Algebra Prelim Written Exam *Fall 2008*

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your **codename***, not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.*

[1] Exhibit four mutually non-isomorphic groups of order 8, and prove they are not isomorphic.

[2] Prove that if a polynomial f in $k[x]$ with a field k has a repeated irreducible factor g in $k[x]$, then g divides the greatest common divisor of f and its derivative. Be sure to explain what *derivative* can mean without limits.

[3] Let S, T be $\mathbb{C}$-linear operators on a finite-dimensional complex vector space V, with $ST = TS$. Show that S, T have a common eigenvector.

[4] Let $K = \mathbb{Q}(\omega)$ be a field generated over $\mathbb{Q}$ by a primitive 21^{st} root of unity ω. Describe all intermediate fields between K and $\mathbb{Q}$ (by giving monic irreducibles in $\mathbb{Q}[x]$ whose roots generate the extensions).

[5] Given a 3-by-3 matrix M with integer entries, show that there exist integer-entry 3-by-3 matrices A, B, with integer-entry *inverses*, such that AMB is *diagonal*.

[6] Exhibit an irreducible polynomial of degree 5 over a field with 5 elements.