Algebra Prelim Written Exam *Spring 2009*

Questions are equally weighted. Give essential explanations and justifications: a large part of each question is demonstration that you understand the context and understand which issues are primary. Do not choose assumptions or contexts making the problems silly. Coherent writing is essential: your paper should not be a puzzle for the grader.

Write your **codename**, *not actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, bluetooth, or other communication devices may be used during the exam.*

[1] Classify *abelian* groups of order 48.

[2] Let k be a field, x an indeterminate. For distinct $a_1, \ldots, a_n$ in a field k, prove that there are unique $A_1, \ldots, A_n \in k$ such that in the rational function field $k(x)$

$$\frac{1}{(x-a_1)(x-a_2)\ldots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}$$

[3] Let V be a finite-dimensional complex vector space. Prove that a *commutative* ring R of complex-linear maps of V to itself has a common eigenvector.

[4] Count the 3-dimensional subspaces of a 7-dimensional vector space over a finite field with p elements.

[5] Show that the ideal generated by 13 and $x^3 - 2$ in $\mathbb{Z}[x]$ is *maximal*.

[6] Let R be a commutative ring with unit. Show that the collection of all *nilpotent* elements of R is an ideal.

[7] Let ζ be a primitive eighth root of unity. Find all intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$.

[8] Show that $x^5 - 2$ is irreducible in $\mathbb{F}_{11}[x]$.