

Complex Analysis Preliminary Exam

August 28, 2007

Write your codename, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Questions are equally weighted. Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems silly.

1. Assume that $g(z)$ is holomorphic with a double zero at $a \in \mathbb{C}$. Assume that $f(z)$ is holomorphic at $z = a$ and that $f(a) \neq 0$. Express $\operatorname{res}_a \left(\frac{f(z)}{g(z)} \right)$ in terms of the data $\{f(a), f'(a), f''(a), g''(a), g'''(a)\}$.

2. Evaluate $\int_0^\infty \frac{x^{2m} dx}{1+x^{2n}}$, where m and n are nonnegative integers with $m < n$.

3. Assume that h is entire and maps the real axis to the real axis and the imaginary axis to the imaginary axis. Prove that

$$h(-z) = -h(z), \quad \text{for all } z \in \mathbb{C}.$$

4. Assume that $f(z) \in \mathbb{C}[z]$ (the space of polynomials in z with complex coefficients) and that $\int_{\partial D(0,1)} f(z) \bar{z}^j dz = 0$, for all $j = 0, 1, 2, \dots$. Show that f is identically zero.

5. a) For $0 < r < 1$, use the trig substitution $\cos \frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$ to show that

$$\int \frac{d\theta}{1-2r \cos \theta + r^2} = \frac{2}{1-r^2} \tan^{-1} \left(\frac{1+r}{1-r} \tan \frac{\theta}{2} \right).$$

- b) Solve the Dirichlet problem on the unit disk with piecewise continuous boundary data

$$f(\theta) = \begin{cases} 100 & \text{for } 0 \leq \theta \leq \pi, \\ 0 & \text{for } \pi < \theta < 2\pi. \end{cases}$$

6. Find the number of zeros in the first quadrant of $z^5 + z^4 + 13z^3 + 10$.

7. Determine the number of zeros of $e^{z^2} - 4z^2$ in the open unit disk.
8. Find an explicit conformal equivalence from the region R between the two circles $|z| < 1$ and $|z + \frac{1}{2}| < \frac{1}{2}$ and the upper half plane.
9. Prove that, if u is a non-vanishing real-valued harmonic function on an open connected subset of the complex plane and if $1/u$ is also harmonic, then u is a constant function.
10. Prove

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2} \right).$$