OPTIMAL ARBITRAGE

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1 Abstract

In a Markovian model for an equity market with mean rates of return $\mathrm{b}_i(\mathfrak{X}(t))$ and covariance rates $\mathrm{a}_{ij}(\mathfrak{X}(t))$, $1 \le i, j \le n$ for its asset prices $\mathfrak{X}(t) = (X_1(t), \cdots, X_n(t))' \in (0,\infty)^n$ at time t, what is the smallest relative amount of initial capital starting with which, and using non-anticipative investment strategies, one can match or exceed the performance of the market portfolio by the end of a given time-horizon $[0,T]$? What are the weights in the different assets of an investement strategy that accomplishes this?

Answers: under appropriate conditions, $U(T, \mathfrak{X}(0))$ and

$$X_i(t)\, D_i \log U\big(T-t, \mathfrak{X}(t)\big) \;+\; \frac{X_i(t)}{X_1(t)+\cdots+X_n(t)}\,, \qquad i=1,\cdots,n\,, \quad t\in[0,T]$$

respectively, where $U : [0,\infty)\times(0,\infty)^n \to (0,1]$ is the smallest non-negative solution of the parabolic partial differential inequality

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) \;\ge\; \widehat{\mathcal{L}}\, U(\tau,\mathbf{x})\,, \quad (\tau,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n$$

subject to the initial condition $U(0+, \cdot\,) \equiv 1$, for the operator

$$\widehat{\mathcal{L}} f \;:=\; \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\, \mathrm{a}_{ij}(\mathbf{x})\, D^2_{ij} f \;+\; \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{x_j\, \mathrm{a}_{ij}(\mathbf{x})}{x_1+\cdots+x_n}\right) x_i D_i f\,.$$

Furthermore, $U(T,\mathbf{x})$ is the probability that the $[0,\infty)^n-$valued diffusion process $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \cdots, Y_n(\cdot))'$ with infinitesimal generator $\widehat{\mathcal{L}}$ as above, and starting with the initial configuration $\mathfrak{Y}(0) = \mathfrak{X}(0) = \mathbf{x} \in (0,\infty)^n$, does not hit the boundary of the non-negative orthant $[0,\infty)^n$ by time $t = T$.

It is perhaps worth noting that the answers involve only the covariance structure of the market, not the actual mean rates of return; the rôle of these latter is limited to ensuring that the diffusion $\mathfrak{X}(\cdot)$ lives in $(0,\infty)^n$.

Strong arbitrage relative to the market portfolio exists on the horizon $[0,T]$, if and only if $U(T,\mathfrak{X}(0)) < 1$; this amounts to failure of uniqueness for the Cauchy problem

$$\frac{\partial U}{\partial \tau}(\tau,\mathbf{x}) = \widehat{\mathcal{L}}\, U(\tau,\mathbf{x}), \quad (\tau,\mathbf{x}) \in (0,\infty)\times(0,\infty)^n \qquad \text{and} \qquad U(0+,\,\cdot\,) \equiv 1 .$$

As suggested by results in Fernholz & Karatzas (2005, 2008), sufficient condition for such failure of uniqueness is that there exist a real constant $h>0$, such that

$$(x_1+\cdots+x_n)\sum_{i=1}^n x_i \mathrm{a}_{ii}(\mathbf{x}) - \sum_{i=1}^n\sum_{j=1}^n x_i x_j \mathrm{a}_{ij}(\mathbf{x}) \ge h\,(x_1+\cdots+x_n)^2, \qquad \forall\ \mathbf{x}\in(0,\infty)^n;$$

another sufficient condition is that there exist a real constant $h>0$ with

$$\big(x_1\cdots x_n\big)^{1/n}\left[\sum_{i=1}^n \mathrm{a}_{ii}(\mathbf{x}) - \frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n \mathrm{a}_{ij}(\mathbf{x})\right] \ge h\,(x_1+\cdots+x_n), \qquad \forall\ \mathbf{x}\in(0,\infty)^n.$$

As far as we know, this is the first instance sufficient conditions for non-uniqueness (equivalently, necessary conditions for uniqueness) are obtained for such Cauchy problems.

Consider an "auxiliary market", whose asset prices are given by $\mathfrak{Y}(\cdot) = (Y_1(\cdot),\cdots,Y_n(\cdot))'$. The probabilistic significance of the change of drift inherent in the definition of the operator $\widehat{\mathcal{L}}$, from $\mathrm{b}_i(\mathbf{x})$ for $\mathfrak{X}(\cdot)$ to $\sum_{j=1}^n (x_j\,\mathrm{a}_{ij}(\mathbf{x}))/(x_1+\cdots+x_n)$ for $\mathfrak{Y}(\cdot)$, is that it corresponds to a change of measure which makes the weights $\nu_i(\cdot) := Y_i(\cdot)/(Y_1(\cdot)+\cdots+Y_n(\cdot))$, $i=1,\cdots,n$ of the auxiliary market portfolio martingales. The financial significance of this change of measure is that it bestows to the auxiliary market portfolio $\underline{\nu}(\cdot) = \big(\nu_1(\cdot),\cdots,\nu_1(\cdot)\big)'$ the so-called *numéraire property:* the ratio of any strategy's performance, relative to the new market with prices $\mathfrak{Y}(\cdot)$, is a supermartingale. This change of measure does not come necessarily from a Girsanov-type (absolutely continuous) transformation; rather, it corresponds to, and represents, the *exit measure* of Föllmer (1972) for an appropriate supermartingale.

The questions raised in this work can be traced back to Fernholz (2002). They are related to the results of Delbaen & Schachermayer (1995), and bear an even closer connection with issues raised in the Finance literature under the general rubric of "bubbles". The literature on this letter topic is large, so let us mention the papers by Lowenstein & Willard (2000), Pal & Protter (2007) and, most significantly, Heston et al. (2007), as the closest in spirit to our approach here. Let us also call attention to the recent preprint by Hugonnier (2007), which demonstrates that arbitrage opportunities can arise also in equilibrium models; we also refer to this preprint and to Heston et al. (2007) for an up-to-date survey of the literature on this and related topics. *(Joint work with Daniel Fernholz, Department of Computer Science, University of Texas at Austin.)*

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