

Complex Analysis Preliminary Exam

April 22, 2009

Write your **codename**, not your actual name, on each booklet. No notes, books, calculators, computers, cell phones, wireless, Bluetooth, or other communication devices may be used during the exam.

Give the essential explanations and justifications: a large part of each question is demonstration that you understand the context, and understand which issues are important. Do not make assumptions or choose contexts which make the problems trivial. If you use a theorem, state it fully and concisely, or identify it clearly. To receive full credit for a problem, the answer must be complete and correct. The scorers are not expected to supply any missing parts of any answer.

All problems have equal weight. For each problem, the parts have equal weights.

The problems are divided into Parts I and II. Please use separate bluebooks for the two parts. Clearly label which problems are in which bluebooks.

Part I

1. Suppose that f is entire that that $\frac{f(z)}{1+|z|^{1/2}}$ is bounded as $z \rightarrow \infty$. Prove that f is constant.
2. Suppose that f is a meromorphic function on $\{z: |z| \leq 2\}$ with double zeros at $1+i$ and $1+2i$, a pole at $\frac{1}{2}-i$ with residue equal to $2i$, and a simple pole at 1 with residue $2-i$. Compute the complex line integral

$$\oint_{\gamma} z \frac{f'(z)}{f(z)} dz,$$

where $\gamma = \left\{ z \in \mathbb{C} : |z| = \frac{3}{2} \right\}$ and where γ is traversed counterclockwise.

3. Find an explicit conformal equivalence which maps the open set bounded by $|z - \frac{1}{2}i| = \frac{1}{2}$ and $|z - i| = 1$ onto the open upper half plane.
4. How many zeros does the function $f(z) = z^6 + 4z^2 e^{z+1} - 3$ have in the unit disk $D(0,1)$?

Part II

5. Let D be a bounded open connected subset of \mathbb{C} , and let h be a continuous real-valued function on ∂D . Recall the Perron family for D and h ,

$$\mathcal{P}(D, h) = \left\{ u : u \text{ is subharmonic on } D, \limsup_{\substack{z \rightarrow \zeta \\ z \in D}} u(z) \leq h(\zeta), \forall \zeta \in \partial D \right\},$$

and the Perron function,

$$P_{D,h}(z) = \sup \{ u(z) : u \in \mathcal{P}(D, h) \}.$$

Find the Perron function when D is the annulus $1 < |z| < 2$ and h is the function

$$h(\zeta) = \begin{cases} 12 & \text{for } |\zeta| = 1, \\ 17 & \text{for } |\zeta| = 2. \end{cases}$$

6. Let L be a lattice in \mathbb{C} .
- Can an even doubly periodic function with respect to L of order 2 have two distinct simple poles mod L ? If so, construct one. If not, give a proof that such functions do not exist.
 - Can an even doubly periodic function with respect to L have odd order? If so, construct one. If not, give a proof that such functions do not exist.
7. (Use only definitions.) Let D be an open disk.
- Prove that any analytic function $f(z)$ on D has a primitive $F(z)$ on D . (I.e., $F'(z) = f(z)$.)
 - Suppose that f is analytic and nonvanishing on D . Prove that a branch of $\log f(z)$ is defined and analytic on D .
8. Suppose that $p_k(z)$ is a polynomial of degree k satisfying $p_k(0) = 1$ and that $p_k(z)$ has no zeros in the disk $\{z : |z| \leq k^3\}$. Prove that $\prod_{k=1}^{\infty} p_k(z)$ converges normally on \mathbb{C} .