty} < \infty\,. \tag{65.1}$$

Noting that in the situation above we must have

$$||u(t)||_{L^\infty} \geq \frac{\varepsilon_1}{\sqrt{T-t}} \tag{65.2}$$

for some $\varepsilon_1 > 0$, as discussed in lecture 61, we see that in Type I blow-up the solution grows with the minimal possible exponent,

$$||u(t)||_{L^\infty} \sim \frac{1}{\sqrt{T-t}}\,. \tag{65.3}$$

A blow-up not satisfying (65.1) is said to be *Type II*. A typical case would be

$$||u(t)||_{L^\infty} \sim \frac{1}{(T-t)^{\frac{1}{2}+\varepsilon}} \tag{65.4}$$

for some $\varepsilon > 0$. As discussed in lecture 13, this is usually called *slow blow-up*, although some authors also use the term "fast blow-up" in this case, presumably due to the increased exponent $\frac{1}{2} + \varepsilon$. See lecture 13 for a discussion of the terminology in the context of Euler's equation.

Usually the Type II blow-up is associated with a situation where the solution exhibits some hesitation in the blow-up, as if there were some mechanism which is trying to prevent the blow-up, although in the end the blow-up tendency wins out. On the other hand, Type I blow up means that the solution blows up at the first allowable opportunity, without much hesitation.

There are model equations where every blow-up is of type one, and other model equations where every blow-up is of type two, and there are also equations which can exhibit both type one and type two blow-ups.

Example 1 (The complex viscous Burgers equation)[277]
We consider functions $u\colon \mathbf{R}\times(0,\infty) \to \mathbf{C}$ (the complex plane) and the equation

$$u_t + uu_x = u_{xx}\,. \tag{65.5}$$

[277]See the paper "Zeros of complex caloric functions and singularities of complex viscous Burgers equations" by P. Polacik and VS

with the initial condition

$$u(x,0) = u_0(x)\,. \tag{65.6}$$

One can easily construct the local-in-time mild solutions for various classes of u_0, for example for $u_0 \in L^\infty$, by following the procedure we used for the Navier-Stokes equations in lecture 61. For each $u_0 \in L^\infty$ there is a maximal mild solution of (65.6) which is defined in $\mathbf{R} \times (0,T)$, where $(0,T)$ is the maximal time interval on which the solution is defined. It is easy to see that when u is real-valued (which will be the case when u_0 is real valued), then the solution satisfies a maximum principle, and hence $||u(t)||_{L^\infty}$ stays bounded and cannot blow up. Therefore $T = \infty$.

When u_0 is complex-valued, the situation is more complicated. The maximum principle is no longer satisfied. Fortuitously, the equation can be solved quite explicitly by the use of the so-called *Cole-Hopf transformation*. This is a "change of variables"

$$u = -\frac{2v_x}{v} \tag{65.7}$$

which turns (65.5) into the heat equation

$$v_t = v_{xx}\,. \tag{65.8}$$

The solutions of (65.8) can be analyzed in great detail, and hence the same is true about the solutions of (65.5). Formula (65.7) shows that the singularities of u are related to the zeroes of v. With some calculations[278] one can show that for complex-valued u_0 equation (65.5) can develop a singularity starting from smooth, compactly supported function u_0 and that the blow up is always of Type II, with

$$||u(t)||_{L^\infty} \sim \frac{1}{(T-t)^{1+\frac{k}{2}}} \tag{65.9}$$

for some $k \in \{0,1,2,\dots\}$, with $k = 0$ in the "generic case", but with any k as above occurring for some solution. In the language introduced above, the blow-up can be arbitrarily slow, and it is never of Type I. In particular, there are no self-similar singularities. We note that, by contrast, the blow-up for the Navier-Stokes equation, if it exists, cannot be arbitrarily slow, at least in the case when when $u_0 \in L^2$. In that case it can be shown[279]

$$\int_0^T ||u(t)||_{L^\infty}\, dt < +\infty\,. \tag{65.10}$$

This is one of the consequences of the energy estimate. For the complex-valued solutions of (65.5) the energy estimate fails (whereas for the real valued solution it is easy to establish).

[278] See the paper Li, Lu, Isolated singularities of the 1D complex viscous Burgers equation J. Dynam. Differential Equations 21 (2009), no. 4, 623-630.

[279] See the paper "Localisation and compactness properties of the Navier-Stokes global regularity problem" by T. Tao, arXiv:1108.1165

Example II (Harmonic map heat flow)[280]
We consider the 3d sphere $S^3 \subset \mathbf{R}^4$, functions $u\colon \mathbf{R}^n \times (0,\infty) \to S^3$, and the equation

$$u_t - \Delta u = |\nabla u|^2 u \tag{65.11}$$

for such functions. The reader can check that this equation is the L^2-gradient flow of the functional

$$I(u) = \int_{\mathbf{R}^n} |\nabla u|^2 \, dx. \tag{65.12}$$

In other words, we can write (65.11) as

$$\dot{u} = -\mathrm{grad}_{L^2} I(u)\,, \tag{65.13}$$

where, as indicated by the notation, the linear functional $I'(u)$ in the tangent space of maps $\mathbf{R}^n \to S^3$ at a given u is identified with a vector in that tangent space by the means of the L^2-scalar product. The Navier-Stokes equation is in many respects very far from being a gradient flow. In fact, its non-linear part, the Euler equation, can be considered as a Hamiltonian system and such systems are in some sense "opposite" to gradient flows. For example, gradient flows never admit periodic or quasi-periodic solutions, whereas for simple Hamiltonian systems the periodic/quasi-periodic behavior is in some sense expected. (This does not quite apply to the Euler equation and many other infinite-dimensional systems for which the periodic/quasi-periodic solutions are expected to be non-generic.) Nevertheless, it is interesting to compare (65.11) with Navier-Stokes. (Note that the linear part of the Navier-Stokes equation, the linear Stokes system, is a gradient flow of the functional $\int_{\mathbf{R}^n} |\nabla u|^2$ on the space H of div-free vector fields.)

Equation (65.11) is preserved by the scaling

$$u(x,t) \to u(\lambda x, \lambda^2 t)\,, \qquad \lambda > 0\,. \tag{65.14}$$

Therefore the quantity which should be compared to the Navier-Stokes velocity field is ∇u. One can see immediately from (65.13) that the quantity $\int_{\mathbf{R}^n} |\nabla u|^2 \, dx$ is non-increasing and that one has an a-priori estimate[281] for $\int_0^T \int_{\mathbf{R}^n} |u_t|^2 \, dx \, dt$. On the other hand, one does not have an a-priori estimate of $\int_0^T \int_{\mathbf{R}^n} |\nabla^2 u|^2 \, dx \, dt$, which would correspond to the dissipation in the Navier-Stokes energy estimate. In this sense the dissipation in (65.11) is weaker than that in Navier-Stokes. At the same time, the gradient nature of the flow implies many nice properties of the flow. In particular, for $n \geq 3$ one has the so-called *monotonicity formula* (discovered by M. Struwe): denoting by Γ the heat kernel, the quantity

$$t \to (t_0 - t) \int_{\mathbf{R}^n} \Gamma(x - x_0, t_0 - t)\, |\nabla u(x,t)|^2 \, dx \tag{65.15}$$

[280] See for example the expository paper "Geometric Evolution problems" by M. Struwe, AMS 1996

[281] The estimate can be thought of in terms of the finite-dimensional system $\dot{x} = -\nabla f(x)$. The equation implies $|\dot{x}|^2 = \frac{d}{dt} f(x)$ and integration of this identity over t gives an estimates for $\int_{t_1}^{t_2} |\dot{x}|^2 \, dt$.

is non-increasing for $t < t_0$, $x_0 \in \mathbf{R}^n$ (for sufficiently regular solutions). The formula is behind many remarkable results concerning the heat flow in dimensions $n \geq 3$, such as the partial regularity and general results about existence of singularities for topological reasons. Similar monotonicity formulae are in many cases at the base of our understanding of a various geometrical flows. As an exercise the reader can check that the formula is scale-invariant. Therefore even in higher dimension it provides an a-priori bound on a scale-invariant quantity. No similar estimate is known for the Navier-Stokes equation. Usually the monotonicity formulae are tied to the gradient flow structure of the corresponding PDE and therefore it is not clear if an analogue exists for the Navier-Stokes equation. Of course, an a-priori bound on any positive non-trivial scale-invariant quantity would most likely represent a breakthrough in the Navier-Stokes theory.

One can "change coordinates" in (65.11) as follows. We think of S^3 as unit quaternions and for $q \in S^3$ we denote by $\overline{q}$ the conjuate quaternion, so that $q\overline{q} = \overline{q}q = 1$. Alternatively, we can identify S^3 with the group $SU(2)$ and use the notation $\overline{q}$ for q^{-1}. For a map $u\colon \mathbf{R}^n \times (t_1, t_2) \to S^3 \subset \mathbf{R}^4$ we set

$$U_\alpha = \overline{u}\frac{\partial u}{\partial x_\alpha}\,. \tag{65.16}$$

Note that for each (x, t) the vector $U_\alpha(x, t)$ belongs to the tangent space of S^3 at the unit element 1. The tangent space can be identified with $\mathbf{R}^3$, and hence U_α can be thought of as an $\mathbf{R}^3-$ valued function. The functions U_α generated this way are not arbitrary, they have to satisfy

$$U_{\alpha,\beta} - U_{\beta,\alpha} + 2U_\beta \wedge U_\alpha = 0, \tag{65.17}$$

a nonlinear analogue of the identity $\operatorname{curl}\nabla f = 0$. Here and in what follows we use the notation $a \wedge b$ for the cross product in $\mathbf{R}^3$. The reader can check that in the U_α variables equation (65.11) becomes

$$U_{\alpha t} - \Delta U_\alpha = 2U_\beta \wedge U_{\alpha,\beta}\,, \qquad \text{(summation over repeated indices)} \tag{65.18}$$

which is at superficially reminiscent of the Navier-Stokes equation. At the level of U_α the scaling (65.14) becomes

$$U_\alpha(x, t) \to \lambda U_\alpha(\lambda x, \lambda^2 t)\,. \tag{65.19}$$

What is known about singularities? In dimension $n \geq 3$ the equation does develop singularities from smooth data. There are self-similar singularities (still for $n \geq 3$) and presumably also Type II singularities, although I am not sure if the existence of Type II blow-up has been rigorously proved for $n \geq 3$, except in the case when the data is 2-dimensional.

In dimension $n = 2$ the equation is critical. However, as we already noted, the dissipation in (65.11) is weaker than in the Navier-Stokes, and it turns out that even in the case $n = 2$ the equation can develop a singularity.[282] In this two-dimensional case the singularities will always be of Type II. In particular, there

[282] Chang, K., Ding, W.-Y., Ye, R., Journal of Diff. Geom., 36 (1992), 507-515.

are no self-similar singularities in this case, which can be easily seen directly from the energy identity.

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Model equations (continued)

Example III (a modification of Burgers/Navier-Stokes)[283]
We consider time-dependent vector fields $u(x,t)$ in R^n (which we think of as a function $u\colon \mathbf{R}^n \times (0,T) \to \mathbf{R}^n$). Let $a \in [0,1]$ and $\varkappa > 0$ be given parameters. Our equation for u will be as follows

$$u_t + au\nabla u + \frac{1}{2}(1-a)\nabla|u|^2 + \frac{1}{2}u \operatorname{div} u = \Delta u + \varkappa \nabla \operatorname{div} u\,. \tag{66.1}$$

We can think of it in several ways. For example, we can start with the viscous Burgers equation

$$u_t + u\nabla u - \Delta u = 0\,. \tag{66.2}$$

One can develop the local-in-time well-posedness theory for mild solution of this equation for initial data $u_0 \in L^p$ for some fixed $p \geq 3$ in a similar way as for the Navier-Stokes. (In fact, the estimates are somewhat more complicated as the non-linearity is not in divergence form, but the complications are not serious.) For example, for $u_0 \in L^\infty$ one has a local-in-time solution u defined on a maximal interval of existence $(0,T)$, and for $T < 0$ we would have to have $||u(t)||_{L^\infty} \to +\infty$ as $t \to T_-$. However, we notice that equation (66.2) implies a maximum principle for each component of u. For each index i we have

$$u_{it} + u\nabla u_i - \Delta u_i = 0\,, \tag{66.3}$$

and hence $||u_i(t)||_{L^\infty}$ is non-increasing in time for each i. Therefore $T = +\infty$ and the solution exists globally for all time (and remains globally bounded).

If we modify (66.2) to

$$u_t + u\nabla u + \frac{1}{2}u \operatorname{div} u - \Delta u = 0\,, \tag{66.4}$$

we destroy the maximum principle but will have an energy inequality instead (which we do not have for (66.2)):

$$\int_{\mathbf{R}^n} \frac{1}{2}|u(x,t)|^2\,dx + \int_0^T \int_{\mathbf{R}^n} |\nabla u|^2\,dx\,dt = \int_{\mathbf{R}^n} \frac{1}{2}|u_0(x)|^2\,dx\,. \tag{66.5}$$

One can think of equations of the form

$$u_t + u\nabla u + \frac{1}{q}u \operatorname{div} u - \Delta u = 0\,, \tag{66.6}$$

and look at the evolution of $||u(t)||_{L^q}$. The Burgers equation corresponds to $q = \infty$ and (66.4) corresponds to $q = 2$. Equation (66.1) can be viewed as an

[283] P. Plechac, V. Sverak, "Singular and regular solutions of a nonlinear parabolic system", Nonlinearity **16** (2003), 2083-2097.

additional generalization of (66.4). We note that for $a = \frac{1}{2}$ the non-linearity is in divergence form, and hence most of the proofs we know from the Navier-Stokes theory (such as the local in time existence of mild solutions, or regularity theory) can be applied to (66.1) with $a = \frac{1}{2}$ practically without change. It is in fact to some degree the case also for the other values of $a \in [0, 1]$, but the case $a \neq \frac{1}{2}$ may sometimes need slight modifications of the Navier-Stokes arguments. The role of the parameter $\varkappa$ is related to compressibility. The energy inequality for (66.1) is

$$\int_{\mathbf{R}^n} \frac{1}{2}|u(x,t)|^2\,dx + \int_0^T \int_{\mathbf{R}^n} \left(|\nabla u|^2 + \varkappa|\operatorname{div} u|^2\right)\,dx\,dt = \int_{\mathbf{R}^n} \frac{1}{2}|u_0(x)|^2\,dx \tag{66.7}$$

and therefore $\operatorname{div} u$ is expected to converge to 0 in $L^2_{t,x}$ when $\varkappa \to \infty$. It is therefore plausible to expect that as $\varkappa \to \infty$, the model (66.1) will approach and incompressible limit.[284]

Heuristically, it seems that the fields u which are most likely to lead to a finite-time blow-up from smooth initial data are the solutions of the form

$$u(x,t) = -v(r,t)x\,, \qquad r = |x|\,. \tag{66.8}$$

For the fields of this form (66.1) becomes

$$v_t = (1+\varkappa)\left(v_{rr} + \frac{n+1}{r}v_r\right) + 3rvv_r + (n+2)v^2\,. \tag{66.9}$$

This equation can be studied in some detail.[285]

Here are some interesting facts which can be established:

- For $n \geq 5$ the solutions can develop finite-time singularities from smooth, compactly supported initial data.
- For $n = 5, 6, 7, 8, 9$ the equation has self-similar singularities of the form

$$v(r,t) = \frac{1}{2\kappa(T-t)}w\left(\frac{r}{\sqrt{2\kappa(T-t)}}\right)\,, \tag{66.10}$$

  with $w(r) \sim \frac{1}{r^2}$ as $r \to \infty$. The number of different solutions of this form is at least (and probably equal to) $\nu(n) - 2$, where $\nu(n)$ is the smallest integer which is bigger or equal to $\frac{n+2}{n-4}$.
- For $n \leq 4$ the solutions of (66.9) starting with smooth functions with sufficiently fast decay to 0 at ∞ stay bounded for all time and cannot develop a singularity. On the other hand.

[284] This limit has been studied by W. Rusin (in preparation).

[285] See the paper of P. Plechac and V.S. quoted above, and also the Thesis of Dapeng Du, University of Minnesota, 2005.

- On the other hand, even for $n = 3, 4$ it is likely that solutions with slow decay at $r = \infty$ can develop singularities at finite time, although this has not been proved rigorously.
- It is conjectured that for $n \geq 5$ there are solutions starting from compactly supported smooth data with type II blow up at a finite time.

The global regularity for solutions with smooth compactly initial data in dimensions $n = 3, 4$ for general solutions of (66.1) or (66.4) seems to be open.

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Model equations (continued)

Example IV (the complex Ginzburg-Landau equation)

Let us first return to equation (63.11) and consider its time-dependent version

$$U_s - \Delta U + \kappa U + \kappa y \nabla U + U \nabla U + \nabla P = 0, \qquad \operatorname{div} U = 0. \tag{67.1}$$

We now assume $U = U(y, s)$. If we set

$$u(x,t) = \frac{1}{\sqrt{2\kappa(T-t)}} U\left(\frac{x}{\sqrt{2\kappa(T-t)}}, s\right), \qquad ds = \frac{dt}{2\kappa(T-t)}, \tag{67.2}$$

the velocity field u will still solve the Navier-Stokes equation. To produce a reasonable singularity, we need a solution of (67.1) which, say, does not approach 0 as $s \to \infty$ and satisfies

$$|U(y,s)| = O\left(\frac{1}{|y|}\right), \qquad |y| \to \infty, \quad s \to \infty. \tag{67.3}$$

One of the simplest examples of such a solution would be a solution of (67.1) which is periodic in s and satisfies (67.3). It is not known if such solutions exist. In fact, it is probably not even known if there are solutions of this type where the dependence on s is just due to rotation about, say, the x_3-axis with uniform angular velocity. The reader can check that, taking $T = 0$, the periodicity of $U(y, s)$ in s corresponds to the condition

$$u_\lambda = u \tag{67.4}$$

for some fixed $0 < \lambda < 1$. This should be contrasted with the condition that (67.4) is valid for *all* $\lambda \in (0, 1)$ which we need for self-similar solutions. In other words, a self-similar singularity discussed in lecture 63 has a continuous group of scaling symmetries, whereas the singularity (67.2) with a U periodic in s has only a discrete group of scaling symmetries, given by the scalings by λ^k, $k \in \mathbf{Z}$. Striking examples of singularities with discrete group of scaling symmetries have been found (numerically) in the context of gravitational collapse in general relativity, see for example the paper "Universality and Scaling in Gravitational Collapse of a Massless Scalar Field", Phys. Rev. Lett. 70, 9–12 (1993) by M. W. Choptuik.

We will now present a model equation which has singularities of such form, although it probably does not have the purely self-similar singularities. The Complex Ginzburg-Landau equation (CGL) is an equation for $u = u(x, t)$ which is a function on $\mathbf{R}^n \times (0, T)$ with values in $\mathbf{C}$ (the complex numbers). We will write it in the form

$$iu_t + (1 - i\varepsilon)\Delta u + |u|^{2\sigma} u = 0. \tag{67.5}$$

We assume that $\varepsilon > 0$ and that also $\sigma > 0$. It is easy to see that sufficiently regular solutions of (67.5) with sufficient decay at ∞ satisfy the energy identity

$$\int_{\mathbf{R}^n} \frac{1}{2}|u(x,t)|^2 + \int_0^t \int_{\mathbf{R}^n} \varepsilon |\nabla u|^2 \, dx \, dt' = \int_{\mathbf{R}^n} \frac{1}{2}|u(x,0)|^2 \, dx \, . \tag{67.6}$$

When $\sigma \leq \frac{2}{n}$, this estimate is sufficient to show that the solution cannot blow-up from smooth initial data with sufficient decay at ∞. Note that the equation has a non-trivial scaling symmetry

$$u(x,t) \to \lambda^{\frac{1}{\sigma}} u(\lambda x, \lambda^2 t) \tag{67.7}$$

and the "critical case" $\sigma = \frac{2}{n}$ correspond to the situation when $\int \frac{1}{2}|u_0(x)|^2 \, dx$ is invariant under the scaling $u_0(x) \to \lambda^{\frac{1}{\sigma}} u_0(\lambda x)$, as we can expect. This corresponds to the 2d Navier-Stokes equation. For $\frac{2}{n} < \sigma < \frac{2}{n} + \frac{1}{2}$ the equation is super-critical, but the non-linearity is still manageable to allow a construction of weak solutions, similar to what we did for 3d Navier-Stokes. In what follows we will assume that our parameters are in this range and will refer to this situation as super-critical, even though we will not consider the case $\sigma \geq \frac{2}{n} + \frac{1}{2}$. The reader can focus on the case $n = 3$, $\sigma = 1$, for example. We can make the change of variables

$$u(x,t) = [2\kappa(T-t)]^{-\frac{1}{2\sigma}} U(\frac{x}{\sqrt{2\kappa(T-t)}}, s) \, , \qquad ds = \frac{dt}{2\kappa(T-t)} \, . \tag{67.8}$$

This gives

$$iU_s + \kappa \frac{U}{\sigma} + \kappa y \nabla U + (1 - i\varepsilon)\Delta U + |U|^{2\sigma} U = 0 \, . \tag{67.9}$$

This equation most likely does not have good steady states which would give self-similar singularities of (67.5), although this may not been proved rigorously in the literature. On the other hand, there is extremely strong numerical evidence that the equation does have good solutions $U(y,s)$ which are periodic in s. In fact, the solutions are of the form

$$U(y,s) = Q(r) e^{-i\omega s} \, , \qquad r = |y| \, , \tag{67.10}$$

so that the solution $u(x,t)$ will be

$$u(x,t) = [2\kappa(T-t)]^{-\frac{1}{2}(\frac{1}{\sigma} - \frac{i\omega}{\kappa})} Q\left(\frac{|x|}{\sqrt{2\kappa(T-t)}}\right) \, . \tag{67.11}$$

In quantum mechanical interpretation (which, strictly speaking, is precise only for $\varepsilon = 0$) we can think of (67.10) as a solution corresponding to a pure energy state Q of the equation (67.9), which in this interpretation can be thought of as a linear equation with potential $-|Q|^{2\sigma}(r) = V(r)$.
We note that for the field $u(x,t)$ to have a good limit as $t \to T_-$ the function Q should satisfy

$$Q(r) \sim r^{-\frac{1}{\sigma} + i\frac{\omega}{\kappa}} \, , \qquad r \to \infty. \tag{67.12}$$

The equation for Q reduces to an ODE in $(0, \infty)$ with the "boundary condition" (67.12) as $r \to \infty$ and the boundary condition $Q'(0) = 0$. It turns out the ODE has many interesting solutions (which were calculated by a combination of rigorous analysis and numerical calculation) and therefore we can conclude with significant confidence that (67.5) does have solutions which blow up in finite time. The reader is referred to the the paper "On self-similar Singular Solutions of the Complex Ginzburg-Landau equation", CPAM, Vol. LIV, 1215–1242, (2006), by P. Plechac an V. S. The paper is also available on the arXiv preprint server.

We should emphasize that a fully rigorous proof of the existence of singularities which would not partly rely on a numerical calculations does not seem to be available. In fact, even for the case $\varepsilon = 0$ (the non-linear Schrödinger equation) with σ in the range considered here the existence of solutions (67.11) probably has not been established rigorously, although based on numerical evidence it is universally believed that such solutions exist.

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4/4/2012

68.1 Stability for of steady solutions of Euler's equation

Let $\overline{u}$ be a solution of steady Euler's equation

$$\overline{u}\nabla\overline{u} + \nabla\overline{p} = 0, \quad \operatorname{div}\overline{u} = 0 \tag{68.1}$$

in a domain Ω, with the usual boundary condition $\overline{u}\cdot n = 0$ at $\partial\Omega$. We will be interested in stability of the solution $\overline{u}$. We consider solutions

$$u = \overline{u} + v \tag{68.2}$$

with initial data close to $\overline{u}$. The closeness of the initial datum $v_0(x) = v(x,0)$ to 0 will be measured in suitable functions spaces which will be specified later. The equation for v is

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + v\nabla v + \nabla q = 0, \qquad \operatorname{div} v = 0\,, \tag{68.3}$$

where q is defined by $p = \overline{p} + q$, with p being a pressure function associated to the solution u. The boundary condition for v is $v\cdot n = 0$ at $\partial\Omega$. The linear part of (68.3) is

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + \nabla q = 0, \qquad \operatorname{div} v = 0 \tag{68.4}$$

with the same boundary condition $v\cdot n = 0$ at $\partial\Omega$ and with the understanding the the q in (68.4) is not the same as the q in (68.3). We can write this equation schematically as

$$v_t = Lv\,, \tag{68.5}$$

where v is a linear operator defined on the div-free fields v in Ω with $v\cdot n = 0$. More precisely, we have

$$L = -P\,(\overline{u}\nabla v + v\nabla\overline{u}) \tag{68.6}$$

where P is the Helmholtz projection onto div-free fields with vanishing normal component at $\partial\Omega$. We will denote the space of all smooth vector fields of this type by X. (We assume that the domain Ω has smooth boundary.) When we consider linear equations such as (68.4), (68.5), the vector fields can be considered complex-valued, with the usual understanding that it is the real part or the imaginary part which describes the physics.

Quite naturally, when we can find $v \in X$ such that for some $\lambda \in \mathbf{C}$ with $\mathrm{Re}\lambda > 0$ we have

$$Lv = \lambda v \tag{68.7}$$

we can say that $\overline{u}$ is linearly unstable. If we can show that there is no $\lambda > 0$ with this property, would it we natural to say that the system is linearly stable? If X were a finite-dimensional space and the operator L was similar to a diagonal matrix, this might be a reasonable definition of a weak form of

stability. However, the operator L above may not be normal (in the sense that $L^*L = LL^*$) and its spectral properties may be quite tricky, depending on which space it is considered. Therefore it turns out that even at the linearized level, the study of stability for steady solutions of Euler's equation is quite difficult, including the issue of coming up with good definitions. For example, we have already seen in lecture 20 that the steady states quite naturally come in infinite-dimensional families. Therefore we mostly do not really expect the solution v above to converge to 0, because even in the most favorable circumstances the solution $\overline{u} + v$ may not be typically approaching $\overline{u}$, but rather some steady state $\overline{\overline{u}}$ close to $\overline{u}$. Moreover, the convergence to $\overline{\overline{u}}$ may be quite weak. In fact, we should view the Euler equation essentially as an infinite-dimensional Hamiltonian system (even though this is not quite precise, it is really an infinite-dimensional *Poisson system*).[286] In finite-dimensional Hamiltonian systems

$$\dot{q}^{\,i} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{68.8}$$

the equilibria are given by the critical points of the Hamiltonian H. If $(\overline{q}, \overline{p})$ is such an equilibrium and H has a (non-degenerate) local minimum/maximum at $(\overline{q}, \overline{p})$, it is natural to consider $(\overline{q}, \overline{p})$ to be stable, but the solution starting close to it will not approach it. Rather, they will oscillate about it. This is clear even at the linearized level. In the context of Euler's equation this situation is complicated by the fact that the equilibrium $(\overline{q}, \overline{p})$ is not isolated. Typically it will belong to an infinite dimensional "manifold" of other equilibria. The infinite dimension of X can cause effects which are not possible in finite dimension. For example, a trajectory $v(t)$ may approach a limit in one space, but may not converge on another space, in a similar sense as the sequence $f_k(t) = \frac{\sin kt}{k}$ approaches 0 in $\mathcal{C}^0$ but not in $\mathcal{C}^1$. All these issues make the topic of stability of Euler's solutions much more subtle than the topic of stability for the Navier-Stokes solutions. The Navier-Stokes case is much closer to the finite-dimensional systems $\dot{x} = f(x)$ with hyperbolic equilibria.[287] Therefore, oversimplifying somewhat, in the Navier-Stokes case the issue of stability of a steady state solution $\overline{u}$ (in a bounded domain, say) typically does reduce to the question whether the linearized operator has eigenvalues λ with $\operatorname{Re}\lambda > 0$. In fact, due to the parabolic nature of the equation, this remains to be true even at the non-linear level: the linearized stability implies the (local) non-linear stability. For Euler's equation the situation is much more complicated, and even at the level of the linearized equations there are many open problems.

Historically, the investigations of stability of Euler steady states started in the 19th century with the study of the eigenvalue problem (68.7), and the subtleties of the whole picture became apparent only gradually.

[286] The exact definition will be given later.

[287] An equilibrium $\overline{x}$ of $\dot{x} = f(x)$ is *hyperbolic* if the eigenvalues of the matrix $Df(\overline{x})$ stay away from the imaginary axis.

We will start with a very simple example, which nevertheless leads to important questions which remain unanswered.

We consider the 2d Euler's equations in a domain $\{(x_1, x_2),\, 0 < x_2 < a\}$, with the x_1 direction being periodic of period L, so that we can think of a strip on a cylinder. The steady solution $\overline{u}$ will be a sheer flow

$$\overline{u}(x) = \begin{pmatrix} \overline{u}_1(x_2) \\ 0 \end{pmatrix} . \tag{68.9}$$

As we proceed, we will impose further assumptions on $\overline{u}$ as needed.

Let us first consider the linearized problem (68.4). In the case under consideration this will be a linear equation with coefficient independent of x_1. We can therefore use one of an important techniques used for linear equations – the decomposition in the Fourier modes in the directions where the coefficients are constants. We seek the solution v, q in the form

$$v(x,t) = \sum_{k \in 2\pi\mathbf{Z}/L} \hat{v}(k, x_2, t) e^{ikx_1}, \qquad q(x,t) = \sum_{k \in 2\pi\mathbf{Z}/L} \hat{q}(k, x_2, t) e^{ikx_1} . \tag{68.10}$$

We obtain an independent equation for each Fourier mode. By slight abuse of notation we will still write v for $\hat{v}(k, x_2, t)$ and q for $\hat{q}(k, x_2, t)$, with the understanding that k is fixed. Also, we will write $\overline{u}(x_2)$ or even $\overline{u}$ for $\overline{u}_1(x_2)$. The equations become

$$\begin{aligned} v_{1t} + ik\overline{u}v_1 + v_2\overline{u}' + ikq &= 0\,, \\ v_{2t} + ik\overline{u}v_2 + q' &= 0\,, \\ ikv_1 + v_2' &= 0\,, \end{aligned} \tag{68.11}$$

where we denote the partial derivative in the x_2 direction by $'$. The boundary conditions are $v_2 = 0$ at $\partial\Omega$. We are interested in the solutions of (68.11) for a given initial condition $v(x_2, 0)$ satisfying the boundary condition $v_2 = 0$ at $\partial\Omega$. The reader can check that the case $k = 0$ is trivial. We will assume $k \neq 0$ in what follows. We can eliminate v_1 and q from the system (68.11), by using the first and the third equations to express q, v_1 in terms of v_2. We obtain

$$(k^2 v_2 - v_2'') + ik\overline{u}(k^2 v_2 - v_2'') + ik\overline{u}'' v_2 = 0\,. \tag{68.12}$$

Letting $f = -k^2 v_2 + v_2'')$ and letting denoting by $G = G_k$ the operator $f \to v_2$ (note that v_2 is uniquely determined by f and the boundary condition $v_2(0) = v_2(a) = 0$)[288], we can re-write (68.12) as

$$f_t + ik\overline{u}f - ik\overline{u}'' Gf = 0\,. \tag{68.13}$$

One can get the same equation more directly if one works with the vorticity for of Euler's equation

$$\omega_t + \{\psi, \omega\} = 0\,, \tag{68.14}$$

[288]The reader can easily check that G is an integral operator given by $Gf(x) = \int_0^a G(x,y) f(y)\, dy$, with an explicitly calculable kernel $G(x,y)$.

as discussed in lecture 14. Writing

$$\overline{\omega} = \operatorname{curl} \overline{u}, \qquad \eta = \operatorname{curl} v, \qquad v = \nabla^{\perp} \varphi \,, \tag{68.15}$$

$$\eta_t + \overline{u}\eta_{,1} + \varphi_{,1}\overline{\omega}' = 0 \,. \tag{68.16}$$

Using Fourier series in x_1 for each unknown function, the equation at the fourier mode with wave number $k \neq 0$ is

$$\eta_t + ik\overline{u}\eta + ik\varphi\overline{\omega}' = 0 \,, \qquad \varphi'' - k^2\varphi = \eta \,, \quad \varphi(0) = \varphi(a) = 0 \,, \tag{68.17}$$

which is the same as (68.13). When $\overline{\omega}$ is constant, which is the same as assuming that $\overline{u}$ is linear in x_2, equation (68.13) becomes

$$f_t + ik\overline{u}f = 0 \,. \tag{68.18}$$

This can be solved explicitly:

$$f(x_2, t) = f(x_2, 0)e^{ikt\overline{u}(x_2)} \,. \tag{68.19}$$

Recalling that $\overline{u}(x_2)$ is linear in x_2 and assuming it is not constant, we see that for each Fourier mode $k \neq 0$ the function $f(x_2, t)$ converges weakly (but not strongly) to 0. For these Fourier modes we see that $\hat{v}(k, x_2, t) \to 0$ as $t \to \infty$. Combining this with the simple analysis of the mode $k = 0$, we see that in the case $\overline{u}'' = 0$ the solutions of (68.11) always approach another shear flow $\overline{v} = \overline{v}(x_2)$ as $t \to \infty$. It is a difficult open problem if for small, smooth, compactly supported initial data this behavior prevails even at the non-linear level, for the solutions of (68.3) (still in the case $\overline{u}'' = 0$).

Equation (68.13) already shows quite well the nature of the linearized problem (68.3). We can write the equation as

$$if_t = kHf \tag{68.20}$$

where H is the operator

$$f \to \overline{u}(x_2)f - \overline{u}''Gf \,. \tag{68.21}$$

The operator $f \to \overline{u}(x_2)f$ can be considered as a bounded self-adjoint operator on L^2. It has a continuous spectrum, with no eigenvalues unless $\overline{u}$ is constant on a set of positive measure. The operator $f \to Gf$ is compact and self-adjoint in L^2. The operator $f \to -\overline{u}''Gf$ is still compact, but may fail to be self-adjoint. The operator H is therefore a possibly non-self-adjoint compact perturbation of an operator with a continuous spectrum. The behavior of operators with continuous spectrum can be in many respects quite different from that of finite dimensional operators.[289] Therefore some aspects of question (68.13) have no analogy in finite dimension, and the infinite-dimensional effects are in some sense much more pronounced than what we see for the Navier-Stokes equation in a similar situation, where the corresponding linear operator will be quite closer to the finite-dimensional case.

[289] A good reference for this topic is Chapter 10 of T. Kato's book "Perturbation Theory of Linear Operators".

Homework Assignment 5
Due April 26

Consider the cubic CGL equation (lecture 67)

$$iu_t + (1 - i\varepsilon)\Delta u + |u|^2 u = 0\,, \tag{68.22}$$

where $\varepsilon > 0$. Consider the equation on a 3d torus Ω, such as $\Omega = \mathbf{R}^3/\mathbf{Z}^3$. We are interested in the Cauchy problem of finding a solution of (68.22) in $\Omega \times (0, \infty)$ with initial condition $u(x, 0) = u_0(x)$, where $u_0 \in L^2(\Omega)$ is given. Use Galerkin approximations to show that this problem has at least one weak solution which can be defined in a way similar to the Leray-Hopf weak solution of the Navier-Stokes equation constructed in lecture 57. In particular, show that one can construct u satisfying a suitable version of energy inequality (67.6), with $u(t) \to u_0$ in L^2 as $t \to 0_+$, and with the equation being satisfied in the sense of distributions.

As an optional part, you can also study the case of dimension $n = 2$. In this case the solutions are unique and smooth for $t > 0$. It takes some effort to establish the uniqueness and regularity in all detail,[290] you can only outline the main reasons why it is possible to obtain this result.

[290] Still for $n = 2$. In the case $n = 3$ one does expect that singularities can develop from smooth data, and uniqueness is not clear.

69

4/6/2012

Stability of steady solutions of Euler's equation (continued) We continue with our investigations of the stability of the shear flow (68.9). Recall that the linearized stability problem for this situation leads to the study of equation (68.20), where the operator H is defined by (68.21). As we already discussed (see (68.7)), the first natural task is to investigate the discrete spectrum. In the context of (68.20) we will be looking at the solutions of

$$Hf = cf\,, \tag{69.1}$$

where $c \in \mathbf{C}$ and f belongs to a suitable class of functions X where the linear equation (68.20) is considered. Natural choices of X include $X = L^2(0,a)$ or $X = \mathcal{C}^\alpha([0,a])$. If (69.1) has a non-trivial solution $f \in X$ when $\operatorname{Im} c > 0$, then obviously the steady state (68.9) is not linearly stable as $t \to \infty$. With $\operatorname{Im} c < 0$ we have instability for $t \to -\infty$.

In general, the problem of finding the eigenvalues of H is non-trivial and except in some special cases there is no simple way of determining them. Here we will only briefly recall some known results.[291]

Let us start with the following simple observation: if $\overline{u}''$ does not vanish in $[0,a]$, then we can define a scalar product

$$(f,g) = (f,g)_{\overline{u}''} = \int_0^a \frac{1}{|\overline{u}''(x)|} f(x)g(x)\,dx \tag{69.2}$$

and it is easy to check that H is self-adjoint with respect to this scalar product, i. e.

$$(Hf,g) = (f,Hg)\,. \tag{69.3}$$

For simplicity we can assume that $\overline{u}''$ is continuous in $[0,a]$ and since we assume that $\overline{u}'' \neq 0$ in $[0,a]$, the norm obtained from scalar product (69.2) is equivalent to the usual L^2-norm. From (69.3) we see that any eigenvalue c of H has to be real, at least when H is considered as an operator on $L^2(0,a)$. From this one can see a (weak version of) the well-known *Rayleigh stability criterion* for shear flows (68.9).

If $\overline{u}$ is either uniformly convex or uniformly concave, then (69.1) has no solutions with $\operatorname{Im} c \neq 0$ (and $f \in L^2$).

There are many ways in which this can be seen and we will also later discuss various extensions of this result.

[291] The reader interested in more details can consult for example papers by Zhiwu Lin, such as "Instability of some ideal plane flows", SIAM J. Math. Anal. 35 (2003), no. 2. Many ideas are also contained in the paper by L. D. Faddeev "On the theory of the stability of stationary plane parallel flows of an ideal fluid", (Russian) Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 21 (1971), 164–172.

In the study of the eigenvalues of H, it is useful to allow changing the parameter k continuously, which corresponds to allowing the circumference L the cylinder in which we consider our "strip" $\{0 < x_2 < a\}$ to change. We also recall that the function G in the definition of H depends on k, so that we can write $G = G_k$. It is easy to see that $G_k \to 0$ as $k \to \infty$ (in suitable norms).

We can rewrite (69.1) as

$$(\overline{u} - c)f - \overline{u}'' G_k f = 0\,. \tag{69.4}$$

Letting $g = (\overline{u} - c)f$, we re-write this as

$$g - \overline{u}'' G_k \frac{g}{\overline{u} - c} = 0\,. \tag{69.5}$$

The operator

$$g \to \overline{u}'' G_k \frac{g}{\overline{u} - c} \tag{69.6}$$

is obviously well defined for $c \notin \overline{u}([0, a])$. When $\overline{u}$ is sufficiently regular and $\overline{u}' > 0$, it turns out the operator has a well-defined limits $M_k(c \pm i0)$ when $c \pm i\varepsilon$, with $\varepsilon \to 0_+$. Here we use the notation

$$M_k(c + i0)g = \lim_{\varepsilon \to 0_+} M_k(c + i\varepsilon)g\,. \tag{69.7}$$

The precise investigation of the conditions under which such limits exists (and in which sense they are taken) will not be discussed at this point, we only wish to illustrate some of the main ideas.

Assuming f solves (69.4), we set

$$f = \frac{g_\varepsilon}{\overline{u} - c - i\varepsilon} \tag{69.8}$$

and let $\varepsilon \to 0_+$ to obtain

$$g = M_k(c + i0)g\,. \tag{69.9}$$

One can also show that $M_k(c + i0)g \to 0$ as $k \to \infty$, and therefore (69.4) has no solutions when k is large. In terms of the operator $H = H_k$ this means that $H = H_k$ has no discrete spectrum when k is sufficiently large. Note that this can also be seen directly from (69.4). However, the form (69.9) of (69.4) is useful also for other purposes.

We can take a large k so that H_k has no discrete spectrum and start lowering until some discrete spectrum possibly appears. Under the assumption $\overline{u}' > 0$ or $\overline{u}' < 0$ (which we assume in what follows) it can be shown that if a discrete spectrum appears at a certain k, then it has to appear at a points $c = \overline{u}(x_1) \in (\overline{u}(0), \overline{u}(a))$. It turns out that when $\overline{u}'' \neq 0$, there can be no discrete spectrum. In this case the operator H only has the same continuous spectrum as the simple multiplication operator $f \to \overline{u}f$. If $\overline{u}''$ has exactly one zero in $(0, a)$, say, x_1, then the discrete spectrum (assuming it appears) has to appear at $c = \overline{u}(x_1)$.

Let us discuss this case in more detail. Letting $\phi = \phi_k = G_k f$, one can re-write equation (69.4) as

$$-\phi'' + \frac{\overline{u}''}{\overline{u} - c}\phi = -k^2\phi\,, \tag{69.10}$$

which can be thought of as an equation for eigenvalues of the operator

$$L = -\frac{d^2}{dx^2} + V, \qquad V = -\frac{\overline{u}''}{\overline{u} - c}\,, \tag{69.11}$$

with Dirichlet boundary condition $\phi(0) = \phi(a) = 0$ at the endpoints. Note that when $\overline{u}''$ is sufficiently regular, then V is a continuous, as $\overline{u}''(x_1) = 0$. When $V \geq 0$, the operator L obviously cannot have negative eigenvalues, and hence the discrete spectrum does not appear. On the other hand, when V is sufficiently negative, there will a non-trivial eigenfunction ϕ with $L\phi = -k^2\phi$ for some $k \neq 0$. If this is the case, what happens to the solutions of (69.4) if we further decrease k? Since the spectrum of L is discrete, the discrete spectrum of H_k cannot stay at $c = u(x_1)$, and it has to move away from the real axis, into the complex region. In the simplest situation a pair of complex conjugate points in the spectrum of H_k is created. As we have seen above, this means that (for L corresponding the the k) the linearized equation (68.20) will have solutions with exponential growth. The reader interested in further details of this topic is referred to the papers of L. Faddeev and Z. Lin quoted above. The situation when $\overline{u}' \neq 0$ and $\overline{u}''$ has finitely many zeroes is more complicated as the complex eigenvalues c of H_k can potentially "travel" between the zeroes of $\overline{u}''$ as k is lowered, appearing first at some k and then either disappearing again later, or being joined by another pair. When $\overline{u}$ is not monotone, additional complications appear. We see that even for shear flows (68.9) the discrete spectrum of the linearized operator can exhibit quite non-trivial behavior.

70

4/16/2012

70.1 Internal waves in rotating fluid

Let us consider a steady state of Euler's equation given by

$$\overline{u} = \Omega\, e_3 \times x\,, \tag{70.1}$$

where e_3 is the unit vector in the direction of the x_3-axis and Ω is angular velocity. (The reader can calculate the pressure as an easy exercise.) We will again consider the solutions $u(x,t)$ which are small perturbations of this steady state solution. Let us consider the perturbations in the coordinate frame which rotates with the fluid. Denoting the perturbation in this frame by v, assuming that the density of the fluid is $\rho = 1$, the equation for v is

$$v_t + 2\Omega e_3 \times v + v\nabla v + \nabla p = 0\,, \qquad \operatorname{div} v = 0\,. \tag{70.2}$$

The linear part of this equation (which can be considered as a linearization of Euler's equation about the solution (70.1)) is

$$v_t + 2\Omega e_3 \times v + \nabla q = 0\,, \qquad \operatorname{div} v = 0\,. \tag{70.3}$$

This equation is sometimes called the Poincaré-Sobolev system. Its a linear system with constant coefficients, and therefor its solutions can be analyzed by using the Fourier transformation. We let

$$v(x,t) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \hat{v}(k,t) e^{ikx}\, dk \tag{70.4}$$

and obtain from (70.3)

$$\hat{v}_t + 2\Omega e_3 \times \hat{v} + ik\hat{q} = 0\,, \qquad k_\alpha \hat{v}_\alpha = 0\,. \tag{70.5}$$

Let $X_k \subset \mathbf{C^3}$ be given by $\{\hat{v},\, k_\alpha \hat{v}_\alpha = 0\}$ and let $P_k \colon \mathbf{C^3} \to X_k$ be the (complex) linear projection with $P_k k = 0$. Equation (70.5) can be written as

$$\hat{v}_t + 2\Omega P_k(e_3 \times \hat{v}) = 0\,, \qquad k_\alpha \hat{v}_\alpha = 0\,. \tag{70.6}$$

the matrix of P_k is[292]

$$P_k = \mathrm{I} - n \otimes n\,, \qquad n = \frac{k}{|k|}\,. \tag{70.7}$$

The let M be the matrix of the map $\hat{v} \to e_3 \times \hat{v}$ and let

$$A = A_k = P_k M\,, \tag{70.8}$$

[292] We denote by I the unit matrix and by $a \otimes b$ the matrix $a_\alpha b_\beta$.

so that (70.6) can be written as

$$\hat{v}_t + 2\Omega A\hat{v} = 0\,, \qquad k_\alpha \hat{v}_\alpha = 0\,. \tag{70.9}$$

Evaluating A, we obtain

$$A = \begin{pmatrix} -n_1 n_2 & -1+n_1^2 & 0 \\ 1-n_2^2 & n_1 n_2 & 0 \\ -n_3 n_2 & n_3 n_1 & 0 \end{pmatrix}. \tag{70.10}$$

Let

$$e = \begin{pmatrix} -\frac{n_2}{\sqrt{n_1^2+n_2^2}} \\ \frac{n_1}{\sqrt{n_1^2+n_2^2}} \\ 0 \end{pmatrix}, \qquad f = \begin{pmatrix} -\frac{n_3 n_1}{\sqrt{n_1^2+n_2^2}} \\ -\frac{n_3 n_2}{\sqrt{n_1^2+n_2^2}} \\ \sqrt{n_1^2+n_2^2} \end{pmatrix}. \tag{70.11}$$

We note that the vectors e, f, n form an orthonormal frame e, f is a basis of X_k, provided $|n_3| \neq 1$, which we assume in what follows. One easily checks

$$Ae = n_3 f, \qquad Af = -n_3 e\,, \tag{70.12}$$

and

$$A(e+if) = -in_3(e+if), \qquad A(e-if) = in_3(e-if)\,. \tag{70.13}$$

Hence the general solution of (70.9) is

$$\hat{v} = a(e+if)e^{i2\Omega n_3 t} + b(e-if)e^{-i2\Omega n_3 t}\,, \tag{70.14}$$

where $a, b \in \mathbf{C}$. We see that the solutions of (70.3) can be represented as

$$v(x,t) = \frac{1}{(2\pi)^3}\int_{\mathbf{R}^3} \Big[a(k)(e_k+if_k)e^{i(kx+2\Omega n_3 t)} + b(k)(e_k-if_k)e^{i(kx-2\Omega n_3 t)}\Big]\, dk\,, \tag{70.15}$$

where $a(k)$, $b(k)$ are complex valued functions of k belonging to, say, $L^2(\mathbf{R}^3)$. Even with this explicit formula, there are many natural questions which are not trivial. For example, you can try to determine under which conditions the solutions converge to zero in L^2_{loc}, in the sense that $\int_{B_R} |v(x,t)|^2\, dx \to 0$ as $t \to \infty$ for any bounded ball B_R.

From (70.15) we see that we can consider the solutions of (70.3) to be composed of waves with the dispersion relations[293]

$$\omega_+(k) = 2\Omega\frac{k_3}{|k|}, \qquad \omega_-(k) = -2\Omega\frac{k_3}{|k|}\,. \tag{70.16}$$

The corresponding "group velocities" of the wave packets will be $\nabla\omega_\pm(k)$. We see high-frequency wave packet propagate slower than low-frequency wave packets, quite contrary to what one has in classical dispersive equations such as the Schrödinger equation, Airy equation[294], the wave equation, etc.

[293] See for example the notes from the last years course Theory of PDEs, lecture 68

[294] $u_t = u_{xxx}$, the linearization of KdV.

71

4/18/2012

71.1 Kelvin's waves on vortices

We mention another situation where classical stability calculations have been made in some detail, starting with a well-known 1880 paper "Vibrations of a columnar vortex" by Kelvin.[295] Here we will not go into detailed calculations,[296] our goal is essentially just to point an interesting situation.

We consider steady-state solutions of the 3d incompressible Euler equation of the form

$$\overline{u}(x) = V(r)e_\theta, \qquad r = \sqrt{x_1^2 + x_2^2}, \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0)\,. \tag{71.1}$$

The linearized equation

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + \nabla q = 0, \qquad \operatorname{div} v = 0\,, \tag{71.2}$$

is most naturally analyzed in the cylindrical coordinates, as the symmetries of the solution $\overline{u}$ reduce to a simple invariance under shifts in these coordinates. As usual, the cylindrical coordinates are given by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z\,. \tag{71.3}$$

We will write

$$v = v^r e_r + v^\theta e_\theta + v^z e_z\,, \tag{71.4}$$

where $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)$ and $e_z = e_3$.
In polar coordinates (71.2) becomes

$$\begin{aligned} v^r_{,t} + \tfrac{V}{r}v^r_{,\theta} - \tfrac{2V}{r}v^\theta + q_{,r} &= 0\,,\\ v^\theta_{,t} + \tfrac{V}{r}v^\theta_{,\theta} + (V' + \tfrac{V}{r})v^r + \tfrac{q_\theta}{r} &= 0\,,\\ v^z_{,t} + \tfrac{V}{r}v^z_{,\theta} + q_{,z} &= 0\,,\\ \tfrac{(rv^r)_{,r}}{r} + \tfrac{v^\theta_{,\theta}}{r} + v^z_{,z} &= 0\,. \end{aligned} \tag{71.5}$$

The coefficients in this linear system are independent of θ and z, and hence it is natural to decompose the solution into the Fourier modes along those directions. In other words, we can assume that

$$(v^r, v^\theta, v^z, q) = \big(\hat v^r(r,t), \hat v^\theta(r,t), \hat v^z(r,t), \hat q(r,t)\big)\, e^{im\theta + ikz}\,, \tag{71.6}$$

[295] Phil. Mag. 10, p. 155, 1880

[296] The reader interested in details can consult for example to the book "Vortex Dynamics" by P. G. Saffman or to the paper "Kelvin waves and the singular modes of the Lamb-Oseen vortex" J. Fluid Mech. 551 (2006), 235–274, by D. Fabre, D. Sipp, and L. Jacquin

where $m \in \mathbf{Z}$ and $k \in \mathbf{R}$. We now change notation and write

$$(\hat{v}^r, \hat{v}^\theta, \hat{v}^z, \hat{q}) = (u, v, w, q). \tag{71.7}$$

Formally the use of v for $\hat{v}^\theta$ is not quite correct as we already used v above in a different sense, but we will see that this slight abuse of notation brings no problems. We also set

$$\Omega = \frac{V}{r}, \qquad B = V' + \frac{V}{r}\,. \tag{71.8}$$

With the Ansatz (71.6) and notation (71.7), (71.8) system (71.5) becomes

$$\begin{array}{rcl} u_t + im\Omega u - 2\Omega v + q' &=& 0\,, \\ v_t + im\Omega v + Bu + \frac{imq}{r} &=& 0\,, \\ w_t + im\Omega w + ikq &=& 0\,, \\ \frac{(ru)'}{r} + \frac{imv}{r} + ikw &=& 0\,, \end{array} \tag{71.9}$$

where the derivative $\frac{\partial}{\partial r}$ is denoted by $'$. This can be viewed as a system of equation for functions of $(r,t) \in [0,\infty) \times \mathbf{R}$. The functions u, v, w, q have to satisfy certain "boundary conditions" at $r = 0$ so that the coordinate singularity at $r = 0$ is compensated for and the "intrinsic" velocity field and pressure are smooth at the x_3-axis.[297] System (71.9) can be thought of as a more complicated version of (68.11). Important special cases of V include

$$V(r) \sim \begin{cases} r\,, & r < 1\,, \\ \frac{1}{r}\,, & r \geq 0\,, \end{cases} \tag{71.10}$$

which was already considered by Kelvin (and where some explicit calculations can be performed) and the so-called Lamb-Oseen vortex given by

$$V(r) \sim \frac{1 - e^{-r^2}}{r}\,, \tag{71.11}$$

which was investigated numerically in the above quoted paper by Fabre et al.

In general, the study of solutions of (71.9) seems to be quite difficult. For example, the determination of the discrete spectrum, corresponding to solutions where

$$(u(r,t), v(r,t), w(r,t)) = (u(r), v(r), w(r))e^{\lambda t} \tag{71.12}$$

leads to spectral problem for a linear ODE which appears to be hard to investigate without some help from numerical simulations. It seems that even for the Lamb-Oseen vortex (71.11) it has not been rigorously proved that there are no exponentially growing modes (corresponding to $\lambda > 0$ in (71.12)), although numerically this appears to be the case. The study of solutions of (71.9) and (71.6) for large times is also difficult. For example, one can speculate that in the case

[297] The reader can work out these conditions as an exercise.

of the Lamb-Oseen vortex a solution of (71.6) starting from smooth, rapidly decaying data will converge to zero on in L^2 on compact subsets of $\mathbf{R}^3$. As far as the author knows, this has not been proved rigorously.

As in the case of rotating fluids, the "waves" generated by the system (71.6) exhibit unusual dispersion. The dispersion appears to be too weak for implying strong results about the full non-linear problem

$$v_t + \overline{u}\nabla v + v\nabla\overline{u} + v\nabla v + \nabla p = 0, \qquad \operatorname{div} v = 0 \tag{71.13}$$

by methods used in the theory of dispersive equations. Nevertheless, in the case of rotating fluids the dispersive effects can be used to improve existence results for the Navier-Stokes equation (where the viscosity will rapidly damp high frequencies for which dispersion effects are particularly weak), see for example the work of Babil, Mahalov and Nicolaenko.[298] Similar effects can presumably be expected near more general equilibria which generate some dispersion.

[298]Makhalov, A. S., Nikolaenko, V. P., Global solvability of three-dimensional Navier-Stokes equations with uniformly high initial vorticity. Russian Math. Surveys 58 (2003), no. 2, 287–318

72

4/20/2012

72.1 Non-linear Stability

Euler's equation describes a motion of continuum subject to certain restrictions (incompressibility, the boundary conditions, etc.). As such, it can be viewed as an infinite-dimensional version of classical Hamiltonian systems. This point of view was emphasized in the 1960s by V. I. Arnold, who has obtained important insights into the behavior of solutions of the Euler's equation via considerations related to various geometrical interpretations of the equations and analogies with phenomena of finite-dimensional Classical Mechanics.[299] Among those insights are certain stability criteria for 2d steady state solutions of incompressible Euler's equations, which we plan to discuss.

We start by a few simple classical observations. Let us consider a Classical Mechanics system described by a phase space $x_1, \dots, x_n, p_1, \dots, p_n$ and Hamiltonian function $H = H(x, p)$. (In the orthodox notation one should write $x^1, \dots, x^n, p_1, \dots, p_n$, but using the lower indices everywhere will be OK for our needs.) The equations of motions are

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}\,. \tag{72.1}$$

Let $(\overline{x}, \overline{p})$ be an equilibrium of the system. Obviously, the equilibria can be identified with the critical points of H. As is customary in this context, we introduce a $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{72.2}$$

and write (72.1) as

$$\dot{z} = J\nabla H\,, \qquad z = (x, p)\,. \tag{72.3}$$

The linearized equations about the equilibrium $\overline{z} = (\overline{x}, \overline{p})$ are

$$\dot{z} = J\nabla Q(z)\,, \qquad Q(z) = \frac{1}{2} H_{,z_\alpha z_\beta}(\overline{z})\, z_\alpha z_\beta\,. \tag{72.4}$$

Here the indices $\alpha, \beta, \dots$ run from 1 to $2n$, and the summation convection is understood. This can be also written as

$$\dot{z} = Az\,, \tag{72.5}$$

where A is the matrix $J\nabla^2 Q$.

[299] The reader is refereed to the books "Mathematical Methods of Classical Mechanics" by V. I. Arnold, and "Topological Methods in Hydromechanics" by V. I. Arnold and B. Khesin.

Due to the way the matrix A arises, it is not a "generic" matrix. If we denote by ω the "canonical symplectic form", which is the bi-linear form given by

$$\omega(z', z'') = J_{\alpha\beta} z'_\beta z''_\alpha \,, \tag{72.6}$$

it is easy to check that

$$\omega(Az', z'') + \omega(z', Az'') = 0 \,, \qquad z',\, z'' \in R^{2n} \,. \tag{72.7}$$

The set of such matrices A form a Lie algebra which is usually denoted by $\mathrm{sp}(2n, \mathbf{R})$. It is a Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbf{R})$ which consists of all $2n \times 2n$ matrices M which preserve the from ω, in the sense that $\omega(Mz', Mz'') = \omega(z', z'')$ for each $z', z'' \in \mathbf{R}^{2n}$. The matrices $A \in \mathrm{sp}(2n, \mathbf{R})$ can be written as

$$A = JS \,, \qquad S \text{ is symmetric.} \tag{72.8}$$

Clearly, the matrices the matrices (72.8) are describes by $n(2n+1)$ parameters, whereas a "generic" $2n \times 2n$ matrix is described by $(2n)^2$ parameters.

In the simples case $n = 1$ we have

$$\mathrm{sp}(2, \mathbf{R}) = \{A \in M^{2\times 2},\ \mathrm{Tr\,A} = 0\} = \mathrm{sl}(2, \mathbf{R}) \,. \tag{72.9}$$

The important consequence of these observation is the following. Whereas for finite dimensional truncations of the Navier-Stokes equations (of dimension n, say) it seems to be reasonable to assume that the linearization of the equation at a "generic equilibrium" is given by a linear system

$$\dot{x} = Ax \tag{72.10}$$

where A is a "generic" $n \times n$ matrix, the linearization of a Hamiltonian system (in dimension n, with the phase space of dimension $2n$) is given by a matrix $A \in \mathrm{sp}(2n, \mathbf{R})$. A generic real $n \times n$ matrix A has n distinct eigenvalues (some of them possibly complex, coming in complex-conjugate pairs), with no eigenvalues of the imaginary axis. In this case the space R^n can be written as a direct sum of a stable and unstable subspaces,

$$\mathbf{R}^n = X_s + X_u \,, \tag{72.11}$$

with trajectories starting in X_s being exponentially convergent to 0, and trajectories starting in X_u being exponentially repelled from 0.

The spectral behavior of a generic matrix $A \in \mathrm{sp}(2n, \mathbf{R})$ is different. For example, in the simplest case $n = 1$ the reader check that any small perturbation of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{72.12}$$

in $\mathrm{sp}(2, \mathbf{R})$ has purely imaginary eigenvalues. Therefore even in the generic case we cannot assume that the spectrum of A is away from the imaginary axis. [300]

[300] The reader can consult for example the book "Classical Mechanics" by V. I. Arnold concerning more information about the spectral properties of matrices in $\mathrm{sp}(2n, \mathbf{R})$.

Also, when A has a real eigenvalue λ, then $-\lambda$ is also an eigenvalue, due to the condition TrA $= 0$. Based on these simple observations we can expect that the stability theory for "generic equilibria" for Hamiltonian systems will be different from the case when the linearization can be considered as a generic $n \times n$ matrix.

Let consider the simplest case of a 1d Hamiltonian system with $(x, p) \in \mathbf{R}^2$ and

$$H = \frac{1}{2}p^2 + V(x)\,, \qquad V(x) = (1 - x^2)^2\,. \tag{72.13}$$

The Hamiltonian H has three critical points

$$z_1 = (x_1, p_1) = (-1, 0), \quad z_2 = (x_2, p_2) = (0, 0), \quad z_3 = (x_3, p_3) = (1, 0)\,. \tag{72.14}$$

The reader can calculate the linearized equations at these equilibria. At z_1, z_3 the eigenvalues of the linearized system will by on the imaginary axis, whereas at z_2 the eigenvalues will be real. The points x_1 and x_3 represent non-degenerate local minimal of the potential V and solutions obtained by small perturbations of the equilibria z_1 or z_3 will oscillate about these equilibria, staying close to them, but not approaching them. The equilibrium z_2 is easily seen to be unstable both linearly and non-linearly. The type of stability exhibited by z_1 and z_3 is, roughly speaking, the best we can hope for in the case of (finite-dimensional) Hamiltonian systems. Next time we will consider more complicated versions of this situation.

73

4/23/2012

Non-linear Stability (continued)

The simple fact that local minima/maxima of the Hamilton function H are stable equilibria of the Hamiltonian system

$$\dot{z} = J\nabla H(z) \tag{73.1}$$

can be combined with conservation laws to obtain information about stability of certain solutions. Assume that the hamiltonian is H and that $f = f(x,p)$ is a quantity conserved by the evolution, i. e.

$$\dot{f} = f_x\dot{x} + f_p\dot{p} = f_{x_j}H_{p_j} - f_{p_j}H_{x_j} = 0\,. \tag{73.2}$$

We introduce the usual notation

$$\{f, H\} = f_{x_j}H_{p_j} - f_{p_j}H_{x_j}\,. \tag{73.3}$$

This expression is called the *Poisson bracket* of the functions $f = f(x,p)$ and $H = H(x,p)$. The conservation of a quantity $f = f(x,p)$ under the evolution generated by H is equivalent to $\{f, H\} = 0$.

Let us fix $c \in \mathbf{R}$ and consider the surface $\Sigma_c = \{f = c\}$. Assume Σ is a smooth surface on which $\nabla f \neq 0$. Let $\overline{z}$ be a local minimum of H on Σ_c. The point z may not be an equilibrium of the system (73.1), as we may have

$$\nabla H(\overline{z}) = \lambda\nabla f(\overline{z}) \tag{73.4}$$

for some Lagrange multiplier λ which may not vanish. Clearly the set $\{H = H(\overline{z})\,,\ f = c\}$ is invariant under the evolution, and so is the set $\{f = c\,,\ H < H(\overline{z}) + \varepsilon\}$. In some situations this can be used to show stability properties of certain solutions (which in this context can be called "orbits"). Note that we can also consider local minima of f on the surfaces $\{H = E\}$. One can also consider a situation with several conserved quantities $f_1, \ldots, f_m$.

Rather than going into the general theory related to these themes, we will present a simple example which will illustrate the main points relevant for the situation arising the in context of Euler's equations.

We consider the classical problem of a particle of unit mass moving in $\mathbf{R}^2$ in the potential $V = V(r)$, with $r = \sqrt{x_1^2 + x_2^2}$. The Hamiltonian is

$$H(x,p) = \frac{1}{2}|p|^2 + V(r)\,. \tag{73.5}$$

The system is invariant under rotations

$$(x,p) \to (R(\theta)x, R(\theta)p)\,, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{73.6}$$

Let us denote by X the phase-space $\mathbf{R}^4$ and by $G = S^1$ the rotation group. For a function f on X and $R \in G$ we will write

$$Rf(z) = f(R^* z)\,. \tag{73.7}$$

We easily verify

$$\{Rf, Rg\} = R\{f, g\}\,, \qquad RH = H\,, \tag{73.8}$$

and therefore the whole system "descends" on the manifold[301] of the $G-$orbits $Y = X/G$. The functions on Y can be identified with the functions f on X satisfying $Rf = f$. The Poisson bracket naturally "descends" to functions on Y, due to (73.8): given f, g on Y, we identify them with invarient functions $\tilde{f}, \tilde{g}$ in X, take $\{\tilde{f}, \tilde{g}\}$ in X, which will be an invariant function on X, and can therefore be (uniquely) identified with a function $\{f, g\}$ on Y. A natural choice of local coordinates[302] on Y is r, p_r, p_θ where r, θ are the polar coordinates in the (x_1, x_2) plane and

$$p_1 dx_1 + p_2 dx_2 = p_1 d(r\cos\theta) + p_2 d(r\sin\theta) = p_r dr + p_\theta d\theta\,. \tag{73.9}$$

These coordinates are usually obtained (in this example) as follows: we consider the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - V(r) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\,. \tag{73.10}$$

Then

$$p_r = \frac{\partial L}{\partial \dot{r}}\,, \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}}\,. \tag{73.11}$$

The action of the group $G = S^1$ on X in the coordinates r, θ, p_r, p_θ is

$$R(\theta_0)(r, \theta, p_r, p_\theta) = (r, \theta - \theta_0, p_r, p_\theta)\,, \tag{73.12}$$

and hence (r, p_r, p_θ) are natural (local) coordinates in (a large part of) Y, as stated. For two functions $f = f(r, p_r, p_\theta), g = g(r, p_r, p_\theta)$ their Poisson bracket, as defined above, is easily seen to be

$$\{f, g\} = f_r g_{p_r} - f_{p_r} g_r\,. \tag{73.13}$$

In particular, this means that

$$\{p_\theta, f\} = 0 \quad \text{for each } f = f(r, p_r, p_\theta)\,. \tag{73.14}$$

In general, functions C with the property that $\{C, f\} = 0$ for each f are called *Casimir functions*, and we see that in our case the function p_θ is an example of

[301] In our case Y can be identified with $\mathbf{R}^3$ (with zero being a "branch point" of the projection $X \to Y$) although this may not be immediately obvious if you have not seen these considerations before. However, for our purposes here we do not really need to identify Y very precisely. In general, if a groups acts of X, the set of orbits X/G may not be a manifold in a neighborhood of each point.

[302] which do not necessarily parametrize the whole Y in a one-to-one manner, only "most of it" - the reader can do the exact analysis of the situation as an exercise

a Casimir function (on the space Y). The function p_θ can be identified with the angular momentum $p \times x$ and equation (73.14) expresses the conservation of the angular momentum[303] in the situation we consider here, and any Hamiltonian invariant under the action (73.6) (or (73.12)). The Hamiltonian (73.5) can be expressed in the $(r, \theta, p_r, p_\theta)$ coordinates as

$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + V(r)\,. \tag{73.15}$$

The evolution of the variables r, p_r, p_θ is given by

$$\dot{r} = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial r}, \quad \dot{p}_\theta = 0\,. \tag{73.16}$$

The equilibria of this system correspond to circular orbits of the point around the origin, and they are given by critical points of H for a given fixed p_θ. In other words, we are looking for critical points of H constrained to $p_\theta = \text{const}$. Let us assume that for some fixed $p_\theta = \overline{p}_\theta$ the Hamiltonian $H(r, p_r, p_\theta)$ has a strict local minimum on the surface $p_\theta = \overline{p}_\theta$ at $r = \overline{r}, p_r = \overline{p}_r$. Note that this means that $\overline{p}_r = 0$. Let us in fact assume that the minimum is non-degenerate, in the sense that the Hessian matrix (with respect to (r, p))

$$\begin{pmatrix} H_{,rr} & H_{,rp_r} \\ H_{,p_r r} & H_{p_r p_r} \end{pmatrix} \tag{73.17}$$

at $(\overline{r}, \overline{p}_r, \overline{p}_\theta)$ is (strictly) positive definite. One can then see that for any initial condition $(r, p_r, \overline{p}_\theta)$ close to $(\overline{r}, \overline{p}_r, \overline{p}_\theta)$ the solution of (73.16) will stay close to $(\overline{r}, \overline{p}_r, \overline{p}_\theta)$. In fact, in the situation we are considering, with strictly positive definite Hessian (73.17), it is not hard to see from the Implicit Function Theorem that for each $\tilde{p}_\theta$ close to $\overline{p}_\theta$ the surface $p_\theta = \tilde{p}_\theta$ has a local minimum $(\tilde{r}, \tilde{p}_r)$ of H close to $(\overline{r}, \overline{p}_r)$, which is unique in a small neighborhood of $(\overline{r}, \overline{p}_r)$ and has similar stability properties as $(\overline{r}, \overline{p}_r)$. Here we can see everything quite explicitly, the equilibria correspond to the circular trajectories, and their stability means that a small perturbation of a circular trajectory will be close to a circular trajectory (assuming that the Hessian (73.17) is strictly positive definite, or strictly negative definite).

For the Newtonian potentials $V(r) = -\frac{\alpha}{r}$ (with $\alpha > 0$) we know that the orbits are ellipses, and from this it is easy to see that in fact any (bound) trajectory is stable in the above sense. For potential which are not newtonian the trajectories which are not circular may not be closed.

[303] This is related to Noether's theorem and also to the fact that if we take the function p_θ as a hamiltonian, the motion we get is exactly the one generated by the rotations. This can be seen easily from the equation in the $(r, \theta, p_r, p_\theta)$ coordinates, which are

$$\dot{r} = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial r}, \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}\,.$$

Note that for $H = p_\theta$ the motion will give the rotations by the group. These themes are studied in detail in the context of *momentum maps* on symplectic manifolds with a group action.

We see from the above picture that for the stability of circular orbits it is necessary that the function[304]

$$V_{\text{eff}} = \frac{p_\theta^2}{2r^2} + V(r) \tag{73.18}$$

has strict local minima in r. (The case when V has a local maximum corresponds to an unstable circular orbit, as the Hessian (73.17) will be indefinite in this case.) The newtonian potential in dimension $n \geq 3$ is

$$V(r) = -\frac{\alpha}{r^{n-2}}, \tag{73.19}$$

where $\alpha > 0$. (Our assumption that the motion takes place in a 2d subspace is justified even in $\mathbf{R}^n$, due to the conservation of the n-dimensional momentum.) The reader can check that for Newtonian potentials in dimension $n = 3$ the potential V_{eff} does have a unique strict local minimum for each $p_\theta \neq 0$. The corresponding circular trajectories will be stable. For $n \geq 4$, the potential V_{eff} does not have local minima, and therefore there are no stable circular orbits (nor any other reasonable stable bound orbits, as the reader can check). The case of dimension $n = 2$ is left to the reader as an exercise.

Similar considerations can be used for studying stability of certain solutions of 2d Euler's equations, as observed by V. I. Arnold in the 1960s.

[304] sometimes called the *effective potential*

74

4/25/2012

Non-linear stability (continued)

The simple example of planetary orbits we calculated last time can be compared with the situation we have for the 2d Euler's equation. We have seen (see e. g. lecture 20) that the Euler evolution

$$\omega_t + u\nabla\omega = 0 \tag{74.1}$$

preserves the orbits

$$\mathcal{O}_{\omega_0} = \{\omega \ , \ \omega = \omega_0 \circ \phi^{-1} \ , \ \phi \in \mathrm{SDiff}(\Omega)\} \ , \tag{74.2}$$

where $\mathrm{SDiff}(\Omega)$ is the group of the volume-preserving diffeomorphisms of Ω. (In fact, we can take just the connected component of the identity.) The fact that $\omega(t)$ stays in $\mathcal{O}_{\omega_0}$ (with $\omega_0 = \omega(0)$) is closely related to the conservation of the quantities

$$I_f(\omega) = \int_\Omega f(\omega)\, dx \tag{74.3}$$

by the Euler evolution. One can introduce a notion of Poisson bracket of the space of the vorticities so that the Euler equation becomes

$$\dot{\omega} = \{\omega, E\}\,, \tag{74.4}$$

where E is the energy functional (20.11) and

$$\{I_f, F\} = 0 \tag{74.5}$$

for any (sufficiently regular) functional F on the space of vorticities. This is analogous to the equations (73.16), with the variables r, p_r, p_θ playing the role of ω the function H playing the role of E and the condition $p_\theta = \text{const.}$ playing the role of $I_f(\omega) = \text{const}$. The surfaces $p_\theta = \text{const.}$ play the role of the orbits $\mathcal{O}_{\omega_0}$. Just as we could get some stability of equilibria which are local minima of $H|_{\{p_\theta = c\}}$ for a fixed c, we can expect (following V. I. Arnold) to get some stability for equilibria of Euler's equations which are local minima/maxima of the energy E on an orbit.

We start with a formula (due to V. I. Arnold) for the second variation of the energy E restricted to an orbit $\mathcal{O}_{\omega_0}$ at an equilibrium $\omega \in \mathcal{O}_{\omega_0}$. (Note that for $\omega \in \mathcal{O}_{\omega_0}$ we trivially have $\mathcal{O}_{\omega_0} = \mathcal{O}_\omega$.) As we have seen in lecture 20, the steady states in $\mathcal{O}_{\omega_0}$ formally correspond to the critical points of the restriction of E to $\mathcal{O}_{\omega_0}$.

We will consider the orbit $\mathcal{O}_{\omega_0}$ formally as a manifold (which is a sub-manifold of the space of the vorticities). In reality the sets $\mathcal{O}_{\omega_0}$ are not really sub-manifolds

of the usual functions spaces, but at this point we will not worry about this. The tangent space $T_\omega \mathcal{O}_{\omega_0}$ at ω will be (formally) identified with the space of functions

$$\eta = \xi \nabla \omega \tag{74.6}$$

where ξ is a smooth div-free field on Ω which is tangent to $\partial\Omega$. We will express the second variation

$$\delta^2 E(\omega) \tag{74.7}$$

as a quadratic form in η.

There is one point which is perhaps worth commenting on before doing the calculation. Let M be a manifold and $f\colon M \to \mathbf{R}$ a function on M. Assume that $x^1, \ldots, x^n$ are local coordinates on M. The differential df is given by

$$df = \frac{\partial f}{\partial x^i} dx^i \qquad \text{(summation understood)}\,. \tag{74.8}$$

and $df(x)$ is intrinsically defined as a linear form of the tangent space $T_x M$. The usual definition is as follows. If $\xi \in T_x M$ and $\gamma(t)$ is a curve in M with $\gamma(0) = x$ and $\frac{d\gamma}{dt}|_{t=0} = \xi$, then

$$df(x)\xi = \frac{d}{dt}|_{t=0} f(\gamma(t))\,. \tag{74.9}$$

In general, the second variation of f does not have an intrinsic meaning as a quadratic form on $T_x M$. For example, in the above situation, when we take

$$\frac{d^2}{dt^2}|_{t=0} f(\gamma(t)) = \frac{\partial f}{\partial x^i}(x)\ddot{\gamma}^i(0) + \frac{\partial^2 f}{\partial x^i \partial x^j}(x)\xi^i \xi^j\,, \tag{74.10}$$

we see that the term on the left-hand side is intrinsic, but the first term is not a quadratic form of $T_x M$, as it depends of $\ddot{\gamma}(0)$. However, when $df(x) = 0$ (which is of course an intrinsic condition), then we see from (74.10) that the expression

$$\frac{\partial^2 f}{\partial x^i \partial x^j}(x)\xi^i \xi^j \tag{74.11}$$

is well-defined intrinsically as a quadratic form on $T_x M$. We will carry out the calculation of (74.7) next time.

Homework assignment 6

due May 10, 2012

Consider a 2d ideal incompressible fluid with inhomogeneous density $\rho = \rho(x,t)$. Assume the fluid occupies a two-dimensional region Ω which is periodic of period L in the x_1 direction, and $x_2 \in (0,a)$, where $a > 0$. The equations of motion are

$$\begin{aligned} \rho u_t + \rho u \nabla u + \nabla p &= -\rho g e_2 \,, \\ \rho_t + u \nabla \rho &= 0 \,, \\ \operatorname{div} u &= 0 \,. \end{aligned}$$

where e_2 is the unit vector in the x_2 direction, $g > 0$ is a constant describing the acceleration due to gravity, u is the velocity field of the fluid, and p is the pressure in the fluid. The boundary condition are $u_2 = 0$ at $\{x_2 = 0\}$ and $\{x_2 = a\}$. Show that any (sufficiently regular) configuration with $u = 0$ and $\rho = \rho(x_2)$ is a steady-state solution (for a suitable p). Analyze the linearized stability of this solution.

75

4/27/2012

75.1 The second variation of energy on a vorticity orbit at a steady solution of 2d Euler

We consider the 2d (incompressible) Euler's equations in a domain Ω. We assume that Ω is smooth, with possibly several boundary components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$. We describe the velocity field u with the help of the stream function ψ, see lecture 14, (14.9). We will consider the vorticity ω as our primary variable. The stream function ψ is obtained from ω by solving

$$\Delta\psi = \omega, \quad \psi|_{\partial\Omega} \text{ is locally constant and } \int_{\Gamma_j} \frac{\partial\psi}{\partial n} = \gamma_j \,, \tag{75.1}$$

where γ_j are given. We can also assume $\psi|_{\Gamma_0} = 0$ without loss of generality. The condition that ψ be locally constant at $\partial\Omega$ expresses the assumption that no fluid flows through the boundary, which means that the boundary components Γ_j are streamlines. If ω and γ_j are given,[305] then ψ (which is determined up to a constant[306]) can be obtained by minimizing the functional

$$\int_\Omega \frac{1}{2}|\nabla\psi|^2 - \sum_j \int_{\Gamma_j} \gamma_j \frac{\partial\psi}{\partial n} \tag{75.2}$$

over the functions ψ with $\nabla\psi \in L^2(\Omega)$. From the Kelvin Circulation Theorem we see that the quantities

$$\int_{\Gamma_j} \frac{\partial\psi}{\partial n} \tag{75.3}$$

are constants of motion. The "vorticity coordinates" of the velocity field $\nabla^\perp\psi$ are given by determining ω and the constants γ_j. The energy of the velocity field $\nabla^\perp\psi$ (given by a vorticity field ω and the circulations γ_j) is

$$\mathcal{E}(\omega) = \int_\Omega \frac{1}{2}|\nabla\psi|^2 \; = \int_{\partial\Omega} \frac{1}{2}\psi\frac{\partial\psi}{\partial n} + \int_\Omega -\frac{1}{2}\omega\psi \; . \tag{75.4}$$

We will denote by η the variations of ω and by φ the variations of ψ (for the given γ_j). Clearly $\Delta\varphi = \eta$ and since the boundary condition that $\varphi|_{\partial\Omega}$ is locally constant and $\int_{\Gamma_j} \frac{\partial\varphi}{\partial n} = 0$. We demand that the variations η belong to the (formal) tangent space $T_\omega\mathcal{O}_{\omega_0}$, which is given by the functions of the form

$$\eta = \xi\nabla\omega \; , \tag{75.5}$$

where ξ is a any smooth div-free field tangent to $\partial\Omega$ at the boundary (i. e. $\xi\, n = 0$ at $\partial\Omega$).

[305] Note that $\int_\Omega \omega = \int_{\partial\Omega} \frac{\partial\psi}{\partial n}$, so that we need to know only m of the $m+1$ numbers $\gamma_0, \gamma_1, \dots, \gamma_m$.

[306] unless we fix the constant for example by demanding that $\psi|_{\Gamma_0} = 0$.

The first variation of $\mathcal{E}$ on Ω in the direction of η given by (75.5) is

$$\delta\mathcal{E}(\omega)\eta = \mathcal{E}'(\omega)\eta = \int_\Omega \nabla\psi\nabla\varphi = \int_\Omega -\eta\psi = \int_\Omega -(\xi\nabla\omega)\psi = \int_\Omega (\omega\nabla\psi)\xi \ . \tag{75.6}$$

We can assume that $\xi = \nabla\alpha$ for some smooth α which is locally constant at $\partial\Omega$. In terms of α we have

$$\mathcal{E}'(\omega)\eta = \mathcal{E}'(\omega)(\nabla^\perp\alpha) = \int_\Omega \{\psi,\omega\}\alpha\,, \tag{75.7}$$

where, as usual,

$$\{\psi,\omega\} = \psi_{,1}\omega_{,2} - \psi_{,2}\omega_{,1}\,. \tag{75.8}$$

We see that ω is a critical point of $\mathcal{E}$ on the orbit if and only if

$$\{\psi,\omega\} = 0\,. \tag{75.9}$$

We also reached this conclusion in lecture 20.

Assuming $\delta\mathcal{E}(\omega) = 0$, we now wish to calculate the second variation

$$\delta^2\mathcal{E}(\omega)(\eta,\eta) = \mathcal{E}''(\omega)(\eta,\eta)\,. \tag{75.10}$$

For this we take some curve ω_ε in the orbit $\mathcal{O}_{\omega_0}$ with

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\omega = \eta \tag{75.11}$$

and calculate

$$\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\mathcal{E}(\omega+\epsilon\eta)\,. \tag{75.12}$$

Last time we have seen that this definition is "intrinsic" (independent of the choice of ω_ε as long as (75.11) is satisfied) if $\delta\mathcal{E}(\omega)\eta = 0$. We can take ω_ε as follows. We assume ξ is as above that $\eta = \xi\nabla\omega$. Let ϕ^ε be the flow generated in Ω by ξ, with $\phi^\varepsilon(x) = x$. We let

$$\omega_\varepsilon = \omega\circ\phi^\varepsilon\,. \tag{75.13}$$

We have

$$\phi^\varepsilon(x) = x + \varepsilon\xi + \frac{1}{2}\varepsilon^2\xi\nabla\xi\,. \tag{75.14}$$

In what follows we use the notation

$$\dot\omega = \frac{d}{d\varepsilon}|_{\varepsilon=0}\ \omega_\varepsilon, \qquad \ddot\omega = \frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\ \omega_\varepsilon\,, \tag{75.15}$$

and similarly for ψ_ε (the stream function of ω_ε with our boundary conditions in (75.1)).

We have

$$\frac{d^2}{d\varepsilon^2}\mathcal{E}(\omega_\varepsilon) = \int_\Omega \nabla\dot\psi\nabla\dot\psi + \nabla\psi\nabla\ddot\psi \ = \int_\Omega |v|^2 - \psi\ddot\omega\,, \tag{75.16}$$

where v is the velocity field generated by $\eta = \dot{\omega}$ (with the stream function $\dot{\psi}$ and the boundary conditions $\int_{\Gamma_j} \frac{\partial \dot{\psi}}{\partial n} = 0$). We have

$$\ddot{\omega} = \frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\ \omega(x+\varepsilon\xi+\frac{1}{2}\varepsilon^2\xi\nabla\xi+\dots) = \omega_{,il}\xi_i\xi_j+\omega_i\xi_j\xi_{i,j} = (\omega_{,i}\xi_j\xi_i)_{,j}\,. \quad (75.17)$$

Hence

$$\int_\Omega -\psi\ddot{\omega} = \int_\Omega -\psi(\omega_{,i}\xi_j\xi_i)_{,j} = \int_\Omega \psi_{,j}\omega_i\xi_j\xi_i\,. \quad (75.18)$$

For a steady state solution ω we know that the vectors $\nabla\psi$ and $\nabla\omega$ are parallel, and hence, assuming $\nabla\omega \neq 0$, we have $\psi_{,j} = a(x)\omega_{,j}$ for some functions $a(x)$. Sometimes the notation

$$a(x) = \frac{d\psi}{d\omega} \quad (75.19)$$

is used, so that we can write

$$\int_\Omega -\psi\ddot{\omega} = \int_\Omega a(x)\eta^2 = \int_\Omega \frac{d\psi}{d\omega}\ \eta^2 \quad (75.20)$$

For example, when $\omega = F(\psi)$ for $F'(\psi) \neq 0$, we have $\frac{d\psi}{d\omega} = \frac{1}{F'(\psi)}$, and hence

$$\delta^2\mathcal{E}(\omega)(\eta,\eta) = \int_\Omega |v|^2 + \frac{1}{F'(\psi)}\eta^2\ . \quad (75.21)$$

We see that, in this situation, when $F'(\psi) > 0$, the second variation $\delta^2\mathcal{E}(\omega)$ is positive definite, and hence the equilibrium ω will be linearly stable. Under some assumptions one can also establish non-linear stability results, as we will see next time.

76

4/30/2012

76.1 More on critical point under constraints

Let us consider two smooth functions $f, g\colon \mathbf{R}^n \to \mathbf{R}$, and the classical problem of finding an extremum of f under the constraint $g = c$. (We assume that the reader is familiar with the basic facts about the Lagrange multipliers.) We will denote by $\Sigma = \Sigma_c$ the set $\{g = c\}$. We assume that the derivative g' (which can be identified with the gradient vector ∇g) does not vanish on Σ (and hence Σ is locally a smooth $(n-1)$-dimensional sub-manifold of $\mathbf{R}^n$). Let $\overline{x}$ be a critical point of f on Σ. We know that

$$f'(\overline{x}) = \lambda g'(\overline{x})\,, \tag{76.1}$$

where f', g' denote the differentials of respectively f, g in $\mathbf{R}^n$. If $\overline{x} = \overline{x}_c$ gives a global maximum of f on Σ_c and

$$S(c) = f(\overline{x}_c) = \max_{\{g=c\}} f\,, \tag{76.2}$$

then one has

$$\frac{dS}{dc} = \lambda\,. \tag{76.3}$$

Let us now consider the second variation of $f|_\Sigma$ at $\overline{x}$. We write $x = (x', x_n)$, with $x' \in \mathbf{R}^{n-1}$. We will assume without loss of generality that we have $\overline{x} = 0, c = 0$, and

$$f(x) = \alpha x_n + \frac{1}{2} a_{ij}\, x_j x_i, \qquad g(x) = \beta x_n + \frac{1}{2} b_{ij}\, x_j x_i\,. \tag{76.4}$$

The surface $\{g = 0\}$ near $x = 0$ is easily seen to be given by

$$x_n = -\frac{1}{2\beta} b_{pq} x'_q x'_p + O(|x'|^3)\,, \quad \text{(with summation over } p, q = 1, \dots n-1). \tag{76.5}$$

Using x' as local coordinates on Σ near 0, we see that on the surface Σ we have

$$f|_\Sigma = -\frac{\alpha}{2\beta} b_{pq} x'_q x'_p + \frac{1}{2} a_{pq} x'_q x'_p + O(|x'|^3)\,. \tag{76.6}$$

This is also immediately seen from

$$f|_\Sigma = (f - \lambda g)|_\Sigma\,, \qquad f'(0) - \lambda g'(0) = 0\,. \tag{76.7}$$

We see that the second differential of the function $f|_\Sigma$ coincides with the restriction to $\{x_n = 0\}$ of the second differential of the function $f - \lambda g$ considered in $\mathbf{R}^n$. Hence $f|_\Sigma$ has a non-degenerate local minimum[307] at $x = 0$ if and only

[307] By a non-degenerate local of a function we mean a point $\overline{x}$ where $f'(\overline{x}) = 0$ and the quadratic form $f''(\overline{x})$ is strictly positive definite.

if the restriction of $f - \lambda g$ to $\{x_n = 0\}$ has a non-degenerate local minimum at $x = 0$.

We note that as long as $\alpha \neq 0, \beta \neq 0$, the role of f, g can be interchanged. We can seek the extremum of g subject to the constraint $\{f = 0\}$. Then λ is changed to $\frac{1}{\lambda}$ and $f - \lambda g$ is changed to $g - \frac{1}{\lambda} f = -\frac{1}{\lambda}(f - \lambda g)$. We see that *when $\lambda > 0$ (resp. $\lambda < 0$) and $f'(\overline{x}) \neq 0, g'(\overline{x}) \neq 0$, then f attains a non-degenerate minimum on $\{g = c\}$ at $\overline{x}$ with $f(\overline{x}) = m$ if and only if g attains a non-degenerate maximum (resp. minimum) at $\{f = m\}$ at $\overline{x}$.*

Let us now return to the 2d Euler equations. We consider a bounded 2d domain Ω with boundary components $\Gamma_0, \Gamma_1, \ldots \Gamma_r$. We consider smooth vorticity functions ω in Ω. The stream function ψ is obtained from ω as in (75.1), where the fluxes $\gamma_1, \ldots, \gamma_r$ are fixed, the integral $\int_\Omega \omega = m$ is fixed, and $\gamma_0 = \psi|_{\Gamma_0} = 0$. We note that $\gamma_0 + \gamma_1 + \ldots \gamma_r = m$. The set of all vorticity function ω in Ω with $\int_\Omega \omega = m$ will be denoted by Y_m. The energy functional $\mathcal{E}$ is defined as usual by

$$\mathcal{E}(\omega) = \int_\Omega \frac{1}{2} |\nabla \psi|^2 \, dx \,, \tag{76.8}$$

where ψ is given by (75.1). Let $f \colon \mathbf{R} \to \mathbf{R}$ be a concave function and consider the entropy functional[308]

$$I_f(\omega) = \int_\Omega f(\omega) \, dx \ . \tag{76.9}$$

We can now consider the problem of maximizing I_f over Y_m subject to the constraint $\mathcal{E}(\omega) = E$. This produces a solution of the equation

$$f'(\omega) + \lambda \psi - \mu = 0 \,, \tag{76.10}$$

where λ, μ are lagrange multipliers generates by the constraints $\mathcal{E}(\omega) = E$ and $\int_\Omega \omega = m$ respectively. If f is sufficiently regular and uniformly concave, the function f' can be inverted and we obtain

$$\omega = g'(-\lambda \psi + \mu) \,, \tag{76.11}$$

for the maximizer, where g is the Legendre transform of f (defined by[309] $g(y) = \inf_x(yx - f(x))$).

The quantity

$$S_f = S_f(E, m, \gamma_1, \ldots, \gamma_r) = \sup\{I_f(\omega), \ \int_\Omega \omega = m, \ \mathcal{E}(\omega) = E\} \tag{76.12}$$

can be considered as a version of entropy. From (76.3) we expect

$$\frac{\partial S_f}{\partial E} = \lambda \,, \tag{76.13}$$

[308] See lecture 36, (36.14), where we used s for instead of f

[309] Some mild assumptions about the growth of f' are needed to endure that f' is globally invertible.

and λ can be considered as the inverse of "temperature",[310] see also lecture 38. By analogy with the finite-dimensional situation considered above, we expect that the second variation of I_f on the "manifold" given by the constraints $\mathcal{E}(\omega) = E$ and $\int_\Omega \omega = m$ is

$$\delta^2(I_f|_{\mathcal{E}(\omega)=E,\omega\in Y_m})(\eta,\eta) = \int_\Omega f''(\omega)\eta^2 - 2\lambda\mathcal{E}_0(\eta)\,, \tag{76.14}$$

where $\mathcal{E}_0$ is calculated similarly as $\mathcal{E}$, except that one takes $\int_\Omega \eta = 0$ and $\int_{\Gamma_j} \frac{\partial\psi}{\partial n} = 0$. By the above remark, if $\lambda > 0$, then maximizing I_f with a given E and m should be the same as minimizing $\mathcal{E}$ with a given $I_f = S$. Moreover, we expect

$$\delta^2\left(\mathcal{E}|_{I_f=S,\int_\Omega\omega=m}\right)(\eta,\eta) = -\frac{1}{\lambda}\int_\Omega f''(\omega)\eta^2 - \lambda\mathcal{E}_0(\eta) = \mathcal{E}_0(\eta) - \frac{1}{\lambda}f''(\omega)\eta^2\,. \tag{76.15}$$

This is in fact a version of formula (75.21), but now the second variation is taken over a much bigger "sub-manifold", namely the set $\{I_f(\omega) = S\,,\ \int_\Omega \omega = m\}$.

[310] This temperature has nothing to do with the usual temperature of the fluid

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77.1 Arnold's stability criterion

We use the same notation as in the last lecture. For simplicity let us assume the function in (76.9) is smooth with $f'' \leq -c_1$ where $c_1 > 0$. (One can work with weaker assumptions, but our goal is just to illustrate the main points.) The meaning of $m, Y_m, \gamma_1, \ldots, \gamma_r$ is the same as in the last lecture, as is the map $\omega, \gamma_1, \ldots, \gamma_r \to \psi$, which associates to each $\omega \in Y_m$ a stream function (for given $\gamma_1, \ldots, \gamma_r$.)

Let us consider a solution $\overline{\omega}$ of (76.10) obtained by maximizing I_f over Y_m subject to $\mathcal{E}(\omega) = E$, and let us assume that the Lagrange multiplier λ in (76.10) satisfies

$$\lambda > 0 \; . \tag{77.1}$$

Let

$$l(\omega) = \int_\Omega \omega \tag{77.2}$$

and let us consider the functional

$$J(\omega) = I_f(\omega) - \lambda \mathcal{E}(\omega) - \mu l(\omega) \; , \tag{77.3}$$

where $\lambda > 0$ is now fixed (to the specific value given by $\overline{\omega}$). This is a uniformly concave functional defined on the space of vorticities. (For example, J is well-defined on $L^2(\Omega)$, with the understanding that $J(\omega)$ can be $-\infty$ if f does not satisfy appropriate growth conditions.[311]) We have

$$J'(\overline{\omega}) = 0 \, , \qquad J(\overline{\omega}) - J(\overline{\omega}+\eta) = J(\overline{\omega}) - J(\overline{\omega}+\eta) + J'(\overline{\omega})\eta \geq \int_\Omega \frac{1}{2} c_1 \eta^2 \, . \tag{77.4}$$

Moreover, for any solution $\omega(t)$ of the Euler equation, the quantity $J(\omega(t))$ is conserved:

$$J(\omega(t)) = J(\omega(0)) \, . \tag{77.5}$$

Therefore a solution $\omega(t)$ starting at ω_0 satisfying

$$J(\overline{\omega}) - J(\omega_0) < \varepsilon \tag{77.6}$$

with satisfy

$$J(\overline{\omega}) - J(\omega(t)) < \varepsilon \, , \tag{77.7}$$

In view of of (77.4) this means that the solution $\omega(t)$ will stay L^2- close to $\overline{\omega}$ (when ε is small, of course). All the solutions we consider here satisfy $\int_{\Gamma_j} \frac{\partial \psi}{\partial n} = \gamma_j \, , \; j = 1, 2, \ldots, r$. We emphasize that we only know the solution $\omega(t)$ is well-defined when the initial value ω_0 is in $L^\infty(\Omega)$. It is not known if one has a well-defined unique time evolution when, say, $\omega_0 \in L^2(\Omega)$.

[311] Here and below we assume that the volume of Ω is finite.

The case $\lambda < 0$ is more complicated as the functional (77.3) is then not transparently concave. In fact, even its boundedness from above is not completely obvious and some assumptions on f are needed to establish it. Of course, if the function $I_f - \lambda\mathcal{E} - \mu\omega$ still attains a strict local maximum at $\overline{\omega}$, one can use the same argument as above to establish the stability of $\overline{\omega}$.

Using Jensen's inequality, we note that for a given m the function S_f attains its maximum in Y_m (without any other constraints) at

$$\omega_m = \frac{m}{|\Omega|}\,, \tag{77.8}$$

where $|\Omega|$ denotes the measure of Ω. We have

$$I_f(\omega_m) = |\Omega|\, f\left(\frac{m}{|\Omega|}\right) = S_m\,. \tag{77.9}$$

Let

$$E_m = \mathcal{E}(\omega_m)\ . \tag{77.10}$$

Then clearly

$$S_f(E_m) = S_m, \qquad \frac{\partial S_f}{\partial E}(E_m) = 0\,. \tag{77.11}$$

In Statistical Mechanics the entropy function $S = S(E)$ is a concave function of E. Assumptions under which the same is true for the function S_f above have not been much investigated, it seems. We will do a few calculation relevant to this question, which will show its connection to the stability of the corresponding equilibria. For simplicity we will fix m and work on the space Y_m in what follows. For a given E a maximizer of I_f on $\{\mathcal{E} = E\}$ will be denoted by $\omega = \omega_E$. In general ω_E may perhaps not be unique, but we will assume that we are in a situation when we can choose ω_E to depend on E in a way which is sufficiently regular for the calculations below. We will use the notation

$$\dot{\omega} = \dot{\omega}_E = \frac{\partial \omega_E}{\partial E}\,. \tag{77.12}$$

As we assume that $\omega \in Y_m$, we have $\int_\Omega \dot{\omega} = 0$. We have

$$I_f'(\omega) - \lambda\mathcal{E}'(\omega) = 0\,. \tag{77.13}$$

The Lagrange multiplier μ is now not necessary since we are working in Y_m. The multiplier $\lambda = \lambda_E$ depends on E, and we will write

$$\dot{\lambda} = \dot{\lambda}_E = \frac{d\lambda_E}{dE}\,. \tag{77.14}$$

Taking derivative of (77.13) with respect to E, we obtain

$$I_f''(\omega)\dot{\omega} - \lambda\mathcal{E}''(\omega)\dot{\omega} = \dot{\lambda}\mathcal{E}'(\omega)\ . \tag{77.15}$$

Here we view the second differentials $I_f'', \mathcal{E}''$ as quadratic forms on Y_m and the first differentials $I_f', \mathcal{E}'$, together with $I_f''(\omega)\dot{\omega}$ and $\mathcal{E}''(\omega)\dot{\omega}$, are viewed as linear functionals on Y_m. We have

$$\mathcal{E}'(\omega)\dot{\omega} = \frac{d}{dE}\mathcal{E}(\omega) = \frac{d}{dE}\, E \, = 1. \tag{77.16}$$

Recalling (76.3), we see that

$$\dot{\lambda} = \frac{\partial}{\partial E}\frac{\partial S_f}{\partial E} = \frac{\partial^2 S_f}{\partial E^2}\,. \tag{77.17}$$

Therefore we see from (77.15) that

$$I_f''(\omega)(\dot{\omega},\dot{\omega}) - \lambda\mathcal{E}''(\omega)(\dot{\omega},\dot{\omega}) = \dot{\lambda} = \frac{\partial^2 S_f}{\partial E^2}\,. \tag{77.18}$$

The direction $\dot{\omega}$ is perpendicular to the tangent space of the surface $\{\mathcal{E} = E\}$ at ω with respect to the quadratic form $Q = I_f''(\omega) - \lambda\mathcal{E}''(\omega)$. To see it, take a function η in the tangent space, which is characterized by $\mathcal{E}'(\omega)\eta = 0$. From (77.15) we obtain

$$I_f''(\omega)(\dot{\omega},\eta) - \lambda\mathcal{E}''(\omega)(\dot{\omega},\eta) = 0\,. \tag{77.19}$$

Since the quadratic form Q is clearly negative[312] on the tangent space of $\{\mathcal{E} = E\}$ at ω, we see that it is negative on Y_m if and only if

$$\frac{\partial^2 S_f}{\partial E^2} \le 0\,. \tag{77.20}$$

If the functional $I_f - \lambda\mathcal{E}$ has a strict local maximum at $\overline{\omega}$ in Y_m, the evolution by Euler's equation starting near $\overline{\omega}$ will stay near $\overline{\omega}$, in the same sense as in the case $\lambda > 0$ considered above. We see that the condition (77.20), or more precisely, it strict version $\frac{\partial^2 S_f}{\partial E^2} < 0$, is related to the stability of the relevant equilibria. From these consideration it is not hard to infer that the function S_f is indeed concave for $E \le E_m$ and descreasing for $E > E_m$.

[312] Here we use negative to mean non-positive.