Option pricing model based on telegraph processes

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Background

Option pricing models based on the geometric Brownian motion, e.g. Black-Scholes model,

$$S(t) = S_0 \mathrm{e}^{\mu t + v w(t)},\ 0 \le t \le T,$$

have well known limitations. These models (Black-Scholes and its derivatives) have infinite propagation velocities, independent log-returns increments on separated time intervals among others.

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have well known limitations. These models (Black-Scholes and its derivatives) have infinite propagation velocities, independent log-returns increments on separated time intervals among others.

To avoid these drawbacks, the existing models become more and more sophisticated. The main goal of this work is to propose the basic financial model (instead of the Black-Scholes) which is simple and at the same time, is free from the shortcomings diffusion based models.

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To construct the hyperbolic model we begin with a (right-continuous) Markov process, $\sigma = \sigma(t),\ t \geq 0$ taking values ± 1 with alternating intensities $\lambda_{\pm} > 0$, more precisely,

$$\mathbb{P}(\sigma(t + \Delta t) = 1 | \sigma(t) = -1) = \lambda_{-} \Delta t + o(\Delta t),$$

$$\mathbb{P}(\sigma(t + \Delta t) = -1 | \sigma(t) = 1) = \lambda_{+} \Delta t + o(\Delta t),$$

as $\Delta t \to 0$.

Telegraph processes

The time intervals $\tau_j - \tau_{j-1},\ j = 1,\ 2,\ \ldots$, separated by moments $\tau_j,\ j = 1,\ 2,\ \ldots$, of value changes are independent and exponentially distributed $\mathrm{Exp}(\lambda_{s(-1)^{j-1}})$.

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The process $\sigma(t),\ t \geq 0$ can be viewed as a Markov flow of random times $0 = \tau_0 < \tau_1 < \tau_2 < \ \ldots$.

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Let $N_s(t)$ be the number of switches on $[0,\ t]$ of the process $\sigma(t)$. Note that N_s are Poisson processes with alternating intensities $\lambda_s,\ \lambda_{-s},\ \lambda_s, \ldots,\ s = \pm 1$. Moreover, $\sigma(t) = s(-1)^{N_s(t)}$.

Telegraph processes (2)

For given numbers $c_- \le c_+$ we define the processes

$$X_s(t) = \int_0^t c_{\sigma_s(\tau)} \mathrm{d}\tau, \ t \ge 0. \tag{1}$$

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The processes $X_s, \ s = \pm$ are called (inhomogeneous) telegraph processes with states $(c_-, \ \lambda_-), \ (c_+, \ \lambda_+)$ with the beginning at s. In a symmetric case (for $\lambda_- = \lambda_+$ and $-c_- = c_+ = c$), the processes $X_\pm = \pm c \int\limits_0^t (-1)^{N(\tau)} \mathrm{d}\tau$ are usually referred to as (integrated) telegraph process.

Markov-modulated dynamics

It is easy to see that stock price of the form

$$\mathrm{d}S(t) = S(t)c_{\sigma(t)}\mathrm{d}t \Leftrightarrow S(t) = S_0 \exp(X_s(t))$$

admits arbitrage opportunities.

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To avoid arbitrage this model can be modified as Markov-modulated dynamics

$$\mathrm{d}S(t) = S(t)[c_{\sigma(t)}\mathrm{d}t + v_{\sigma(t)}\mathrm{d}W(t)]$$

$$\Updownarrow$$

$$S(t) = S_0 \exp(\tilde{X}_s(t) + \int_0^t v_{\sigma(\tau)}\mathrm{d}W(\tau)),$$

where $W(t)$ is a standard Brownian motion (X.Guo, L.C.G.Rogers).

Jump-telegraph model

For given numbers $h_{\pm} > -1$ we define jump process

$$J_s(t) = \sum_{j=1}^{N_s(t)} h_{\sigma_s(\tau_j -)}, \; t \geq 0$$

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$$J_s(t) = \sum_{j=1}^{N_s(t)} h_{\sigma_s(\tau_j-)}, \ t \geq 0$$

and consider the stock price dynamics of the form

$$\mathrm{d}S(t) = S(t-)[c_{\sigma_s(t)}\mathrm{d}t + \mathrm{d}J_s(t)]$$

$$\Updownarrow$$

$$S(t) = S_0\mathcal{E}_t\{X_s + J_s\} = S_0 \exp(X_s(t)) \cdot \prod_{j=1}^{N_s(t)} (1 + h_{\sigma_s(\tau_j-)}). \quad (2)$$

Picture

Sample path of $X + J$

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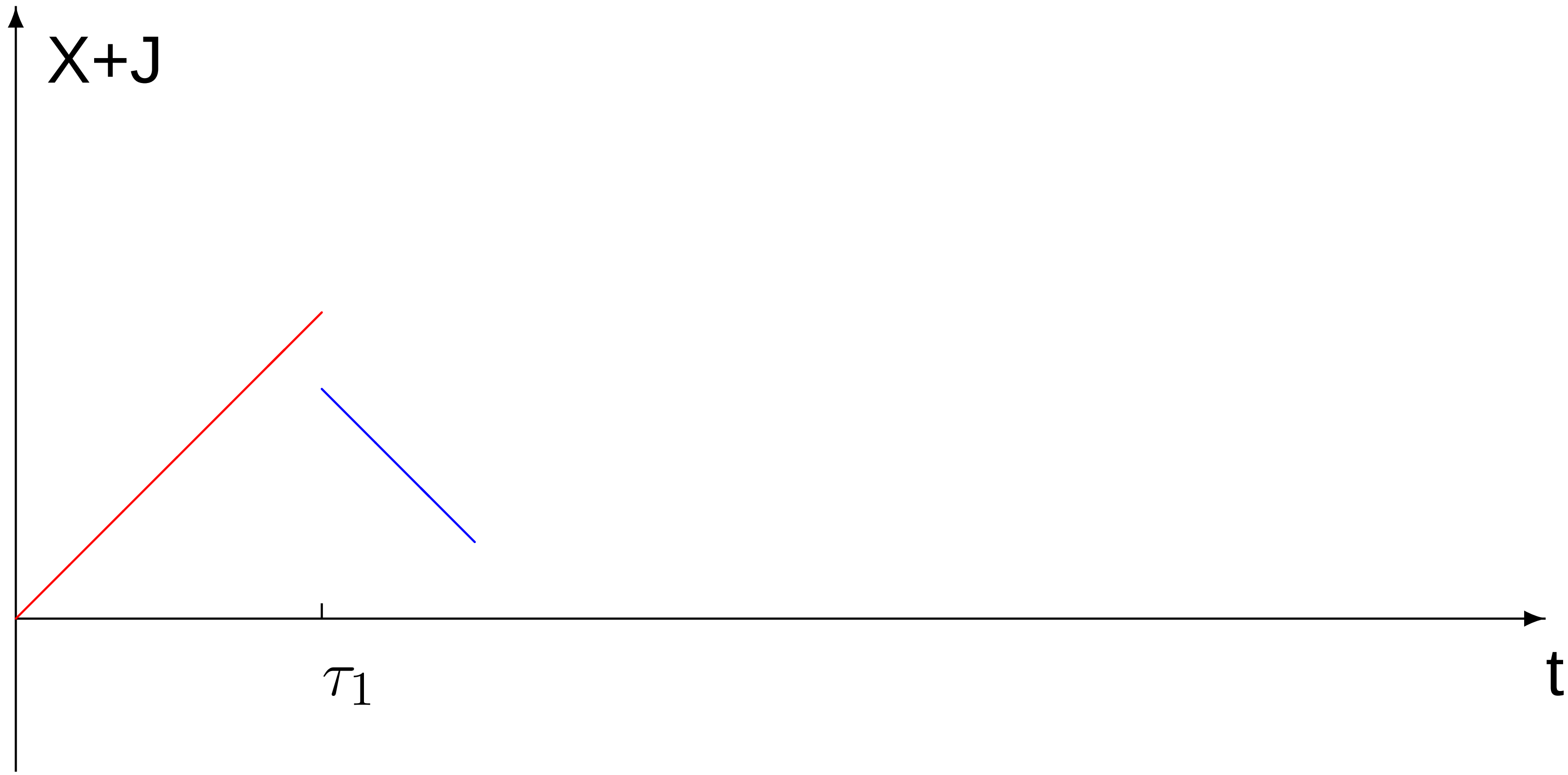

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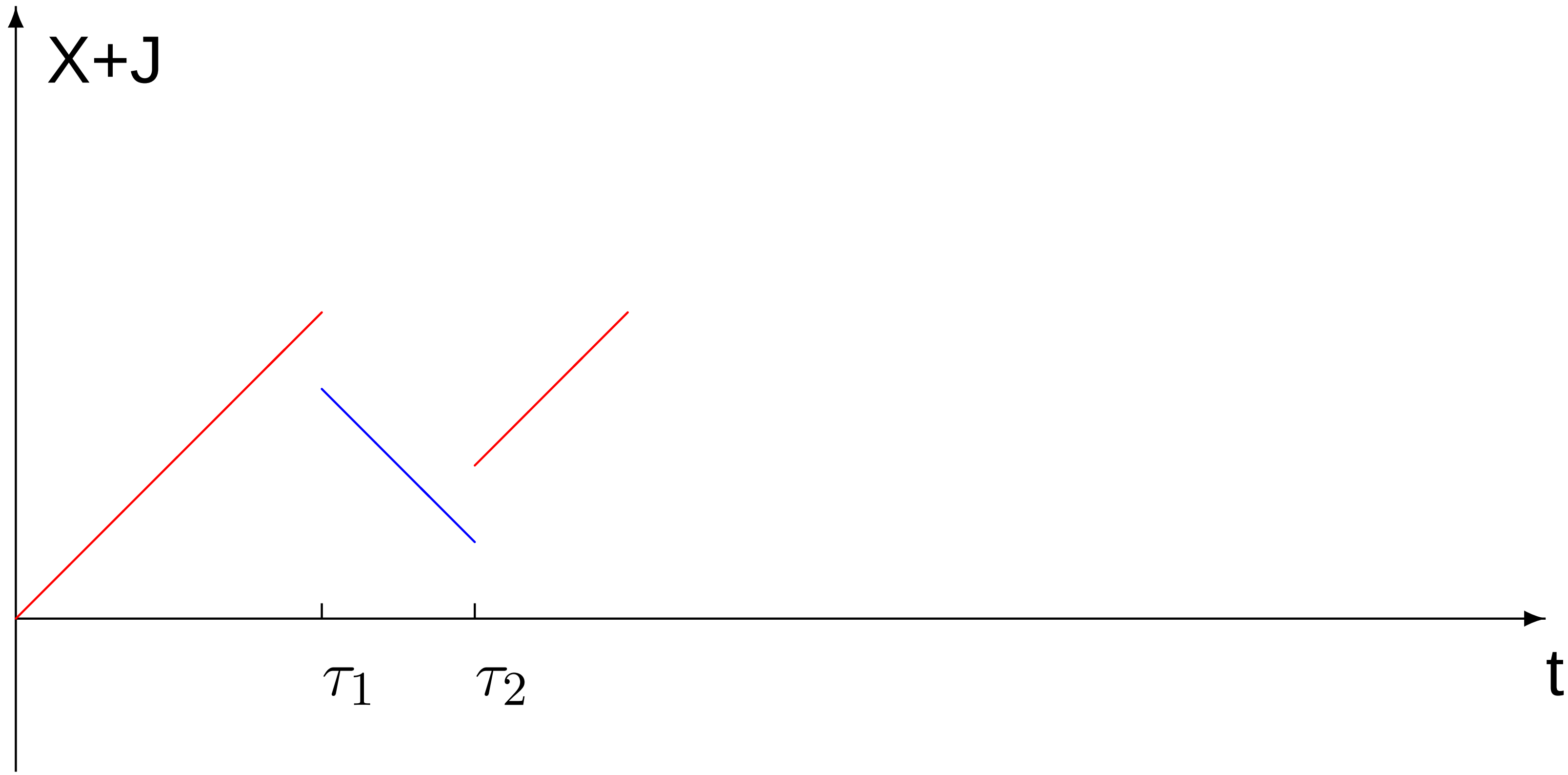

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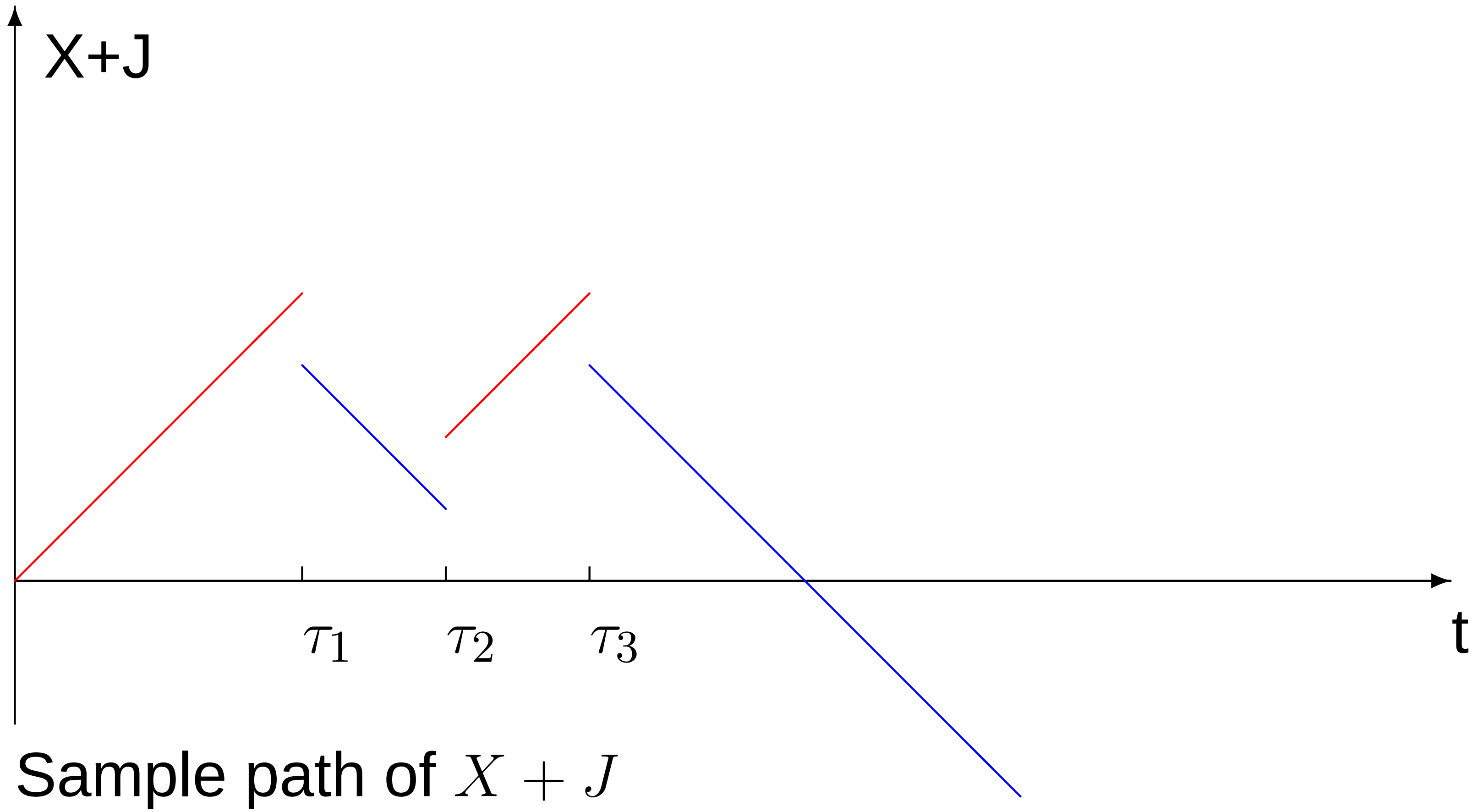

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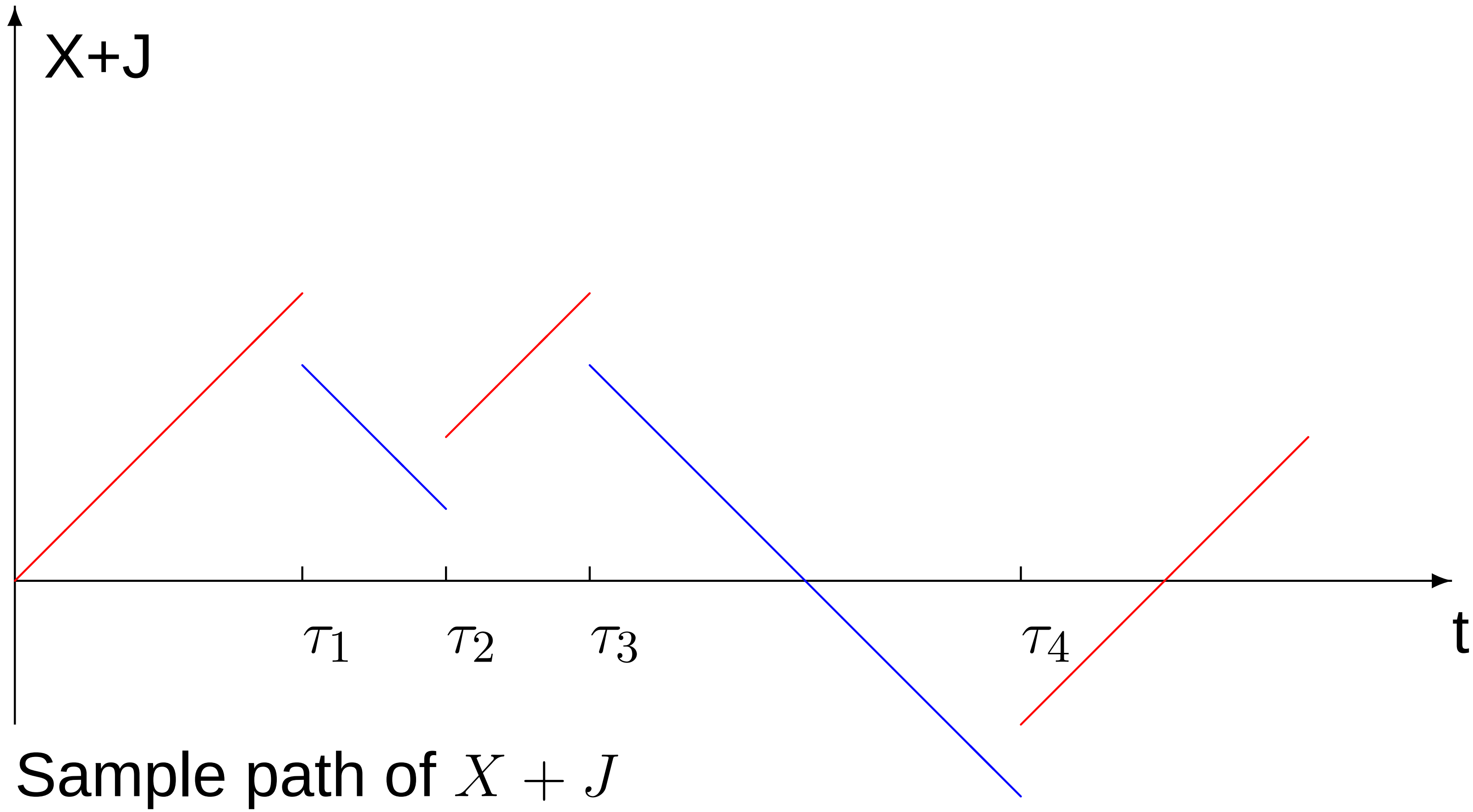

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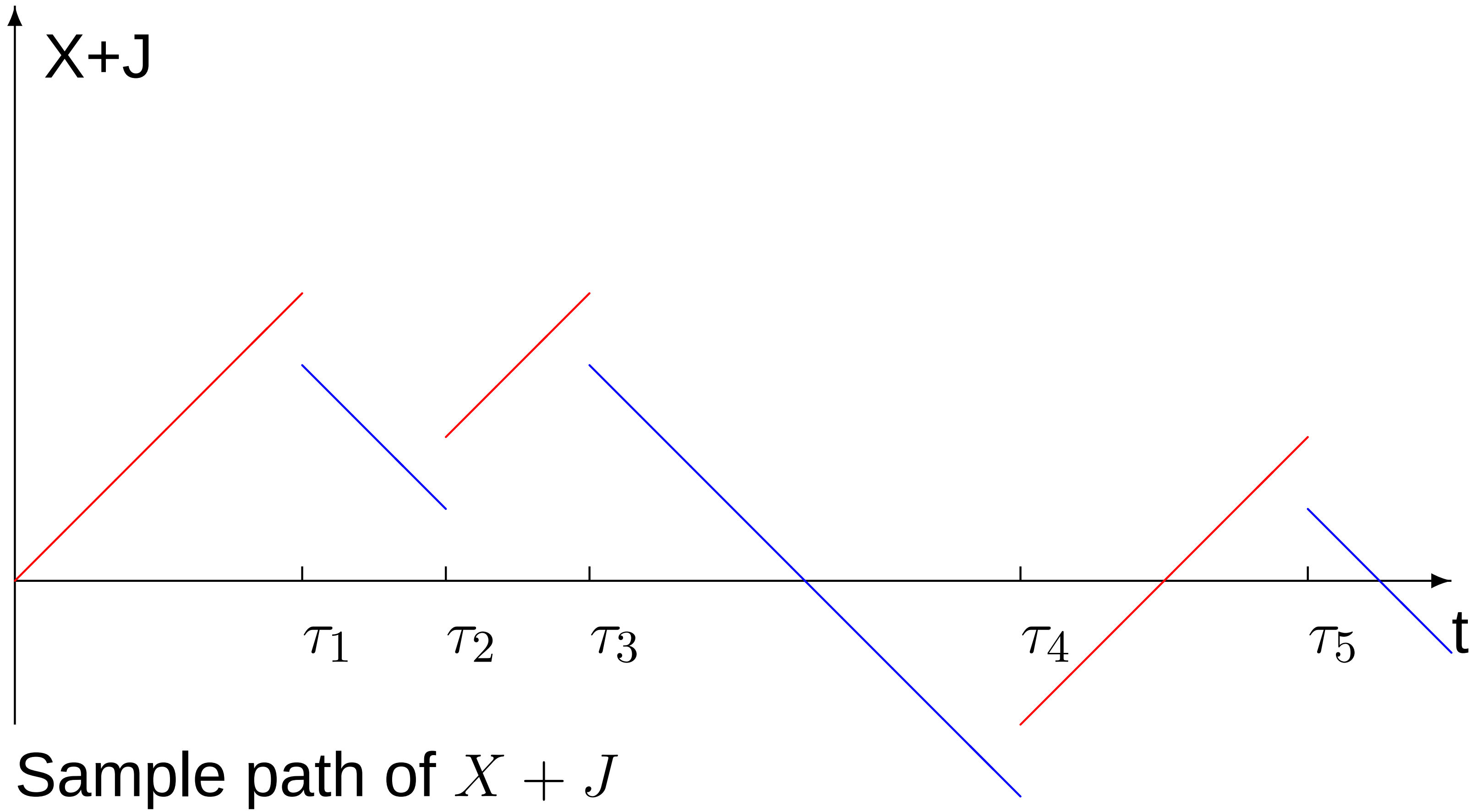

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Jump-telegraph model

The price of the non-risky asset has the form

$$B(t) = \mathrm{e}^{Y_s(t)}, \qquad Y_s(t) = \int\limits_0^t r_{\sigma(\tau)} \mathrm{d}\tau, \qquad r_-, \ r_+ > 0. \qquad (3)$$

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We assume that the parameters of model (2)-(3) satisfy the conditions

$$\lambda_-^* := \frac{r_- - c_-}{h_-} > 0, \qquad \lambda_+^* := \frac{r_+ - c_+}{h_+} > 0 \tag{4}$$

Telegraph martingales

The next theorem could be considered as a version of the Doob-Meyer decomposition for telegraph processes with alternating intensities.

Theorem 1. *Let X_s be the telegraph process with states $(c_-,\ \lambda_-)$ and $(c_+,\ \lambda_+)$, and J_s be the jump process with jump values $h_\pm > -1$, which are defined in (1), $s = \pm$. Then $X_s + J_s$ is a martingale if and only if*

$$\lambda_- h_- = -c_-, \qquad \lambda_+ h_+ = -c_+.$$

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Remark. *In particular, it means that any (nontrivial) telegraph process without jumps (i.e. if* $h_\pm = 0$*) never possess a martingale measure.*
Jump component here is supplied not only by reasons of adequacy. Jumps serves as the instrument to avoid arbitrage opportunities.

Proof

The proof follows from representations:

$$\mathbb{E}(J_s(t) \mid \mathbf{F}_\tau^s) = J_s(\tau) + \gamma H(t-\tau) + \lambda_{\sigma(\tau)} a_{\sigma(\tau)} \frac{1 - \mathrm{e}^{-\Lambda(t-\tau)}}{\Lambda},$$

$$\mathbb{E}(X_s(t) \mid \mathbf{F}_\tau^s) = X_s(\tau) + g(t-\tau) + \lambda_{\sigma(\tau)} d_{\sigma(\tau)} \frac{1 - \mathrm{e}^{-\Lambda(t-\tau)}}{\Lambda},$$

with $H = h_- + h_+,\ \Lambda = \lambda_- + \lambda_+,\ \gamma = \frac{\lambda_-\lambda_+}{\Lambda},\ g = \frac{c_+\lambda_- + c_-\lambda_+}{\Lambda}$,
and $a_\pm = \pm\frac{\lambda_+ h_+ - \lambda_- h_-}{\Lambda}$, $d_\pm = \pm\frac{c_+ - c_-}{\Lambda}, s = \pm$.
Here $\mathbf{F} = (\mathbf{F}_t^\pm)_{t\geq 0}$ denotes the filtration ($\mathbf{F}_0^\pm = \{\emptyset,\ \Omega\}$), generated by $\sigma(t),\ t \geq 0$, starting at $s,\ s = \sigma(0) = \pm$.

Change of measure

Let X_s^* be the telegraph process with the states $(c_\pm^*,\ \lambda_\pm)$, and $J_s^* = -\sum\limits_{j=1}^{N_s(t)} c^*_{\sigma(\tau_j-)}/\lambda_{\sigma(\tau_j-)}$ be the jump process with jump values $h_\pm^* = -c_\pm^*/\lambda_\pm > -1$. Consider a probability measure $\mathbb{P}_s^*$ with a local density Z_s with respect to $\mathbb{P}_s,\ s = \pm$:

$$Z_s(t) = \frac{\mathrm{d}\mathbb{P}_s^*}{\mathrm{d}\mathbb{P}_s}\,|_t = \mathcal{E}_t(X_s^* + J_s^*), \qquad 0 \le t \le T.$$

Using properties of stochastic exponentials, we obtain

$$Z_s(t) = \mathrm{e}^{X_s^*(t)} \times \prod_{j=1}^{N_s(t)} \left(1 + h^*_{\sigma(\tau_j-)}\right).$$

Girsanov theorem

Theorem 2. *Under the probability measure* $\mathbb{P}_s^*$*,*

- *process* $N_s = N_s(t),\ 0 \le t \le T$ *is a Poisson process with intensities* $\lambda_-^* = \lambda_- - c_-^* = \lambda_-(1 + h_-^*)$ *and* $\lambda_+^* = \lambda_+ - c_+^* = \lambda_+(1 + h_+^*)$.
- *process* $X_s = X_s(t),\ 0 \le t \le T$ *is a telegraph process with states* (c_-, λ_-^*) *and* (c_+, λ_+^*).

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Theorem 3. *Measure* $\mathbb{P}_s^*$ *is the martingale measure for the process* $B(t)^{-1}S(t),\ t \ge 0$ *if and only if*

$$c_-^* = \lambda_- - \frac{r_- - c_-}{h_-}, \quad c_+^* = \lambda_+ - \frac{r_+ - c_+}{h_+}$$

Moreover, under the probability measure $\mathbb{P}_s^*$, *the process* N_s *is the Poisson process with alternating intensities* $\lambda_-^* = \frac{r_- - c_-}{h_-}$ *and* $\lambda_+^* = \frac{r_+ - c_+}{h_+}$.

Fundamental equation (1)

Consider a European option with maturity time T and payoff function $f(S(T))$. We assume f is a continuous and piecewise smooth function. To price these options, we need to study the function

$$F(t,\ x,\ s) = \mathbb{E}_s^* \left[\mathrm{e}^{-Y_s(T-t)} f(x\mathrm{e}^{X_s(T-t)} \kappa_s(T-t)) \right], \qquad (5)$$

$$s = \pm,\ 0 \le t \le T,$$

where $\mathbb{E}_s^*$ denotes the expectation with respect to the martingale measure $\mathbb{P}_s^*$.

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where $\mathbb{E}_s^*$ denotes the expectation with respect to the martingale measure $\mathbb{P}_s^*$.
$F_t := F(t,\ S(t),\ \sigma(t))$ is the strategy value at time $t,\ 0 \le t \le T$ of the option with claim $f(S(T))$ at the maturity time T.

Fundamental equation (2)

Theorem 4. *Function F is a solution of the following hyperbolic system: for $0 < t < T$,*

$$\frac{\partial F}{\partial t}(t,x,s) + c_s x \frac{\partial F}{\partial x}(t,x,s)$$

$$= (r_s + \lambda_s^*) F(t,x,s) - \lambda_s^* F(t, x(1+h_s), -s),\ s = \pm \qquad (6)$$

with the terminal condition $F(T,\ x,\ s) = f(x)$. Here $\lambda_s^* = (r_s - c_s)/h_s$.

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This system plays the same role for our model as the fundamental Black-Scholes equation:

$$\frac{1}{2}v^2 x^2 \frac{\partial^2 F}{\partial x^2} + rx \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} = rF \tag{7}$$

Fundamental equation (3)

In contrast with classical theory, system (6) is hyperbolic. In particular, it implies the finite velocity of propagation, which corresponds better to the intuitive understanding of financial markets and to the viewpoint of technical analysis.

Note that these equations do not depend on $\lambda_\pm$, just as the equation (7) in the Black-Scholes model does not depend on the drift parameter.

Convergence to Black-Scholes (1)

It is known that (homogeneous) telegraph process $X = X(t),\ t \geq 0$ converges to the standard Brownian motion $w(t),\ t \geq 0$, if $c,\ \lambda \to \infty$, $c^2/\lambda \to 1$.

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The following theorem provides a similar connection (under respective scaling) between stock prices driven by geometric jump telegraph processes and geometric Brownian motion.

Let $c_+ - c_- \to \infty$, $\lambda_-, \lambda_+ \to \infty,\ h_-, h_+ \to 0$ and

$$\frac{c_+ - c_-}{\sqrt{\lambda_+} + \sqrt{\lambda_-}} \to \sigma, \qquad \sqrt{\frac{\lambda_+}{\lambda_-}} \to \gamma, \qquad \sqrt{\lambda_\pm} h_\pm \to \alpha_\pm, \quad (8)$$

$$\frac{\sqrt{\lambda_+}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}(c_- + \lambda_- h_-) + \frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}(c_+ + \lambda_+ h_+) \to \delta. \quad (9)$$

Convergence to Black-Scholes (2)

Theorem 5. *Under the scaling conditions (8)-(9) model (2) converges to the Black-Scholes:*

$$S(t) \xrightarrow{D} S_0 \exp\{vw(t) + (\delta - \beta^2/2)t\},$$

where $v = \sqrt{(\sigma + (\gamma\alpha_+ - \alpha_-)/(1+\gamma))^2 + \beta^2}$ *and* $\beta^2 = \frac{\alpha_+^2 + \gamma\alpha_-^2}{1+\gamma}$.

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Remark. *Under the martingale measure* $\mathbb{P}^*$ *transition intensities take a form* $-c_\pm/h_\pm$ *(if* $r_\pm = 0$*). Thus the drift vanishes,*

$$\Delta = \frac{\sqrt{\lambda_+}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_- + \lambda_- h_-) + \frac{\sqrt{\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(c_+ + \lambda_+ h_+) = 0.$$

Moreover, in this case

$\sigma = \lim \frac{c_+ - c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} = -\lim \frac{\lambda_+ h_+ - \lambda_- h_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} = -\frac{\gamma\alpha_+ - \alpha_-}{1+\gamma}$. *The limiting volatility* v *in this case coincides with* β*:* $v = \beta = \sqrt{\frac{\alpha_+^2 + \gamma\alpha_-^2}{1+\gamma}}$.

Convergence to Black-Scholes (3)

Remark. *Condition (11) in this theorem means that the total drift* $\Delta \equiv \frac{\sqrt{\lambda_+}c_+ + \sqrt{\lambda_-}c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}} + \frac{\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-)$ *is asymptotically finite. Here* $\frac{\sqrt{\lambda_+}c_+ + \sqrt{\lambda_-}c_-}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}$ *is generated by the velocities of the telegraph process, and the summand* $\frac{\sqrt{\lambda_+\lambda_-}}{\sqrt{\lambda_+}+\sqrt{\lambda_-}}(\sqrt{\lambda_+}h_+ + \sqrt{\lambda_-}h_-)$ *represents the drift component (possibly with infinite asymptotics) that is motivated only by jumps. If here the limits of* $\lambda_\pm h_\pm$ *are finite, then* $\alpha_\pm = \lim \sqrt{\lambda_\pm}h_\pm = 0$*. In this case the volatility of limit is* $v = \sigma = \lim(c_+ - c_-)/(\sqrt{\lambda_+} + \sqrt{\lambda_-})$*.*

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Convergence to Black-Scholes (4)

In general, the limiting volatility $v = \sqrt{\left(\sigma + (\gamma\alpha_+ - \alpha_-)/(1+\gamma)\right)^2 + \beta^2}$ depends both on "telegraph" and jump components.

Convergence to Black-Scholes (4)

In general, the limiting volatility $v = \sqrt{(\sigma + (\gamma\alpha_+ - \alpha_-)/(1+\gamma))^2 + \beta^2}$ depends both on "telegraph" and jump components.
So it is natural to define volatility of jump telegraph market as

$$\mathrm{vol}^2 = \left(\frac{c_+ - c_-}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}\right)^2 \left(1 + \frac{\lambda_+ h_+ - \lambda_- h_-}{c_+ - c_-}\right)^2 + \frac{\sqrt{\lambda_-}\lambda_+ h_+^2 + \sqrt{\lambda_+}\lambda_- h_-^2}{\sqrt{\lambda_+} + \sqrt{\lambda_-}}.$$

Pricing call options (1)

According to the theory on option pricing, we have

$$\mathbb{c}^s = \mathbb{E}_s^* \left[B(T)^{-1} (S(T) - K)^+ \right],$$

where K is the strike price and $\mathbb{E}_s^*(\cdot)$ is the expectation with respect to the martingale measure $\mathbb{P}_s^*$. In case of the model (2)-(3), one can rewrite $\mathbb{c}^s$ as

$$\mathbb{c}^s = S_0 U^{(s)}(y, T) - K u^{(s)}(y, T), \qquad s = \pm \tag{10}$$

Pricing call options (2)

with

$$u^{(s)}(y,T) = \sum_{n=0}^{\infty} u_n^{(s)}(y - b_n^{(s)}, T),$$

$$U^{(s)}(y,T) = \sum_{n=0}^{\infty} U_n^{(s)}(y - b_n^{(s)}, T),$$

where $y = \ln K/S_0$, $b_n^{(s)} = \sum_{j=1}^{n} \ln(1 + h_{\sigma(\tau_j -)})$, and functions $u_n^{(s)}, U_n^{(s)},\ n \geq 0, s = \pm$ can be directly calculated.

Pricing call options (3)

Here

$$u_n^{(s)}(y,t) = \mathbb{E}_s^* \left[B(t)^{-1} \mathbf{1}_{\{X(t)>y, N(t)=n\}} \right],$$

$$U_n^{(s)}(y,t) = \mathbb{E}_s^* \left[B(t)^{-1} \mathcal{E}_t(X+J) \mathbf{1}_{\{X(t)>y, N(t)=n\}} \right].$$

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Lemma. $U_n^{(s)}(y,\ t;\ \lambda_\pm^*,\ c_\pm,\ r_\pm) = u_n^{(s)}(y,\ t;\ \lambda_\pm^*(1+h_\pm),\ c_\pm,\ 0).$

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Functions $(u_n^{(s)},\ n \geq 1)$ are continuous and piece-wise continuously differentiable.

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$$u_n^{(s)}(y,t) = \mathbb{E}_s^* \left[B(t)^{-1} \mathbf{1}_{\{X(t)>y, N(t)=n\}} \right],$$

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Lemma. $U_n^{(s)}(y,\ t;\ \lambda_\pm^*,\ c_\pm,\ r_\pm) = u_n^{(s)}(y,\ t;\ \lambda_\pm^*(1+h_\pm),\ c_\pm,\ 0)$. Functions ($u_n^{(s)},\ n \geq 1$) are continuous and piece-wise continuously differentiable. Moreover, $\forall n, u_n^{(s)} \equiv 0$ if $y > c_+t$, and $u_n^{(s)}(y,t) \equiv \rho_n^{(s)}(t) = \mathrm{e}^{-b_r t} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-a_r x} p_{n,*}^{(s)}(x,t) \mathrm{d}x$ if $y < c_-t$.

Pricing call options (4)

Finally, these functions can be calculated as

$$u_n^{(s)} = \begin{cases} 0, & y > c_+ t, \\ w_n^{(s)}(p,\ q), & c_- t \le y \le c_+ t, \\ \rho_n^{(s)}(t), & y < c_- t, \end{cases} \qquad \sigma = \pm 1,$$

$p = \frac{c_+ t - y}{c_+ - c_-}$, $q = \frac{y - c_- t}{c_+ - c_-}$, where

$$w_n^{(s)} = \mathrm{e}^{-(\lambda_+^* + r_+)q - (\lambda_-^* + r_-)p} \Lambda_n^{(s)} v_n^{(s)}(p,\ q),$$

$\rho_n^{(s)}(t) = \mathrm{e}^{-(\lambda_-^* + r_-)t} \Lambda_n^{(s)} P_n^{(s)}(t)$ with

$\Lambda_n^{(s)} = (\lambda_s^*)^{[(n+1)/2]} (\lambda_{-s}^*)^{[n/2]}$, $s = \pm,\ n \ge 0.$

Pricing call options (5)

Functions $P_n^{(\pm)}$ and $v_n^{(\pm)}$ are defined as follows:

$P_0^{(+)} = \mathrm{e}^{-at}, P_0^{(-)} \equiv 1,$

$P_n^{(\pm)} = P_n^{(\pm)}(t) = \frac{t^n}{n!}\left[1 + \sum_{k=1}^{\infty} \frac{(m_n^{(\sigma)}+1)_k}{(n+1)_k} \cdot \frac{(-at)^k}{k!}\right]$, where

$m_n^{(+)} = [n/2]\, , m_n^{(-)} = [(n-1)/2]$,

$(m)_k = m(m+1)\ \ldots\ (m+k-1)$, $a = \lambda_+^* - \lambda_-^* + r_+ - r_-$;

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$v_0^{(-)} \equiv 0$, $v_0^{(+)} = \mathrm{e}^{-ap},\ \ v_1^{(\pm)} = P_1(p)$ and for $n \geq 1$

$$v_{2n+1}^{(\pm)} = v_{2n+1}^{(\pm)}(p,\ q) = P_{2n+1}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k,n}(p),$$

$$v_{2n}^{(-)} = v_{2n}^{(-)}(p,\ q) = P_{2n}^{(-)}(p) + \sum_{k=1}^{n-1} \frac{q^k}{k!}\varphi_{k+1,n}(p),$$

$$v_{2n}^{(+)} = v_{2n}^{(+)}(p,\ q) = P_{2n}^{(+)}(p) + \sum_{k=1}^{n} \frac{q^k}{k!}\varphi_{k-1,n-1}(p).$$

Pricing call options (6)

Here $\varphi_{0,n} = P_{2n+1}$,

$$\varphi_{k,n} = \sum_{j=0}^{k-1} a^{k-j-1} \beta_{k,j} P^{(-)}_{2n-j}, \; 1 \le k \le n,$$

where $\beta_{k,j} = \frac{(k-j)_{[j/2]}}{[j/2]!}$.

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In particular case $\lambda_+^* = \lambda_-^* = \lambda,\ r_+ = r_- = r$ these functions have a more simple form $\rho_n^{(\pm)}(t) = \mathrm{e}^{-(\lambda+r)t}\frac{(\lambda t)^n}{n!}$,

$w_n^{(\pm)} = \mathrm{e}^{-(\lambda+r)t}\frac{\lambda^n}{n!}\sum_0^{m_n^{(\pm)}} \begin{pmatrix} n \\ k \end{pmatrix} q^k p^{n-k}$. Here

$m_n^{(+)} = [n/2]\,, m_n^{(-)} = [(n-1)/2]$.

Memory effects and historical volatility

Historical volatility is defined as

$$\mathrm{HV}(t) = \sqrt{\frac{\mathrm{Var}\{\log S(t+\tau)/S(\tau)\}}{t}}. \tag{1}$$

For classical Black-Scholes model $\log S(t+\tau)/S(\tau) \overset{D}{=} at + vw(t)$ the historical volatility is constant: $\mathrm{HV}_{\mathrm{BS}}(t) \equiv v$.

Memory effects and historical volatility

Historical volatility is defined as

$$\mathrm{HV}(t) = \sqrt{\frac{\mathrm{Var}\{\log S(t+\tau)/S(\tau)\}}{t}}. \tag{2}$$

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In a moving-average type model

$$\log S(t)/S(0) = at + vw(t) - v\int\limits_0^t \mathrm{d}\tau \int\limits_{-\infty}^{\tau} p\mathrm{e}^{-(q+p)(\tau-u)}\mathrm{d}w(u),$$

where $v,\ q,\ q+p>0$ the historical volatility is

Memory and historical volatility (2)

$$\mathrm{HV} = \frac{\sigma}{2\lambda}\sqrt{q^2 + p(2q+p)\Phi_\lambda(t)}$$

with $2\lambda = q + p$ and $\Phi_\lambda(t) = \frac{1-\mathrm{e}^{-2\lambda t}}{2\lambda t}$. Recently this type of models have been applied to capture memory effects of the market.

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Historical volatility in the jump telegraph model
(in particular case $\lambda_0 = \lambda_1 := \lambda$) is

$$\mathrm{HV}_i(t) = \sqrt{\sigma^2 + \kappa^2\Phi_{2\lambda}(t)/\lambda + \gamma_i\Phi_\lambda(t) - 2B\kappa(-1)^i\mathrm{e}^{-2\lambda t}}$$

with $\sigma^2 = a^2/\lambda + \lambda B^2$, $\kappa = a + \lambda b$, $\gamma_i = -2a(\kappa - (-1)^i\lambda B)/\lambda$.
Here $b = \frac{1}{2}\ln\frac{1+h_1}{1+h_0}$, $B = \frac{1}{2}\ln(1+h_1)(1+h_0)$, $c = (c_1 - c_0)/2$,
$a = (c_1 + c_0)/2$.

Memory and historical volatility (3)

Historical volatility in jump telegraph model has the following very natural limiting behaviour:

$$\lim_{t\to 0} \mathrm{HV}_{\pm}(t) = \sqrt{\lambda_{\pm}}\ln(1+h_{\pm}),$$

$$\lim_{t\to\infty} \mathrm{HV}_{\pm}(t) = \sqrt{\frac{\lambda_+\lambda_-}{2\Lambda^3}[(\lambda_- B - c)^2 + (\lambda_+ B + c)^2]}$$

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($B = \frac{1}{2}\ln(1+h_+)(1+h_-)$, $c = (c_+ - c_-)/2$).
These limits look reasonable: the limit at 0 is engendered by jumps only, the limit at ∞ contains both “velocity" component and a long term influence of jumps.

Memory and historical volatility(3)

The limits of historical volatility under a standard diffusion scaling are more complicated. Nevertheless, in the symmetric case $\lambda_+ = \lambda_- = \lambda$, we have under the scaling conditions $\lambda,\ a \to \infty,\ h_\pm \to 0,\ a^2/\lambda \to \sigma^2,\ \sqrt{\lambda} h_\pm \to \alpha_\pm$ that the historical volatility $\mathrm{HV}_\pm(t)$ converges to $\sqrt{\sigma^2 + (\alpha_+ + \alpha_-)^2/4}$.

Notice, that under the martingale measure $\mathbb{P}^*$, we have $\lambda = -c_\pm/h_\pm$, $\sigma = (-\alpha_+ + \alpha_-)/2$, and the diffusion limit of historical volatility equals to $v = \sqrt{(\alpha_+^2 + \alpha_-^2)/2}$, which coincides with the volatility expression for the diffusion scaling.

Implied volatility (1)

Define the Black-Scholes call price function $f(\mu, v)$, $\mu = \log K$ by

$$f(\mu, v) = \begin{cases} F\left(\frac{-\mu}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - \mathrm{e}^{\mu} F\left(\frac{-\mu}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right), & \text{if } v > 0, \\ (1 - \mathrm{e}^{\mu})^{+}, & \text{if } v = 0, \end{cases}$$

where F is the standard Gaussian distribution function.

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where F is the standard Gaussian distribution function. The processes $V_{\pm}(\mu, t),\ t \geq 0,\ \mu \in \mathbb{R}$ defined by the equation

$$\mathbb{E}\left[(S(t+\tau)/S(\tau) - \mathrm{e}^{\mu})^{+} | \mathcal{F}_{\tau}\right] = f(\mu,\ V_{\sigma(\tau)}(\mu, t))$$

are referred to as implied variance processes.

Implied volatility (2)

The implied volatilities $\mathbb{IV}_{\pm}(\mu, t)$ are

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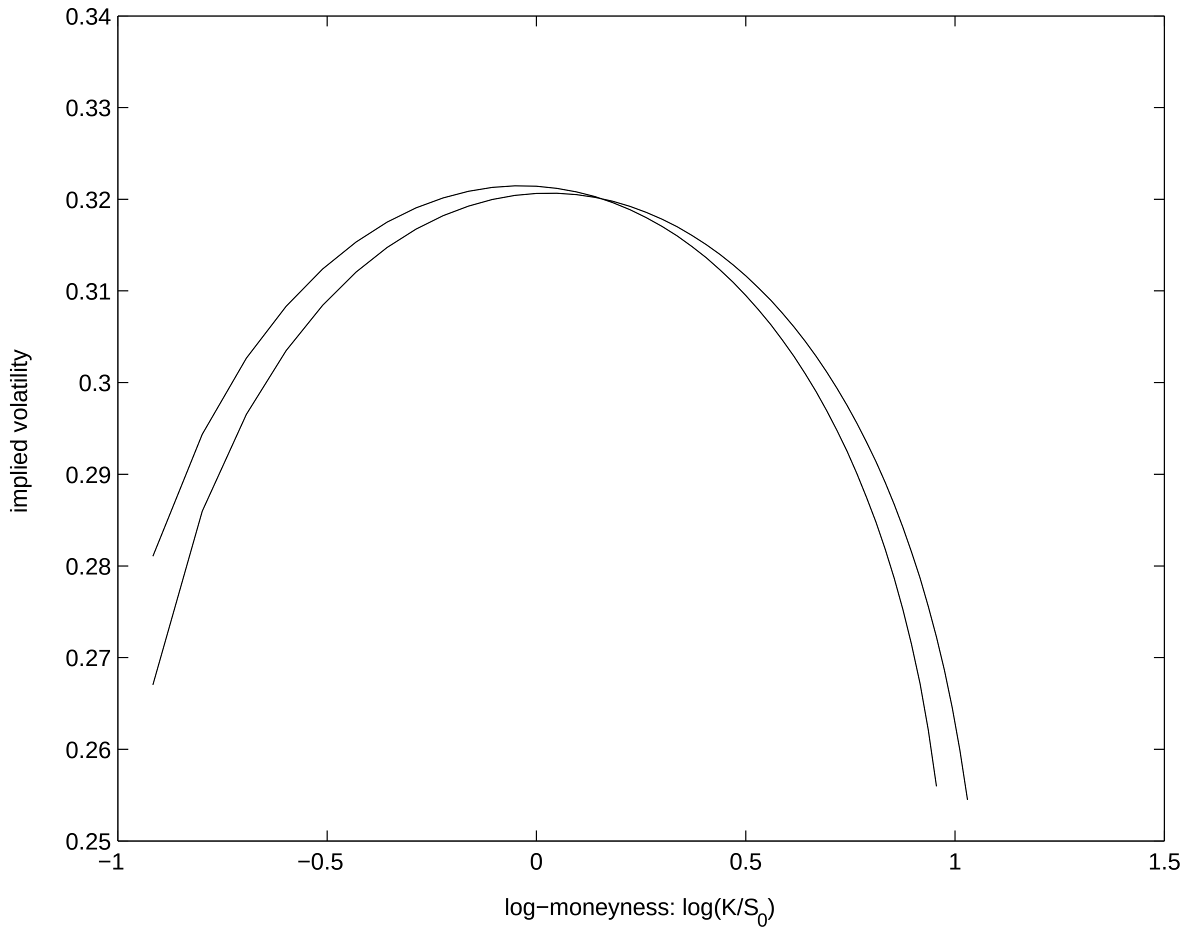
We assume $S_0 = 100, T = 1$.
First, consider the symmetric case: $\lambda_\pm = 10, c_\pm = \pm 1$ and $h_\pm = \mp 0.1$. Notice that these frowned smiles of implied volatilities $\mathbb{IV}_-$ and $\mathbb{IV}_+$ intersect at $K/S_0 = 1.17$.

Volatility smile (1)

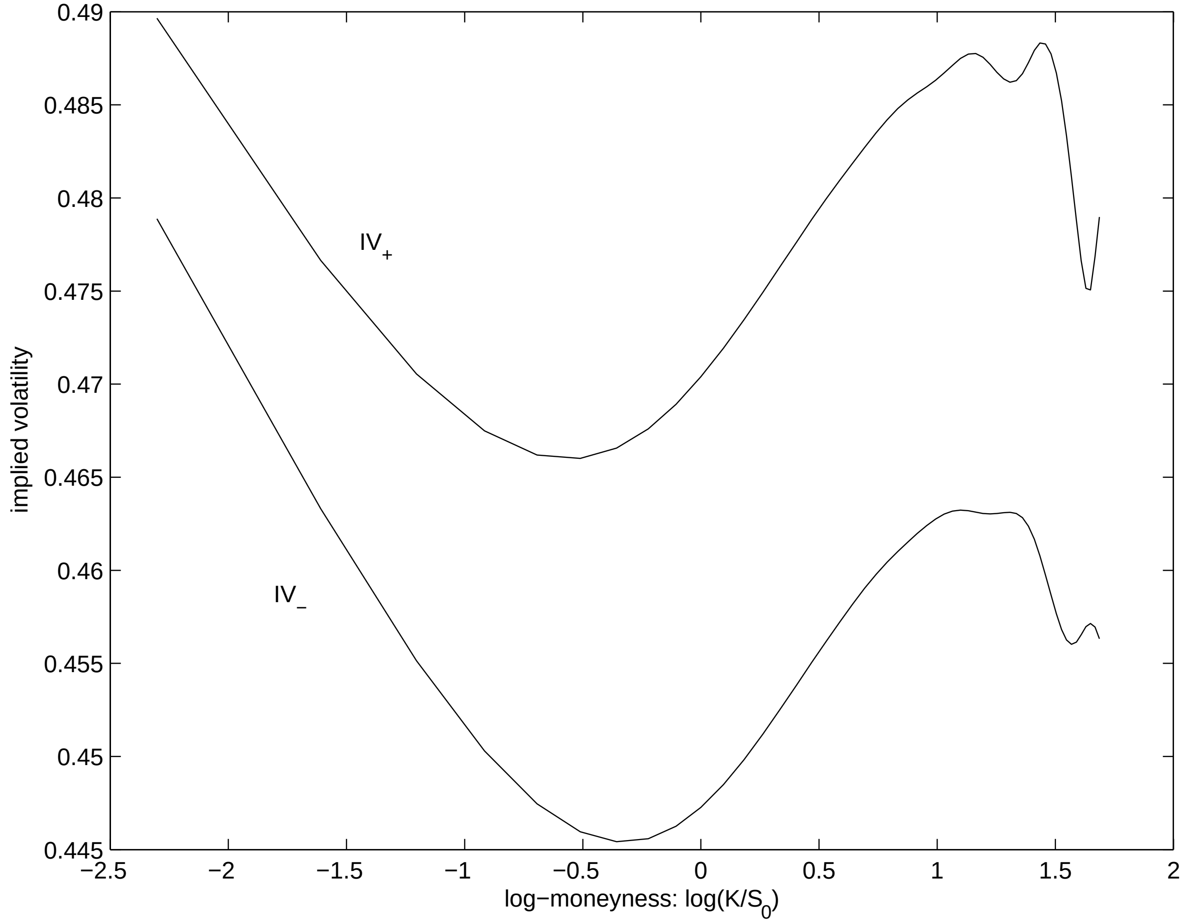
$\mathrm{HV}_{\pm} = 0.3162$, jump telegraph volatility=0.3162



$\lambda_{\pm} = 10, c_{\pm} = \pm 1, h_{\pm} = \mp 0.1$

Volatility smile (2)

$HV_- = 0.4198, HV_+ = 0.4402$; tel. volatility=0.4301

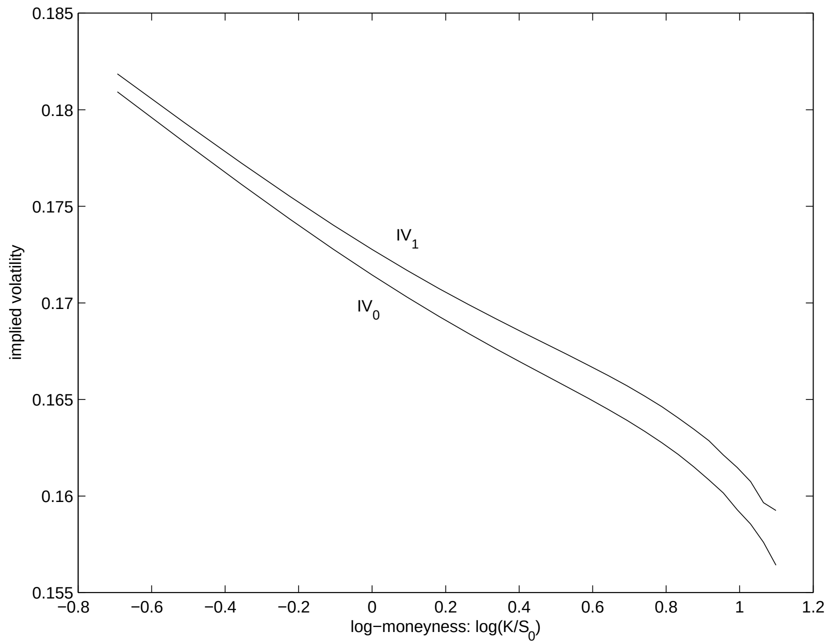


$T = 1,\ S_0 = 100,\ r = 0,\ c_- = 0.3,\ c_+ = 1.9,$

$\lambda_\pm = 10, h_- = -0.03, h_+ = -0.19$

Volatility smile (3)

Dow-Jones industrial average July 1971-Aug 1974



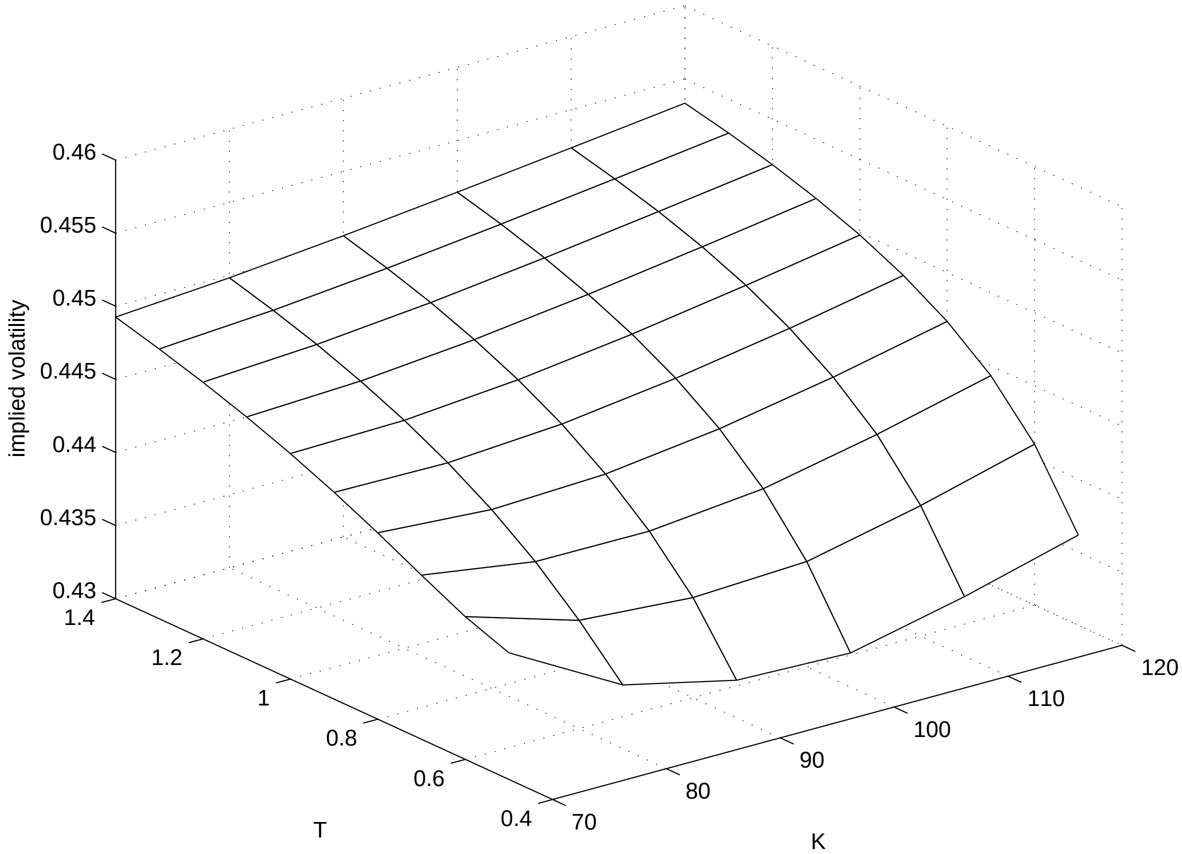
$\lambda_0 = 48.53, \lambda_1 = 34.61, h_0 = -0.0126, h_1 = -0.0358,$
$c_0 = 0.61, c_1 = 1.24; \mathrm{HV}_0 = 0.1630, \mathrm{HV}_1 = 0.1642;$

jump telegraph volatility=0.1661

A. De Gregorio and S. Iacus, Parametric estimation for the standard and geometric telegraph process observed at discrete times. Preprint, Milan, 2006

Volatility smile (4)

IV_-



$$t = 1, S_0 = 100, \lambda_\pm = 10, h_- = -0.03, h_+ = -0.19, c_- = 0.3, c_+ = 1.9$$

Problems and perspectives

- (a) Velocities move through a binary tree (as in CRR-model);
  (b) Calibration of the parameters of jump telegraph model according to real market data;
  (c) inhomogeneous case:
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Thank you for your kind attention!

- N. Ratanov, *Telegraph evolutions in inhomogeneous media*, Markov Processes Relat. Rields **5** (1999), 53-68
- N. Ratanov, *Pricing options under telegraph processes*, Rev. Econ. Ros., **8** (2005), no.2, 131-150.
- A. Melnikov, N. Ratanov, *Inhomogeneous telegraph processes and their application to financial market modeling.* Doklady Mathematics, **75** (2007), No 1, 115-117
- N. Ratanov, *A jump telegraph model for option pricing*, Quantitative Finance **7**, (2007), No 5, 575-583
- N. Ratanov, *Telegraph models of financial markets,* Rev. Col. Matem., **41**, (2007), 247-252

- N. Ratanov, *Jump telegraph models and financial markets with memory.* J. Appl. Math. Stoch. Anal., vol. 2007, Article ID 72326, 2007
- N. Ratanov, *An option pricing model based on jump telegraph processes.* Proc. Appl. Math. Mech. (PAMM), **7**, Issue 1, 2007
- N. Ratanov and A. Melnikov, *On financial markets based on telegraph processes.* Stochastics, **80**, No. 2-3, 2008, 247-268
- N. Ratanov, *Jump Telegraph Processes and Volatility Smile*, submitted