Algebra Prelim Written Exam *Fall 2007*

Questions are equally weighted. Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems silly.

Write your **codename***, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.*

[1] Let G be a group of order 105. Suppose that G acts *transitively* on a set X. What are the possible cardinalities of the set X?

[2] Show that a group of order 15 is necessarily cyclic.

[3] Show that $x^5 - 12x + 6$ is irreducible in $\mathbb{Q}[x]$.

[4] Let S, T be k-linear endomorphisms of a finite-dimensional vector space V over an algebraically closed field k. Suppose that at $ST = TS$. Show that S and T have a *simultaneous* eigenvector.

[5] Show that the ideal in $\mathbb{Z}[x]$ generated by $x^2 + 1$ and 11 is *maximal.*

[6] Let ζ be a primitive 9^{th} root of unity inside an algebraic closure of $\mathbb{Q}$. *Determine the intermediate fields* between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$ in the following sense: for each intermediate field k between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$, determine an irreducible polynomial $f(x)$ with rational coefficients such that $k = \mathbb{Q}(\alpha)$, where $f(\alpha) = 0$.

[7] Find a polynomial condition on the parameter $a \in k$ to guarantee that the equation $x^5 - 5ax + 1 = 0$ has *distinct roots* in an algebraically closed field k of characteristic 0.

[8] Let k be a finite field, and K a finite extension. Show that the Galois norm map $K \longrightarrow k$ is surjective.
