

# Complex Analysis Preliminary Exam

August 28, 2008

Write your **codename**, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems silly. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

All problems are worth 10 points. The points allocated to parts **a** and **b** are indicated.

The problems are divided into Parts I and II. Please use separate bluebooks for the two parts. Clearly label which problems are in which bluebooks.

## Part I

1. Let  $P(z) = z^7 + z^3 + \frac{1}{16}$ .
  - (5) **a.** Show that  $P$  has no multiple roots.
  - (5) **b.** Determine the number of zeros of  $P$  that lie in the closed disc  $\{z \in \mathbb{C} : |z| \leq 1/2\}$ .
  
2. Find a conformal map from the open upper half of the unit disc to the entire open unit disc.
  
3. Show that, if  $f$  is analytic and one-to-one on an open neighborhood of  $z = 0$  and if  $f(0) = 0$ , then  $f'(0) \neq 0$ .
  
4. Suppose that  $u(x, y)$  is harmonic.
  - (2) **a.** If  $v$  is a harmonic conjugate of  $u$ , show that  $uv$  is harmonic.
  - (8) **b.** Show that, if  $xu(x, y)$  is harmonic, then  $u(x, y) = ay + b$ , where  $a$  and  $b$  are constants.

## Part II

5. Evaluate  $\int_0^\pi \frac{a d\theta}{a^2 + \sin^2 \theta}$ .

6. Let  $F$  be a holomorphic function on the punctured open unit disc:  $\{z \in \mathbb{C} : 0 < |z| < 1\}$ , and set

$$g(r) = \int_0^{2\pi} F(re^{i\theta}) d\theta, \quad \text{for } 0 < r < 1.$$

Show that  $g$  is independent of  $r$ .

7. Recall the following definition. If  $f(z)$  is doubly periodic meromorphic function on  $\mathbb{C}$ , then its *order* is the number of poles (counted with multiplicity) in a fundamental domain. Recall also that, if  $f$  is doubly periodic, then so is  $f'$ .

Assume that  $f$  is doubly periodic of order  $m$  and that  $f'$  is doubly periodic of order  $n$ . Show that

$$m + 1 \leq n \leq 2m.$$

8. Suppose that  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  is entire. For  $r > 0$ , let  $j(r) = j$  be the unique index such that

$$\begin{aligned} |c_j| r^j &\geq |c_k| r^k, & \text{for all } k, \text{ and} \\ |c_j| r^j &> |c_k| r^k, & \text{for } k > j. \end{aligned}$$

Show that  $j$  is a nondecreasing function of  $r$ .