OPTIMAL ARBITRAGE

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“ YOU CANNOT BEAT THE MARKET ”

Observation?

Postulate?

Aphorism?

Theorem?

A possible approach: Try to find conditions under which you cannot, and conditions under which you might... and then, show how.

1. THE MODEL

Equity market model of the form

$$dB(t) = B(t)r(t)\,dt, \qquad B(0) = 1,$$

$$dX_i(t) = X_i(t)\left(\beta_i(t)dt + \sum_{k=1}^{K} \sigma_{ik}(t)\,dW_k(t)\right), \qquad i = 1, \ldots, n.$$

Money-market $B(\cdot)$ and n stocks with prices $X_1(\cdot), \cdots, X_n(\cdot)$ strictly positive, driven by the Brownian motion $W(\cdot) = (W_1(\cdot), \cdots, W_K(\cdot))'$ with $K \geq n$.

All processes are assumed to be progressively measurable with respect to a filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which represents the "flow of information" in the market. *Right-continuous*, and $\mathcal{F}(0) = \{\emptyset, \Omega\}$.

We shall take $r(\cdot) \equiv 0$: investing in the money-market will amount to hoarding, whereas borrowing from the money-market will incur no interest.

The vector process $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))'$ of *mean rates of return* for the various stocks, and the $(n \times n)-$matrix process $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{1 \le i \le n,\, 1 \le k \le K}$ of stock-price *volatilities*, will satisfy for every $T \in (0, \infty)$ the integrability condition

$$\sum_{i=1}^{n} \int_0^T \Big(\big| \beta_i(t) \big| + \alpha_{ii}(t) \Big)\, dt < \infty\,, \qquad \text{a.s.}$$

Here $\alpha(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ is the matrix of *covariance rates* of the stocks in the market:

$$\alpha_{ij}(t) := \sum_{k=1}^{K} \sigma_{ik}(t)\sigma_{jk}(t) = \frac{1}{X_i(t)X_j(t)} \cdot \frac{d}{d\,t} \langle X_i, X_j \rangle (t)\,.$$

We shall assume, finally, that there exists a *market price of risk* (or "relative risk") $\vartheta : [0,\infty) \times \Omega \rightarrow \mathbb{R}^K$, an $\mathbf{F}-$progressively measurable process that satisfies for each $T \in (0,\infty)$

$$\sigma(t)\vartheta(t) = \beta(t)\,, \quad \forall \quad 0 \le t \le T \qquad \text{and} \qquad \int_0^T \|\vartheta(t)\|^2\, dt < \infty\,.$$

The existence of this $\vartheta(\cdot)$ allows us to introduce the exponential local martingale and supermartingale

$$Z(t) := \exp\left\{-\int_0^t \vartheta'(s)\, dW(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2\, ds\right\}, \quad 0 \le t < \infty\,.$$

This is a martingale, if and only if $\mathbb{E}(Z(T)) = 1$, $\forall\ T \in (0,\infty)$.

- For the purposes of this talk, it is important to allow $Z(\cdot)$ to be a strict local martingale, that is, not to exclude the possibility $\mathbb{E}(Z(T)) < 1$ for some (perhaps all...) $T \in (0,\infty)$.

2. STRATEGIES and PORTFOLIOS

Let us consider now a small investor who selects, at each time t and for every $i = 1, \cdots, n$, which proportion $\pi_i(t)$ of his current wealth $V(t)$ to invest in the i^{th} stock.

We require that $\pi_i(t)$ be $\mathcal{F}(t)-$measurable. The proportion $1 - \sum_{i=1}^{n} \pi_i(t)$ gets invested in the money market.

Thus, the wealth $V(\cdot) \equiv V^{v,\pi}(\cdot)$ corresponding to an initial capital $v \in (0, \infty)$ and an investment strategy $\pi(\cdot) = \big(\pi_1(\cdot), \cdots, \pi_n(\cdot)\big)'$ satisfies $V(0) = v$ and the linear equation

$$\frac{dV(t)}{V(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)} + \left(1 - \sum_{i=1}^{n} \pi_i(t)\right) \frac{dB(t)}{B(t)}$$

$$= \pi'(t)\Big[\beta(t)dt + \sigma(t)\, dW(t)\Big] = \pi'(t)\sigma(t)\Big[dW(t) + \vartheta(t)dt\Big].$$

- We shall call *investment strategy* an $\mathbb{F}-$progressively measurable process $\pi : [0, \infty) \times \Omega \to \mathbb{R}^n$ which satisfies for each $T \in (0, \infty)$ the integrability condition

$$\int_0^T \left(\Big| \pi'(t)\beta(t) \Big| + \pi'(t)\alpha(t)\pi(t) \right) dt < \infty\,, \quad \text{a.s.}$$

The collection of all investment strategies will be denoted by $\mathcal{H}$.

An investment strategy $\pi(\cdot)$ with

$$\sum_{i=1}^{n} \pi_i(t) = 1\,, \qquad \forall \;\; 0 \le t < \infty$$

almost surely, will be called *portfolio*. A portfolio never invests in the money market, and never borrows from it. We shall say that a portfolio is *bounded*, if there exists a real constant $K > 0$ such that $\|\pi(t, \omega)\| \le K$ holds for all $(t, \omega) \in [0, \infty) \times \Omega$.

We shall call *long-only portfolio* one that satisfies almost surely

$$\pi_1(t) \geq 0, \cdots, \pi_n(t) \geq 0, \qquad \forall \;\; 0 \leq t < \infty;$$

such a portfolio never sells any stock short.

Every long-only portfolio is also bounded.

- Corresponding to an investment strategy $\pi(\cdot)$ and an initial capital $v > 0$, the associated wealth process, is given as

$$V^{v,\pi}(t) = v \exp\left\{\int_0^t (\pi'\beta)(s)ds + \int_0^t (\pi'\sigma)(s)dW(s) - \frac{1}{2}\int_0^t (\pi'\alpha\pi)(s)ds\right\}.$$

Note that $Z(\cdot)V^{v,\pi}(\cdot)$ is also a positive local martingale and a supermartingale, namely

$$Z(t)V^{v,\pi}(t) = v + \int_0^t Z(s)V^{v,\pi}(s)\,\Big(\sigma'(s)\pi(s) - \vartheta(s)\Big)'\,dW(s).$$

3. THE MARKET PORTFOLIO

Consider now the *market portfolio*

$$\mu_i(t) := \frac{X_i(t)}{X(t)}\,, \qquad i = 1, \ldots, n\,,$$

where

$$X(t) := X_1(t) + \ldots + X_n(t)\,.$$

This invests in all stocks in proportion to their relative capitalization weights. This amounts to "owning the entire market": the wealth process becomes

$$V^{v,\mu}(\cdot) = vX(\cdot)/X(0)\,.$$

4. RELATIVE ARBITRAGE (RA)

Given a real number $T > 0$ and any two investment strategies $\pi(\cdot)$ and $\rho(\cdot)$, we shall say that $\pi(\cdot)$ is an *arbitrage relative to* $\rho(\cdot)$ over $[0, T]$, if we have

$$\mathbb{P}\Big(V^{1,\pi}(T) \geq V^{1,\rho}(T)\Big) = 1 \quad \text{and} \quad \mathbb{P}\Big(V^{1,\pi}(T) > V^{1,\rho}(T)\Big) > 0\,.$$

If in fact

$$\mathbb{P}\Big(V^{1,\pi}(T) > V^{1,\rho}(T)\Big) = 1$$

we call such relative arbitrage *strong*.

5. REMARKS

A few remarks are in order:

- Suppose that the covariance process $\alpha(\cdot)$ satisfies the a.s. boundedness condition

$$\xi'\alpha(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \le L\|\xi\|^2, \quad \forall \;\; t \in [0,\infty)\,, \quad \xi \in \mathbb{R}^n \tag{1}$$

for some real number $L > 0$. Then, if $\pi(\cdot)$ is arbitrage relative to $\rho(\cdot)$, and both are bounded portfolios, the processes $Z(\cdot)$ and $Z(\cdot)V^{v,\rho}(\cdot)$ are strict local martingales:

$$\mathbb{E}\Big(Z(T)\Big) < 1\,, \qquad \mathbb{E}\,[\,Z(T)V^{v,\rho}(T)\,] \,<\, v\,.$$

. In particular, if there exists a bounded portfolio $\pi(\cdot)$ which is arbitrage relative to $\mu(\cdot)$, we have

$$\mathbb{E}\Big(Z(T)\Big) < 1\,, \quad \mathbb{E}\,[\,Z(T)X(T)\,] \,<\, X(0)\,, \quad \mathbb{E}\,[\,Z(T)X_i(T)\,] \,<\, X_i(0)\,.$$

- With this same notation, suppose that there exists a real constant $h > 0$ for which the condition

$$\boxed{\sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{i=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t) \geq h\,,} \qquad \forall \;\; 0 \leq t < \infty \tag{2}$$

holds almost surely. The long-only *entropic portfolio*

$$\pi_i(t) = \frac{\mu_i(t)\Big(c - \log \mu_i(t)\Big)}{\sum_{j=1}^{n} \mu_j(t)\Big(c - \log \mu_j(t)\Big)}\,, \qquad i = 1, \cdots, n$$

is then, for a sufficiently large real constant $c > 0$, a strong arbitrage relative to the market portfolio $\mu(\cdot)$ over any time-horizon $[0, T]$ with $T > (2 \log n)/h$.

It is still an open question, whether such relative arbitrage can be constructed over *arbitrary* time-horizons under (2).

Interpretation of the quantity

$$\gamma^*(t) := \sum_{i=1}^{n} \mu_i(t)\alpha_{ii}(t) - \sum_{i=1}^{n}\sum_{i=1}^{n} \mu_i(t)\alpha_{ij}(t)\mu_j(t)$$

on the previous slide, as **Intrinsic Volatility** of the market:

$$\boxed{\gamma^*(t) = \sum_{i=1}^{n} \mu_i(t)\,\mathfrak{A}^{\mu}_{ii}(t)}\,,$$

where $\sigma^{\mu}_k(t) := \sum_{i=1}^{n} \mu_i(t)\sigma_{ik}(t)$ and

$$\mathfrak{A}^{\mu}_{ij}(t) := \sum_{k=1}^{K} \Big(\sigma_{ik}(t) - \sigma^{\mu}_k(t)\Big)\Big(\sigma_{jk}(t) - \sigma^{\mu}_k(t)\Big)\,.$$

Variances of assets relative to the market, averaged according to their individual market weights.

. Posits "sufficient intrinsic volatility" as generatrix of arbitrage.

- Another condition guaranteeing the existence of arbitrage relative to the market, is that there exist a real constant $h > 0$ with

$$\boxed{\Big(\mu_1(t)\cdots\mu_n(t)\Big)^{1/n}\left[\sum_{i=1}^{n}\alpha_{ii}(t)-\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)\right]\ \geq\ h\,,}\quad \forall\ t\geq 0 \tag{3}$$

almost surely; then for a sufficiently large real constant $c > 0$, the long-only *modified equally-weighted portfolio*

$$\pi_i(t)\;=\;\frac{c}{c+Q(t)}\cdot\frac{1}{n}+\frac{Q(t)}{c+Q(t)}\cdot\mu_i(t)\,,\qquad i=1,\cdots,n\,,$$

$Q(t) := (\mu_1(t)\cdots\mu_n(t))^{1/n}$ (convex combination of the equally-weighted and market portfolios) is a strong arbitrage relative to the market over any time-horizon $[0,T]$ with $T > \Big(2n^{1-(1/n)}\Big)/h$.

- Consider now the a.s. strong non-degeneracy condition

$$\xi'\alpha(t)\xi = \xi'\sigma(t)\sigma'(t)\xi \geq \varepsilon\|\xi\|^2, \quad \forall \;\; t \in [0,\infty)\,, \quad \xi \in \mathbf{R}^n \tag{4}$$

for some real number $\varepsilon > 0$, on the covariance process $\alpha(\cdot)$.

If (1), (4) and (2) all hold, then for any given constant $p \in (0,1)$, the long-only portfolio

$$\boxed{\pi_i(t) \;=\; \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}}\,, \qquad i = 1, \cdots, n$$

is again a strong arbitrage relative to the market portfolio over sufficiently long time-horizons.

Appropriate modifications of this (diversity-weighted) portfolio yield such arbitrage over *any* time-horizon $[0,T]$.

- *Please note that all these portfolios (entropic, modified equally-weighted, diversity-weighted) are determined entirely from the market weights* $\mu_1(t), \cdots, \mu_n(t)$ *.*

These are perfectly easy to observe and measure.

Construction of these portfolios does not assume *any* knowledge about the exact structure of market parameters, such as the mean rates of return $\beta_i(\cdot)$'s, or the local covariance rates $\alpha_{ij}(\cdot)$'s.

- The choice of covariance matrix

$$\boxed{\alpha_{ij}(t) \,=\, \frac{1}{\mu_i(t)}\,\delta_{ij}}\ , \qquad 1 \le i,j \le n \tag{5}$$

has been studied under the rubric of "stabilization by volatility".

With this choice and $n \geq 2$, both (2) and (3) hold with $h = n-1$ (the first as equality), so the *entropic portfolio* leads to strong arbitrage relative to the market over any time-horizon of length

$$T > (2 \log n)/(n-1)\,;$$

and the *modified equally-weighted portfolio* leads to strong arbitrage relative to the market over any time-horizon of length

$$T > \Big(2n^{1-(1/n)}\Big)/(n-1)\,.$$

It was shown recently by A.Banner & D.Fernholz (2008) that such arbitrage is then possible over *any* time horizon, through appropriate modifications of the entropic portfolio.

6. QUESTIONS

All of which begs, of course, some fairly obvious questions:

- IF THERE IS ARBITRAGE ON $[0,T]$ RELATIVE TO THE MARKET, WHAT IS THE "BEST" SUCH ARBITRAGE

$$\mathbf{u}(T) := \inf\Big\{v > 0 \mid \exists\, \pi(\cdot) \in \mathcal{H} \ \text{ s.t. } V^{\,vX(0),\pi}(T) \geq X(T), \ \text{a.s.}\Big\} \quad ?$$

- On a given time-horizon $[0,T]$, what is the maximal return

$$\mathbf{w}(T) := \sup\Big\{w > 0 \mid \exists\, \pi(\cdot) \in \mathcal{H} \ \text{ s.t. } V^{\,X(0),\pi}(T) \geq w\, X(T), \ \text{a.s.}\Big\}\ ,$$

that can be achieved relative to the market ?

- For a given level $w \geq 1$, what is the shortest length of time

$$\mathbf{T}(w) := \inf\Big\{T > 0 \mid \exists\, \pi(\cdot) \in \mathcal{H} \ \text{ s.t. } V^{\,X(0),\pi}(T) \geq w\, X(T), \ \text{a.s.}\Big\}$$

required to guarantee the existence of an investment strategy with return of at least w times the market ?

Turns out, if you can answer the first question, you can answer the other two as well:

$$\mathbf{w}(T) = 1/\mathbf{u}(T)\,; \qquad \mathbf{T}(w) = \inf\{T > 0 \mid \mathbf{u}(T) \leq 1/w\}\,.$$

And the answer to the first question is

$$\underbrace{\mathbf{u}(T) \;=\; \frac{1}{X(0)} \cdot \mathbb{E}\Big[\, Z(T)X(T) \,\Big]}\ \in (0,1]\,.$$

CAN THIS BE COMPUTED MORE EXPLICITLY ?

WHAT ABOUT THE STRATEGY $\widehat{\pi}(\cdot) \in \mathcal{H}$ THAT ATTAINS

$$V^{\,\mathbf{u}(T)X(0),\,\widehat{\pi}}(T) \;=\; X(T), \quad \text{a.s.} \quad ?$$

7. MARKOVIAN MODEL:

$\beta_i(t) = \mathsf{b}_i(\mathfrak{X}(t))$, and $\sigma_{ik}(t) = \mathsf{s}_{ik}(\mathfrak{X}(t))$, $1 \le i,k \le n$, $0 \le t < \infty$.

Here $\mathfrak{X}(t) = (X_1(t), \cdots, X_n(t))'$ is the vector of stock prices at time t, and $\mathsf{b}_i : (0,\infty)^n \to \mathbb{R}$, $\mathsf{s}_{ik} : (0,\infty)^n \to \mathbb{R}$ suitable (Hölder) continuous functions.

We shall denote by $\mathsf{b}(\cdot) = (\mathsf{b}_1(\cdot), \cdots, \mathsf{b}_n(\cdot))'$, $\mathsf{s}(\cdot) = (\mathsf{s}_{ik}(\cdot))_{1 \le i,k \le n}$ the vector and matrix, respectively, of these local rate and volatility functions, and the covariance rate structure by

$$\mathsf{a}_{ij}(\mathbf{x}) := \sum_{k=1}^{n} \mathsf{s}_{ik}(\mathbf{x})\, \mathsf{s}_{jk}(\mathbf{x}) \qquad \text{for} \quad 1 \le i,j \le n\,.$$

Let $\mathbb{P}^{\mathbf{x}}$ stand for the distribution of this process, and note

$$\mathbf{u}(T) \equiv U(T,\mathbf{x}) = \frac{1}{x_1 + \cdots + x_n} \cdot \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\Big[\, Z(T) X(T) \Big]\,.$$

- With this structure, the process $\mathfrak{X}(\cdot) = (X_1(\cdot), \cdots, X_n(\cdot))'$ becomes a diffusion with values in $(0,\infty)^n$, initial configuration $\mathfrak{X}(0) = \mathbf{x} = (x_1, \cdots, x_n)'$, and infinitesimal generator

$$\boxed{\mathcal{L}f := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, x_j\, \mathsf{a}_{ij}(\mathbf{x})\, D^2_{ij}f + \sum_{i=1}^{n} x_i\, \mathsf{b}_i(\mathbf{x})\, D_i f}\,.$$

- AUXILIARY PROCESS $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \cdots, Y_n(\cdot))'$: a diffusion with values in $[0,\infty)^n \setminus \{\mathbf{0}\}$, initial configuration $\mathfrak{Y}(0) = \mathbf{x} = (x_1, \cdots, x_n)' \in (0,\infty)^n$, and infinitesimal generator

$$\widehat{\mathcal{L}}f := \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, x_j\, \mathsf{a}_{ij}(\mathbf{x})\, D^2_{ij}f + \sum_{i=1}^{n} x_i\, \underbrace{\left(\sum_{j=1}^{n} \frac{x_j\, \mathsf{a}_{ij}(\mathbf{x})}{x_1 + \cdots + x_n}\right)}\, D_i f\,.$$

Denote by $\mathbb{Q}^{\mathbf{x}}$ the distribution of this process.

The auxiliary diffusion process has Markovian dynamics

$$dY_i(t) \,=\, Y_i(t) \, \left[\widehat{\mathsf{b}}_i\big(\mathfrak{Y}(t)\big)\, dt \,+\, \sum_{k=1}^{n} \mathsf{s}_{ik}\big(\mathfrak{Y}(t)\big)\, dW_k(t) \right]$$

with drifts

$$\boxed{\widehat{\mathsf{b}}_i(\mathbf{x}) \,=\, \sum_{j=1}^{n} \left(\frac{x_j}{x_1 + \cdots + x_n} \right) \mathsf{a}_{ij}(\mathbf{x})} \,, \qquad i = 1, \cdots, n\,.$$

These are determined *entirely* from the local covariance structure. The resulting process has state-space $[0, \infty)^n \setminus \{\mathbf{0}\}$; with initial configuration $\mathfrak{Y}(0) = \mathbf{x} = (x_1, \cdots, x_n)' \in (0, \infty)^n$, consider the first time one of its components becomes zero:

$$\boxed{\mathcal{T} \,:=\, \inf \Big\{ t \geq 0 \mid \mathfrak{Y}(t) \notin (0, \infty)^n \Big\}} \;.$$

8. FINANCIAL SIGNIFICANCE:

Consider the relative weights

$$\nu_i(t) := Y_i(t)/Y(t)\,, \qquad Y(t) := Y_1(t) + \cdots + Y_n(t)\,, \quad i = 1, \cdots, n$$

of the "fictitious market" with stock-prices $\mathfrak{Y}(\cdot) = (Y_1(\cdot), \cdots, Y_n(\cdot))'$; to wit, local covariance rates $\mathsf{a}_{ij}(\mathbf{x})$ the same as in the actual market, and local appreciation rates

$$\underbrace{\widehat{\mathsf{b}}_i(\mathbf{x}) = \sum_{j=1}^{n} \left(\frac{x_j}{x_1 + \cdots + x_n} \right) \mathsf{a}_{ij}(\mathbf{x})}\,, \qquad i = 1, \cdots, n\,.$$

These weights $\underline{\nu}(\cdot) = (\nu_1(\cdot), \cdots, \nu_n(\cdot))'$ are martingales:

$$\frac{d\nu_i(t)}{\nu_i(t)} = \sum_{k=1}^{n} \left(\mathsf{s}_{ik}\Big(\mathfrak{Y}(t)\Big) - \sum_{j=1}^{n} \nu_j(t)\, \mathsf{s}_{jk}\Big(\mathfrak{Y}(t)\Big) \right) dW_k(t)$$

$$= \Big(e_i - \nu(t)\Big)' \mathsf{s}\Big(\mathfrak{Y}(t)\Big)\, dW(t)\,.$$

. *A change of drift that bestows the "Numéraire Property" to this 'fictitious-market' portfolio*:

$$\boxed{\frac{\mathcal{V}^{1,\pi}(\cdot)}{\mathcal{V}^{1,\nu}(\cdot)} \quad \text{is a } \mathbb{Q}^{\mathbf{x}} - \text{supermartingale}}$$

for any trading strategy $\pi(\cdot)$.

9. ANALYTICAL SIGNIFICANCE: The function $U(T, \mathbf{x})$ solves the *Cauchy Problem* (CP): $U(0+, \cdot\,) \equiv 1$ and

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) = \widehat{\mathcal{L}}\, U(\tau, \mathbf{x})\,, \quad (\tau, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n\,.$$

¶ The existence of arbitrage relative to the market, for instance

$$U(T, \mathbf{x}) < 1\,, \qquad \forall \;\; \mathbf{x} \in (0, \infty)^n \tag{6}$$

for some $T \in (0, \infty)$, amounts to *lack of uniqueness* for CP.

♠ And if for every compact subset K of $(0, \infty)^n$ there exists an $\varepsilon = \varepsilon_K > 0$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i\, x_j\, \mathsf{a}_{ij}(\mathbf{x})\, \xi_i\, \xi_j \geq \varepsilon \|\xi\|^2\,, \qquad \forall \;\; \mathbf{x} \in K\,, \quad \xi \in \mathbb{R}^n\,,$$

then (6) above holds *for all* $T \in (0, \infty)$. *(Maximum Principle.)*

¶ Sufficient condition for such lack of uniqueness: there exists a real constant $h > 0$, such that for all $\mathbf{x} \in (0, \infty)^n$ we have

$$\boxed{(x_1 + \cdots + x_n) \sum_{i=1}^{n} x_i \mathsf{a}_{ii}(\mathbf{x}) - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \mathsf{a}_{ij}(\mathbf{x}) \geq h\,(x_1 + \cdots + x_n)^2}$$

(reverse-engineered from condition (2).

♣ Another sufficient condition: there exists a real constant $h > 0$, such that for all $\mathbf{x} \in (0, \infty)^n$ we have

$$\boxed{\Big(x_1 \cdots x_n\Big)^{1/n} \left[\sum_{i=1}^{n} \mathsf{a}_{ii}(\mathbf{x}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathsf{a}_{ij}(\mathbf{x}) \right] \geq h\,(x_1 + \cdots + x_n)}$$

(reverse-engineered from condition (3).

Characterization: The function $U(T, \mathbf{x})$ is the $\mathbf{smallest}$ non-negative solution of the parabolic differential inequality

$$\frac{\partial U}{\partial \tau}(\tau, \mathbf{x}) \geq \widehat{\mathcal{L}}\, U(\tau, \mathbf{x})\,, \qquad (\tau, \mathbf{x}) \in (0, \infty) \times (0, \infty)^n$$

subject to

$$U(0+, \cdot\,) \equiv 1\,.$$

- Strategy that realizes this "optimal arbitrage" is *functionally generated:*

$$\underbrace{\widehat{\pi}_i(t) \,=\, X_i(t)\, D_i \log U\Big(T - t, \mathbf{X}(t)\Big) \,+\, \frac{X_i(t)}{X(t)}}\,, \qquad i = 1, \cdots, n\,.$$

- Everything is determined *entirely* form the covariance structure of the market..... The rest is just observables.

10. PROBABILISTIC SIGNIFICANCE: The function $U(T, \mathbf{x})$ equals

$$\boxed{U(T, \mathbf{x}) \,=\, \mathbb{Q}^{\mathbf{x}}\big(\mathcal{T} > T\big)}\,,$$

the probability that the diffusion process $\mathfrak{Y}(\cdot)$ with initial configuration $\mathfrak{Y}(0) = \mathbf{x}$ does not hit the boundary of the positive orthant $(0, \infty)^n$ before time T.

¶ This allows a representation of Föllmer's "exit measure" (also Delbaen-Schachermayer, Pal-Protter)

$$\mathfrak{P}^{\mathbf{x}}\Big((T,\infty] \times A\Big) \,=\, \frac{1}{X(0)}\,\mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\Big[\,Z(T)X(T) \cdot 1_A\Big]\,, \qquad A \in \mathcal{F}(T),\;\; T \in [0, \infty)$$

for the supermartingale $Z(\cdot)X(\cdot)$ in this case. In particular,

$$\mathfrak{P}^{\mathbf{x}}\Big((T,\infty] \times \Omega\Big) \,=\, U(T, \mathbf{x}) \,=\, \mathbb{Q}^{\mathbf{x}}\Big(\,\mathcal{T} > T\,\Big)\,.$$

More specifically: the $\mathbb{P}^{\mathbf{x}}-$supermartingale $Z(\cdot)X(\cdot)$ is a $\mathbb{P}^{\mathbf{x}}-$

- *martingale,* if and only if

$$\mathbb{Q}^{\mathbf{x}}\Big(\mathcal{T} = \infty\Big) = 1$$

(no relative arbitrage is possible, on any time-horizon, if $Y(\cdot)$ never hits the boundary of the positive orthant);

- *strict local* (and super)*martingale* on $[0, T]$, if and only if

$$\mathbb{Q}^{\mathbf{x}}\Big(\mathcal{T} \le T\Big) > 0$$

(arbitrage relative to the market *is* possible on $[0, T]$, if and only if $\mathfrak{Y}(\cdot)$ hits the boundary of the positive orthant with positive probability by time T);

- *potential* (*i.e.*, $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\mathbf{x}}}\Big(Z(T)X(T)\Big) = 0$), if and only if $\mathbb{Q}^{\mathbf{x}}\Big(\mathcal{T} < \infty\Big) = 1$.

Here is an example where not only $\mathbb{Q}^{\mathbf{x}}\big(\mathcal{T} < \infty\big) = 1$, but in fact

$$\mathbb{E}^{\mathbb{Q}^{\mathbf{x}}}\big(\mathcal{T}\big) < \infty\,.$$

11. Example: STABILIZATION BY VOLATILITY.

$$d\,(\log X_i(t)) \;=\; \frac{\zeta}{2\mu_i(t)}\,dt \;+\; \frac{dW_i(t)}{\sqrt{\mu_i(t)}}\;, \qquad i = 1, \cdots, n$$

for some $\zeta \in [0,1]$, $n \geq 2$, or equivalently

$$dX_i(t) \;=\; \frac{1+\zeta}{2}\Big(X_1(t) + \cdots + X_n(t)\Big)\,dt$$

$$+\;\sqrt{X_i(t)\,\Big(X_1(t) + \cdots + X_n(t)\Big)}\;dW_i(t)\,.$$

This system of SDE's admits a unique in distribution weak solution, with state-space $(0,\infty)^n$. In this case

$$\mathsf{b}_i(\mathbf{x}) = \frac{1+\zeta}{2}\left(\frac{x_1+\cdots+x_n}{x_i}\right), \qquad \mathsf{a}_{ij}(\mathbf{x}) = \delta_{ij}\left(\frac{x_1+\cdots+x_n}{x_i}\right),$$

In addition, $\widehat{\mathsf{b}}_i(\mathbf{x}) \equiv 1$, so the auxiliary process $\mathbb{Y}(\cdot)$ satisfies

$$\underline{dY_i(t) \,=\, Y_i(t)\,dt + \sqrt{Y_i(t)\Big(Y_1(t)+\cdots+Y_n(t)\Big)}\,dW_i(t)}\;.$$

This system of SDE's also admits a unique (in distribution) weak solution, but with state-space $[0,\infty)^n \setminus \{\mathbf{0}\}$ (Bass & Perkins (2002), Dawson & Perkins (2006)). Weights have dynamics

$$d\nu_i(t) \,=\, \sqrt{\nu_i(t)}\,dW_i(t) \,-\, \nu_i(t)\sum_{k=1}^{n}\sqrt{\nu_k(t)}\,dW_k(t)$$

$$=\, \sqrt{\nu_i(t)(1-\nu_i(t))}\,dW_i^{\sharp}(t), \qquad i=1,\cdots,n.$$

Both conditions (2), (3) are satisfied with $h = n - 1$, the first in fact as equality. From what we have already seen: `relative arbitrage exists over sufficiently long time-horizons.`

Each $\nu_i(\cdot)$ hits eventually one of the boundary points of the unit interval, with probability one (in fact, the time it takes for this to happen, has finite expectation). But this means that, eventually, all but one of the $Y_i(\cdot)$'s die out, that is, are absorbed at zero, and only one of them remains positive. From that time onwards, their sum $Y(\cdot)$ behaves like a geometric Brownian motion with drift, to wit $dY(t) = Y(t)(dt + dW(t))$, and the diffusion $\mathfrak{Y}(\cdot)$ never hits the origin of the positive orthant.

It develops from all this, that $Z(\cdot)X(\cdot)$ is a potential: `relative arbitrage exists over` *any* `time-horizon of positive length.`

12. KNIGHTIAN UNCERTAINTY: Suppose that at any given time we only know that the covariance matrix satisfies

$$\alpha(t) \in \mathcal{A}(\mathfrak{X}(t))\,, \quad \text{for some} \quad \mathbb{A} = \{\mathcal{A}(\mathbf{y})\}_{\mathbf{y} \in [0,\infty)^n \setminus \{\mathbf{0}\}}$$

a given family of sets.

Under suitable conditions on this family, the function $U(T, \mathbf{x})$ – that describes the optimal arbitrage w.r.t. any possible scenario – satisfies then a *fully non-linear* partial differential equation of parabolic (Hamilton-Jacobi-Bellman) type: The maximal probability that a diffusion process of the type $\mathfrak{Y}(\cdot)$, with $\mathfrak{Y}(0) = \mathbf{x}$ and covariance structure which obeys this constraint, does not hit the boundary of the positive orthant by time T.

Project in active development....

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