Algebra Prelim Written Exam *Spring 2008*

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your **codename**, *not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.*

[**1**] Show that a group of order $3 \cdot 5 \cdot 17$ has a *normal* subgroup of order 17.

[**2**] Given distinct primes p, q, r, show that there are integers a, b, c such that

$$\frac{1}{pqr} = \frac{a}{p} + \frac{b}{q} + \frac{c}{r}$$

[**3**] Give an example (with proof) of a torsion-free $\mathbb{Z}$-module that is not free.

[**4**] Let T be a complex n-by-n matrix so that $T^* = T$, where $*$ is conjugate-transpose. Show that there is an n-by-n matrix U with $U^*U = 1$ such that U^*TU is *diagonal*.

[**5**] Show that for $a \neq 0$ the polynomial $x^p - x + a$ is irreducible in $\mathbb{F}_p[x]$, where $\mathbb{F}_p$ is the finite field with p elements, p a prime.

[**6**] With ζ a primitive 12^{th} root of unity, describe each field intermediate between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$ as $\mathbb{Q}(\alpha)$ with α a root of an explicit irreducible monic polynomial in $\mathbb{Q}[x]$.

[**7**] Find the determinant of the *circulant* matrix

$$\begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_3 & a_4 & \dots & a_1 & a_2 \\ a_2 & a_3 & \dots & a_n & a_1 \end{pmatrix}$$

Hint: Let ζ be an n^{th} root of 1. Note that the determinant vanishes when $a_{i+1} = \zeta \cdot a_i$ for all $i = 1, \dots, n-1$.

[**8**] Show that $x^5 + y^5 + z^5$ is irreducible in $\mathbb{C}[x, y, z]$.